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"HAUSDORFF DISTANCE" VIA CONICAL COCOMPLETION

by Isar STUBBE*

À Francis Borceux, qui m'a tant appris et qui m'apprend toujours

Résumé. Dans le contexte des catégories enrichies dans un quantaloïde, nous expliquons comment toute classe de poids saturée définit, et est définie par, une unique sous-KZ-doctrine pleine de la doctrine pour la cocomplétion libre. Les KZ-doctrines qui sont des sous-KZ-doctrines pleines de la doctrine pour la cocomplétion libre, sont caractérisées par deux conditions simples de "pleine fidélité". Les poids coniques forment une classe saturée, et la KZ-doctrine correspondante est exactement (la généralisation aux catégories enrichies dans un quantaloïde de) la doctrine de Hausdorff de [Akhvlediani *et al.*, 2009].

Abstract. In the context of quantaloid-enriched categories, we explain how each saturated class of weights defines, and is defined by, an essentially unique full sub-KZ-doctrine of the free cocompletion KZ-doctrine. The KZ-doctrines which arise as full sub-KZ-doctrines of the free cocompletion, are characterised by two simple "fully faithfulness" conditions. Conical weights form a saturated class, and the corresponding KZ-doctrine is precisely (the generalisation to quantaloid-enriched categories of) the Hausdorff doctrine of [Akhvlediani *et al.*, 2009].

Keywords. Enriched category, cocompletion, KZ-doctrine, Hausdorff distance
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1. Introduction

At the meeting on "Categories in Algebra, Geometry and Logic" honouring Francis Borceux and Dominique Bourn in Brussels on 10–11 October 2008, Walter Tholen gave a talk entitled "On the categorical meaning of Hausdorff and Gromov distances", reporting on joint work with Andrei Akhvlediani and Maria Manuel Clementino [2009]. The term 'Hausdorff distance' in his title refers to the following construction: if (X,d) is a metric space and $S,T \subseteq X$, then

$$\delta(S,T) := \bigvee_{s\in S} \bigwedge_{t\in T} d(s,t)$$

defines a (generalised) metric on the set of subsets of X. But Bill Lawvere [1973] showed that metric spaces are examples of enriched categories, so one can aim at

* Postdoctoral Fellow of the Research Foundation Flanders (FWO)

suitably generalising this 'Hausdorff distance'. Tholen and his co-workers achieved this for *categories enriched in a commutative quantale* $\mathcal{V}$. In particular they devise a KZ-doctrine on the category of $\mathcal{V}$-categories, whose algebras – in the case of metric spaces – are exactly the sets of subsets of metric spaces, equipped with the Hausdorff distance.

We shall argue that the notion of Hausdorff distance can be developed for *quantaloid-enriched categories* too, using *enriched colimits* as main tool. In fact, very much in line with the work of [Albert and Kelly, 1988; Kelly and Schmitt, 2005; Schmitt, 2006] on cocompletions of categories enriched in a symmetric monoidal category and the work of [Kock, 1995] on the abstraction of cocompletion processes, we shall see that, for quantaloid-enriched categories, each saturated class of weights defines, and is defined by, an essentially unique KZ-doctrine. The KZ-doctrines that arise in this manner are the full sub-KZ-doctrines of the free cocompletion KZ-doctrine, and they can be characterised with two simple "fully faithfulness" conditions. As an application, we find that the conical weights form a saturated class and the corresponding KZ-doctrine is precisely (the generalisation to quantaloid-enriched categories of) the Hausdorff doctrine of [Akhvlediani *et al.*, 2009].

In this paper we do not speak of 'Gromov distances', that other metric notion that Akhvlediani, Clementino and Tholen [2009] refer to. As they analyse, Gromov distance is necessarily built up from *symmetrised* Hausdorff distance; and because their base quantale $\mathcal{V}$ is commutative, they can indeed extend this notion too to $\mathcal{V}$-enriched categories. More generally however, symmetrisation for quantaloid-enriched categories makes sense when that quantaloid is involutive. Preliminary computations indicate that 'Gromov distance' ought to exist on this level of generality, but quickly got too long to include them in this paper: so we intend to work this out in a sequel.

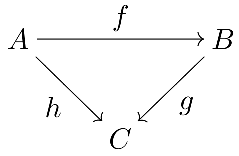
2. Preliminaries

2.1 Quantaloids

A **quantaloid** is a category enriched in the monoidal category Sup of complete lattices (also called sup-lattices) and supremum preserving functions (sup-morphisms). A quantaloid with one object, i.e. a monoid in Sup, is a **quantale**. Standard references include [Rosenthal, 1996; Paseka and Rosicky, 2000] .

Viewing $\mathcal{Q}$ as a locally ordered category, the 2-categorical notion of *adjunction in* $\mathcal{Q}$ refers to a pair of arrows, say $f\colon A\longrightarrow B$ and $g\colon B\longrightarrow A$, such that $1_A \leq g \circ f$ and $f \circ g \leq 1_B$ (in which case f is left adjoint to g, and g is right adjoint to f, denoted $f \dashv g$).

Given arrows



in a quantaloid $\mathcal{Q}$, there are adjunctions between sup-lattices as follows:

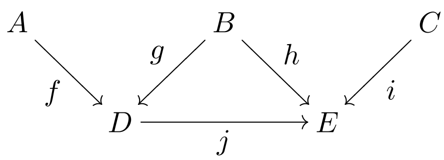
$$\mathcal{Q}(B,C) \underset{\{f,-\}}{\overset{-\circ f}{\underset{\perp}{\rightleftarrows}}} \mathcal{Q}(A,C), \quad \mathcal{Q}(A,B) \underset{[g,-]}{\overset{g\circ -}{\underset{\perp}{\rightleftarrows}}} \mathcal{Q}(A,C),$$

$$\mathcal{Q}(A,B) \underset{[-,h]}{\overset{\{-,h\}}{\underset{\perp}{\rightleftarrows}}} \mathcal{Q}(B,C)^{\mathsf{op}}.$$

The arrow $[g,h]$ is called the **lifting** of h trough g, whereas $\{f,h\}$ is the **extension** of h through f. Of course, every left adjoint preserves suprema, and every right adjoint preserves infima. For later reference, we record some straightforward facts:

Lemma 2.1 *If $g\colon B \longrightarrow C$ in a quantaloid has a right adjoint g^*, then $[g,h] = g^* \circ h$ and therefore $[g,-]$ also preserves suprema. Similarly, if $f\colon A \longrightarrow B$ has a left adjoint $f_!$ then $\{f,h\} = h \circ f_!$ and thus $\{f,-\}$ preserves suprema.*

Lemma 2.2 *For any commutative diagram*



in a quantaloid, we have that $[i,h] \circ [g,f] \leq [i, j \circ f]$. If all these arrows are left adjoints, and g moreover satisfies $g \circ g^* = 1_D$, then $[i,h] \circ [g,f] = [i, j\circ f]$.

Lemma 2.3 *If $f\colon A \longrightarrow B$ in a quantaloid has a right adjoint f^* such that moreover $f^* \circ f = 1_A$, then $[f\circ x, f \circ y] = [x,y]$ for any $x,y\colon X \rightrightarrows A$.*

2.2 Quantaloid-enriched categories

From now on $\mathcal{Q}$ denotes a *small* quantaloid. Viewing $\mathcal{Q}$ as a (locally ordered) bicategory, it makes perfect sense to consider categories enriched in $\mathcal{Q}$. Bicategory-enriched categories were invented at the same time as bicategories by Jean Bénabou [1967], and further developed by Ross Street [1981, 1983]. Bob Walters [1981] particularly used quantaloid-enriched categories in connection with sheaf theory. Here we shall stick to the notational conventions of [Stubbe, 2005], and refer to that paper for additional details, examples and references.

