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INFINITARY LINEAR COMBINATIONS IN REDUCED COTORSION MODULES

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Dedicated to Francis Borceux on the occasion of his sixtieth birthday

Abstract

We investigate sets with infinitary linear combinations subject to the usual axioms with coefficients in a suitable ring, e.g. a complete valuation ring. They are Eilenberg-Moore algebras for a monad of countable arity. Moreover, they are always modules; surprisingly infinitary linear combinations yield a *property*. This is quite different from real or the complex case studied by Pumplün and Röhrl.

These modules were called cotorsion modules and defined by a cohomological property by Matlis. They form a reflective subcategory; the reflection also has a cohomological description. This yields some insight, particularly if the first Ulm functor does not vanish.

Nous étudions des ensembles avec combinaisions linéaires infinies, qui satisfont aux axiomes ordinaires, ayant des coefficients dans un anneau avec certaines propriétés, p.ex. un anneau complet d'évaluation. Ici, il s'agit d'algèbres d'Eilenberg-Moore pour une monade d'arité dénombrable. En plus elles sont toujours des modules; de manière inattendue combinaisions linéaires infinies impliquent une *propriété*. C'est tout à fait différent du cas réel ou complexe considéré par Pumplün et Röhrl.

Ces modules de cotorsion étaient définit par une propriété cohomologique par Matlis. Ils constituent une sous-catégorie réflexive; la réflexion a une description cohomologique. Cela nous ouvre des perspectives, en particulier si le premier foncteur d'Ulm ne disparaît pas.

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1 Introduction

Pumplün and Röhrl [10] introduced the category **TC** of *totally convex spaces* as the *Eilenberg-Moore category* of the monad induced by the adjunction between the unit-ball functor $\mathbf{Ban}_1 \longrightarrow \mathbf{Set}$ and its left adjoint, where $\mathbf{Ban}_1$ is the category of Banach spaces and linear operators of norm ≤ 1 over the field $\mathbb{R}$ or $\mathbb{C}$. For Banach spaces over a complete field with a *Krull-valuation* subject to some mild conditions the monad and its Eilenberg-Moore-algebras can be formed analogously and form a *locally countably presentable category*, but they look quite different. As opposed to the real and to the complex situation, the Eilenberg-Moore-algebras admit an *addition* subject to the usual rules; this leads to an *additive* and even *abelian* category. The algebras are modules over the valuation ring, and finitary linear combinations are formed as in the module. The existence of infinitary linear combinations excludes the existence of non-trivial divisible submodules; following some authors we call a module with this property *reduced.* The module carries a canonical topology, which turns out to be bounded, and every infinitary linear combination is the limit of the finitary sub-combinations in this topology. If the topology is Hausdorff, this limit is unique; this happens if and only if the module is *division-free*, i.e. the first *Ulm functor* vanishes. But there also exist division elements in Eilenberg-Moore-algebras; surprisingly, then the infinitary linear combinations are still determined by the finitary ones. The Eilenberg-Moore-category is even a *full* subcategory of the category of all modules. Its objects were already investigated by Matlis [9] in a different context and called *cotorsion modules*; according to new terminology we call them *reduced* Matlis-cotorsion modules. It is well-known that Matlis-cotorsion is equivalent to completeness in the division-free situation, but surprisingly, our results remain valid if there exist non-trivial division elements. Then infinitary linear combinations are still uniquely determined, though the limit of the corresponding finitary linear combinations is not unique. Infinitary linear combinations can also be characterized as unique solutions of systems of linear equations. Though always some Ulm functor vanishes, the chain of Ulm functors can be arbitrarily long. This easily implies that the category of reduced Matlis-cotorsion modules has no cogenerator.

2 Eilenberg-Moore-algebras

In this section we consider a field K with a *Krull–valuation*, i.e. a surjective map $v : K \longrightarrow \overline{\Gamma}$, where $\overline{\Gamma} := \Gamma \cup \{\infty\}$, Γ a *totally ordered* (additively written) *abelian group* and ∞ is an additional largest element, subject to the following axioms:

(V1) $v(\alpha) = \infty$ if and only if $\alpha = 0$.

(V2) $v(\alpha\beta) = v(\alpha) + v(\beta)$ for all $\alpha, \beta \in K$.

(V3) $v(\alpha + \beta) \geq \min(v(\alpha), v(\beta))$ for all $\alpha, \beta \in K$.

Surjectivity of v can always be achieved by codomain restriction. We want to include the case that the value group Γ is not archimedean; the valuation is non-archimedean anyway. The totally ordered group Γ is archimedean if and only if it can be embedded into $\mathbb{R}$; the most famous examples are p-adic valuations. In order to avoid some trivial cases, we assume that Γ contains a countable unbounded subset; this guarantees the existence of convergent subsequences which are not eventually constant. Since the single-element group is obviously bounded, Γ has at least two elements; this makes the canonical topology non-discrete. The above conditions are always satisfied for non-trivial real-valued valuations. Moreover, we assume that K is *complete*, i.e. sequence $(\alpha_n)_{n\in\mathbb{N}}$ in K converges if $(\alpha_n - \alpha_{n+1})_{n\in\mathbb{N}}$ converges to 0; we always assume $0 \in \mathbb{N}$. Here a sequence $(\alpha_n)_{n\in\mathbb{N}}$ converges to $\alpha \in K$ if $v(\alpha_n - \alpha) \to \infty$ for

$n \to \infty$. The *valuation ring* $R := \{\alpha \in K \mid v(\alpha) \geq 0\}$ is a local ring with *maximal ideal* $\{\alpha \in K \mid v(\alpha) > 0\}$.

Let us now define (K, v)-Banach spaces; in the case of a real-valued field they were defined by A.F. Monna and studied by van Rooij [11]. This leads to an easy generalization to this case; assuming $||\alpha|| := e^{-v(\alpha)}$ for $\alpha \neq 0$ and $||0|| := 0$. Then the (K, v)-Banach spaces form a complete and cocomplete category; the construction of limits and colimits sometimes requires suprema and infima which exist by completeness of $\mathbb{R}$. For non-archimedean Γ this is not possible, because then Γ is not complete and cannot even be embedded into a complete totally ordered group. Therefore, we define the Banach structure on a K-vector space E by a binary relation $\dashv$ on $E \times \Gamma$; here $x \dashv g$ should be thought of as $u(x) \geq g$ for some (valuation) map $u : E \to \overline{\Gamma}$; for a real-valued v we can define (K, v)-Banach spaces by v in this way. Another problem occurs: If Γ has no least positive element, then 0 is the infimum of all positive elements, and every element of Γ is the supremum of all strictly smaller elements and the infimum of all strictly larger elements. If Γ has a smallest positive element, this is not the case. This requires a distinction of two cases; we come to the following definition:

A (K, v)-*Banach space* is a K-vector space E together with a binary relation $\dashv$ on $E \times \Gamma$ with:

(KB0) For every $x \in E$ there exists a $g \in \Gamma$ with $x \dashv g$. $0 \dashv g$ holds for all $g \in \Gamma$, but for every $x \in E \setminus \{0\}$ there exists a $g \in \Gamma$ with $x \not\dashv g$.

