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COVERING MORPHISMS AND NORMAL EXTENSIONS IN GALOIS STRUCTURES ASSOCIATED WITH TORSION THEORIES

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Dedicated to Francis Borceux on the occasion of his 60th birthday

ABSTRACT. Nous étudions les revêtements et les extensions normales relatives aux structures galoisiennes munies de ce que nous appelons *foncteurs test*. Ces foncteurs apparaissent naturellement dans les structures galoisiennes associées aux théories de torsion dans les catégories homologiques. Sous des hypothèses additionnelles appropriées, tout morphisme à noyau sans torsion est un revêtement, et tout revêtement est une extension normale, pourvu qu'il soit un morphisme de descente effective. Nos contre-exemples, qui montrent l'importance de ces conditions supplémentaires, sont semi-abéliens, et proviennent de la théorie des groupes, en faisant intervenir des produits semi-directs de groupes cycliques. Nous comparons nos nouveaux résultats avec ceux connus pour les revêtements localement semi-simples et pour les extensions centrales généralisées.

ABSTRACT. We study covering morphisms and normal extensions with respect to Galois structures equipped with what we call test functors. These test functors naturally occur in Galois structures associated with torsion theories in homological categories. Under suitable additional conditions, every morphism with a torsion free kernel is a covering, and every covering is a normal extension whenever it is an effective descent morphism. Our counter-examples showing the relevance of those additional conditions are semi-abelian, and moreover, group-theoretic, involving semidirect products of cyclic groups. We also briefly compare our new results with what is known for the so-called locally semi-simple coverings and for generalized central extensions.

Introduction

The purpose of *categorical Galois theory* is to study covering morphisms in general categories defined with respect to so-called Galois structures,

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which sometimes are merely abstract semi-left-exact reflections in the sense of [CHK] (see e.g. [J1], [BorJ], [J2]). Apart from classical cases, where the covering morphisms become (quasi-) separable algebras over commutative rings, ordinary covering maps of locally connected topological spaces, étale coverings of schemes in algebraic geometry, etc., there are non-trivial examples far away from commutative algebra and algebraic topology and geometry, such as generalized central extensions in congruence modular varieties of universal algebras. Another "non-classical" case is the Galois theory associated with a torsion theory; it was briefly examined in [CJKP] in the abelian case, and then in [GR] for non-abelian torsion theories in the sense of [BG]. While being Galois theory of the torsion-free reflection, it also substantially uses the torsion coreflection, and clearly suggests considering a more general situation of an abstract Galois structure equipped with what we call a *test functor* because such a functor T is required to "test" trivial covering morphisms via the equivalence

$$(A,f) \text{ is a trivial covering} \Leftrightarrow T(f) \text{ is an isomorphism.}$$

As explained in Section 2 below, in the case of a torsion theory this condition essentially follows from *Bourn protomodularity*.

The purpose of the present paper is to continue the study of Galois theories associated with torsion theories, and, specifically, to prove/explain/clarify the following:

- Under suitable additional conditions, every morphism with a torsion free kernel is a covering, and every covering is a normal extension whenever it is a monadic extension (=an effective descent morphism).

- There are simple (counter-)examples showing the relevance of those additional conditions. The ones we consider have varieties of groups as their ground categories, and all groups used in the covering morphisms we construct are nothing but semidirect products of cyclic groups.

- The ground structure needed to obtain our main results is far more general than a torsion theory: it is a *finitely complete admissible Galois* *structure* equipped with the above-mentioned test functor: in fact it is a new notion we introduce.

• Coverings defined via torsion theories are to be compared with central extension defined via Birkhoff subcategories [JK] (see also [J2], [G2], and references there), and with locally semisimple coverings in the sense of [JMT1].

The paper is divided into six sections as follows:

1. Coverings and normal extensions in general categories. It recalls basic notions of categorical Galois theory; the next sections freely use them and their simple properties.

2. Covering morphisms under the presence of a test functor. Test functors are introduced and our main results are presented as simple propositions on a Galois structure equipped with a test functor.

3. Galois structures of torsion theories. The main results are translated into the context of a torsion theory.

Sections 4 and 5 present our examples, and Section 6 makes brief comparisons with central extensions and locally semisimple coverings.