A **$\mathcal{Q}$-category** $\mathbb{A}$ consists of a set of objects $\mathbb{A}_0$, a type function $t\colon \mathbb{A}_0 \longrightarrow \mathcal{Q}_0$, and $\mathcal{Q}$-arrows $\mathbb{A}(a',a)\colon ta \longrightarrow ta'$; these must satisfy identity and composition axioms, namely:

$$1_{ta} \leq \mathbb{A}(a,a) \ \text{ and } \ \mathbb{A}(a'',a') \circ \mathbb{A}(a',a) \leq \mathbb{A}(a'',a).$$

A **$\mathcal{Q}$-functor** $F\colon \mathbb{A} \longrightarrow \mathbb{B}$ is a type-preserving object map $a \mapsto Fa$ satisfying the functoriality axiom:

$$\mathbb{A}(a',a) \leq \mathbb{B}(Fa',Fa).$$

And a **$\mathcal{Q}$-distributor** $\Phi\colon \mathbb{A} \mathrel{-\!\!\!\circ\!\!\!\rightarrow} \mathbb{B}$ is a matrix of $\mathcal{Q}$-arrows $\Phi(b,a)\colon ta \longrightarrow tb$, indexed by all couples of objects of $\mathbb{A}$ and $\mathbb{B}$, satisfying two action axioms:

$$\Phi(b,a') \circ \mathbb{A}(a',a) \leq \Phi(b,a) \ \text{ and } \ \mathbb{B}(b,b') \circ \Phi(b',b) \leq \Phi(b,a).$$

Composition of functors is obvious; that of distributors is done with a "matrix" multiplication: the composite $\Psi \otimes \Phi\colon \mathbb{A} \mathrel{-\!\!\!\circ\!\!\!\rightarrow} \mathbb{C}$ of $\Phi\colon \mathbb{A} \mathrel{-\!\!\!\circ\!\!\!\rightarrow} \mathbb{B}$ and $\Psi\colon \mathbb{B} \mathrel{-\!\!\!\circ\!\!\!\rightarrow} \mathbb{C}$ has as elements

$$(\Psi \otimes \Phi)(c,a) = \bigvee_{b \in \mathbb{B}_0} \Psi(c,b) \circ \Phi(b,a).$$

Moreover, the elementwise supremum of parallel distributors $(\Phi_i\colon \mathbb{A} \mathrel{-\!\!\!\circ\!\!\!\rightarrow} \mathbb{B})_{i \in I}$ gives a distributor $\bigvee_i \Phi_i\colon \mathbb{A} \mathrel{-\!\!\!\circ\!\!\!\rightarrow} \mathbb{B}$, and it is easily checked that we obtain a (large) quantaloid $\mathsf{Dist}(\mathcal{Q})$ of $\mathcal{Q}$-categories and distributors. Now $\mathsf{Dist}(\mathcal{Q})$ is a 2-category, so we can speak of adjoint distributors. In fact, any functor $F\colon \mathbb{A} \longrightarrow \mathbb{B}$ determines an adjoint pair of distributors:

$$\mathbb{A} \underset{\mathbb{B}(F-,-)}{\overset{\mathbb{B}(-,F-)}{\underset{\perp}{\rightleftarrows}}} \mathbb{B}. \tag{1}$$

Therefore we can sensibly order parallel functors $F,G\colon \mathbb{A} \rightrightarrows \mathbb{B}$ by putting $F \leq G$ whenever $\mathbb{B}(-,F-) \leq \mathbb{B}(-,G-)$ (or equivalently, $\mathbb{B}(G-,-) \leq \mathbb{B}(F-,-)$) in $\mathsf{Dist}(\mathcal{Q})$. Doing so, we get a locally ordered category $\mathsf{Cat}(\mathcal{Q})$ of $\mathcal{Q}$-categories and functors, together with a 2-functor

$$i\colon \mathsf{Cat}(\mathcal{Q}) \longrightarrow \mathsf{Dist}(\mathcal{Q})\colon \Big(F\colon \mathbb{A} \longrightarrow \mathbb{B}\Big) \mapsto \Big(\mathbb{B}(-,F-)\colon \mathbb{A} \mathrel{-\!\!\!\circ\!\!\!\rightarrow} \mathbb{B}\Big). \tag{2}$$

(The local order in $\mathsf{Cat}(\mathcal{Q})$ need not be anti-symmetric, i.e. it is not a partial order but rather a preorder, which we prefer to call simply an order.)

This is the starting point for the theory of quantaloid-enriched categories, including such notions as:

- **fully faithful functor**: an $F: \mathbb{A} \longrightarrow \mathbb{B}$ for which $\mathbb{A}(a', a) = \mathbb{B}(Fa', Fa)$, or alternatively, for which the unit of the adjunction in (1) is an equality,
- **adjoint pair**: a pair $F: \mathbb{A} \longrightarrow \mathbb{B}$, $G: \mathbb{B} \longrightarrow \mathbb{A}$ for which $1_{\mathbb{A}} \leq G \circ F$ and also $F \circ G \leq 1_{\mathbb{B}}$, or alternatively, for which $\mathbb{B}(F-, -) = \mathbb{A}(-, G-)$,
- **equivalence**: an $F: \mathbb{A} \longrightarrow \mathbb{B}$ which are fully faithful and essentially surjective on objects, or alternatively, for which there exists a $G: \mathbb{B} \longrightarrow \mathbb{A}$ such that $1_{\mathbb{A}} \cong G \circ F$ and $F \circ G \cong 1_{\mathbb{B}}$,
- **left Kan extension**: given $F: \mathbb{A} \longrightarrow \mathbb{B}$ and $G: \mathbb{A} \longrightarrow \mathbb{C}$, the left Kan extension of F through G, written $\langle F, G \rangle: \mathbb{C} \longrightarrow \mathbb{B}$, is the smallest such functor satisfying $F \leq \langle F, G \rangle \circ G$,

and so on. In the next subsection we shall recall the more elaborate notions of presheaves, weighted colimits and cocompletions.

2.3 Presheaves and free cocompletion

If X is an object of $\mathcal{Q}$, then we write $*_X$ for the one-object $\mathcal{Q}$-category, whose single object $*$ is of type X, and whose single hom-arrow is 1_X.

Given a $\mathcal{Q}$-category $\mathbb{A}$, we now define a new $\mathcal{Q}$-category $\mathcal{P}(\mathbb{A})$ as follows:

- objects: $(\mathcal{P}(\mathbb{A}))_0 = \{\phi: *_X \mathrel{-\!\!\circ\!\!\to} \mathbb{A} \mid X \in \mathcal{Q}_0\}$,
- types: $t(\phi) = X$ for $\phi: *_X \mathrel{-\!\!\circ\!\!\to} \mathbb{A}$,
- hom-arrows: $\mathcal{P}(\mathbb{A})(\psi, \phi) =$ (single element of) the lifting $[\psi, \phi]$ in $\mathsf{Dist}(\mathcal{Q})$.

Its objects are **(contravariant) presheaves** on $\mathbb{A}$, and $\mathcal{P}(\mathbb{A})$ itself is the **presheaf category** on $\mathbb{A}$.