(KB1) $x \dashv g'$ whenever $x \dashv g$ and $g \geq g'$.

(KB2) $\alpha x \dashv v(\alpha) + g$ whenever $x \dashv g$, $\alpha \in K \setminus \{0\}$.

(KB3) $x + y \dashv g$ if $x \dashv g$ and $y \dashv g$.

(KB4) If $(x_n)_{n \in \mathbb{N}}$ is a sequence in E such that for every $g \in \Gamma$ there exists an $n_0 \in \mathbb{N}$ with $x_n - x_{n+1} \dashv g$ for all $n \geq n_0$, then there exists an $x \in V$ such that for every $g \in \Gamma$ there exists an $n_1 \in \mathbb{N}$ with $x_n - x \dashv g$ for all $n \geq n_1$.

If Γ has a least positive element (but only in this case) we also assume:

(KB5) $x \dashv 0$ whenever $x \dashv g$ for all positive $g \in \Gamma$.

A *morphism* $f : E_0 \to E_1$ of (K, v)-Banach spaces is a K-linear map $E_0 \to E_1$ such that $x \dashv g$ implies $f(x) \dashv g$.

Observe that (KB4) is a *completeness* condition; it means that every Cauchy sequence in the *canonical topology* converges; the canonical topology has the sets $U_g := \{x; x \dashv g\}$ as a basis. The Banach structure is already given by the *unit ball* $\bigcirc E := \{x \in E | x \dashv 0\}$ of E; observe $x \dashv v(\alpha)$ if and only if $\alpha^{-1}x \in \bigcirc E$ for $\alpha \neq 0$. $\bigcirc E$ is no longer a K-vector space, but still an R-module. Observe that the base-field K is always a (K, v)-Banach space in the canonical way with $\bigcirc K = R$. By the existence of a countable unbounded set in Γ this topology is always first-countable and hence sequential, i.e. every sequentially closed set is closed.

The set-valued functor $\bigcirc$ on the category of (K, v)-Banach spaces has a left adjoint and induces a monad on the category of sets. This left adjoint maps every set X to the K-vector space $\ell_1(X)$ of all families $(\xi_x)_{x\in X} \in K^X$ such that for each $g \in \Gamma$ there are only finitely many $x \in X$ with $v(\xi_x) < g$; we define $(\xi_x)_{x\in X} \dashv g :\Leftrightarrow \forall x \in X\ v(\xi_x) \geq g$ for $(\xi_x)_{x\in X} \in \ell_1(X)$ and $g \in \Gamma$. An Eilenberg-Moore-algebra is a non-empty set M together with maps $M^{\mathbb{N}} \to M$ for $(\alpha_n)_{n\in\mathbb{N}} \in \Omega$, which we shall write as $x_\bullet := (x_n)_{n\in\mathbb{N}} \mapsto \sum_{n=0}^{\infty} \alpha_n x_n$, where $\Omega := \ell_1(\mathbb{N}) = \{\alpha_\bullet = (\alpha_n)_{n\in\mathbb{N}} |\ \forall n \in \mathbb{N}\ \alpha_n \in R,\ v(\alpha_n) \to \infty \text{ for } n \to \infty\}$. For $\alpha_\bullet = (\alpha_n)_{n\in\mathbb{N}} \in \Omega$ we put $\alpha_\bullet^\odot := (\alpha_{n+1})_{n\in\mathbb{N}} \in \Omega$; likewise we define $x_\bullet^\odot := (x_{n+1})_{n\in\mathbb{N}} \in M^{\mathbb{N}}$ for $x_\bullet := (x_n)_{n\in\mathbb{N}} \in M^{\mathbb{N}}$. Moreover, for $\beta \in K$ and $\alpha_\bullet \in \Omega$ we define $\beta\alpha_\bullet := (\beta\alpha_n)_{n\in\mathbb{N}}$; it even belongs to Ω if $\beta\alpha_n \in R = \bigcirc K$ for all $n \in \mathbb{N}$; this is always satisfied for $\beta \in R$. Since every $\alpha_\bullet \in \Omega$ converges to 0 by hypothesis, for each $\beta \in R \setminus \{0\}$ there are only finitely many $n \in \mathbb{N}$ with $\beta^{-1}\alpha_n \notin R$; therefore iterated application of $^\odot$ to $\beta^{-1}\alpha_\bullet$ finally leads to an element of Ω. There is always a distinguished element $0 = \sum_{n=0}^{\infty} 0x_n \in M$, which does not depend on the $x_n \in M$. The Eilenberg-Moore-algebras for the adjunction given by the above unit ball functor $\bigcirc$ and its left adjoint ℓ_1 can be characterized in our situation in the same way as Pumplün and Röhrl did in the real and in the complex situation (cf. [10]).

Theorem 2.1 *The Eilenberg-Moore-category of the monad induced by the set-valued functor $\bigcirc$ is the category of non-empty sets M together with maps $(x_n)_{n\in\mathbb{N}} \mapsto \sum_{n=0}^{\infty} \alpha_n x_n$ for $(\alpha_n)_{n\in\mathbb{N}} \in \Omega$ subject to the follow-*

ing axioms:

(LC1) $\sum_{n=0}^{\infty} \delta_{n,m} x_n = x_m$ *for all* $m \in \mathbb{N}$*, where* δ *is the Kronecker symbol with values in* R*, i.e.* $\delta_{n,n} = 1$ *and* $\delta_{n,k} = 0$ *for* $n \neq k$*.*

(LC2) $\sum_{n=0}^{\infty} \alpha_n (\sum_{m=0}^{\infty} \beta_{n,m} x_m) = \sum_{m=0}^{\infty} (\sum_{n=0}^{\infty} \alpha_n \beta_{n,m}) x_m$ *for all* $\alpha_\bullet \in \Omega$*, and* $\beta_{n,\bullet} \in \Omega$ *for all* $n \in \mathbb{N}$*, where* $x_m \in M$ *for all* $m \in \mathbb{N}$*.* □

The proof is completely analogous to the real and in the complex situation; nevertheless, the algebras look quite different. As in the real and in the complex case, the monad has rank $\aleph_1$; all operations are countable; therefore the category is locally countably presentable (cf. [6]). Moreover Ω contains all sequences in R with only finitely many non-zero entries; this defines *finitary linear operations* in every algebra M; so M becomes an *R-module*. In this way the Eilenberg-Moore-category is *additive*; it even turns out to be *abelian*; it is also a symmetric monoidal closed category, even an autonomous category in the sense of Linton [8]. The tensor product in this category can be constructed by applying the reflection to the usual tensor product of modules. The set-theoretical image of every morphism is a subalgebra, in particular a submodule. This submodule can always be divided out; so we see that all epimorphisms are surjective. All this is different from the real and the complex case because $1 + 1 = 2 > 1$ holds there; thus the category is no longer additive. These algebras were studied in more detail in the *habilitationsschrift* of the second author [7].