1. Coverings and normal extensions in general categories

A *finitely complete admissible Galois structure* $\Gamma = (\mathbf{C},\mathbf{X},I,H,\eta,\varepsilon,\mathbf{F},\mathbf{\Phi})$ (as defined in [J2], slightly differently from the original definition in [J1]) on a category $\mathbf{C}$ consists of an adjunction

$$(I,H,\eta,\varepsilon) : \mathbf{C} \to \mathbf{X} \qquad (1.1)$$

between categories with finite limits, and two classes $\mathbf{F}$ and $\mathbf{\Phi}$ of morphisms in $\mathbf{C}$ and $\mathbf{X}$ respectively, whose elements are called fibrations; the following conditions on fibrations are required:

• the classes of fibrations are pullback stable;

• the classes of fibrations are closed under composition and contain all isomorphisms;

• the functors I and H preserve fibrations;

• for every object C in $\mathbf{C}$ and every fibration $\varphi : X \to I(C)$ in $\mathbf{X}$, the composite

$$I(C\times_{HI(C)}H(X)) \to IH(X) \to X \qquad (1.2)$$

of canonical morphisms is an isomorphism. This last condition is called *admissibility*.

We assume such a structure to be fixed, and, for an arbitrary object C in $\mathbf{C}$, write

$$(I^C,H^C,\eta^C,\varepsilon^C) : \mathbf{F}(C) \to \mathbf{\Phi}(I(C))$$

for the usual induced adjunction, in which:

• $\mathbf{F}(C)$ is the full subcategory in $(\mathbf{C}\downarrow C)$ with objects all (A,f) in $(\mathbf{C}\downarrow C)$, in which $f : A \to C$ is a fibration;

• similarly $\mathbf{\Phi}(I(C))$ is the full subcategory in $(\mathbf{X}\downarrow I(C))$ with objects all (X,φ) in $(\mathbf{X}\downarrow I(C))$, in which $\varphi : X \to I(C)$ is a fibration;

• $I^C(A,f) = (I(A),I(f))$, $H^C(X,\varphi) = (C\times_{HI(C)}H(X),pr_1)$, and η^C and ε^C are defined accordingly; in particular $(\varepsilon^C)_{(X,\varphi)}$ is determined by the composite (1.2) and so the admissibility condition simply says that ε^C is an isomorphism for each C in $\mathbf{C}$.

Let us recall (e.g. again from [J2]):

Definition 1.1. (a) For a fibration $p : E \to B$ in $\mathbf{C}$, the object (E,p) in $\mathbf{F}(B)$ is said to be a *monadic extension* of B if the pullback functor

$$p^* : \mathbf{F}(B) \to \mathbf{F}(E)$$

is monadic, or, equivalently, p is an effective descent morphism with respect to the class $\mathbf{F}$.

(b) An object (A,f) in $\mathbf{F}(B)$ is said to be a *trivial covering* of B if the diagram

$$\begin{array}{ccc} A & \xrightarrow{\eta_A} & HI(A) \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle HI(f)} \\ B & \xrightarrow[\eta_B]{} & HI(B) \end{array}$$

is a pullback, or, equivalently, the morphism $(\eta^B)_{(A,f)} : (A,f) \to H^B I^B(A,f)$ is an isomorphism.

(c) An object (A,f) in $\mathbf{F}(B)$ is said to be *split* over a monadic extension (E,p) of B, if $p^*(A,f)$ is a trivial covering of E.

(d) An object (A,f) in $\mathbf{F}(B)$ is said to be a *covering* of B if it is split over some monadic extension; we then also say that $f : A \to B$ is a covering morphism.

(e) A monadic extension (E,p) is said to be a *normal extension* if it is split over itself.

Remark 1.2. There is a long list of known examples, which we do not recall here. Let us, however, mention that the main ingredient of a Galois structure is of course the adjunction (1.1) that is usually a reflection. In particular the following types of reflections seem to be especially important:

(a) Totally disconnected reflections, where $I(C)$ is the object of connected components of C in a suitable sense. These types of reflections produce classical examples mentioned at the beginning of Introduction.

(b) Reflections of varieties of universal algebras into their subvarieties, or, more generally, reflections of exact categories into their Birkhoff subcategories [JK]. Their covering morphisms are generalized central extensions in the sense of [JK], and in particular central extensions of Ω-groups in the sense of A. S.-T. Lue [L], who also refers to A. Fröhlich's work. A further generalization is developed in [G2].

(c) Torsion-free reflections associated with torsion theories, whose covering morphisms are studied in this paper, continuing [GR].