The presheaf category $\mathcal{P}(\mathbb{A})$ **classifies distributors** with codomain $\mathbb{A}$: for any $\mathbb{B}$ there is a bijection between $\mathsf{Dist}(\mathcal{Q})(\mathbb{B}, \mathbb{A})$ and $\mathsf{Cat}(\mathcal{Q})(\mathbb{B}, \mathcal{P}(\mathbb{A}))$, which associates to any distributor $\Phi: \mathbb{B} \mathrel{-\!\!\circ\!\!\to} \mathbb{A}$ the functor $Y_\Phi: \mathbb{B} \longrightarrow \mathcal{P}(\mathbb{A}): b \mapsto \Phi(-, b)$, and conversely associates to any functor $F: \mathbb{B} \longrightarrow \mathcal{P}(\mathbb{A})$ the distributor $\Phi_F: \mathbb{B} \mathrel{-\!\!\circ\!\!\to} \mathbb{A}$ with elements $\Phi_F(a, b) = (Fb)(a)$. In particular is there a functor, $Y_{\mathbb{A}}: \mathbb{A} \longrightarrow \mathcal{P}(\mathbb{A})$, that corresponds with the identity distributor $\mathbb{A}: \mathbb{A} \mathrel{-\!\!\circ\!\!\to} \mathbb{A}$: the elements in the image of $Y_{\mathbb{A}}$ are the **representable presheaves on** $\mathbb{A}$, that is to say, for each $a \in \mathbb{A}$ we have $\mathbb{A}(-, a): *_{ta} \mathrel{-\!\!\circ\!\!\to} \mathbb{A}$. Because such a representable presheaf is a left adjoint in $\mathsf{Dist}(\mathcal{Q})$, with right adjoint $\mathbb{A}(a, -)$, we can verify that

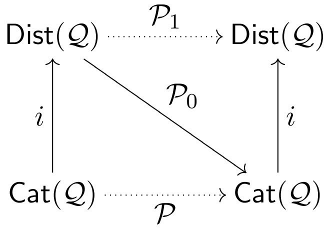
$$\mathcal{P}(\mathbb{A})(Y_{\mathbb{A}}(a), \phi) = [\mathbb{A}(-, a), \phi] = \mathbb{A}(a, -) \otimes \phi = \phi(a).$$

This result is known as **Yoneda's Lemma**, and implies that $Y_{\mathbb{A}}: \mathbb{A} \longrightarrow \mathcal{P}(\mathbb{A})$ is a fully faithful functor, called the **Yoneda embedding** of $\mathbb{A}$ into $\mathcal{P}(\mathbb{A})$.

By construction there is a 2-functor

$$\mathcal{P}_0: \mathsf{Dist}(\mathcal{Q}) \longrightarrow \mathsf{Cat}(\mathcal{Q}): (\Phi: \mathbb{A} \mathrel{\circ\!\!\!\longrightarrow} \mathbb{B}) \mapsto (\Phi \otimes -: \mathcal{P}(\mathbb{A}) \longrightarrow \mathcal{P}(\mathbb{B})),$$

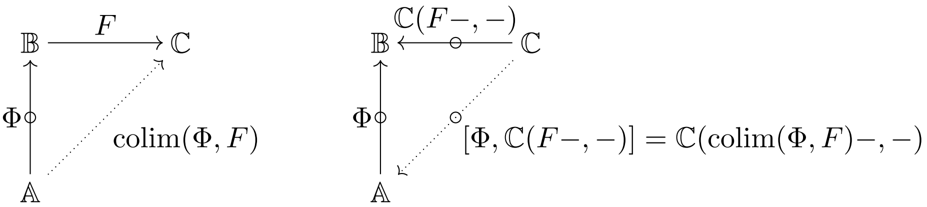
which is easily seen to preserve local suprema. Composing this with the one in (2) we define two more 2-functors:



(3)

In fact, $\mathcal{P}_1$ is a Sup-functor (a.k.a. a homomorphism of quantaloids). Later on we shall encounter these functors again.

For a distributor $\Phi: \mathbb{A} \mathrel{\circ\!\!\!\longrightarrow} \mathbb{B}$ and a functor $F: \mathbb{B} \longrightarrow \mathbb{C}$ between $\mathcal{Q}$-categories, the **Φ-weighted colimit of** F is a functor $K: \mathbb{A} \longrightarrow \mathbb{C}$ such that $[\Phi, \mathbb{B}(F-,-)] = \mathbb{C}(K-,-)$. Whenever a colimit exists, it is essentially unique; therefore the notation $\mathrm{colim}(\Phi, F): \mathbb{A} \longrightarrow \mathbb{C}$ makes sense. These diagrams picture the situation:



A functor $G: \mathbb{C} \longrightarrow \mathbb{C}'$ is said to **preserve** $\mathrm{colim}(\Phi, F)$ if $G \circ \mathrm{colim}(\Phi, F)$ is the Φ-weighted colimit of $G \circ F$. A $\mathcal{Q}$-category admitting all possible colimits, is **cocomplete**, and a functor which preserves all colimits which exist in its domain, is **cocontinuous**. (There are, of course, the dual notions of limit, completeness and continuity. We shall only use colimits in this paper, but it is a matter of fact that a $\mathcal{Q}$-category is complete if and only if it is cocomplete [Stubbe, 2005, Proposition 5.10].)

For two functors $F: \mathbb{A} \longrightarrow \mathbb{B}$ and $G: \mathbb{A} \longrightarrow \mathbb{C}$, we can consider the $\mathbb{C}(G-,-)$-weighted colimit of F. Whenever it exists, it is $\langle F, G\rangle: \mathbb{C} \longrightarrow \mathbb{B}$, the left Kan extension of F through G; but not every left Kan extension need to be such a colimit. Therefore we speak of a **pointwise left Kan extension** in this case.

Any presheaf category $\mathcal{P}(\mathbb{C})$ is cocomplete, as follows from its classifying property: given a distributor $\Phi\colon \mathbb{A} \multimap \mathbb{B}$ and a functor $F\colon \mathbb{B} \longrightarrow \mathcal{P}(\mathbb{C})$, consider the unique distributor $\Phi_F\colon \mathbb{B} \longrightarrow \mathbb{C}$ corresponding with F; now in turn the composition $\Phi_F \otimes \Phi\colon \mathbb{A} \multimap \mathbb{C}$ corresponds with a unique functor $Y_{\Phi_F \otimes \Phi}\colon \mathbb{A} \longrightarrow \mathcal{P}(\mathbb{C})$; the latter is $\mathrm{colim}(\Phi, F)$.

In fact, the 2-functor

$$\mathcal{P}\colon \mathsf{Cat}(\mathcal{Q}) \longrightarrow \mathsf{Cat}(\mathcal{Q})$$

is the **Kock-Zöberlein-doctrine**[1] **for free cocompletion**; the components of its multiplication $M\colon \mathcal{P} \circ \mathcal{P} \Longrightarrow \mathcal{P}$ and its unit $Y\colon 1_{\mathsf{Cat}(\mathcal{Q})} \Longrightarrow \mathcal{P}$ are

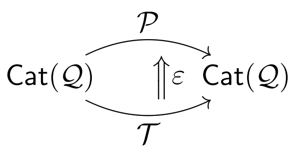
$$\mathrm{colim}(-, 1_{\mathcal{P}(\mathbb{C})})\colon \mathcal{P}(\mathcal{P}(\mathbb{C})) \longrightarrow \mathcal{P}(\mathbb{C}) \ \text{ and } \ Y_{\mathbb{C}}\colon \mathbb{C} \longrightarrow \mathcal{P}(\mathbb{C}).$$

This means in particular that $(\mathcal{P}, M, Y)$ is a monad on $\mathsf{Cat}(\mathcal{Q})$, and a $\mathcal{Q}$-category $\mathbb{C}$ is cocomplete if and only if it is a $\mathcal{P}$-algebra, if and only if $Y_{\mathbb{C}}\colon \mathbb{C} \longrightarrow \mathcal{P}(\mathbb{C})$ admits a left adjoint in $\mathsf{Cat}(\mathcal{Q})$.