We can also define (K, v)*-normed vector spaces* as above by omitting the completeness condition (KB4); we do not need the completeness of K. Instead of Ω we get the set of all sequences in R with only finitely many non-zero entries. Now the operations can be viewed as *finitary* linear combinations; in particular, we have a binary addition and a multiplication with an element of R on each Eilenberg-Moore-algebra M. Then we easily see that M is an R-module under these operations, and arbitrary linear combinations are just as in this module. Conversely, in every R-module the finitary linear combinations satisfy (LC1) and (LC2). Therefore the Eilenberg-Moore-category is just the category of R-modules in this situation.

If $\sigma : \Gamma \to \Gamma'$ is a surjective order-preserving group homomorphism, and if we still have $\Gamma' \neq \{0\}$, then the valuation $v' := \sigma \circ v$ also satisfies the above conditions. The kernel of σ is then bounded, but the valuation ring R' of v' is strictly larger than R unless σ is bijective, because there are elements $g < 0$ in Γ with $\sigma(g) = 0$. Every (K, v)-Banach space E is a (K, v')-Banach space in the obvious way; it carries the same canonical topology, but as a (K, v')-Banach space it has a larger unit ball. In particular, R is the unit ball of K over R and admits infinitary linear combinations with (LC1) and (LC2) over R, but not over R'.

3 Finitary and infinitary linear combinations

In the remainder of this paper we shall consider a more general situation. The ring R need not be a valuation ring. But we assume that R is an integral domain and $K \neq R$ is its quotient field. More generally, every R-module M carries a *canonical topology*, whose basis are all sets αM with $\alpha \in R \setminus \{0\}$. For a valuation v it coincides with our previous definition. Since R is not a field, the canonical topology on R is Hausdorff. This happens for valuations because we consider surjective Krull-valuations rather than (possibly non-surjective) valuations into $\mathbb{R}$; for the latter approach, a trivial valuation would lead to the discrete topology. Moreover, we assume R to be *first countable* in the canonical topology; this is equivalent to saying that R is *powerful* in the sense of Matlis [9], i.e. K is countably generated as an R-module; then K has homological dimension 1, as we shall see: K is a union of an increasing chain of countably many submodules $\gamma_n^{-1}R$ with $\gamma_n \in R \setminus \{0\}$ for all $n \in \mathbb{N}$; we also can assume $\gamma_0 = 1$. This can always be achieved by choosing γ_n as the product of the denominators of the first n generators. Each one of the modules $\gamma_n^{-1}R$ is isomorphic to R. If R is a valuation ring for a valuation $v : K \to \overline{\Gamma}$ and if the set of integer multiples of $v(\gamma)$ is unbounded in R for some $\gamma \in R \setminus \{0\}$, then we can choose $\gamma_n := \gamma^n$ for all $n \in \mathbb{N}$; this yields $\gamma_n^{-1}\gamma_{n+1} = \gamma$ for each $n \in \mathbb{N}$. In the non-powerful case the situation may be much more complicated.

In particular, we have a short exact sequence $0 \to R^{(\mathbb{N})} \to R^{(\mathbb{N})} \to$

$K \to 0$, where the morphism $R^{(\mathbb{N})} \to R^{(\mathbb{N})}$ maps the n-th unit vector e_n to $e_n - (\gamma_n^{-1}\gamma_{n+1})e_{n+1}$; the other non-trivial map is $R^{(\mathbb{N})} \to K$, $e_n \mapsto \gamma_n^{-1}$. M is also *sequential* in the canonical topology; i.e. every sequentially closed subset of M is closed. Finally we assume R to be *(sequentially) complete* in the canonical topology; this means that every sum in R converges in the canonical topology provided its members converges to 0. This is more general as the case of a valuation ring, but it cannot always be achieved by completion; e.g. the completion of the powerful integral domain $\mathbb{Z}$ has zero-divisors; it is the product of all rings of p-adic integers for all primes p.

For valuations v, v' and a surjection σ as above, every R'-module is an R-module by restriction of the operations, and one easily sees that the canonical topologies coincide for both valuations.

The first *Ulm functor* $U = U^1$ is defined by $UM := \bigcap_{\alpha \in R\setminus\{0\}} \alpha M$ for every R-module M; this is an R-module again. Moreover, U^0 is the identity functor for R-modules; for each ordinal number κ we define $U^{\kappa+1} := UU^{\kappa}$, and for a limit ordinal λ we set $U^{\lambda}M := \bigcap_{\kappa<\lambda} U^{\kappa}M$. Moreover, we put $U^{\infty}M := \bigcap_{\kappa \,\text{ordinal}} U^{\kappa}M$. Then M is *divisible* if and only if $UM = M$ holds; M is called *division-free* if and only if $UM = \{0\}$ holds. If M admits infinitary linear combinations with (LC1) and (LC2), then all $U^{\kappa}M$, in particular $U^{\infty}M$, are closed under these combinations.

Theorem 3.1 *The division-free R-modules form a full reflective subcategory of the category of all R-modules; the reflection maps M to M/UM.* □

Proposition 3.2 *For an R-module M the following statements are equivalent:*

(i) *M is reduced.*

(ii) $U^{\infty}M = \{0\}$.

(iii) $\mathrm{Hom}(K, M) = \{0\}$.

Proof. (i) ⇒ (ii): The $U^{\kappa}M$ form a decreasing sequence of submodules of M; thus it must become constant somewhere. Hence there exists an ordinal κ with $U^\kappa M = U^{\kappa+1}M = UU^\kappa M$; therefore $U^\infty M = U^\kappa M$ is divisible. Now (i) yields $U^\infty M = \{0\}$.

(ii) $\Rightarrow$ (iii): For an R-linear map $f : K \to M$ we obtain $f(K) = f(U^\infty K) \subset U^\infty M = \{0\}$.

