CAHIERS DE TOPOLOGIE ET GEOMETRIE DIFFERENTIELLE CATEGORIQUES *Vol. LI-1 (2010)*

ON QUASI-EQUATIONS IN LOCALLY PRESENTABLE CATEGORIES II: A LOGIC

by Jirí ADÁMEK and Lurdes SOUSA

Dedicated to Francis Borceux on the occasion of his sixtieth birthday

Résumé

Les quasi-équations, données par des paires parallèles de morphismes finitaires, représentent des propriétés des objets: un objet satisfait la propriété si son foncteur hom contravariant fusionne les morphismes de la paire. Récemment Adámek et Hébert ont caractérisé les sous-catégories des catégories localement de présentation finie spécifiées par des quasi-équations. Nous présentons ici une logique de quasi-équations proche de la logique classique équationnelle de Birkhoff. Nous prouvons qu'elle est consistante et complète dans toute catégorie localement présentation finie avec relations d'équivalence effectives.

Abstract

Quasi-equations, given by parallel pairs of finitary morphisms, represent properties of objects: an object satisfies the property if its contravariant hom-functor merges the parallel pair. Recently Adámek and Hébert characterized subcategories of locally finitely presentable categories specified by quasi-equations. We now present a logic of quasi-equations close to Birkhoff's classical equational logic. We prove that it is sound and complete in all locally finitely presentable categories with effective equivalence relations.

Mathematics Subject Classification: 18A15, 08B05, 18C35, 18E10.

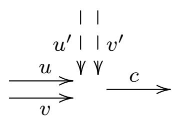
Key words: quasi-equations, finitary morphisms, locally finitely presentable categories, exact categories, equational logic, logic of quasi-equations.

The authors acknowledge partial financial support from the Center of Mathematics of the University of Coimbra/FCT.

1 Introduction

It was Bill Hatcher who first considered a representation of properties of objects via a parallel pair $u, v : R \to X$ of morphisms in the sense that an object A has the property iff every morphism $f : X \to A$ fulfils $f \cdot u = f \cdot v$, see [11]. Later Bernhard Banaschewski and Horst Herrlich [5] considered the related concept of injectivity w.r.t. a regular epimorphism $c : X \to Y$: this is just the step from parallel pairs to their coequalizers. For regular epimorphisms which are finitary, that is, have finitely presentable domain and codomain, Banaschewski and Herrlich [5] characterized full subcategories of "suitable" categories which can be specified by such injectivity: they are precisely the subcategories closed under products, subobjects, and filtered colimits. Recently the same result was proved for all locally finitely presentable categories, see [2], where parallel pairs of morphisms u, v with finitely presentable domain and codomain are called *quasi-equations*. Notation: $u \equiv v$.

In the present paper we introduce a logic of quasi-equations: for every set Q of quasi-equations we characterize its *consequences*, that is, quasi-equations $u \equiv v$ which hold in every object satisfying every quasi-equation in Q. In fact, we introduce two logics. The first one is sound and complete in every locally finitely presentable category. Moreover, this logic is extremely simple: it states that (1) $u \equiv u$ always holds, (2) if $u \equiv v$ holds, then also $q \cdot u \equiv q \cdot v$ holds, and (3) if $u \equiv v$ holds and c is a coequalizer of u and v



then for all pairs with $c \cdot u' = c \cdot v'$ we have that $u' \equiv v'$ holds. However this last rule makes the logic disputable in applications. Think of Birkhoff's Equational Logic in the category $\mathbf{Alg}\,\Sigma$: its aim is to describe the fully invariant congruence generated by (u, v), whereas the coequalizer rule takes the congruence that (u, v) generates for granted.

We therefore present our main logic, called the Quasi-Equational Logic, without the coequalizer rule. Instead, we work with the parallel pairs alone. This logic is a bit more involved than (1)-(3) above, but is much nearer to

Birkhoff's classical result [7]. We prove its completeness in

(i) every locally finitely presentable category with effective equivalence relations

and

(ii) in $\mathbf{Mod}\,\Sigma$, the category of Σ-structures for every (many-sorted) first-order signature.

However, we also present an example of a regular, locally finitely presentable category in which the Quasi-Equational Logic is not complete.

Related Work Satisfaction of a quasi-equation $u \equiv v$ is equivalent to injectivity w.r.t. the coequalizer of u and v. Our simple logic is just a translation of the injectivity logic w.r.t. epimorphisms presented in [4]. The full logic we introduce below is based on a description of the kernel pairs which for regular, locally finitely presentable categories was presented by Pierre Grillet [10], and the generalization to all locally finitely presentable categories we use stems from [1].

Acknowledgements The authors are grateful to the referees for a number of useful suggestions.

2 The Coequalizer Logic

Here we present a (surprisingly simple) deduction system for quasi-equations which is sound and complete in all locally finitely presentable categories. Its only disadvantage is that it uses the concept of coequalizer, and this makes the usufelness in applications a bit questionable.

Throughout the paper we assume that a locally finitely presentable category is given, see [9] or [3].

2.1. Definition A *finitary morphism* is one whose domain and codomain are finitely presentable objects. A *quasi-equation* is a parallel pair of finitary morphisms $u, v : R \to X$. We use the notation $u \equiv v$. An object A *satisfies* $u \equiv v$ if $f \cdot u \equiv f \cdot v$ holds for all $f : X \to A$. A quasi-equation $u \equiv v$ is said to be a *consequence* of a set Q of quasi-equations, written $Q \models u \equiv v$, if every object satisfying all members of Q also satisfies $u \equiv v$.

2.2. Observation Let the diagram

$$\begin{array}{ccccc} & & R' & & \\ & & {\scriptstyle u'}\downarrow\downarrow{\scriptstyle v'} & & \\ R & \overset{u}{\underset{v}{\rightrightarrows}} & X & \xrightarrow{\ c\ } & C \end{array}$$

be such that we have

$$c \cdot u' = c \cdot v' \quad \text{and} \quad c = \text{coeq}(u, v).$$

Then the quasi-equation $u' \equiv v'$ is a consequence of $u \equiv v$. In fact, if A satisfies $u \equiv v$ then for every $f : X \to A$ we see that f factors through c, consequently, $f \cdot u' \equiv f \cdot v'$.

This suggests the following

2.3. Definition The *Coequalizer Logic* uses the following deduction rules:

Reflexivity:
$$\frac{}{u \equiv u}$$

Left Composition:
$$\frac{u \equiv v}{q \cdot u \equiv q \cdot v} \qquad \text{given} \ \overset{u}{\underset{v}{\rightrightarrows}} \xrightarrow{\ q\ }$$

Coequalizer:
$$\frac{u \equiv v \quad c \cdot u' = c \cdot v'}{u' \equiv v'} \qquad \text{for } c = \text{coeq}(u, v)$$

2.4. Remark (i) The Coequalizer Deduction System is obviously sound: whenever we can prove a quasi-equation $u \equiv v$ from a given set Q by using the above three deduction rules, it follows that $u \equiv v$ is a consequence of Q.