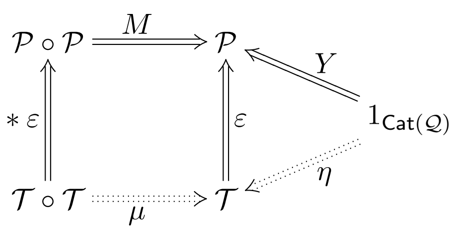
2.4 Full sub-KZ-doctrines of the free cocompletion doctrine

The following observation will be useful in a later subsection.

Proposition 2.4 *Suppose that $\mathcal{T}\colon \mathsf{Cat}(\mathcal{Q}) \longrightarrow \mathsf{Cat}(\mathcal{Q})$ is a 2-functor and that*



is a 2-natural transformation, with all components $\varepsilon_{\mathbb{A}}\colon \mathcal{T}(\mathbb{A}) \longrightarrow \mathcal{P}(\mathbb{A})$ fully faithful functors, such that there are (necessarily essentially unique) factorisations



[1] A Kock-Zöberlein-doctrine (or KZ-doctrine, for short) $\mathcal{T}$ on a locally ordered category $\mathcal{K}$ is a 2-functor $\mathcal{T}\colon \mathcal{K} \longrightarrow \mathcal{K}$ for which there are a multiplication $\mu\colon \mathcal{T} \circ \mathcal{T} \Longrightarrow \mathcal{T}$ and a unit $\eta\colon 1_{\mathcal{K}} \Longrightarrow \mathcal{T}$ making $(\mathcal{T}, \mu, \eta)$ a 2-monad, and satisfying moreover the "KZ-inequation": $\mathcal{T}(\eta_K) \leq \eta_{\mathcal{T}(K)}$ for all objects K of $\mathcal{K}$. The notion was invented independently by Volker Zöberlein [1976] and Anders Kock [1972] in the more general setting of 2-categories. We refer to [Kock, 1995] for all details.

Then $(\mathcal{T},\mu,\eta)$ *is a sub-2-monad of* $(\mathcal{P},M,Y)$*, and is a KZ-doctrine. We call the pair* $(\mathcal{T},\varepsilon)$ *a* **full sub-KZ-doctrine** *of* $\mathcal{P}$*.*

Proof : First note that, because each $\varepsilon_{\mathbb{A}}\colon\mathcal{T}(\mathbb{A})\longrightarrow\mathcal{P}(\mathbb{A})$ is fully faithful, for each $F,G\colon\mathbb{C}\longrightarrow\mathcal{T}(\mathbb{A})$,

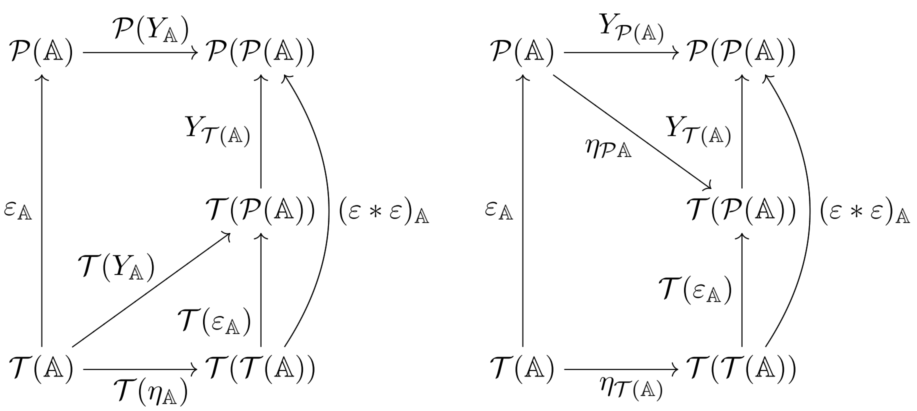
$$\varepsilon_{\mathbb{A}}\circ F\leq\varepsilon_{\mathbb{A}}\circ G\Longrightarrow F\leq G,$$

thus in particular $\varepsilon_{\mathbb{A}}$ is (essentially) a monomorphism in $\mathsf{Cat}(\mathcal{Q})$: if $\varepsilon_{\mathbb{A}}\circ F\cong\varepsilon_{\mathbb{A}}\circ G$ then $F\cong G$. Therefore we can regard $\varepsilon\colon\mathcal{T}\Longrightarrow\mathcal{P}$ as a subobject of the monoid $(\mathcal{P},M,Y)$ in the monoidal category of endo-2-functors on $\mathsf{Cat}(\mathcal{Q})$. The factorisations of M and Y then say precisely that $(\mathcal{T},\mu,\eta)$ is a submonoid, i.e. a 2-monad on $\mathsf{Cat}(\mathcal{Q})$ too.

But $\mathcal{P}\colon\mathsf{Cat}(\mathcal{Q})\longrightarrow\mathsf{Cat}(\mathcal{Q})$ maps fully faithful functors to fully faithful functors, as can be seen by applying Lemma 2.3 to the left adjoint $\mathbb{B}(-,F-)\colon\mathbb{A}\ \rightarrow\!\!\!\!\!\!\circ\ \ \mathbb{B}$ in $\mathsf{Dist}(\mathcal{Q})$, for any given fully faithful $F\colon\mathbb{A}\longrightarrow\mathbb{B}$. Therefore each

$$(\varepsilon*\varepsilon)_{\mathbb{A}}\colon\mathcal{T}(\mathcal{T}(\mathbb{A}))\longrightarrow\mathcal{P}(\mathcal{P}(\mathbb{A}))$$

is fully faithful: for $(\varepsilon*\varepsilon)_{\mathbb{A}}=\mathcal{P}(\varepsilon_{\mathbb{A}})\circ\varepsilon_{\mathcal{T}\mathbb{A}}$ and by hypothesis both $\varepsilon_{\mathbb{A}}$ and $\varepsilon_{\mathcal{T}\mathbb{A}}$ are fully faithful. The commutative diagrams



thus imply, together with the KZ-inequation for $\mathcal{P}$, the KZ-inequation for $\mathcal{T}$. $\square$

Some remarks can be made about the previous Proposition. Firstly, about the fully faithfulness of the components of $\varepsilon\colon\mathcal{T}\Longrightarrow\mathcal{P}$. In any locally ordered category $\mathcal{K}$ one defines an arrow $f\colon A\longrightarrow B$ to be *representably fully faithful* when, for any object X of $\mathcal{K}$, the order-preserving function

$$\mathcal{K}(f,-)\colon\mathcal{K}(X,A)\longrightarrow\mathcal{K}(X,B)\colon x\mapsto f\circ x$$

is order-reflecting – that is to say, $\mathcal{K}(f,-)$ is a fully faithful functor between ordered sets viewed as categories – and therefore f is also essentially a monomorphism in $\mathcal{K}$. But the converse need not hold, and indeed does not hold in $\mathcal{K} = \mathsf{Cat}(\mathcal{Q})$: not every monomorphism in $\mathsf{Cat}(\mathcal{Q})$ is representably fully faithful, and not every representably fully faithful functor is fully faithful. Because the 2-functor $\mathcal{P}\colon \mathsf{Cat}(\mathcal{Q}) \longrightarrow \mathsf{Cat}(\mathcal{Q})$ preserves representable fully faithfulness as well, the above Proposition still holds (with the same proof) when the components of $\varepsilon\colon \mathcal{T} \Longrightarrow \mathcal{P}$ are merely *representably* fully faithful; and in that case it might be natural to say that $\mathcal{T}$ is a "sub-KZ-doctrine" of $\mathcal{P}$. But for our purposes later on, the interesting notion is that of *full* sub-KZ-doctrine, thus with the components of $\varepsilon\colon \mathcal{T} \Longrightarrow \mathcal{P}$ being fully faithful.