(iii) $\Rightarrow$ (i): Let $D \subset M$ be a divisible submodule and assume $x_0 \in D$. We have an increasing representation $K = \bigcup_{n=0}^\infty \gamma_n^{-1} R$ with $\gamma_0 = 1$. Since D is divisible, we can find a sequence $(x_n)_{n\in\mathbb{N}}$ in D with $(\gamma_n^{-1}\gamma_{n+1})x_{n+1} = x_n$ for all $n \in \mathbb{N}$. Now for each $n \in \mathbb{N}$ we consider the map $\gamma_n^{-1}R \to M$, $\xi \mapsto \gamma_n \xi x_n$. This map is R-linear, and the map for $n+1$ extends the map for n. So they can be merged to a linear map $f : K \to M$, which is trivial by (iii). This implies $x_0 = f(1) = 0$, proving $D = \{0\}$. $\square$

The statements (i) and (ii) are always equivalent, and they imply (iii), but the proof of (iii) $\Rightarrow$ (i) needs the hypothesis that R be powerful. Modules satisfying (iii) are called h-reduced, e.g. by Matlis [9]. The assumption is necessary in the case of a valuation ring R for a Krull valuation $v : K \to \overline{\Gamma}$. Indeed, if R is not powerful, i.e. if in $\Gamma \neq \{0\}$ every countable subset is bounded; then the homological dimension of K as an R-module is ≥ 2 by VI,3.4 of [3], and from VII, 2.8 of [3] we see that there exists a reduced R-module which is not h-reduced and hence an h-divisible R-module which is not divisible.

The canonical topology of an R-module M has the set of all γM with $\gamma \in R \setminus \{0\}$ as a basis of 0-neighbourhoods; it is Hausdorff if and only if M is *division-free*. The finite sums $\sum_{n=0}^m \alpha_n x_n$ converge to the infinitary linear combination $\sum_{n=0}^\infty \alpha_n x_n$ in the canonical topology for all $\alpha_\bullet \in \Omega$. In the division-free case, the limit is unique, (LC1) and (LC2) are clearly satisfied, and we can split up the infinitary linear combinations into module operations and limits. In general, an infinitary linear combination is *one* operation and the *coefficients* are crucial, not just the *summands*. In particular, it cannot be split into module operations and some unique limits, maybe in a finer topology, as we see in the following

Theorem 3.3 *If an R-module M admits infinitary linear combinations satisfying* (LC1) *and* (LC2)*, then an element x_0 of M belongs to UM if* *and only if it is of the form* $x_0 = \sum_{n=0}^{\infty} \alpha_n y_n$ *with* $\alpha_\bullet \in \Omega$ *and* $\alpha_n y_n = 0$ *for all* $n \in \mathbb{N}$.

Proof. For x_0 of the given form and for $\gamma \in R \setminus \{0\}$ there is an $m \in \mathbb{N}$ such that m applications of $^\odot$ to $\gamma^{-1}\alpha_\bullet$ yield an element of Ω. Then we have

$$\begin{aligned} x_0 = \quad & \sum_{n=0}^{\infty} \alpha_n y_n = \quad \sum_{n=0}^{m-1} \alpha_n y_n + \sum_{n=m}^{\infty} \alpha_n y_n \\ = \quad \sum_{n=0}^{m-1} 0 + & \sum_{n=m}^{\infty} (\gamma^{-1}\alpha_n)(\gamma y_n) = \quad \gamma \sum_{n=m}^{\infty} (\gamma^{-1}\alpha_n) y_n \in \gamma M. \end{aligned}$$

This yields $x_0 \in UM$.

Conversely, we have an increasing representation $K = \bigcup_{n=0}^{\infty} \alpha_n^{-1} R$ with $\alpha_0 = 1$; in particular we have $\alpha_\bullet \in \Omega$. Since $x_0 \in UM$, there are x_n in M with $x_0 = \alpha_n x_n$ for all $n \in \mathbb{N}$. Then for $y_n := x_n - (\alpha_n^{-1}\alpha_{n+1}) x_{n+1}$ ($n \in \mathbb{N}$) we obtain

$$\begin{aligned} x_0 = \quad x_0 + \sum_{n=1}^{\infty} \alpha_n x_n - \sum_{n=1}^{\infty} \alpha_n x_n = \quad & \sum_{n=0}^{\infty} \alpha_n x_n - \sum_{n=1}^{\infty} \alpha_n x_n \\ = \quad \sum_{n=0}^{\infty} \alpha_n x_n - \sum_{n=0}^{\infty} \alpha_{n+1} x_{n+1} = \quad & \sum_{n=0}^{\infty} \alpha_n y_n \end{aligned}$$

with $\alpha_n y_n = \alpha_n x_n - \alpha_{n+1} x_{n+1} = x_0 - x_0 = 0$. $\square$

A topological R-module M is called *bounded* if for every 0–neighbourhood $W \subset M$ there is an $\alpha \in R \setminus \{0\}$ with $\alpha M \subset W$. M is called sequentially complete if every Cauchy sequence converges; a sequence $(x_n)_{n \in \mathbb{N}}$ is called a Cauchy sequence if for every 0–neighbourhood $W \subset M$ there exists an $n_0 \in \mathbb{N}$ with $x_{n_1} - x_{n_2} \in W$ for all $n_1, n_2 \geq n_0$; since we are in the non-archimedean situation this property of n_0 is equivalent to $x_{n+1} - x_n \in W$ for all $n \geq n_0$. If M is a sequentially complete R-module, then for every $\alpha_\bullet \in \Omega$ the partial sums $\sum_{n=0}^{m} \alpha_n x_n$ converge to $\sum_{n=0}^{\infty} \alpha_n x_n$ for $m \to \infty$.

Proposition 3.4 *Let* M *be a topological* R*–module such that* $(\sum_{n=0}^{m} \alpha_n x_n)_{m \in \mathbb{N}}$ *converges for every* $\alpha_\bullet \in \Omega$, $x_\bullet \in M^{\mathbb{N}}$. *Then* M *is bounded.*

Proof. Assume the contrary. Then there is a 0–neighbourhood $W \subset M$ with $\gamma M \not\subset W$ for all $\gamma \in R \setminus \{0\}$. Since R is powerful, we can find $\alpha_n \in R$, $n \in \mathbb{N}$ with $\bigcup_{n\in\mathbb{N}} \alpha_n^{-1} R = K$; we can even achieve $\alpha_{n+1} \in \alpha_n R$ for all $n \in \mathbb{N}$. Then for every $n \in \mathbb{N}$ there is an $x_n \in M$ with $\alpha_n x_n \notin W$; hence $(\alpha_n x_n)_{n\in\mathbb{N}}$ does not converge to 0 in M, therefore $(\sum_{n=0}^{m} \alpha_n x_n)_{m\in\mathbb{N}}$ cannot converge in M, though $\alpha_\bullet \in \Omega$. □

The following observation is crucial for the further discussion: We shall see later that infinitary linear combinations do not guarantee that the canonical topology is Hausdorff, i.e. that the module is *division-free*. But it is still true that the modules are *reduced*, i.e. they may contain non-trivial divisible *elements*, but not divisible *submodules* (which would be closed under these operations by our previous remark).