2. Covering morphisms under presence of a test functor

Definition 2.1. Let $\Gamma = (\mathbf{C},\mathbf{X},I,H,\eta,\varepsilon,\mathbf{F},\Phi)$ be as above. A *test functor* is a finite limit preserving functor $T : \mathbf{C} \to \mathbf{Y}$ from **C** to any category **Y** with finite limits, such that the following conditions on a fibration $f : A \to B$ in **C** are equivalent:

(a) (A,f) is a trivial covering of B;

(b) $T(f) : T(A) \to T(B)$ is an isomorphism in **Y**.

We will fix such a test functor T; the reasons for introducing it are the following obvious facts:

Proposition 2.2. The following conditions on a fibration $f : A \to B$ and a monadic (E,p) of B are equivalent:

(a) (A,f) is split over (E,p);

(b) the pullback projection $T(E\times_B A) \approx T(E)\times_{T(B)}T(A) \to T(E)$ is an isomorphism. □

Proposition 2.3. The following conditions on a monadic extension (E,p) of B are equivalent:

(a) (E,p) is a normal extension;

(b) the pullback projections $T(E\times_B E) \approx T(E)\times_{T(B)}T(E) \to T(E)$ are isomorphisms;

(c) $T(p) : T(E) \to T(B)$ is a monomorphism. □

From now on we will assume that the category **C** is pointed, write 0 for its zero object and its zero morphisms, and write $\ker(f) : \mathrm{Ker}(f) \to A$ for a (the) kernel of a morphism $f : A \to B$ in it. We will also assume that all morphisms into 0 are fibrations. Furthermore, since the functor I must preserve zero, the admissibility condition implies that the functor H is fully faithful, and we will identify the category **X** with its replete H-image in **C**.

Proposition 2.4. The following conditions on an object C in **C** are equivalent:

(a) C is in **X**, i.e. the morphism $\eta_C : C \to HI(C)$ is an isomorphism;

(b) the zero morphism $C \to 0$ is a trivial covering;

(b) $T(C) = 0$. □

Proposition 2.5. For a fibration $f : A \to B$ in **C** the implications (a)⇒(b)⇒(c) hold for:

(a) (A,f) is a normal extension;

(b) (A,f) is a covering;

(c) $\mathrm{Ker}(f)$ is in **X**.

Moreover:

(d) if (A,f) is a monadic extension and every morphism in **Y** with zero kernel is a monomorphism, then conditions (a), (b), and (c) are equivalent to each other;

(e) if there exists a monadic extension (E,p) of B with E in **X**, then condition (c) implies condition (b).

Proof. The implication (a)⇒(b) is trivial.

(b)⇒(c): When (A,f) is a covering, $p^*(A,f) = (E\times_B A,\mathrm{pr}_1)$ is a trivial covering for some monadic extension (E,p) of B. Since the class of trivial covering morphisms is (obviously) pullback stable, this makes $\mathrm{Ker}(\mathrm{pr}_1 : E\times_B A \to E) \approx \mathrm{Ker}(f) \to 0$ a trivial covering, and we can apply Proposition 2.4.

(d) follows from Proposition 2.3.

(e): We have $T(E\times_B A) \approx T(E)\times_{T(B)}T(A) = 0\times_{T(B)}T(A) = \mathrm{Ker}(T(f)) \approx T(\mathrm{Ker}(f))$, which tell us that $E\times_B A$ is in **X** if and only if so is $\mathrm{Ker}(f)$. But having E and $E\times_B A$ in **X** implies that (A,f) is split over (E,p) and therefore is a covering.□

3. Galois structures of torsion theories

In this section we construct a finitely complete admissible Galois structure $\Gamma = (\mathbf{C},\mathbf{X},I,H,\eta,\varepsilon,\mathbf{F},\mathbf{\Phi})$ equipped with a test functor T as follows:

• **C** is a homological category in the sense of [BB];

• $(I,H,\eta,\varepsilon) : \mathbf{C} \to \mathbf{X}$ is the torsion-free reflection of a torsion theory $(\mathbf{Y},\mathbf{X})$ on **C** in the sense of [BG] (which generalizes the classical, i.e. abelian, case of S. E. Dickson [D]; we do not consider here the most general context of [JT]);

• **F** and **Φ** are the classes of all morphisms in **C** and **X** respectively;

• $T : \mathbf{C} \to \mathbf{Y}$ is the torsion coreflection of the torsion theory $(\mathbf{Y},\mathbf{X})$ above.