A second remark: in the situation of Proposition 2.4, the components of the transformation $\varepsilon\colon \mathcal{T} \Longrightarrow \mathcal{P}$ are necessarily given by pointwise left Kan extensions. More precisely, $\langle Y_{\mathbb{A}}, \eta_{\mathbb{A}}\rangle\colon \mathcal{T}(\mathbb{A}) \longrightarrow \mathcal{P}(\mathbb{A})$ is the $\mathcal{T}(\mathbb{A})(\eta_{\mathbb{A}}-,-)$-weighted colimit of $Y_{\mathbb{A}}$ (which exists because $\mathcal{P}(\mathbb{A})$ is cocomplete), and can thus be computed as

$$\langle Y_{\mathbb{A}}, \eta_{\mathbb{A}}\rangle\colon \mathcal{T}(\mathbb{A}) \longrightarrow \mathcal{P}(\mathbb{A})\colon t \mapsto \mathcal{T}(\mathbb{A})(\eta_{\mathbb{A}}-, t).$$

By fully faithfulness of $\varepsilon_{\mathbb{A}}\colon \mathcal{T}(\mathbb{A}) \longrightarrow \mathcal{P}(\mathbb{A})$ and the Yoneda Lemma, we can compute that

$$\mathcal{T}(\mathbb{A})(\eta_{\mathbb{A}}-, t) = \mathcal{P}(\mathbb{A})(\varepsilon_{\mathbb{A}} \circ \eta_{\mathbb{A}}-, \varepsilon_{\mathbb{A}}(t)) = \mathcal{P}(\mathbb{A})(Y_{\mathbb{A}}-, \varepsilon_{\mathbb{A}}(t)) = \varepsilon_{\mathbb{A}}(t).$$

Hence the component of $\varepsilon\colon \mathcal{T} \Longrightarrow \mathcal{P}$ at $\mathbb{A} \in \mathsf{Cat}(\mathcal{Q})$ is necessarily the Kan extension $\langle Y_{\mathbb{A}}, \eta_{\mathbb{A}}\rangle$. We can push this argument a little further to obtain a characterisation of those KZ-doctrines which occur as full sub-KZ-doctrines of $\mathcal{P}$:

Corollary 2.5 *A KZ-doctrine $(\mathcal{T}, \mu, \eta)$ on $\mathsf{Cat}(\mathcal{Q})$ is a full sub-KZ-doctrine of $\mathcal{P}$ if and only if all $\eta_{\mathbb{A}}\colon \mathbb{A} \longrightarrow \mathcal{T}(\mathbb{A})$ and all left Kan extensions $\langle Y_{\mathbb{A}}, \eta_{\mathbb{A}}\rangle\colon \mathcal{T}(\mathbb{A}) \longrightarrow \mathcal{P}(\mathbb{A})$ are fully faithful.*

Proof : If $\mathcal{T}$ is a full sub-KZ-doctrine of $\mathcal{P}$, then we have just remarked that $\varepsilon_{\mathbb{A}} = \langle Y_{\mathbb{A}}, \eta_{\mathbb{A}}\rangle$, and thus these Kan extensions are fully faithful. Moreover – because $\varepsilon_{\mathbb{A}} \circ \eta_{\mathbb{A}} = Y_{\mathbb{A}}$ with both $\varepsilon_{\mathbb{A}}$ and $Y_{\mathbb{A}}$ fully faithful – also $\eta_{\mathbb{A}}$ must be fully faithful.

Conversely, if $(\mathcal{T}, \mu, \eta)$ is a KZ-doctrine with each $\eta_{\mathbb{A}}\colon \mathbb{A} \longrightarrow \mathcal{T}(\mathbb{A})$ fully faithful, then – e.g. by [Stubbe, 2005, Proposition 6.7] – the left Kan extensions $\langle Y_{\mathbb{A}}, \eta_{\mathbb{A}}\rangle$ (exist and) satisfy $\langle Y_{\mathbb{A}}, \eta_{\mathbb{A}}\rangle \circ \eta_{\mathbb{A}} \cong Y_{\mathbb{A}}$. By assumption each of these Kan extensions is fully faithful, so we must now prove that they are the components of a natural transformation and that this natural transformation commutes with the multiplications of $\mathcal{T}$ and $\mathcal{P}$. We do this in four steps:

(i) For any $\mathbb{A} \in \mathsf{Cat}(\mathcal{Q})$, there is the free $\mathcal{T}$-algebra $\mu_{\mathbb{A}}\colon \mathcal{T}(\mathcal{T}(\mathbb{A})) \longrightarrow \mathcal{T}(\mathbb{A})$. But the free $\mathcal{P}$-algebra $M_{\mathbb{A}}\colon \mathcal{P}(\mathcal{P}(\mathbb{A})) \longrightarrow \mathcal{P}(\mathbb{A})$ on $\mathcal{P}(\mathbb{A})$ also induces a $\mathcal{T}$-algebra on $\mathcal{P}(\mathbb{A})$: namely, $M_{\mathbb{A}} \circ \langle Y_{\mathcal{P}(\mathbb{A})}, \eta_{\mathcal{P}(\mathbb{A})}\rangle : \mathcal{T}(\mathcal{P}(\mathbb{A}) \longrightarrow \mathcal{P}(\mathbb{A})$. To see this, it suffices to prove the adjunction $M_{\mathbb{A}} \circ \langle Y_{\mathcal{P}(\mathbb{A})}, \eta_{\mathcal{P}(\mathbb{A})}\rangle \dashv \eta_{\mathcal{P}(\mathbb{A})}$. The counit is easily checked:

$$M_{\mathbb{A}} \circ \langle Y_{\mathcal{P}(\mathbb{A})}, \eta_{\mathcal{P}(\mathbb{A})}\rangle \circ \eta_{\mathcal{P}(\mathbb{A})} = M_{\mathbb{A}} \circ Y_{\mathcal{P}(\mathbb{A})} = 1_{\mathcal{P}(\mathbb{A})},$$

using first the factorisation property of the Kan extension and then the split adjunction $M_{\mathbb{A}} \dashv Y_{\mathcal{P}(\mathbb{A})}$. As for the unit of the adjunction, we compute that

$$\begin{aligned}
\eta_{\mathcal{P}(\mathbb{A})} \circ M_{\mathbb{A}} \circ \langle Y_{\mathcal{P}(\mathbb{A})}, \eta_{\mathcal{P}(\mathbb{A})}\rangle &= \mathcal{T}(M_{\mathbb{A}} \circ \langle Y_{\mathcal{P}(\mathbb{A})}, \eta_{\mathcal{P}(\mathbb{A})}\rangle) \circ \eta_{\mathcal{T}(\mathcal{P}(\mathbb{A}))} \\
&\geq \mathcal{T}(M_{\mathbb{A}} \circ \langle Y_{\mathcal{P}(\mathbb{A})}, \eta_{\mathcal{P}(\mathbb{A})}\rangle) \circ \mathcal{T}(\eta_{\mathcal{P}(\mathbb{A})}) \\
&= \mathcal{T}(M_{\mathbb{A}} \circ \langle Y_{\mathcal{P}(\mathbb{A})}, \eta_{\mathcal{P}(\mathbb{A})}\rangle \circ \eta_{\mathcal{P}(\mathbb{A})}) \\
&= \mathcal{T}(1_{\mathcal{P}(\mathbb{A})}) \\
&= 1_{\mathcal{T}(\mathcal{P}(\mathbb{A}))},
\end{aligned}$$

using naturality of η and the KZ inequality for $\mathcal{T}$, and recycling the computation we made for the counit.