Theorem 3.5 *Every R-module that admits infinitary linear combinations satisfying* (LC1) *and* (LC2) *is reduced.*

Proof. We use 3.2 (iii) ⇒ (i). For an R-module M let $f : K \to M$ be R-linear. As above, we can represent K as an increasing union $\bigcup_{n=0}^{\infty} \alpha_n^{-1} R$ of copies of R with $\alpha_0 = 1$. Then in M for every $\xi \in K$ we obtain

$$
\begin{aligned}
f(\xi) + \sum_{n=1}^{\infty} \alpha_n f(\alpha_n^{-1}\xi) &= \sum_{n=0}^{\infty} \alpha_n f(\alpha_n^{-1}\xi) = \\
\sum_{n=0}^{\infty} \alpha_n \left((\alpha_n^{-1}\alpha_{n+1}) f(\alpha_{n+1}^{-1}\xi)\right) &= \sum_{n=0}^{\infty} \alpha_{n+1} f(\alpha_{n+1}^{-1}\xi) = \\
\sum_{n=1}^{\infty} \alpha_n f(\alpha_n^{-1}\xi)\,,
\end{aligned}
$$

hence $f(\xi) = 0$, proving $f = 0$. □

The following observation looks surprising at first glance:

Theorem 3.6 *If M and N admit infinitary linear combinations with* (LC1) *and* (LC2) *and if $f : M \to N$ is an R-module homomorphism, then f preserves these infinitary linear combinations.*

Proof. The set

$$D := \{\sum_{n=0}^{\infty} \alpha_n f(x_n) - f(\sum_{n=0}^{\infty} \alpha_n x_n) | x_n \in M \text{ for } n \in \mathbb{N}, \alpha_\bullet \in \Omega\}$$

is an R-submodule of M; and we have to show $D = \{0\}$; by 3.5 it suffices to show that D is divisible. For an arbitrary

$$y = \sum_{n=0}^{\infty} \alpha_n f(x_n) - f(\sum_{n=0}^{\infty} \alpha_n x_n) \in D,$$

and $m \in \mathbb{N}$ we obtain

$$y = \sum_{n=0}^{m-1} \alpha_n f(x_n) - f(\sum_{n=0}^{m-1} \alpha_n x_n) + \sum_{n=m}^{\infty} \alpha_n f(x_n) - f(\sum_{n=m}^{\infty} \alpha_n x_n) = \sum_{n=m}^{\infty} \alpha_n f(x_n) - f(\sum_{n=m}^{\infty} \alpha_n x_n)$$

because the linear map f preserves finitary linear combinations. For every $\gamma \in R \setminus \{0\}$ there exists an $m \in \mathbb{N}$ with $\gamma^{-1}\alpha_\bullet \star m \in \Omega$. Then we have $z := \sum_{n=m}^{\infty}(\gamma^{-1}\alpha_n) f(x_n) - f(\sum_{n=m}^{\infty}(\gamma^{-1}\alpha_n)x_n) \in D$ and $\gamma z = y$. This proves $y \in UD$, hence D is divisible. □

Corollary 3.7 *On an R-module there is at most one way to introduce infinitary linear combinations with* (LC1) *and* (LC2) *extending the finitary ones.*

Proof. Apply 3.6 to the identity map. □

This is the only case we know, where a full reflective subcategory of an equational locally finitely presentable category is equational (and locally countably presentable), but not locally finitely presentable.

4 Cotorsion Modules

We have seen that infinitary linear combinations are unique in every R-module and are preserved by every R-linear map; so the remaining question is their *existence*. Modules admitting infinitary linear combinations with (LC1) and (LC2) form a full subcategory of the category of all R-modules; we shall see that they coincide with the modules called *cotorsion* modules by Matlis [9], who introduced them. Later it has become common to define this term without reducedness; also the stronger notions of *Enochs-cotorsion* and *Warfield-cotorsion* were introduced; they were studied by Enochs, Fuchs, Harrison, Matlis and Warfield (cf. [2], [3], [5]). M is a *reduced Matlis-cotorsion module* if and only if $\mathrm{Hom}(K, M) = \{0\}$ and $\mathrm{Ext}^1(K, M) = \{0\}$ hold; torsion-free ones are classical examples of splitters (cf. [4]).

The above subcategory is also *reflective*. Once we have characterized it, we can describe the reflection in terms of the long cohomology sequence. But in order to achieve the characterization, we need its existence first and also another property.

Lemma 4.1 *The R-modules admitting infinitary linear combinations with* (LC1) *and* (LC2) *form a reflective subcategory of the category of R-modules. The reflection map is injective if and only if M is reduced.*

Proof. Since the category of modules with infinitary linear combinations with (LC1) and (LC2) is defined by adding more operations and equations to an equationally defined category, the existence of the left adjoint follows from the Adjoint Functor Theorem (cf. [1]).

Obviously, M is reduced by 3.5 if the reflection map $r : M \to M'$ is injective. Conversely, for reduced M consider the pushout of r along the multiplication $M \to M$, $\xi \mapsto \gamma\xi$ for some $\gamma \in R \setminus \{0\}$, i.e. the map $r' : M \to M'' := (M \times M')/N$, $x \mapsto (x, 0) + N$, where $N := \{(\gamma x, -r(x)) | x \in M\} \subset M \times M'$. This yields a module structure on M''; it does not admits infinitary linear combinations with (LC1) and (LC2) for simple categorical reasons. Usually, the definition does not contain the minus sign; but here it does not change the module and it allows to "add the components" in the usual way. The infinitary linear combinations are given by

$$\sum_{n=0}^{\infty} \alpha_n((x_n, y_n) + N) := (\sum_{n=0}^{\infty} \alpha_n r'(x_n, 0)) + (0, \sum_{n=0}^{\infty} \alpha_n y_n) + N$$

and

$$\sum_{n=0}^{\infty} \alpha_n r'(x_n, 0) := (\sum_{n=0}^{m} \alpha_n x_n, \sum_{n=m+1}^{\infty} (\gamma^{-1}\alpha_n) r(x_n)) + N = \sum_{n=0}^{m} \alpha_n r'(x_n, 0) + \gamma \sum_{n=m+1}^{\infty} (\gamma^{-1}\alpha_n) r'(x_n, 0)$$

whenever $\alpha_\bullet \in \Omega$, $x_\bullet \in M^{\mathbb{N}}$, $y_\bullet \in M'^{\mathbb{N}}$,$\gamma \in R \setminus \{0\}$, and $\gamma^{-1}\alpha_n \in R$ for $n > m$; the result does not depend on the choice of m.