(ii) We will prove the completeness of the above deduction system by reducing it to the completeness of the logic presented by Manuela Sobral and the authors in [4]. That logic concerned injectivity w.r.t. finitary epimorphisms $e : X \to Y$. Recall that an object A is injective w.r.t. e if every morphism from X to A factors through e. We say that e is an *injectivity consequence* of a set $\mathcal{E}$ of finitary epimorphisms provided that every object injective w.r.t. members of $\mathcal{E}$ is also injective w.r.t. e. We formulated the following logic of injectivity consisting of one axiom and three deduction rules (where e and e' are finitary epimorphisms):

(A) $\quad \dfrac{}{\mathrm{id}_X}$ $\quad$ for finitely presentable objects X

(P) $\quad \dfrac{e}{e'}$ $\quad$ for every pushout $\begin{array}{ccc} \cdot & \xrightarrow{e} & \cdot \\ \downarrow & & \downarrow \\ \cdot & \xrightarrow{e'} & \cdot \end{array}$

(C) $\quad \dfrac{e \quad e'}{e \cdot e'}$ $\quad$ given $\xrightarrow{e'} \xrightarrow{e}$

(L) $\quad \dfrac{e \cdot e'}{e'}$

And we proved that this represents a sound and complete injectivity logic in every locally finitely presentable category. That is, given a set Q of finitary epimorphisms, then the injectivity consequences e of Q are precisely those which have a (finite) proof applying the above axiom and deduction rules to members of Q.

(iii) Before proceeding with our logic of quasi-equations, we observe an unexpected property of proofs based on the rules above: Let Q be a set of finitary epimorphisms containing all finitary identity morphisms. Then for every injectivity consequence e of Q there exists a proof of the following form

$$\left\{\begin{array}{l} e_1 \\ \vdots \\ e_{k_1} \end{array}\right.$$

$$\text{(P)} \left\{\begin{array}{l} e_{k_1+1} \\ \vdots \\ e_{k_2} \end{array}\right.$$

$$\text{(C)} \begin{cases} e_{k_2+1} \\ \vdots \\ e_{k_3} \end{cases}$$

$$\text{(L)} \begin{cases} e_{k_3+1} \\ \vdots \\ e_{k_4} = e \end{cases}$$

whose first part consists of elements of Q, the second part uses only (P), the third one only (C), and the last one only (L). This follows from the next lemma in which we put

$Q_C = \{e_1 \cdot e_2 \ldots e_k;\ e_i \in Q\}$ (the closure under (C))
$Q_L = \{e';\ e \cdot e' \in Q \text{ for some } e\}$ (the closure under (L))

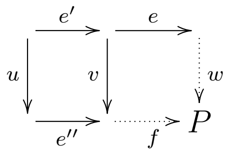
and

$Q_P = \{e;\ e \text{ finitary and opposite to a member of } Q \text{ in a pushout}\}$
(the closure under (P))

2.5. Lemma *Let Q be a set of finitary epimorphisms containing all* id_X, *X finitely presentable. Then $((Q_P)_C)_L$ is closed under pushout, composition and left cancellation.*

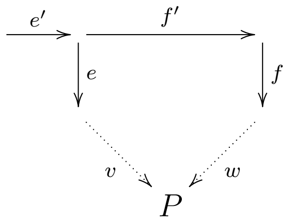
Proof Observe that $(Q_P)_C$ is closed under pushout (and composition) since a pushout of a composite is the composite of pushouts.

To prove the statement, let us first prove that $((Q_P)_C)_L$ is closed under pushout: Given $e' \in ((Q_P)_C)_L$, there exists e finitary such that $ee' \in (Q_P)_C$. Consider the pushout e'' of e' along u



and form a pushout P of e along v to get, by the above, $f \cdot e'' \in (Q_P)_C$, thus, $e'' \in ((Q_P)_C)_L$. Next we prove that $((Q_P)_C)_L$ is closed under composition:

Consider a composite $f' \cdot e'$



where $e \cdot e' \in (Q_P)_C$ and $f \cdot f' \in (Q_P)_C$. Form the pushout P of e and $f \cdot f'$ to get $v \in (Q_P)_C$, thus $v \cdot e \cdot e' = w \cdot f \cdot f' \cdot e' \in (Q_P)_C$. This proves $f' \cdot e' \in ((Q_P)_C)_L$. □

2.6. Theorem *The Coequalizer Deduction System is sound and complete in every locally finitely presentable category. That is, a quasi-equation is a consequence of a set Q of quasi-equations iff it can be deduced from Q.*

Proof We apply the result of [4] mentioned in 2.4: given a set $\mathcal{H}$ of finitary epimorphisms containing all finitary identity morphisms, then the injectivity consequences of e form the closure of $\mathcal{H}$ under composition, pushout, and left cancellation.

Denote by $\mathcal{A}_{fp}$ the full subcategory of all finitely presentable objects in the category $\mathcal{A}$ and by

$$K : \mathcal{A}_{fp}^{\rightrightarrows} \longrightarrow \mathcal{A}_{fp}^{\rightarrow}$$

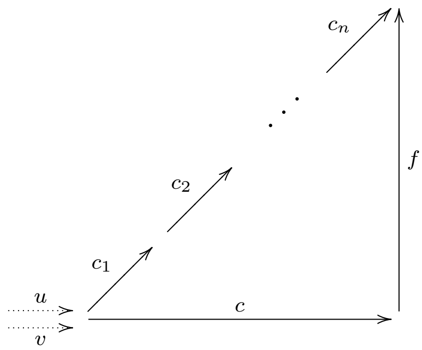
the functor assigning to every quasi-equation its coequalizer. We have

$$Q \models u \equiv v \quad \text{iff} \quad K(u,v) \text{ is an injectivity consequence of } K[Q].$$

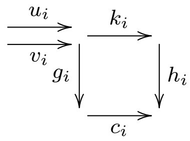
Assume, without loss of generality, that Q contains all pairs $u \equiv u$. Then the above result together with Lemma 2.5 tells us that

$$Q \models u \equiv v \quad \text{iff} \quad K(u,v) \in ((K[Q]_P)_C)_L.$$

Thus, all we need to do is to present a proof of $u \equiv v$ from Q given that the coequalizer $c = K(u,v)$ lies in the left-cancellation hull of $(K[Q]_P)_C$, i.e., it has the form



and for every i we have a pushout



for some $u_i \equiv v_i$ in Q and $k_i = K(u_i, v_i)$. Observe first that c_i is a coequalizer of $u_i' = g_i \cdot u_i$ and $v_i' = g_i \cdot v_i$ and we have

$$\frac{u_i \equiv v_i}{u_i' \equiv v_i'}$$

due to Left Composition. The Coequalizer Rule then yields

$$\frac{c_n c_{n-1} \dots c_1 u = c_n c_{n-1} \dots c_1 v \qquad u_n' \equiv v_n'}{c_{n-1} \dots c_1 u = c_{n-1} \dots c_1 v \qquad u_{n-1}' \equiv v_{n-1}'}$$

$$\vdots$$

$$\frac{\overline{c_1 u = c_1 v \qquad u_1 \equiv v_1}}{u \equiv v}$$

□

3 The Quasi-Equational Logic in Exact Categories

In the present section we introduce the logic of quasi-equations that only works with parallel pairs (and does not use coequalizers). This logic is sound in all locally finitely presentable categories, and we prove here that it is complete whenever the category is *exact*, see [6] or [8], which means that