We need conditions (a) and (b) of Definition 2.1 to be equivalent to each other. For, given a morphism $f : A \to B$ in **C**, consider the diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & T(A) & \xrightarrow{\iota_A} & A & \xrightarrow{\eta_A} & I(A) & \longrightarrow & 0 \\
 & & {\scriptstyle T(f)}\downarrow & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle I(f)} & & \\
0 & \longrightarrow & T(B) & \xrightarrow[\iota_B]{} & B & \xrightarrow[\eta_B]{} & I(B) & \longrightarrow & 0,
\end{array}$$

where ι is the counit of the torsion coreflection. We need to know that the right-hand square is a pullback if and only if the first vertical arrow is an isomorphism. However, since the rows of this diagram are short exact sequences and the category **C** is homological, this follows from Bourn protomodularity (see [BB]).

For this Galois structure Γ associated with the torsion theory (**Y**,**X**), the main result of Section 2 becomes:

Theorem 3.1. For a morphism $f : A \to B$ in **C** the implications (a)⇔(b)⇒(c)⇒(d) hold for:

(a) (A,f) is a monadic extension, and f induces a monomorphism $T(f) : T(A) \to T(B)$ in **Y** from the torsion coreflection of A to the torsion coreflection of B;

(b) (A,f) is a normal extension;

(c) (A,f) is a covering;

(d) Ker(f) is torsion free.

Moreover:

(e) if (A,f) is a monadic extension and every morphism in **Y** with zero kernel is a monomorphism, then conditions (a), (b), (c) and (d) are equivalent to each other;

(f) if there exists a monadic extension (E,p) of B with E in **X**, then condition (d) implies condition (c). □

Such a theorem obviously suggests to:

• Find non-trivial examples where 3.1(e) and 3.1(f) can be applied; here "non-trivial" has a specific meaning: not every covering should be trivial (in the sense of Definition 1.1(b)).

• Show that conditions 3.1(a), 3.1(b), and 3.1(c) in general are not equivalent to each other. More precisely, find an example where (A,f) satisfies 3.1(d) but not 3.1(c), and another example where it is monadic and satisfies 3.1(c) but not 3.1(b).

This will be done in Sections 4 and 5 respectively.

Remark 3.2. Theorem 3.1 certainly applies to protolocalisations of homological categories in the sense of [BCGS]. However, in that case the functor I preserves all pullbacks along regular epimorphisms, which makes every covering trivial.

4. Two "non-trivial" examples where conditions 3.1(e) and 3.1(f) do apply

Example 4.1. The additional assumption on **Y** made in 3.1(e) obviously holds in the following two cases:

(a) When **Y** is hereditary in **C** with respect to normal monomorphisms, i.e. when every normal monomorphism $m : C \to Y$ in **C** with Y in **Y** must have C in **Y**. It follows that Theorem 3.1 includes the main result in [GR], which was obtained in the more restrictive context of quasi-hereditary torsion theories.

(b) When **C** is (regular and) additive – since this forces **Y** to be additive. In particular **C** could be abelian as in [CJKP].

For both of these cases "the simplest" example of a non-trivial covering morphism (which is even a normal extension) $f : A \to B$ has

• **C** = the category of abelian groups;

• **X** = the category of torsion free abelian groups (in the usual sense);

• f = any epimorphism from the additive group of integers to a non-zero cyclic group.

For this ordinary torsion theory of abelian groups 3.1(f) also applies since every abelian group is a quotient of a torsion-free abelian group. However, it does not apply well to e.g. the dual torsion theory (since only torsion abelian groups are subgroups of torsion abelian groups; see [CJKP] for some related remarks).