Now from the universal property of r we obtain an R-module homomorphism $f : M' \to M''$ with $f \circ r = r'$. This implies that the kernel $\tilde{N}$ of r is contained in the kernel $\gamma\tilde{N}$ of r'. As $\gamma \in R \setminus \{0\}$ is arbitrary, $\tilde{N} \subset M$ is divisible; since M is reduced, this implies $\tilde{N} = \{0\}$. □

Theorem 4.2 *For an R-module M the following statements are equivalent:*

(i) *M admits infinitary linear combinations with* (LC1) *and* (LC2)*.*

(ii) *The following system of linear equations has a unique solution $(y_{\alpha_\bullet, x_\bullet})_{\alpha_\bullet \in \Omega, x_\bullet \in M^{\mathbb{N}}}$:*

$$y_{\alpha_\bullet, x_\bullet} = \alpha_0 x_0 + y_{\alpha_\bullet^\odot, x_\bullet^\odot} \text{ for all } \alpha_\bullet \in \Omega,\ x_\bullet \in M^{\mathbb{N}}$$
$$y_{\beta\alpha_\bullet, x_\bullet} = \beta y_{\alpha_\bullet, x_\bullet} \text{ for all } \beta \in R,\ m \in \mathbb{N},\ \alpha_\bullet \in \Omega,\ x_\bullet \in M^{\mathbb{N}}.$$

(iii) *M is a reduced Matlis-cotorsion module, i.e.* $\mathrm{Hom}(K, M) = \{0\}$ *and* $\mathrm{Ext}^1(K, M) = \{0\}$*.*

Proof. (i) ⇔ (ii): If we have infinitary linear combinations, we can solve the system of equations by $y_{\alpha_\bullet, x_\bullet} := \sum_{n=0}^{\infty} \alpha_n x_n$ for all $\alpha_\bullet \in \Omega$, $m \in \mathbb{N}$, $(x_n)_{n\in\mathbb{N}} \in M^{\mathbb{N}}$, $\beta \in R$. The solution is unique, because the corresponding homogenous system has only the zero solution. Indeed, if we have a solution for $x_n = 0$ for all $n \in \mathbb{N}$, then the y_n form a divisible submodule of M, which must vanish by (i) and 3.5. Conversely, if we have a solution of the system of equations, we define $\sum_{n=0}^{\infty} \alpha_n x_n := y_{\alpha_\bullet, x_\bullet}$ for $\alpha_\bullet \in \Omega$, $(x_n)_{n\in\mathbb{N}} \in M^{\mathbb{N}}$. Using the uniqueness, we get (LC1) and (LC2).

(i) $\Rightarrow$ (iii): Consider the *torsion-free cover* F of M; it can be constructed as the module of all homomorphisms f from K into the injective hull J of M with $f(1) \in M$. Then F is torsion-free, and we have the canonical projection $F \to M$; according to the above construction it is the linear map $f \mapsto f(1)$. Its kernel N consists of all $f : K \to J$ with $f(1) = 0$. Moreover, N is torsion-free as a submodule of F.

Now assume that M admits infinitary linear combinations with (LC1) and (LC2); from 3.5 we see that M is reduced. So we still have to show $\mathrm{Ext}^1(K, M) = \{0\}$. For each fixed $\xi \in K$ there is a $\gamma \in R \setminus \{0\}$ with $\gamma\xi \in R$. Then for every $g \in F$ we obtain $\gamma g(\xi) = g(\gamma\xi) = (\gamma\xi)g(1) \in M$. So for ξ and γ as above we can define the map $\sum_{n=0}^{\infty} \alpha_n f_n : K \to J$ in F by $(\sum_{n=0}^{\infty} \alpha_n f_n)(\xi) := \sum_{n=0}^{\infty} \alpha_n(\gamma f_n(\gamma^{-1}\xi))$. This definition does not depend on the choice of γ.

This defines infinitary linear combinations with (LC1) and (LC2) on the torsion-free R-module F. So F is complete in the canonical topology, and from Matlis [9] we obtain $\mathrm{Ext}^1(K, F) = \{0\}$. Since K has projective dimension 1, we have $\mathrm{Ext}^2(K, N) = \{0\}$.

Now we apply the left-exact functor $\mathrm{Hom}(K, -)$ to the short exact sequence

$$0 \to N \to F \to M \to 0.$$

Then the long cohomology sequence contains the part $\mathrm{Ext}^1(K, F) \to \mathrm{Ext}^1(K, M) \to \mathrm{Ext}^2(K, N)$. Since the first and the last module vanish, we can conclude $\mathrm{Ext}^1(K, M) = \{0\}$.

(iii) $\Rightarrow$ (i): Assume $\mathrm{Hom}(K, M) = \{0\}$ and $\mathrm{Ext}^1(K, M) = \{0\}$ and let $r : M \to M'$ be the reflection map from 4.1. Then M is reduced by 3.2, hence r is injective, and we have a short exact sequence

$$0 \to M \to M' \to M'/M.$$

Since M' admits infinitary linear combinations with (LC1) and (LC2), it is also reduced, i.e. $\mathrm{Hom}(K, M') = \{0\}$, and from the long cohomology sequence we see that $\mathrm{Hom}(K, M'/M) = \{0\}$, i.e. M'/M is reduced. But M'/M is also divisible: For all $z \in M'$ and all $\gamma \in R \setminus \{0\}$ we have to show $z + r(M) \in \gamma(M'/r(M)) = \gamma M' + r(M)$. By a routine argument, z is of the form $z = \sum_{n=0}^{\infty} \alpha_n r(x_n)$ for some $\alpha_\bullet \in \Omega$ and some $x_\bullet \in M^{\mathbb{N}}$, because all such elements form a submodule of M' containing $r(M)$ closed under infinitary linear combinations. Then there is an $m \in \mathbb{N}$ with $\gamma^{-1}\alpha_\bullet \star m \in \Omega$, and then we get $z = \sum_{n=0}^{\infty} \alpha_n r(x_n) = \sum_{n=0}^{m-1} \alpha_n r(x_n) + \sum_{n=m}^{\infty} \alpha_n r(x_n) = r(\sum_{n=0}^{m-1} \alpha_n x_n) + \gamma \sum_{n=m}^{\infty} (\gamma^{-1}\alpha_n) r(x_n) \in r(M) + \gamma M'$.

So M'/M is both divisible and reduced, thus we have $M'/M = \{0\}$. This means that r is bijective; since M' admits infinitary linear combinations with (LC1) and (LC2), M also does. □

For an individual $\alpha'_\bullet \in \Omega$ and $x'_\bullet \in M^{\mathbb{N}}$ the linear combination $\sum_{n=0}^{\infty} \alpha'_n x'_n$ can be uniquely characterized by a countable system of linear equations. In order to determine $y_{\alpha_\bullet, x_\bullet}$ it suffices to have the equations in (ii) for countably many cases.

For an increasing representation $K = \bigcup_{n=0}^{\infty} \alpha_n^{-1} R$ we need only the countably many cases where β is α_m for some $m \in \mathbb{N}$ and $\beta\alpha_\bullet$ is obtained from $\alpha'_\bullet$ by finitely many applications of $^\odot$ and $x_\bullet$ is obtained from $x'_\bullet$ by the same number of applications of $^\odot$. This is true because $(\beta^{-1}\alpha_m)_{m\in\mathbb{N}}$ always converges to 0 for $\beta \neq 0$, therefore it contains only finitely many elements outside R. For $\beta = 0$ the statement is trivial anyway.