(a) it is *regular* in the sense of Michael Barr (that is, it has regular factorizations, meaning regular epimorphism followed by a monomorphism, and regular epimorphisms are closed under pullback)

and

(b) it has effective equivalence relations (see 3.5 for details).

We present also important examples (graphs, posets, first-order structures) of categories in which our logic is complete, although they are not exact. However, a counter-example demonstrates that the logic is **not** complete in every regular, locally finitely presentable category.

3.1. Definition The *Quasi-Equational Logic* uses the following deduction rules

Reflexivity: $$\frac{}{u \equiv u}$$

Symmetry: $$\frac{u \equiv v}{v \equiv u}$$

Transitivity: $$\frac{u \equiv v \qquad v \equiv w}{u \equiv w}$$

Union: $$\frac{u \equiv v \qquad u' \equiv v'}{u + u' \equiv v + v'}$$

Composition: $$\frac{u \equiv v}{q \cdot u \cdot p \equiv q \cdot v \cdot p}$$ given $\xrightarrow{p} \underset{v}{\overset{u}{\rightrightarrows}} \xrightarrow{q}$

Epi-Cancellation: $$\frac{u \cdot p \equiv v \cdot p}{u \equiv v}$$ for epimorphisms p

We say that a quasi-equation $u \equiv v$ is *deducible* from a set Q of quasi-equations, in symbols

$$Q \vdash u \equiv v$$

if there exists a (finite) proof of $u \equiv v$ applying the above deduction rules to members of Q.

3.2. Remark The Quasi-Equational Logic is obviously sound: whenever $Q \vdash u \equiv v$, then the quasi-equation $u \equiv v$ is a consequence of Q. That is, every object satisfying all quasi-equations in Q satisfies $u \equiv v$ too.

We will discuss the completeness in this and the next section.

3.3. Remark Every proof in Birkhoff's Equational Logic has an easy translation into the Quasi-Equational Logic: Recall that that logic for a given signature Σ consists of Reflexivity, Symmetry, Transitivity, and the following rules:

Invariance: $$\frac{u \equiv v}{\sigma(u) \equiv \sigma(v)} \quad \text{for all substitutions } \sigma$$

Congruence: $$\frac{u_1 \equiv v_1,\ u_2 \equiv v_2,\ \ldots,\ u_n \equiv v_n}{h(u_1, u_2, \ldots, u_n) \equiv h(v_1, v_2, \ldots, v_n)} \quad \text{for all } n\text{-ary symbols } h \text{ in } \Sigma$$

Let $F : \mathbf{Set} \to \mathbf{Alg}\,\Sigma$ be the left adjoint of the forgetful functor of $\mathbf{Alg}\,\Sigma$. A (finitary) equation $u \equiv v$ (where $u, v : 1 \to FX$ are Σ-terms for some finite set X of variables) may be regarded as a pair of morphisms of $\mathbf{Alg}\,\Sigma$

$$F1 \underset{\overline{u}}{\overset{\overline{v}}{\rightrightarrows}} FX$$

extending u and v. This replacement of equations by quasi-equations, together with a convenient translation of the deduction rules, transforms every formal proof in Birkhoff's equational logic into one in the Quasi-Equational Logic. The Invariance Rule is a special case of Left Composition (recall that a substitution is nothing else than an endomorphism $\sigma : FX \to FX$):

$$\frac{u \equiv v}{\sigma \cdot u \equiv \sigma \cdot v}$$

For the Congruence Rule, consider the homomorphism $\overline{h} : F1 \to Fn$ taking the generator of $F1$ to the term $h(0, \ldots, n-1)$ in Fn. By applying Union we obtain

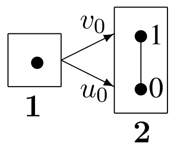
$$\overline{u}_0 + \overline{u}_1 + \cdots + \overline{u}_{n-1} \equiv \overline{v}_0 + \overline{v}_1 + \cdots + \overline{v}_{n-1} : Fn \to FX$$

and then we just compose with $\overline{h}$ from the right and the codiagonal from the left:

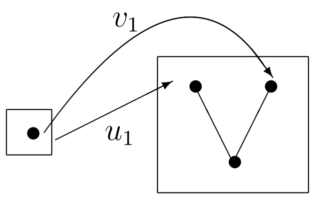
$$F1 \xrightarrow{\overline{h}} Fn \underset{\overline{v}_0+\overline{v}_1+\cdots+\overline{v}_{n-1}}{\overset{\overline{u}_0+\cdots+\overline{u}_{n-1}}{\rightrightarrows}} FX + \cdots + FX \xrightarrow{\nabla} FX$$

3.4. Example In the category of posets deduction of quasi-equations is rather trivial:

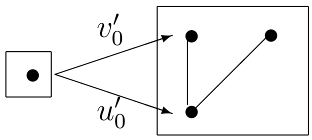
(i) Consider the following quasi-equation

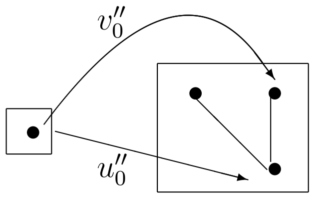


From $u_0 \equiv v_0$ we can deduce the following quasi-equation $u_1 \equiv v_1$:



In fact, by using Composition we deduce from $u_0 \equiv v_0$ the following

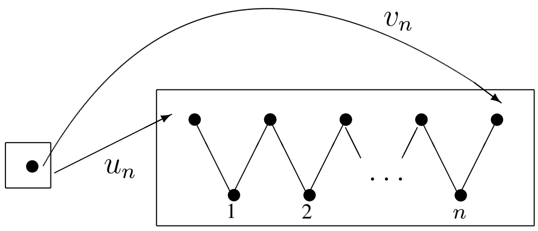


and


Symmetry yields $v'_0 \equiv u'_0$ and, since $u'_0 = u''_0$, Transitivity yields

$$u_0 \equiv v_0 \vdash u_1 \equiv v_1 .$$

(ii) Analogously we deduce from $u_0 \equiv v_0$ the following quasi-equations



(iii) More generally, we will show that the consequences of $u_0 \equiv v_0$ are all quasi-equations $u,\ v : A \to B$ such that

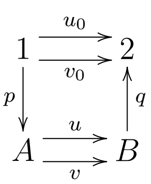
(*) $\qquad u(a)$ and $v(a)$ lie in the same component of B for all $a \in A$.