Example 4.2. Let $\mathcal{T}$ be a semi-abelian algebraic theory, i.e. an algebraic theory whose models form a semi-abelian category, or, equivalently, a pointed *BIT speciale* variety in the sense of A. Ursini [U] (see also [BouJ], [JMT2], and [JMU] for the clarification of the relationship between the categorical and universal-algebraic approaches). The models of $\mathcal{T}$ are universal algebras of a fixed type admitting, among others, a constant 0, n binary terms $s_1, ..., s_n$, and an $(n + 1)$-ary term t, satisfying the identities

$$t(s_1(x,y),...,s_n(x,y),y) = x, \quad s_1(x,x) = ... = s_n(x,x) = 0, \quad u = 0$$

for all 0-ary terms u. Note that these terms also satisfy the implication

$$s_1(x,y) = ... = s_n(x,y) = 0 \Rightarrow x = y. \tag{4.1}$$

We take

• **C** = the category of topological $\mathcal{T}$-algebras (=models of $\mathcal{T}$ in the category of topological spaces), which is known from F. Borceux and M. M. Clementino [BC] to be a homological category;

• **X** = the category of totally disconnected topological $\mathcal{T}$-algebras, hence obtaining the torsion theory (**Y**,**X**), whose torsion objects are connected topological $\mathcal{T}$-algebras.

We claim that every morphism $f : A \rightarrow B$ in the category **Y** of connected topological $\mathcal{T}$-algebras that has the trivial kernel in **Y** is a monomorphism in **Y**. Indeed:

Let g, $h : Y \rightarrow A$ be morphisms in **Y** with $fg = fh$, and $k_1, ..., k_n : Y \rightarrow A$ be maps (which are not necessarily $\mathcal{T}$-algebra homomorphisms) defined by

$$k_i(y) = s_i(g(y),h(y)) \quad (i = 1, ..., n).$$

Our next step requires to compare the kernel of f in **Y** with the kernel of f in **C**, and we will denote these kernels by $\mathrm{Ker}_{\mathbf{Y}}(f)$ and $\mathrm{Ker}_{\mathbf{C}}(f)$ respectively. We observe:

- $fg = fh$ easily implies that $fk_i = 0$ for all $i = 1, ..., n$.

- Since $\mathrm{Ker}_{\mathbf{C}}(f) = \{a \in A \mid f(a) = 0\}$ and $fk_i = 0$ for all $i = 1, ..., n$, the images of Y under all k_i's are in $\mathrm{Ker}_{\mathbf{C}}(f)$.

- Since $\mathrm{Ker}_{\mathbf{Y}}(f)$ is nothing but the connected component of 0 in $\mathrm{Ker}_{\mathbf{C}}(f)$, and since Y is connected, the previous observation implies that the images of Y under all k_i's are in $\mathrm{Ker}_{\mathbf{Y}}(f)$.

- Therefore $\mathrm{Ker}_{\mathbf{Y}}(f) = 0$ implies $k_i = 0$ for all $i = 1, ..., n$, which itself implies $g = h$ by the implication (4.1).

That is $\mathrm{Ker}_{\mathbf{Y}}(f) = 0$ implies that f is a monomorphism.

There are many non-trivial coverings, e.g. the canonical map $\mathrm{R} \rightarrow \mathrm{R/Z}$ in the case when $\mathcal{T}$ is the theory of groups, where R is the topological (additive) group of real numbers and Z is the group of integers. This is, again, a normal extension, and, topologically it is nothing but the classical universal covering of the 1-dimensional sphere $S^1 = \mathrm{R/Z}$ of course.

Remark 4.3. Concerning 3.1(f): We do not know if, in the situation 4.2, for every object B in **C**, there exists a monadic extension (E,p) of B with E in **X**. However, as mentioned in [GR] with a reference to A. Arkhangel'skiĭ [A], this is the case when $\mathcal{T}$ is the theory of groups.

5. Counter-examples for (d)⇒(c)⇒(b) in Theorem 3.1

Example 5.1. Let $\mathbf{B}_n$ be the Burnside variety (of groups) of exponent n, i.e. the variety of all groups G with $x^n = 1$ for all x in G, and C_n the cyclic group of order n. We construct the data described in Section 3 by taking $\mathbf{C} = \mathbf{B}_6$ and $\mathbf{X} = \mathbf{B}_3$, and take f to be the unique epimorphism from the symmetric group S_3 to C_2. Then $\mathrm{Ker}(f) = C_3$ is torsion free, but (S_3,f) is not a covering. Indeed:

Consider the pullback diagram

$$\begin{array}{ccc} P & \xrightarrow{pr_2} & S_3 \\ {\scriptstyle pr_1}\downarrow & & \downarrow{\scriptstyle f} \\ C_6 & \xrightarrow[p]{} & C_2 \end{array}$$

in which p is the unique epimorphism $C_6 \to C_2$. An easy calculation shows that the projection $P \to C_6$ induces an isomorphism $I(P) \approx I(C_6)$ of the torsion-free reflections; since that projection itself is not an isomorphism, it follows that (P,pr_1) is not a trivial covering. On the other hand, since the category $\mathbf{C} = \mathbf{B}_6$ is exact, and since C_6 is a projective object in it with respect to regular epimorphisms, if (S_3,f) were a covering then it would be split over (C_6,p), i.e. (P,pr_1) would be a trivial covering.