Corollary 4.3 *Every torsion-free reduced Matlis-cotorsion module is division-free.*

Proof. For a torsion-free reduced Matlis-cotorsion module M assume $x \in UM$. Then by 3.3, x can be written as $x = \sum_{n=0}^{\infty} \alpha_n y_n$ with $\alpha_n y_n = 0$ for all $n \in \mathbb{N}$. Since M is torsion-free, this implies $y_n = 0$ for all $n \in \mathbb{N}$ with $\alpha_n \neq 0$. But this easily yields $x = 0$. □

Lemma 4.4 $\mathrm{Ext}^1(K/R, M) = \{0\}$ *holds for every divisible R-module M.*

Proof. Represent K as an increasing union of copies $\gamma_n^{-1}R$ of R. It suffices to show that every short exact sequence $0 \to M \to M' \to K/R \to 0$ splits. Since divisible modules are closed under extensions, M' is also divisible. Since the projection $q : M' \to K/R$ in this sequence is surjective, there is an $x_0 \in M'$ with $q(x_0) = \gamma_0^{-1} + R$. If x_n has already been defined with $q(x_n) = \gamma_n^{-1} + R$, the surjectivity of q yields an $y \in M'$ with $q(y) = \gamma_{n+1}^{-1} + R$, hence also $q((\gamma_n^{-1}\gamma_{n+1})y - x_n) = (\gamma_n^{-1}\gamma_{n+1})q(y) - q(x_n) = 0$. Therefore $(\gamma_n^{-1}\gamma_{n+1})y - x_n$ is in the kernel of q, i.e. in the image of the divisible module M'. Thus there is a $z \in M'$ with $q(z) = 0$ and $(\gamma_n^{-1}\gamma_{n+1})z = (\gamma_n^{-1}\gamma_{n+1})y - x_n$, i.e. $x_n = (\gamma_n^{-1}\gamma_{n+1})(y - z)$. Now we define $x_{n+1} := y - z \in M'$, thus $(\gamma_n^{-1}\gamma_{n+1})x_{n+1} = x_n$ and $q(x_{n+1}) = q(y) - q(z) = \gamma_{n+1}^{-1} + R$. Thus there is a unique linear map $K/R \to M'$, which maps $\gamma_n^{-1} + R$ to x_n for all $n \in \mathbb{N}$; this map splits the short exact sequence. □

For R a discrete valuation ring 4.4 is obvious because every divisible module is injective.

Theorem 4.5 *Mapping an arbitrary R-module M to the canonical map $M \to \operatorname{Ext}^1(K/R, M)$ yields the reflection from the category of R-modules to the category of reduced Matlis-cotorsion R-modules.*

Proof. First we claim that the canonical map $\operatorname{Ext}^1(K/R, M) \to \operatorname{Ext}^1(K/R, M/U^\infty M)$ is always an isomorphism. The long cohomology sequence for the functor $\operatorname{Hom}(K/R, -)$ applied to the short exact sequence $0 \to U^\infty M \to M \to M/U^\infty M \to 0$ contains the part $\operatorname{Ext}^1(K/R, U^\infty M) \to \operatorname{Ext}^1(K/R, M) \to \operatorname{Ext}^1(K/R, M/U^\infty M) \to \operatorname{Ext}^2(K/R, U^\infty M)$. The first module vanishes by 4.4, and the last part vanishes, since R has cohomological dimension 1. The assignment in the theorem is obviously a natural transformation. By 3.2 $U^\infty M$ is always in the kernel, so by 4.4 we can restrict our attention to reduced modules. Since K/R is a torsion module and $M/U^\infty M$ is reduced, we have $\operatorname{Hom}(K/R, M/U^\infty M) = \{0\}$. Now by 2.1 of [9] $\operatorname{Ext}^1(K/R, M)$ is a Matlis-cotorsion module.

For a reduced Matlis-cotorsion module M, the long cohomology sequence of $\operatorname{Hom}(-, M)$ applied to $0 \to R \to K \to K/R \to 0$ contains the part $\operatorname{Hom}(K, M) \to \operatorname{Hom}(R, M) \to \operatorname{Ext}^1(K/R, M) \to \operatorname{Ext}^1(K, M)$. Since the first and the last module vanish, the middle arrow is an isomorphism; thus also the canonical arrow $M \to \operatorname{Ext}^1(K/R, M/U^\infty M) \cong \operatorname{Ext}^1(K/R, M)$.

So we have a natural transformation from the identity functor to a functor that maps all R-modules to reduced Matlis-cotorsion R-modules, and for all reduced Matlis-cotorsion R-modules the natural transformation yields an isomorphism. Therefore this must be the reflection. □

5 The Chain of Ulm Functors

We have seen that every R-module M admitting infinitary linear combinations with (LC1) and (LC2) is *reduced*, but we have not seen that it is *division-free*, though we have not given a counterexample up to now. At first glance the difference looks quite harmless; we always have $U^\infty M = \{0\}$, but not necessarily $UM = \{0\}$. We do not see immediately why not all elements of UM have to be divisible, i.e. why we may have $U^2 \neq U^1$. Of course, the U^κ form a decreasing chain of functors; thus it must be eventually constant, and for reduced M this can happen only at $\{0\}$, i.e. there is a κ with $U^\infty M = U^\kappa M = \{0\}$. But the smallest κ with this property can be arbitrarily large, even for a reduced Matlis-cotorsion module. We shall see this below, using the machinery of infinitary linear combinations used above.

Theorem 5.1 *For every torsion-free reduced Matlis-cotorsion module M and for every ordinal κ there exists a reduced Matlis-cotorsion module P with $U^\kappa P \cong M$.*

Proof. We represent K as a union of an increasing sequence of $\alpha_n^{-1}R$, we assume $\alpha_0 = 1$ and we consider the set T of all tuples $t = ((\nu_1, \ldots, \nu_n), (m_1, \ldots, m_n))$, where the ν_k form a strictly decreasing chain of ordinals $< \kappa$, including the empty tuple Λ, and where $m_1, \ldots, m_n$ is a strictly increasing tuple of natural numbers; for such a pair of pairs we use the shorter notation $t = (\nu_1, \ldots, \nu_n; m_1, \ldots, m_n)$, and we put $\nu_0 := \kappa$.