(ii) Next we prove, for each $\mathcal{Q}$-category $\mathbb{A}$, that $\langle Y_{\mathbb{A}}, \eta_{\mathbb{A}}\rangle : \mathcal{T}(\mathbb{A}) \longrightarrow \mathcal{P}(\mathbb{A})$ is a $\mathcal{T}$-algebra homomorphism, for the algebra structures explained in the previous step. This is the case if and only if $\langle Y_{\mathbb{A}}, \eta_{\mathbb{A}}\rangle = (M_{\mathbb{A}} \circ \langle Y_{\mathcal{P}(\mathbb{A})}, \eta_{\mathcal{P}(\mathbb{A})}\rangle) \circ \mathcal{T}(\langle Y_{\mathbb{A}}, \eta_{\mathbb{A}}\rangle) \circ \eta_{\mathcal{T}(\mathbb{A})}$ (because the domain of $\langle Y_{\mathbb{A}}, \eta_{\mathbb{A}}\rangle$ is a free $\mathcal{T}$-algebra), and indeed:

$$\begin{aligned}
& M_{\mathbb{A}} \circ \langle Y_{\mathcal{P}(\mathbb{A})}, \eta_{\mathcal{P}(\mathbb{A})}\rangle \circ \mathcal{T}(\langle Y_{\mathbb{A}}, \eta_{\mathbb{A}}\rangle) \circ \eta_{\mathcal{T}(\mathbb{A})} \\
&= M_{\mathbb{A}} \circ \langle Y_{\mathcal{P}(\mathbb{A})}, \eta_{\mathcal{P}(\mathbb{A})}\rangle \circ \eta_{\mathcal{P}(\mathbb{A})} \circ \langle Y_{\mathbb{A}}, \eta_{\mathbb{A}}\rangle \\
&= M_{\mathbb{A}} \circ Y_{\mathcal{P}(\mathbb{A})} \circ \langle Y_{\mathbb{A}}, \eta_{\mathbb{A}}\rangle \\
&= 1_{\mathcal{P}(\mathbb{A})} \circ \langle Y_{\mathbb{A}}, \eta_{\mathbb{A}}\rangle \\
&= \langle Y_{\mathbb{A}}, \eta_{\mathbb{A}}\rangle.
\end{aligned}$$

(iii) To check that the left Kan extensions are the components of a natural transformation we must verify, for any $F: \mathbb{A} \longrightarrow \mathbb{B}$ in $\mathsf{Cat}(\mathcal{Q})$, that $\mathcal{P}(F) \circ \langle Y_{\mathbb{A}}, \eta_{\mathbb{A}}\rangle = \langle Y_{\mathbb{B}}, \eta_{\mathbb{B}}\rangle \circ \mathcal{T}(F)$. Since this is an equation of $\mathcal{T}$-algebra homomorphisms for the $\mathcal{T}$-algebra structures discussed in step (i) – concerning $\mathcal{P}(F)$, it is easily seen to be a left adjoint and therefore also a $\mathcal{T}$-algebra homomorphism [Kock, 1995, Proposition 2.5] – it suffices to show that $\mathcal{P}(F) \circ \langle Y_{\mathbb{A}}, \eta_{\mathbb{A}}\rangle \circ \eta_{\mathbb{A}} = \langle Y_{\mathbb{B}}, \eta_{\mathbb{B}}\rangle \circ \mathcal{T}(F) \circ \eta_{\mathbb{A}}$. This is straightforward from the factorisation property of the Kan extension and the naturality of $Y_{\mathbb{A}}$ and $\eta_{\mathbb{A}}$.

(iv) Finally, the very fact that $\langle Y_{\mathbb{A}}, \eta_{\mathbb{A}}\rangle\colon \mathcal{T}(\mathbb{A}) \longrightarrow \mathcal{P}(\mathbb{A})$ is a $\mathcal{T}$-algebra homomorphism as in step (ii), means that

$$\begin{array}{ccccc}
\mathcal{T}(\mathcal{T}(\mathbb{A})) & \xrightarrow{\mathcal{T}(\langle Y_{\mathbb{A}}, \eta_{\mathbb{A}}\rangle)} & \mathcal{T}(\mathcal{P}(\mathbb{A})) & \xrightarrow{\langle Y_{\mathcal{P}(\mathbb{A})}, \eta_{\mathcal{P}(\mathbb{A})}\rangle} & \mathcal{P}(\mathcal{P}(\mathbb{A})) \\
\Big\downarrow{\scriptstyle \mu_{\mathbb{A}}} & & & & \Big\downarrow{\scriptstyle M_{\mathbb{A}}} \\
\mathcal{T}(\mathbb{A}) & & \xrightarrow[\langle Y_{\mathbb{A}}, \eta_{\mathbb{A}}\rangle]{} & & \mathcal{P}(\mathbb{A})
\end{array}$$

commutes: it expresses precisely the compatibility of the natural transformation whose components are the Kan extensions, with the multiplications of, respectively, $\mathcal{T}$ and $\mathcal{P}$. $\square$

3. Interlude: classifying cotabulations

In this section it is Proposition 3.3 which is of most interest: it explains in particular how the 2-functors on $\mathsf{Cat}(\mathcal{Q})$ of Proposition 2.4 can be extended to $\mathsf{Dist}(\mathcal{Q})$. It could easily be proved with a direct proof, but it seemed more appropriate to include first some material on classifying cotabulations, then use this to give a somewhat more conceptual proof of (the quantaloidal generalisation of) Akhvlediani *et al.*'s 'Extension Theorem' [2009, Theorem 1] in our Proposition 3.2, and finally derive Proposition 3.3 as a particular case.

A **cotabulation** of a distributor $\Phi\colon \mathbb{A} \mathrel{-\!\!\circ\!\!\rightarrow} \mathbb{B}$ between $\mathcal{Q}$-categories is a pair of functors, say $S\colon \mathbb{A} \longrightarrow \mathbb{C}$ and $T\colon \mathbb{B} \longrightarrow \mathbb{C}$, such that $\Phi = \mathbb{C}(T-, S-)$. If $F\colon \mathbb{C} \longrightarrow \mathbb{C}'$ is a fully faithful functor then also $F \circ S\colon \mathbb{A} \longrightarrow \mathbb{C}'$ and $F \circ T\colon \mathbb{B} \longrightarrow \mathbb{C}'$ cotabulate Φ; so a distributor admits many different cotabulations. But the classifying property of $\mathcal{P}(\mathbb{B})$ suggests a particular one:

Proposition 3.1 *Any distributor $\Phi\colon \mathbb{A} \mathrel{-\!\!\circ\!\!\rightarrow} \mathbb{B}$ is cotabulated by $Y_\Phi\colon \mathbb{A} \longrightarrow \mathcal{P}(\mathbb{B})$ and $Y_{\mathbb{B}}\colon \mathbb{B} \longrightarrow \mathcal{P}(\mathbb{B})$. We call this pair the* **classifying cotabulation** *of $\Phi\colon \mathbb{A} \mathrel{-\!\!\circ\!\!\rightarrow} \mathbb{B}$.*

Proof: We compute for $a \in \mathbb{A}$ and $b \in \mathbb{B}$ that $\mathcal{P}(\mathbb{B})(Y_{\mathbb{B}}(b), Y_\Phi(a)) = Y_\Phi(a)(b) = \Phi(b, a)$ by using the Yoneda Lemma. $\square$

For two distributors $\Phi\colon \mathbb{A} \mathrel{-\!\!\circ\!\!\rightarrow} \mathbb{B}$ and $\Psi\colon \mathbb{B} \mathrel{-\!\!\circ\!\!\rightarrow} \mathbb{C}$ it is easily seen that $Y_{\Psi \otimes \Phi} = \mathcal{P}_0(\Psi) \circ Y_\Phi$, so the classifying cotabulation of the composite $\Psi \otimes \Phi$ relates to those of Φ and Ψ as

$$\Psi \otimes \Phi = \mathcal{P}(\mathbb{C})(\mathcal{P}_0(\Psi) \circ Y_\Phi -, Y_{\mathbb{C}} -). \tag{4}$$