Example 5.2. Now we construct the data described in Section 3 by taking:

- $\mathbf{C}$ = the category of groups;
- $\mathbf{X}$ = the category of torsion-free groups in the usual sense, which will make $\mathbf{Y}$ to be the category of all groups generated by their elements of finite order.

Let us also take $B = C_2$ (=the cyclic group of order 2), $A = B \ltimes Z$ = the infinite dihedral group, and $f : A \to B$ to be the semidirect product projection. In particular f is a split epimorphism and (A,f) is a monadic extension. Consider the pullback diagram

$$\begin{array}{ccc} P & \xrightarrow{pr_2} & A \\ {\scriptstyle pr_1}\downarrow & & \downarrow{\scriptstyle f} \\ \mathrm{Z} & \xrightarrow[p]{} & B \end{array}$$

in which p is the unique epimorphism from the additive group of integers to B. Since Z and P are (obviously) torsion free, and (Z,p) is a monadic extension (since p is a regular epimorphism in an exact category), we conclude that (A,f) is a covering of B. On the other hand, since f is a split epimorphism, if (A,f) were a normal extension it would be a trivial covering. And (A,f) is not a trivial covering since A and B are in **Y**, while f is not an isomorphism.

That is, (A,f) is a covering that is a monadic extension but not a normal extension.

6. Remarks on central extensions and locally semisimple coverings

In this section we compare three contexts, which we will call TT, CE, and LSC for short. They are:

• TT: The context of a torsion theory described in Section 3.

• CE stands for "central extension"; it is the context used in [JK], where the ground Galois structure $\Gamma = (\mathbf{C},\mathbf{X},I,H,\eta,\varepsilon,\mathbf{F},\mathbf{\Phi})$ has **C** an exact category, **X** a Birkhoff subcategory in **C**, and **F** and **Φ** are the classes of regular epimorphisms in **C** and **X** respectively. When **C** is pointed, we could try to define a test functor $T : \mathbf{C} \to \mathbf{Y}$, where **Y** is a suitable subcategory in **C**, by $T(C) = \mathrm{Ker}(\eta_C)$. Moreover, such a functor was actually used in [JK] (called R there) in results similar to ours. However, in general this functor will not preserve finite limits.

• LSC stands for "locally semisimple covering"; it is a context used in [JMT1], where still having (exact) **C** and a full subcategory **X** in it, we do not require the existence of the reflection $I : \mathbf{C} \to \mathbf{X}$. One then replaces trivial coverings with morphisms in **X** and modifies definitions of Section 1 accordingly. So, a locally semisimple covering is a morphism in **C** that is "in **X** up to effective descent" rather than a "trivial covering up to effective descent".

Let us begin our comparisons with the property

$((A,f)$ is a monadic extension and a covering$) \Rightarrow ((A,f)$ is a normal extension$)$, (6.1)

which we obtained in TT under any of the following conditions:

Every morphism in **Y** with zero kernel is a monomorphism, (6.2)

There exists a monadic extension (E,p) of B with E in **X**. (6.3)

The implication (6.1) does not make much sense in LSC (unless A is in **X**), but it holds in CE under additional conditions that are (when **C** is exact) much weaker (see [JK]) than what we require in TT. And (6.2) holds in CE as soon as the category **C** has the similar property. Still, in order to prove it in CE, neither the arguments we used for 3.1(e) nor the arguments we used for 3.1(f) can be applied. The reason is that neither 3.1(a)⇒(b) nor (6.3) can be used.