Now M^T is a torsion-free Matlis-cotorsion module; we write its elements as maps $\phi : T \to M$. Consider the submodule $M^{[T]} \subset M^T$ of all $\phi \in M^T$ such that for each $\gamma \in R \setminus \{0\}$ there are only finitely many $t \in T$ with $\phi(t) \notin \gamma M$; since R is powerful and M is torsion-free, hence also division-free, this implies $\phi(t) = 0$ for all but countably many $t \in T$; we write ϕ as a formally uncountable linear combination. For each $t \in T$ we have an R-module homomorphism $u_t : M \to M^{[T]}$ defined by $u_t(x)(t) := x$ for $x \in M$ and $u_t(x)(t') := 0$ for $t' \in T$ with $t' \neq t$. Let $N \subset M^{[T]}$ be the submodule of all infinitary linear combinations of elements of the form $u_{(\nu_1,\ldots,\nu_{n-1};m_1,\ldots m_{n-1})}(x) - (\alpha_{m_{n-1}}^{-1}\alpha_{m_n})u_{(\nu_1,\ldots,\nu_n;m_1,\ldots,m_n)}(x)$. In such a representation of a $\phi \in N$, we assume w.l.o.g. that all $t :=$

$(\nu_1, \ldots, \nu_n; m_1, \ldots, m_n)$ are different; otherwise we gather all entries with the same t. Moreover we can assume $x \neq 0$ for all summands; otherwise we can omit them. If $\phi \neq 0$, for some t the coefficient β of some $u_t(x)$ does not vanish; we can choose $t = (\nu_1, \ldots \nu_n; m_1 \ldots, m_n)$ in such a way that ν_n is minimal. Since M is torsion-free by hypothesis, this implies $\phi(t)(\beta\alpha_{m_{n-1}}^{-1}\alpha_{m_n})x \neq 0$. So if we have $\phi(t) = 0$ for all $t := (\nu_1, \ldots, \nu_n; m_1, \ldots, m_n)$ with $\nu_n < \mu$ for some ordinal μ, we see that ϕ is a linear combination of generators t as above with $\nu_n \leq \mu$. Let $P := M^{[T]}/N$ be the quotient and let $q : M^{[T]} \to P$ be the canonical projection. Since $N \subset M^{[T]}$ is closed under infinitary linear combinations, P is a reduced Matlis-cotorsion module.

We claim that for each ordinal $\mu \leq \kappa$ the Ulm submodule $U^\mu P$ consists of all $q(x)$, where $x(\nu_1, \ldots, \nu_n; m_1, \ldots m_n) = 0$ holds whenever $\nu_n < \mu$. The limit step is obvious. Now we assume the statement for some $\mu < \kappa$ and prove it for $\mu + 1$. For the first direction consider a $\phi \in M^{[T]}$ with $\phi(\nu_1, \ldots, \nu_n; m_1, \ldots, m_n) = 0$ for $\nu_n \leq \mu$ and a $\gamma \in R\setminus\{0\}$. By $\phi \in M^{[T]}$, the set $S := \{s \in T | \phi(s) \notin \gamma M\}$ is finite, and we have $\phi = \gamma\psi + \sum_{t \in T\setminus S} u_T(\phi(t))$ for some $\psi \in M^{[T]}$ with $\psi(\nu_1, \ldots, \nu_n; m_1, \ldots, m_n) = 0$ for $\nu_n \leq \mu$; by induction hypothesis this implies $\psi \in U^\mu P$. Now there exists a natural number $l > m_n$ with $\alpha_{m_n}^{-1}\alpha_l \in \gamma R$. For $\nu_{n+1} := \mu$ and $m_{n+1} := l$ we get

$$u_{(\nu_1,\ldots,\nu_n;m_1,\ldots,m_n)}(x) - (\alpha_{m_n}^{-1}\alpha_{m_{n+1}})u_{(\nu_1,\ldots,\nu_{n+1};m_1,\ldots,m_{n+1})}(x) \in N,$$

hence

$$q(u_{(\nu_1,\ldots,\nu_n;m_1,\ldots,m_n)}(x)) = (\alpha_{m_n}^{-1}\alpha_{m_{n+1}})q(u_{(\nu_1,\ldots,\nu_{n+1};m_1,\ldots,m_{n+1})}(x)),$$

for all $x \in M$. By induction hypothesis we have

$$q(u_{(\nu_1,\ldots,\nu_{n+1};m_1,\ldots,m_{n+1})}(x)) \in U^\mu P,$$

therefore

$$q(u_{(\nu_1,\ldots,\nu_n;m_1,\ldots,m_n)}(x)) \in (\alpha_{m_n}^{-1}\alpha_{m_{n+1}})U^\mu P \subset \gamma U^\mu P.$$

This implies

$$\phi = \gamma\psi + \sum_{s\in S} u_s(\phi(s)) \in \gamma U^\mu P,$$

proving $\phi \in UU^\mu P = U^{\mu+1}P$.

Conversely, assume $\phi \in M^{[T]}$ with $q(\phi) \in U^{\mu+1}P = UU^\mu P \subset U^\mu P$; by hypothesis we assume $\phi(\nu_1, \ldots, \nu_n; m_1, \ldots, m_n) = 0$ for $\nu_n < \mu$. For every $\gamma \in R \setminus \{0\}$ there is a $\psi \in U^\mu P$ with $\psi(\nu_1, \ldots, \nu_n; m_1, \ldots, m_n) = 0$ for $\nu_n < \mu$ and $q(\phi) = \gamma q(\psi) = q(\gamma\psi)$, hence $\phi - \gamma\psi \in N$. For $t = (\nu_1, \ldots, \nu_n; m_1, \ldots, m_n) \in T$ we have $\phi - \gamma\psi(t) = 0$ whenever $\nu_n < \mu$. Then by our above considerations we can assume that only elements t with $\nu_n \geq \mu$ have non-zero coefficients; therefore we have $\phi - \gamma\psi(t) = 0$ for $\nu_n = \mu$. Thus we get $\phi(t) = \gamma\psi(t) \in \gamma M$; this proves $\phi(t) \in UM = \{0\}$, because M is division-free. □

This proof works only in the torsion-free case; otherwise the transfinite induction breaks down. We do not see whether 5.1 is still true otherwise, even in the division-free case. But of course, $U^\kappa P$ is not always division-free; the above construction also yields modules with prescribed division-free $U^{\kappa+1}P$. The question looks even interesting for $\kappa = 1$; then it would follow for all finite κ. Moreover, maybe one can only prove that every reduced Matlis-cotorsion module occurs as a submodule of some $U^\kappa P$.

Corollary 5.2 *The category of reduced Matlis-cotorsion R-modules has no cogenerator.*

Proof. Assume the contrary, i.e. let C be a cogenerator. Then there exists an ordinal κ with $U^\infty C = U^\kappa C = \{0\}$. Now by 5.1 there exists a reduced Matlis-cotorsion R-module M with $U^\kappa M = R \neq \{0\}$. Then an R-linear map $M \to C$ maps all elements of $U^\kappa M$ to 0, hence it does not separate them. □

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