Given a quasi-equation $u \equiv v$ satisfying (*) then

$$u_0 \equiv v_0 \vdash u \equiv v\,.$$

This is clear from (ii) in case $A = \mathbf{1} = \{0\}$ is the terminal object: since $u(0)$ and $v(0)$ lie in the same component they are connected by a zig-zag. By using Union and Composition (with the codiagonal as q and $p = \mathrm{id}$) we conclude that the statement holds for all $u, v : A \to B$ with $A = \mathbf{1} + \cdots + \mathbf{1}$. And if A is arbitrary use the epimorphism $e : \mathbf{1} + \cdots + \mathbf{1} \to A$ carried by the identity map: since $u_0 \equiv v_0 \vdash u \cdot e \equiv v \cdot e$, Epi-Cancellation yields $u_0 \equiv v_0 \vdash u \equiv v$.

(iv) Conversely, every quasi-equation $u \equiv v$ where $u, v : A \to B$ are distinct implies $u_0 \equiv v_0$. In fact, choose $p \in A$ with $u(p) \neq v(p)$; say, $u(p) \not\geq v(p)$. Then we have an isotone map $q : B \to \mathbf{2} = \{0, 1\}$ where $q(u(p)) = 0$ and $q(v(p)) = 1$. Consequently, $u \equiv v \vdash u_0 \equiv v_0$ by Composition:



(v) Given $u, v : A \to B$ such that (*) does not hold, then $u \equiv v$ implies the quasi-equation $l \equiv r$ for the coproduct injections $l, r : \mathbf{1} \to \mathbf{1} + \mathbf{1}$: use Composition picking $p : \mathbf{1} \to A$ such that $u \cdot p$ and $v \cdot p$ lie in different components and $q : B \to \mathbf{1} + \mathbf{1}$ which maps one of the components to l and the rest to r.

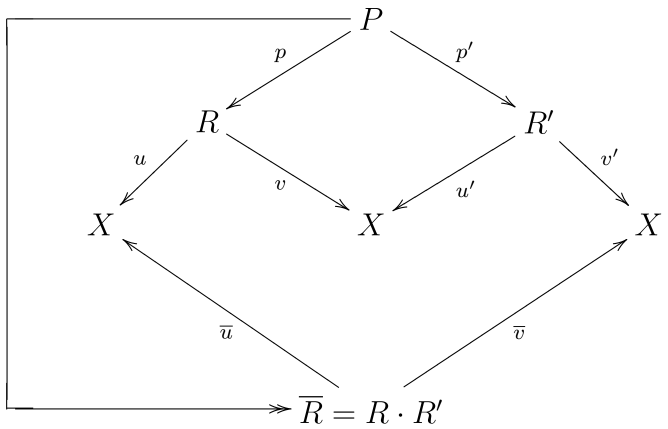
(vi) Conversely, $l \equiv r$ implies every quasi-equation. In fact, by Composition we clearly derive quasi-equations $u, v : \mathbf{1} \to B$. Using Union and Composition this yields all $u, v : \mathbf{1}+\mathbf{1} \cdots + \mathbf{1} \to B$. Finally, use $e : A' \to A$ as in (ii) above.

3.5. Remark Recall from [6] or [10] that in a regular, locally finitely presentable category:

(i) By a *relation* R on an object X is meant a subobject of $X \times X$. We can represent it by a collectively monic pair $u, v : R \to X$.

(ii) The *inverse relation* R^{-1} is represented by $v, u : R \to X$.

(iii) The *relation composite* $R \cdot R'$ of relations represented by collectively monic pairs $u, v : R \to X$ and $u', v' : R' \to X$ is obtained from the pullback P of v and u' via a factorization of $u \cdot p$, $v' \cdot p' : P \to X$:



as a regular epimorphism $e : P \to R \cdot R'$ followed by a collectively monic pair $\overline{u}, \overline{v} : R \cdot R' \to X$. This composition is associative.

(iv) An *equivalence relation* is a relation R which is

a. reflexive, i.e., $\Delta_X \subseteq R$

b. symmetric, i.e., $R = R^{-1}$, and

c. transitive, i.e., $R = R \cdot R$.

Example: every kernel pair is an equivalence relation.

(v) A regular category has *effective equivalence relations* if every equivalence relation $u, v : R \to X$ is a kernel pair (of some morphism – it follows that it is the kernel pair of coeq(u, v)).

(vi) Let R be a reflexive and symmetric relation. Then the smallest equivalence relation containing R is

$$\widehat{R} = R \cup (R \cdot R) \cup (R \cdot R \cdot R) \cup \dots$$

see [10], 1.6.8. That is, we form the chain $R^1 \subseteq R^2 \subseteq R^3 \subseteq \dots$ of subobjects of $X \times X$ defined by $R^1 = R$ and $R^{n+1} = R \cdot R^n$, and the union of this chain (a) is an equivalence relation and (b) is contained in every equivalence relation containing R.

3.6. Examples (i) Sets, presheaves, Σ-algebras (for every finitary, possibly many-sorted signature Σ) and their varieties all form exact, locally finitely presentable categories.

(ii) Every coherent Grothendieck topos is an exact, locally finitely presentable category.

(iii) The category

$$\mathbf{Mod}\,\Sigma$$

of models of a (possibly many-sorted) first-order signature is a regular, locally finitely presentable category. Recall that Σ is given by a set Σ_f of function symbols with prescribed arities $\sigma : s_1 \dots s_n \to s$ (for $s_1 \dots s_n \in S^*$ and $s \in S$) and a set Σ_r of relation symbols with prescribed arities $s_1 \dots s_n$ in S^*. A model of Σ is an S-sorted set $A = (A_s)_{s \in S}$ together with functions $\sigma^A : A_{s_1} \times \dots \times A_{s_n} \to A_s$ for all $\sigma : s_1 \dots s_n \to s$ in Σ_f and relations $\rho^A \subseteq A_{s_1} \times \dots \times A_{s_n}$ for all ρ in Σ_r of arity $s_1 \dots s_n$.

The regularity of $\mathbf{Mod}\,\Sigma$ is due to the fact that a homomorphism $h : A \to B$ is a regular epimorphism iff every sort $h_s : A_s \to B_s\,(s \in S)$ is an epimorphism in $\mathbf{Set}$ and for every relation ρ of arity $s_1 \dots s_n$ the derived function from ρ^A to ρ^B (restricting $h_{s_1} \times \dots \times h_{s_n}$) is an epimorphism in $\mathbf{Set}$.