Now let us consider the property

$(\mathrm{Ker}(f)$ is in $\mathbf{X}) \Rightarrow ((A,f)$ is a covering$)$. (6.4)

It "almost never" holds in CE: for instance it does not hold for the ordinary central extensions of groups, which is a very basic fact in group theory (not every group extension with an abelian kernel is central!); an example where it does hold is given by the Birkhoff subcategory Dis(**C**) of discrete

equivalence relations in the category Grpd(**C**) of internal groupoids in a semi-abelian category **C** studied in [G1] and [EG].
But it holds in many special cases of LSC for essentially the same reasons as for 3.1(f). Moreover, 3.1(f) is a consequence of Proposition 2.3 in [JMT1] (in the case of exact **C**, although as mentioned in [JMT1], exactness is not essential there). Indeed, it is easy to see that **X** in TT is a semisimple class in **C** (when **C** is exact) in the sense of [JMT1] satisfying Condition 2.2 of [JMT1], as required in Proposition 2.3 there.

Finally, note that our examples, especially 5.1 and 5.2, are, in a sense, suggested by these comparisons; in fact Example 5.1 can be used also in the context CE and Example 5.2 can be used also in the context LSC (since (6.3) holds there).

References

[A] A. Arkhangel’skiĭ, Any topological group is a quotient group of a zero-dimensional topological group, Soviet Math. Doklady 23, 1981, 615-618

[BB] F. Borceux and D. Bourn, Mal'cev, Protomodular, Homological and Semi-Abelian Categories, Mathematics and Its Applications, Kluwer, 2004

[BC] F. Borceux and M.M. Clementino, Topological semi-abelian algebras, Advances in Mathematics 190, 2005, 425-453

[BCGR] F. Borceux, M.M. Clementino, M. Gran and L. Sousa, Protolocalisations of homological categories, Journal of Pure and Applied Algebra 212, 2008, 1898-1927

[BorJ] F. Borceux and G. Janelidze, Galois Theories, Cambridge Studies in Advanced Mathematics 72, Cambridge University Press, 2001

[BG] D. Bourn and M. Gran, Torsion theories in homological categories, Journal of Algebra 305, 2006, 18-47

[BouJ] D. Bourn and G. Janelidze, Protomodularity, descent, and semidirect products, Theory and Applications of Categories 4, 1998, 37-46

[CJKP] A. Carboni, G. Janelidze, G. M. Kelly, and R. Paré, On localization and stabilization of factorization systems, Applied Categorical Structures 5, 1997, 1-58

[CHK] C. Cassidy, M. Hébert, and G. M. Kelly, Reflective subcategories, localizations, and factorization systems, Journal of Australian Mathematical Society (Series A), 1985, 287-329

[D] S. E. Dickson, A torsion theory for abelian categories, Transactions of the American Mathemetical Society 121, 1966, 223-235

[EG] T. Everaert and M. Gran, Homology of n-fold groupoids, 2009, preprint.

[G1] M. Gran, Central extensions and internal groupoids in Maltsev categories, Journal of Pure and Applied Algebra 155, 2001, 139-166

[G2] M. Gran, Applications of categorical Galois theory in universal algebra, Fields Institute Communications 43, 2004, 243-280

[GR] M. Gran and V. Rossi, Torsion theories and Galois coverings of topological groups, Journal of Pure and Applied Algebra 208, 2007, 135-151

[J1] G. Janelidze, Pure Galois theory in categories, Journal of Algebra 132, 1990, 270-286

[J2] G. Janelidze, Categorical Galois theory: revision and some recent developments, in: "Galois Connections and Applications", Mathematics and Its Applications 565, Kluwer Academic Publishers B.V., 2004, 139-171

[JK] G. Janelidze and G. M. Kelly, Galois theory and a general notion of a central extension, Journal of Pure and Applied Algebra 97, 1994, 135-161

[JMT1] G. Janelidze, L. Márki, and W. Tholen, Locally semisimple coverings, Journal of Pure and Applied Algebra 128, 1998, 281-289

[JMT2] G. Janelidze, L. Márki, and W. Tholen, Semi-abelian categories, Journal of Pure and Applied Algebra 168, 2002, 367-386

[JMU] G. Janelidze, L. Márki, and A. Ursini, Ideals and clots in universal algebra and in semi-abelian categories, Journal of Algebra 307, 1, 2007, 191-208

[JT] G. Janelidze and W. Tholen, Characterization of torsion theories in general categories, Contemporary Mathematics 431, 2007, 249-256

[L] A. S.-T. Lue, Baer-invariants and extensions relative to a variety, Proceedings of Cambridge Philosophical Society 63, 1967, 569-578

[U] A. Ursini, Osservazioni sulla varietá BIT, Boll. U. M. I. (4) 7, 1973, 205-211

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