For a functor $F\colon \mathbb{A} \longrightarrow \mathbb{B}$ it is straightforward that $Y_{\mathbb{B}(-,F-)} = Y_{\mathbb{B}} \circ F$, so

$$\mathbb{B}(-,F-) = \mathcal{P}(\mathbb{B})(Y_{\mathbb{B}}-, Y_{\mathbb{B}} \circ F-). \tag{5}$$

In particular, the identity distributor $\mathbb{A}\colon \mathbb{A} \mathrel{-\!\!\!\circ\!\!\!\to} \mathbb{A}$ has the classifying cotabulation

$$\mathbb{A} = \mathcal{P}(\mathbb{A})(Y_{\mathbb{A}}-, Y_{\mathbb{A}}-). \tag{6}$$

Given that classifying cotabulations are thus perfectly capable of encoding composition and identities, it is natural to extend a given endo-functor on $\mathsf{Cat}(\mathcal{Q})$ to an endo-functor on $\mathsf{Dist}(\mathcal{Q})$ by applying it to classifying cotabulations. Now follows a statement of the 'Extension Theorem' of [Akhvlediani *et al.*, 2009] in the generality of quantaloid-enriched category theory, and a proof based on the calculus of classifying cotabulations.

Proposition 3.2 (Akhvlediani et al., 2009) *Any 2-functor $\mathcal{T}\colon \mathsf{Cat}(\mathcal{Q}) \longrightarrow \mathsf{Cat}(\mathcal{Q})$ extends to a lax 2-functor $\mathcal{T}'\colon \mathsf{Dist}(\mathcal{Q}) \longrightarrow \mathsf{Dist}(\mathcal{Q})$, which is defined to send a distributor $\Phi\colon \mathbb{A} \mathrel{-\!\!\!\circ\!\!\!\to} \mathbb{B}$ to the distributor cotabulated by $\mathcal{T}(Y_\Phi)\colon \mathcal{T}(\mathbb{A}) \longrightarrow \mathcal{T}(\mathcal{P}(\mathbb{B}))$ and $\mathcal{T}(Y_{\mathbb{B}})\colon \mathcal{T}(\mathbb{B}) \longrightarrow \mathcal{T}(\mathcal{P}(\mathbb{B}))$. This comes with a lax transformation*

$$\begin{array}{ccc} \mathsf{Dist}(\mathcal{Q}) & \xrightarrow{\mathcal{T}'} & \mathsf{Dist}(\mathcal{Q}) \\ {\scriptstyle i}\uparrow & \nwarrow & \uparrow{\scriptstyle i} \\ \mathsf{Cat}(\mathcal{Q}) & \xrightarrow[\mathcal{T}]{} & \mathsf{Cat}(\mathcal{Q}) \end{array} \tag{7}$$

all of whose components are identities. This lax transformation is a (strict) 2-natural transformation (i.e. this diagram is commutative) if and only if $\mathcal{T}'$ is normal, if and only if each $\mathcal{T}(Y_{\mathbb{A}})\colon \mathcal{T}(\mathbb{A}) \longrightarrow \mathcal{T}(\mathcal{P}(\mathbb{A}))$ is fully faithful.

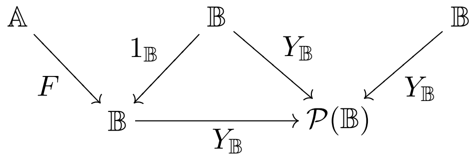
Proof : If $\Phi \leq \Psi$ holds in $\mathsf{Dist}(\mathcal{Q})(\mathbb{A},\mathbb{B})$ then (and only then) $Y_\Phi \leq Y_\Psi$ holds in $\mathsf{Cat}(\mathcal{Q})(\mathbb{A},\mathcal{P}(\mathbb{B}))$. By 2-functoriality of $\mathcal{T}\colon \mathsf{Cat}(\mathcal{Q}) \longrightarrow \mathsf{Cat}(\mathcal{Q})$ we find that $\mathcal{T}(Y_\Phi) \leq \mathcal{T}(Y_\Psi)$, and thus $\mathcal{T}'(\Phi) \leq \mathcal{T}'(\Psi)$.

Now suppose that $\Phi\colon \mathbb{A} \mathrel{-\!\!\!\circ\!\!\!\to} \mathbb{B}$ and $\Psi\colon \mathbb{B} \mathrel{-\!\!\!\circ\!\!\!\to} \mathbb{C}$ are given. Applying $\mathcal{T}$ to the commutative diagram

$$\begin{array}{ccccccc} \mathbb{A} & & & \mathbb{B} & & & \mathbb{C} \\ & {\scriptstyle Y_\Phi}\searrow & {\scriptstyle Y_{\mathbb{B}}}\swarrow & & \searrow{\scriptstyle Y_\Psi} & \swarrow{\scriptstyle Y_{\mathbb{C}}} & \\ & & \mathcal{P}(\mathbb{B}) & \xrightarrow[\mathcal{P}_0(\Psi)]{} & \mathcal{P}(\mathbb{C}) & & \end{array}$$

gives a commutative diagram in $\mathsf{Cat}(\mathcal{Q})$, which embeds as a commutative diagram of left adjoints in the quantaloid $\mathsf{Dist}(\mathcal{Q})$ by application of $i\colon \mathsf{Cat}(\mathcal{Q}) \longrightarrow \mathsf{Dist}(\mathcal{Q})$. Lemma 2.2, the formula in (4) and the definition of $\mathcal{T}'$ allow us to conclude that $\mathcal{T}'(\Psi) \otimes \mathcal{T}'(\Phi) \leq \mathcal{T}'(\Psi \otimes \Phi)$.

Similarly, given $F\colon \mathbb{A} \longrightarrow \mathbb{B}$ in $\mathsf{Cat}(\mathcal{Q})$, applying $\mathcal{T}$ to the commutative diagram



gives a commutative diagram in $\mathsf{Cat}(\mathcal{Q})$. This again embeds as a diagram of left adjoints in $\mathsf{Dist}(\mathcal{Q})$ via $i\colon \mathsf{Cat}(\mathcal{Q}) \longrightarrow \mathsf{Dist}(\mathcal{Q})$. Lemma 2.2, the formula in (5) and the definition of $\mathcal{T}'$ then straightforwardly imply that

$$\begin{aligned}\mathcal{T}'(\mathbb{B}(-,F-)) &= \mathcal{T}(\mathcal{P}(\mathbb{B}))(\mathcal{T}(Y_{\mathbb{B}})-,\mathcal{T}(Y_{\mathbb{B}})-) \otimes \mathcal{T}(\mathbb{B})(-,\mathcal{T}(F)-)\\ &\geq \mathcal{T}(\mathbb{B})(-,\mathcal{T}(F)-),\end{aligned}$$

accounting for the lax transformation in (7).

It further follows from this inequation, by applying it to identity functors, that $\mathcal{T}'$ is in general lax on identity distributors. But Lemma 2.2 also says: (i) if each $\mathcal{T}(Y_{\mathbb{B}})\colon \mathcal{T}(\mathbb{B}) \longrightarrow \mathcal{T}(\mathcal{P}(\mathbb{B}))$ is fully faithful (equivalently, if $\mathcal{T}'$ is normal), then necessarily $(i \circ \mathcal{T})(F) \cong (\mathcal{T}' \circ i)(F)$ for all $F : \mathbb{A} \longrightarrow \mathbb{B}$ in $\mathsf{Cat}(\mathcal{Q})$, asserting that the diagram in (7) commutes; (ii) and conversely, if that diagram commutes, then chasing the identities in $\mathsf{Cat}(\mathcal{Q})$ shows that $\mathcal{T}'$ is normal. $\square$

We shall be interested in extending full sub-KZ-doctrines of the free cocompletion doctrine $\mathcal{P}\colon \mathsf{Cat}(\mathcal{Q}) \longrightarrow \mathsf{Cat}(\mathcal{Q})$ to $\mathsf{Dist}(\mathcal{Q})$; for this we make use of the functor $\mathcal{P}_1\colon \mathsf{Dist}(\mathcal{Q}) \longrightarrow \mathsf{Dist}(\mathcal{Q})$ defined in the diagram in (3).