These categories are not exact in general. A simple example in the category of directed graphs (Σ given by one binary relation): let $u, v : \mathbf{2} \times \mathbf{2} \to \mathbf{2}$ (where $\mathbf{2}$ is the chain $0 < 1$) be the kernel pair of the morphism $\mathbf{2} \to \mathbf{1}$. If R is the subobject of $\mathbf{2} \times \mathbf{2}$ with the same underlying set which has $(0,0) < (1,1)$ as the only strict relation, then $u, v : R \to \mathbf{2}$ is an equivalence relation that is not a kernel pair.

(iv) The category of posets (and monotone maps) is not regular. In fact, let A be a coproduct of two 2-chains $a < a'$ and $b < b'$, and let $e : A \to B$

be the surjection which merges a' with b to get the 3-chain $a < a' < b'$. The map $e : A \to B$ is a regular epimorphism, but its pullback along the embedding of the 2-chain $a < b'$ into B is not: the pullback is the map from the discrete two-point set into a 2-chain.

3.7. Notation Given a parallel pair $u, v : R \to X$ we denote by

$$u_0, v_0 : R_0 \to X$$

the reflexive and symmetric relation it generates in the following sense: factorize the pair

$$[u, v, \text{id}], [v, u, \text{id}] : R + R + X \to X$$

as a regular epimorphism $e_0 : R + R + X \twoheadrightarrow R_0$ followed by a collectively monic pair (u_0, v_0). Then we denote by

$$R_0^n \rightarrowtail \widehat{R}$$

the inclusion of the n-subobject in the union of 3.5(vi), represented by

$$u_n, v_n : R_0^n \to X.$$

3.8. Remark For further use let us recall here that in a locally finitely presentable category every directed union $R = \bigcup_{i \in I} R_i$ of subobjects is the colimit $R = \operatorname{colim} R_i$ of the corresponding diagram of inclusion maps, see [3], 1.62.

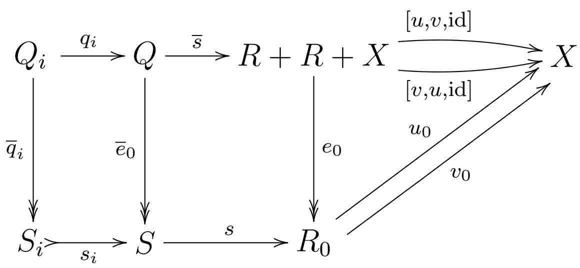
3.9. Theorem *The Quasi-Equational Logic is sound and complete in every exact, locally finitely presentable category. That is, for every set Q of quasi-equations and every quasi-equation $u \equiv v$, $Q \models u \equiv v$ iff $Q \vdash u \equiv v$.*

Proof (1) We prove first that for every quasi-equation $u \equiv v$ the relations $u_n, v_n : R_0^n \to X$ of 3.7 have the following property:

(*) $u \equiv v \vdash u_n \cdot s \equiv v_n \cdot s$ for every $s : S \to R_0^n$ with S finitely presentable.

The proof is by induction in n.

Case $n = 0$: Given $s : S \to R_0$:



we form the pullback Q of s along e_0 and express Q as a filtered colimit of finitely presentable objects with the colimit cocone $q_i : Q_i \to Q\,(i \in I)$. Then we form the regular factorization of $\overline{e}_0 \cdot q_i$ as indicated in the diagram above. The object S is the union of the subobjects $s_i : S_i \to S\,(i \in I)$ because $[s_i] : \coprod_{i \in I} S_i \to S$ is a regular epimorphism. In fact, $[s_i] \cdot \coprod \overline{q}_i = \overline{e}_0 \cdot [q_i]$ obviously is a regular epimorphism (since in the regular category $\overline{e}_0$ is a regular epimorphism), thus, so is $[s_i]$. By 3.8 we have $S = \operatorname{colim} S_i$, therefore, the fact that S is finitely presentable implies that s_j is an isomorphism for some $j \in I$. We now have a derivation of $u_0 \cdot s \equiv v_0 \cdot s$ as follows:

$$\begin{array}{c} u \equiv v \\ \hline u \equiv v \quad v \equiv u \quad \mathrm{id} \equiv \mathrm{id} \\ \hline \\ [u,v,\mathrm{id}] \equiv [v,u,\mathrm{id}] \\ \hline u_0 \cdot s \cdot s_j \quad \overline{q}_j \equiv v_0 \cdot s \cdot s_j \cdot \overline{q}_j \\ \hline u_0 \cdot s \equiv v_0 \cdot s \end{array}$$

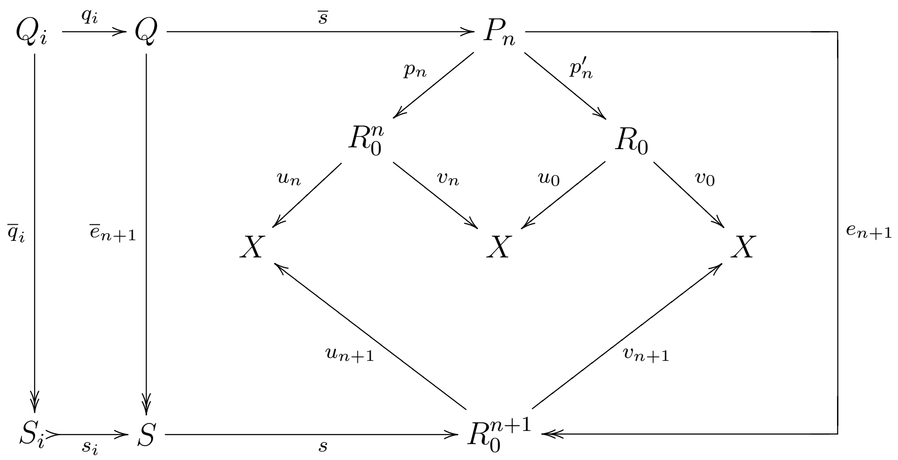
by Symmetry and Reflexivity

by Union and Composition (with $p = \mathrm{id}$, $q = \nabla : X + X + X \to X$)

by Composition ($p = \overline{s} \cdot q_j$, $q = \mathrm{id}$)

by Epi-Cancellation

Induction Case: Suppose (*) holds and $s : S \to R_0^{n+1}$ with S finitely presentable is given.



Analogously to the above case we form the pullback Q of s and e_n and express Q as a filtered colimit of finitely presentable objects Q_i with the colimit cocone $q_i : Q_i \to Q\,(i \in I)$. We then form regular factorizations of $\overline{e}_{n+1} \cdot q_i$ as indicated, and by the above argument we conclude that s_j is an isomorphism for some $j \in I$. Therefore, by induction hypothesis, from $u \equiv v$, we can deduce

$$u_0 \cdot p'_n \cdot \overline{s} \cdot q_j \equiv v_0 \cdot p'_n \cdot \overline{s} \cdot q_j \qquad \text{and} \qquad u_n \cdot p_n \cdot \overline{s} \cdot q_j \equiv u_0 \cdot p'_n \cdot \overline{s} \cdot q_j \quad (3.1)$$

since $v_n \cdot p_n = u_0 \cdot p'_n$. Hence, by Transitivity,

$$u_n \cdot p_n \cdot \overline{s} \cdot q_j \equiv v_0 \cdot p'_n \cdot \overline{s} \cdot q_j$$

that is,

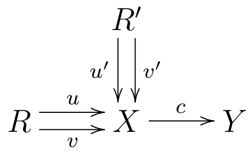
$$u_{n+1} \cdot s \cdot s_j \cdot \overline{q}_j \equiv v_{n+1} \cdot s \cdot s_j \cdot \overline{q}_j \,.$$

Now, by Epi-Cancellation, we conclude

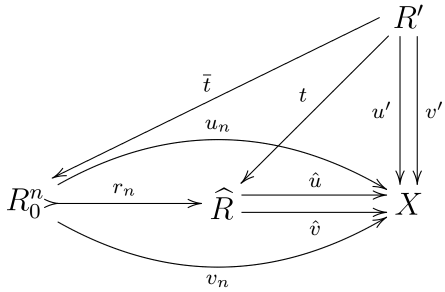
$$u_{n+1} \cdot s \equiv v_{n+1} \cdot s \,.$$

(2) We are ready to prove the completeness of the Quasi-Equational Logic. Since the Coequalizer Deduction System is complete, and the only

deduction rule not contained in 3.1 is the Coequalizer rule, it is sufficient to find a translation of that rule:



Suppose $u \equiv v$ and $u' \equiv v'$ are quasi-equations such that the coequalizer c of u, v fulfils $c \cdot u' = c \cdot v'$. Then we will find a derivation of $u' \equiv v'$ from $u \equiv v$ in the deduction system of 3.1. Let $\hat{u}, \hat{v} : \widehat{R} \to X$ be the kernel pair of c. Then $\widehat{R}$, being an equivalence relation, is the smallest equivalence relation containing R_0 in 3.7, consequently $\widehat{R} = \bigcup_{n \in \mathbb{N}} R_0^n$ by 3.5(vi). Then the pair u', v' factorizes through it via a morphism $t : R' \to \hat{R}$. Now $\widehat{R}$ is a chain colimit by 3.8, and R' is finitely presentable, thus, t factors through one of the colimit morphisms $r_n = [u_n, v_n] : R_0^n \rightarrowtail \widehat{R}$:



That is, we have $\bar{t} : R' \to R_0^n$ such that $u_n \cdot \bar{t} = u'$ and $v_n \cdot \bar{t} = v'$. Thus, we can derive $u' \equiv v'$ from $u \equiv v$, see (1). □

3.10. Remark (i) Observe that the effectivity of equivalence relations was not used in the first part of the proof.

(ii) Observe also that Epi-Cancellation was only used for regular epimorphisms in the above proof. We will use it more generally in 3.12 below.

3.11. Remark The above theorem implies that in categories

$$\mathbf{Alg}\,\Sigma$$

of algebras of an arbitrary finitary S-sorted (algebraic) signature Σ the Quasi-Equational Logic is complete: in fact, $\mathbf{Alg}\,\Sigma$ is an exact, locally finitely presentable category. We want to extend this result to categories $\mathbf{Mod}\,\Sigma$ of 3.6(iii). Although $\mathbf{Mod}\,\Sigma$ does not have effective equivalence relations, we have the following:

3.12. Proposition *The Quasi-Equational Logic is complete in* $\mathbf{Mod}\,\Sigma$*.*

Proof Consider the adjoint situation

$$\mathbf{Mod}\,\Sigma \underset{D}{\overset{W}{\underset{\top}{\rightleftarrows}}} \mathbf{Alg}\,\Sigma_f$$

where W forgets the relations and D defines them to be empty. Both W and D preserve limits, colimits and finitely presentable objects. Consequently, they preserve regular factorizations and composition of relations.

As in the previous proof, we just need to translate the Coequalizer rule: given quasi-equations in $\mathbf{Mod}\,\Sigma$:

$$\begin{array}{ccccc} & & R' & & \\ & & {\scriptstyle u'}\downarrow\downarrow{\scriptstyle v'} & & \\ R & \underset{v}{\overset{u}{\rightrightarrows}} & X & \xrightarrow{c} & Y \end{array}$$

with $c \cdot u' = c \cdot v'$ for $c = \text{coeq}(u, v)$, we will prove that

$$u \equiv v \vdash u' \equiv v'.$$

From the proof of 3.9 and 3.10 we have that $u \equiv v \vdash u_n \cdot s \equiv v_n \cdot s$ for all $s : S \to R_0^n$ with S finitely presentable. Further, since Wc is the coequalizer of Wu, Wv and the kernel pair of Wc is represented by the relation

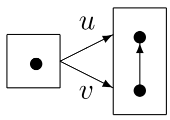
$$W\widehat{R} = \bigcup WR_0^n = \bigcup (WR)_0^n$$

we see that the pair Wu', Wv' factorizes through some Wu_n, $Wv_n : WR_0^n \to WX$ via a morphism $\bar{t} : WR' \to WR_0^n$. In case $R' = DWR'$ we have a morphism $s : R' \to R_0^n$ with $\bar{t} = Ws$, and then $u \equiv v \vdash u' \equiv v'$ because $u' = u_n \cdot s$ and $v' \equiv v_n \cdot s$. In general, the counit of $D \dashv W$ gives an epimorphism $e : DWR' \to R'$ (carried by the identity map) and the above consideration yields $u \equiv v \vdash u' \cdot e \equiv v' \cdot e$. Using Epi-Cancellation, we derive $u \equiv v \vdash u' \equiv v'$. □

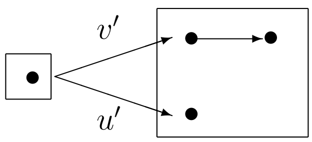
3.13. Example The Quasi-Equational Logic is complete in the category of posets. This follows easily from Example 3.4: If $u \equiv v$ is a consequence of a set Q of quasi-equations, and if some member of Q does not satisfy (*), then $Q \vdash l \equiv r$, and from that $Q \vdash u \equiv v$ follows. If all members of Q satisfy (*) then also $u \equiv v$ does (it is easy to see that the set of all quasi-equations satisfying (*) is closed under the deduction rules of 3.1). Thus, either Q contains a nontrivial quasi-equation, in which case we deduce $u_0 \equiv v_0$ from Q and we also deduce $u \equiv v$ from $u_0 \equiv v_0$. Or Q contains only quasi-equations $w \equiv w$, but then $u = v$.

3.14. Example of incompleteness of the Quasi-Equational Logic. For the language Σ_2 of one binary relation the category $\mathbf{Mod}\,\Sigma_2$ (of directed graphs and homomorphisms) has complete Quasi-Equational Logic by 3.12. Let $\mathcal{A}$ be the full subcategory of all graphs (X, R) which are antireflexive ($R \cap \Delta_X = \emptyset$) with the terminal object added. $\mathcal{A}$ is closed under limits, filtered colimits and regular factorizations in $\mathbf{Mod}\,\Sigma_2$, thus, it is a regular, locally finitely presentable subcategory.

The quasi-equation



is satisfied by precisely those graphs in $\mathcal{A}$ that are discrete or terminal. Therefore, it has as a consequence the quasi-equation



However, we cannot derive $u' \equiv v'$ from $u \equiv v$. In fact, all quasi-equations $\overline{u} \equiv \overline{v}$ that can be deduced from $u \equiv v$ have the property (*) in 3.4, since the quasi-equation $u \equiv v$ fulfils it and the set of all quasi-equations $\overline{u} \equiv \overline{v}$ fulfilling it is closed under all deduction rules. Since $u' \equiv v'$ does not, the proof is concluded.

4 The Quasi-Equational Logic in Non-Exact Categories

In the present section we work in a locally finitely presentable category with effective equivalence relations – but we do not assume regularity. We prove, again, that the Quasi-Equational Logic is complete. However, we need to extend slightly the concept of quasi-equation: we will consider all parallel pairs $u, v : R \to X$ where X is finitely presentable but R only finitely generated. Since finitely generated objects are precisely the strong quotients $e : \overline{R} \twoheadrightarrow R$ of finitely presentable objects $\overline{R}$, the difference is just a small technicality: for the quasi-equations (in the sense of preceding sections) $u' \equiv v'$ where $u' = u \cdot e$, $v' = v \cdot e$ we have $u \equiv v \vdash u' \equiv v'$ by Composition and, conversely, $u' \equiv v' \vdash u \equiv v$ by Epi-Cancellation.

4.1. Definition A *weak quasi-equation* is a parallel pair of morphisms (u, v) whose domain is finitely generated and codomain is finitely presentable. An object A satisfies $u \equiv v$ if $\mathcal{A}(-, A)$ merges u and v.

4.2. Theorem *The Quasi-Equational Logic is complete and sound in every locally finitely presentable category with effective equivalence relations. That is, given a set Q of weak quasi-equations, then a weak quasi-equation $u \equiv v$ is a consequence of Q iff it can be deduced from Q.*

4.3. Remark Before we prove this theorem, we need to modify Remark 3.5. Every locally finitely presentable category has the factorization system (strong epi, mono), see [3], 1.61. By a relation we again understand a subobject of $X \times X$. In the definition of composite, see 3.5 (iii), we just use the (strong epi, mono)-factorization of $u \cdot p$, $v' \cdot p'$. Then the concept of equivalence relation and having effective equivalence relations as in 3.5. However, relation composition is not associative in general.

Let R be a reflexive and symmetric relation. Then the smallest equivalence relation containing R is

$$\widehat{R} = R \cup (R \cdot R) \cup (R \cdot (R \cdot R)) \cup ((R \cdot R) \cdot R) \cup \dots$$

that is, the union

$$\widehat{R} = \bigcup_{i \in I} R_i$$

of the smallest set $R_i\,(i \in I)$ of relations containing R and closed under composition. This is essentially proved in [1]. For the sake of easy reference here is a proof:

(a) $\widehat{R}$ is reflexive since R is (so that R_i is reflexive for every i since a composite of reflexive relations is reflexive).

(b) $\widehat{R}$ is symmetric since R is: the formula

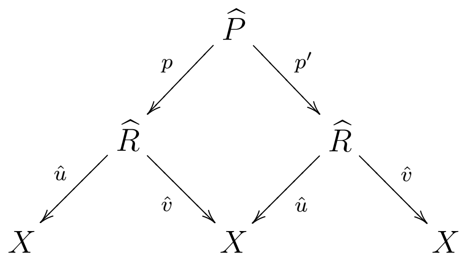
$$(R_j \cdot R_i)^{-1} = R_i^{-1} \cdot R_j^{-1}$$

implies that the set $\{R_i\}_{i\in I}$ is closed under the formation of inverses.

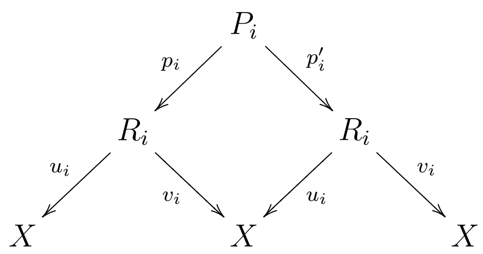
(c) $\widehat{R}$ is transitive because by 3.8

$$\widehat{R} = \operatorname*{colim}_{i\in I} R_i$$

and in locally finitely presentable categories pullbacks commute with filtered colimits. Indeed, let $u_i, v_i : R_i \to X$ be the pair representing r_i and $\hat{u}, \hat{v} : \widehat{R} \to X$ that representing $\hat{r}$. Form the pullback



Transitivity of $\widehat{R}$ means that the pair $\hat{u} \cdot p$, $\hat{v} \cdot p' : \widehat{P} \to X$ factors through $\hat{u}, \hat{v}$. The above pullback is a colimit of the pullbacks



and for each $i \in I$ we have $j \in J$ with $R_j = R_i \cdot R_i$, therefore, the pair $u_i \cdot p_i, v_i \cdot p'_i : P_i \to X$ factors through u_j, v_j. From $p = \operatorname{colim} p_i$ and

$p' = \operatorname{colim} p'_i$ we conclude that the pair $\hat{u} \cdot p$, $\hat{v} \cdot p'$ factors through $\hat{u}, \hat{v}$, as requested.

(d) It is obvious that an equivalence relation S containing R contains each R_i, thus, $\widehat{R} \subseteq S$. Moreover, it is easy to see that for every morphism $c : X \to Y$ we have

$$c \cdot u = c \cdot v \quad \text{iff} \quad c \cdot \hat{u} = c \cdot \hat{v}$$

(since $c \cdot u = c \cdot v$ implies that the set of all relations u', v' with $c \cdot u' = c \cdot v'$ is closed under inverse and relation composite – thus, $c \cdot u_i = c \cdot v_i$ for all $i \in I$.)

4.4. Notation For a weak quasi-equation $u, v : R \to X$ we denote by $u_0, v_0 : R_0 \to X$ the reflexive-and-symmetric hull given by a factorization of $[u, v, \mathrm{id}]$, $[v, u, \mathrm{id}] : R + R + X \to X$ as a strong epimorphism followed by a collectively monic pair (u_0, v_0). Then we have the above subobjects

$$r_i : R_i \to \widehat{R} \;\; (i \in I)$$

forming the least equivalence relation $\widehat{R} = \bigcup_{i \in I} R_i$ containing R_0 represented by pairs $u_i, v_i : R_i \to X$. If the pair $\hat{u}, \hat{v} : \widehat{R} \to X$ represents the equivalence relation $\widehat{R}$, then $u_i = \hat{u} \cdot r_i$ and $v_i = \hat{v} \cdot r_i$.

4.5. Proof of Theorem 4.2 Let $u, v : R \to X$ be a weak quasi-equation which is a consequence of a set Q of weak quasi-equations. We prove $Q \vdash u \equiv v$.

(1) We first prove that for every weak quasi-equation $u \equiv v$ we have

$$u \equiv v \vdash u_i \cdot s \equiv v_i \cdot s \quad \text{for every } s : S \to R_i \text{ with } S \text{ finitely generated}$$

by structural induction on $i \in I$: we verify first the case $s : S \to R_0$ for the reflexive-and-symmetric hull R_0 of R, and then show that if the above holds for R_i and R_j, then it holds for $R_i \cdot R_j$.

Base case: As in 3.9 derive $[u, v, \mathrm{id}] \equiv [v, u, \mathrm{id}]$ from $u \equiv v$, then use Epi-Cancellation to get $u_0 \equiv v_0$. Using Composition $u \equiv v \vdash u_0 \cdot s \equiv v_0 \cdot s$.

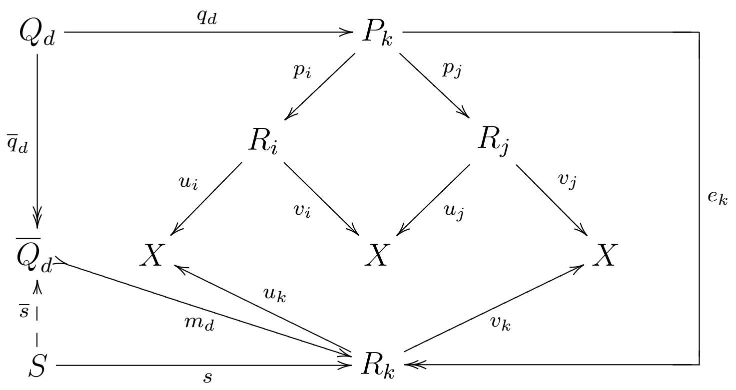
Induction case: Let $R_k = R_i \cdot R_j$ and let

$$u \equiv v \;\vdash\; u_i \cdot s \equiv v_i \cdot s \quad \text{and} \quad u \equiv v \;\vdash\; u_j \cdot s \equiv v_j \cdot s$$

hold for all morphisms s with finitely generated domain and codomain such that the composites are defined. Given

$$s : S \to R_k\,,\ S \text{ finitely generated,}$$

we prove $u \equiv v \vdash u_k \cdot s \equiv v_k \cdot s$. Let us recall the definition of $R_k = R_i \cdot R_j$:



Express P_k as a filtered colimit of finitely presentable objects $Q_d\,(d \in D)$ with the colimit cocone $q_d : Q_d \to P_k\,(d \in D)$ and let the (strong epi, mono)-factorization of $e_k \cdot q_d$ be

$$e_k \cdot q_d = m_d \cdot \overline{q}_d \quad \text{for } m_d : \overline{Q}_d \rightarrowtail R_k\,.$$

Then $R_k = \bigcup_{d \in D} \overline{Q}_d$ because $[m_d] \cdot \coprod_{d \in D} \overline{q}_d = e_k \cdot [q_d]$ is a strong epimorphism, thus, so is $[m_d]$. By 3.8

$$R_k = \operatorname{colim} \overline{Q}_d$$

is a colimit of a directed diagram of monomorphisms. Since S is finitely generated, $\mathcal{A}(S, -)$ preserves this colimit, consequently, $s : S \to \operatorname{colim} \overline{Q}_d$ factors through some m_d:

$$s = m_d \cdot \overline{s} \quad \text{for some } d \in D \text{ and } \overline{s} : S \to \overline{Q}_d\,.$$

By induction hypothesis,

$$u \equiv v \vdash u_i \cdot p_i \cdot q_d \equiv v_i \cdot p_i \cdot q_d \quad \text{and} \quad u \equiv v \vdash u_j \cdot p_j \cdot q_d \equiv v_j \cdot p_j \cdot q_d$$

which by Transitivity and $v_i \cdot p_i = u_j \cdot p_j$ yields

$$u \equiv v \vdash u_i \cdot p_i \cdot q_d = v_j \cdot p_j \cdot q_d \,.$$

In other words,

$$u \equiv v \vdash u_k \cdot e_k \cdot q_d \equiv v_k \cdot e_k \cdot q_d \,.$$

Now from $e_k \cdot q_d = m_d \cdot \overline{q}_d$ we deduce, due to Epi-Cancellation,

$$u \equiv v \vdash u_k \cdot m_d \equiv v_k \cdot m_d$$

and using $s = m_d \cdot \overline{s}$ we get, via Composition,

$$u \equiv v \vdash u_k \cdot s \equiv v_k \cdot s$$

as desired.

(2) The rule Coequalizer (for finitary morphisms) is, due to (1), translated to the rules of 3.1 quite analogously as in the proof of 3.9, part (2).

(3) To prove the completeness, let $u, v : R \to X$ be a weak quasi-equation which is a consequence of the set Q. Since R is finitely generated, it is a strong quotient $e : R^* \twoheadrightarrow R$ of a finitely presentable object R^* and we consider the quasi-equation $u^* \equiv v^*$ obtained from $u \equiv v$ by composition with e. Analogously, for every member $\overline{u} \equiv \overline{v}$ of Q we form a quasi-equation $\overline{u}^* \equiv \overline{v}^*$ in the above manner and get a set Q^* of quasi-equations.

It is clear that $u \equiv v$ is a consequence of Q iff $u^* \equiv v^*$ is a consequence of Q^*: use the soundness of Epi-Cancellation and Composition. By Theorem 2.6, there is a formal proof of $u^* \equiv v^*$ from Q^* using the Coequalizer Deduction System. We see from (2) that this formal proof gives rise to a proof of $u^* \equiv v^*$ from Q^* using the deduction rules of 3.1. Now $Q \vdash u \equiv v$ follows from the fact that $Q \vdash Q^*$ and $u^* \equiv v^* \vdash u \equiv v$. □

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Jiří Adámek
Department of Theoretical Computer Science, Technical University of Braunschweig, Postfach 3329,
38023 Braunschweig, Germany

Lurdes Sousa
School of Technology of Viseu, Campus Politecnico, 3504-510 Viseu, Portugal
CMUC, University of Coimbra, 3001-454 Coimbra, Portugal