Proposition 3.3 *Let $(\mathcal{T},\varepsilon)$ be a full sub-KZ-doctrine of $\mathcal{P}\colon \mathsf{Cat}(\mathcal{Q}) \longrightarrow \mathsf{Cat}(\mathcal{Q})$. The lax extension $\mathcal{T}'\colon \mathsf{Dist}(\mathcal{Q}) \longrightarrow \mathsf{Dist}(\mathcal{Q})$ of $\mathcal{T}\colon \mathsf{Cat}(\mathcal{Q}) \longrightarrow \mathsf{Cat}(\mathcal{Q})$ (as in Proposition 3.2) can then be computed as follows: for $\Phi\colon \mathbb{A} \looparrowright \mathbb{B}$,*

$$\mathcal{T}'(\Phi) = \mathcal{P}(\mathbb{B})(\varepsilon_{\mathbb{B}}-,-) \otimes \mathcal{P}_1(\Phi) \otimes \mathcal{P}(\mathbb{A})(-,\varepsilon_{\mathbb{A}}-). \tag{8}$$

Moreover, $\mathcal{T}'$ is always a normal lax Sup*-functor, thus the diagram in (7) commutes.*

Proof : Let $\Phi\colon \mathbb{A} \looparrowright \mathbb{B}$ be a distributor. Proposition 3.2 defines $\mathcal{T}'(\Phi)$ to be the distributor cotabulated by $\mathcal{T}(Y_\Phi)$ and $\mathcal{T}(Y_{\mathbb{B}})$; but by fully faithfulness of the components of $\varepsilon\colon \mathcal{T} \Longrightarrow \mathcal{P}$, and its naturality, we can compute that

$$\mathcal{T}(\mathcal{P}(\mathbb{B}))(\mathcal{T}(Y_{\mathbb{B}})-,\mathcal{T}(Y_\Phi)-)$$

$$
\begin{aligned}
&= \mathcal{P}(\mathcal{P}(\mathbb{B}))((\varepsilon_{\mathcal{P}(\mathbb{B})} \circ \mathcal{T}(Y_{\mathbb{B}}))-, (\varepsilon_{\mathcal{P}(\mathbb{B})} \circ \mathcal{T}(Y_{\Phi}))-) \\
&= \mathcal{P}(\mathcal{P}(\mathbb{B}))((\mathcal{P}(Y_{\mathbb{B}}) \circ \varepsilon_{\mathbb{B}})-, (\mathcal{P}(Y_{\Phi}) \circ \varepsilon_{\mathbb{A}})-) \\
&= \mathcal{P}(\mathbb{B})(\varepsilon_{\mathbb{B}}-, -) \otimes \mathcal{P}(\mathcal{P}(\mathbb{B}))(\mathcal{P}(Y_{\mathbb{B}})-, \mathcal{P}(Y_{\Phi})-) \otimes \mathcal{P}(\mathbb{A})(-, \varepsilon_{\mathbb{A}}-).
\end{aligned}
$$

The middle term in this last expression can be reduced:

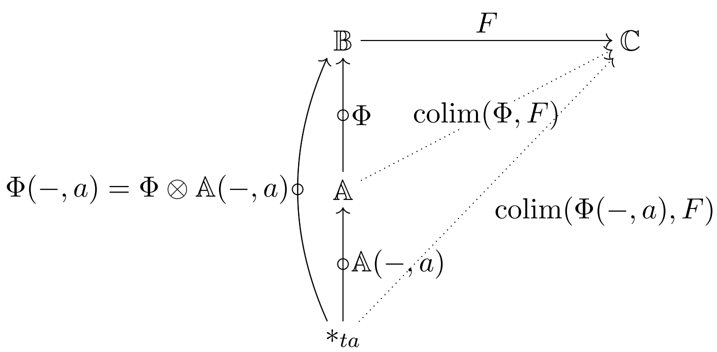
$$
\begin{aligned}
\mathcal{P}(\mathcal{P}(\mathbb{B}))(\mathcal{P}(Y_{\mathbb{B}})-, \mathcal{P}(Y_{\Phi})-) &= [\mathcal{P}(\mathbb{B})(-, Y_{\mathbb{B}}-) \otimes -,\ \mathcal{P}(\mathbb{B})(-, Y_{\Phi}-) \otimes -] \\
&= [-, \mathcal{P}(\mathbb{B})(Y_{\mathbb{B}}-, -) \otimes \mathcal{P}(\mathbb{B})(-, Y_{\Phi}-) \otimes -] \\
&= [-, \mathcal{P}(\mathbb{B})(Y_{\mathbb{B}}-, Y_{\Phi}-) \otimes -] \\
&= [-, \Phi \otimes -] \\
&= \mathcal{P}(\mathbb{B})(-, \mathcal{P}_0(\Phi)-) \\
&= (i \circ \mathcal{P}_0)(\Phi)(-, -) \\
&= \mathcal{P}_1(\Phi)(-, -).
\end{aligned}
$$

Thus we arrive at (8). Because $\mathcal{P}_1$ is a (strict) functor and because each $\varepsilon_{\mathbb{A}}$ is fully faithful, it follows from (8) that $\mathcal{T}'$ is normal. Similarly, because $\mathcal{P}_1$ is a Sup-functor, $\mathcal{T}'$ preserves local suprema too. □

If we apply Proposition 3.2 to the 2-functor $\mathcal{P}\colon \mathsf{Cat}(\mathcal{Q}) \longrightarrow \mathsf{Cat}(\mathcal{Q})$ itself, then we find that $\mathcal{P}' = \mathcal{P}_1$ (and thus it is strictly functorial, not merely normal lax). In general however, $\mathcal{T}'$ does *not* preserve composition.

4. Cocompletion: saturated classes of weights vs. KZ-doctrines

The Φ-weighted colimit of a functor F exists if and only if, for every $a \in \mathbb{A}_0$, $\mathrm{colim}(\Phi(-,a), F)$ exists:



Indeed, $\mathrm{colim}(\Phi, F)(a) = \mathrm{colim}(\Phi(-,a), F)(*)$. But now $\Phi(-,a)\colon *_{ta} \longrightarrow \mathbb{B}$ is a presheaf on $\mathbb{B}$. As a consequence, a $\mathcal{Q}$-category $\mathbb{C}$ is cocomplete if and only if it admits all colimits weighted by presheaves.

It therefore makes perfect sense to fix a class $\mathcal{C}$ of presheaves and study those $\mathcal{Q}$-categories that admit all colimits weighted by elements of $\mathcal{C}$: by definition these are the $\mathcal{C}$**-cocomplete categories**. Similarly, a functor $G\colon \mathbb{C} \longrightarrow \mathbb{C}'$ is $\mathcal{C}$**-cocontinuous** if it preserves all colimits weighted by elements of $\mathcal{C}$.

As [Albert and Kelly, 1988; Kelly and Schmitt, 2005] demonstrated in the case of $\mathcal{V}$-categories (for $\mathcal{V}$ a symmetric monoidal closed category with locally small, complete and cocomplete underlying category $\mathcal{V}_0$), and as we shall argue here for $\mathcal{Q}$-categories too, it is convenient to work with classes of presheaves that "behave nicely":

Definition 4.1 *A class $\mathcal{C}$ of presheaves on $\mathcal{Q}$-categories is* **saturated** *if:*

i. $\mathcal{C}$ contains all representable presheaves,

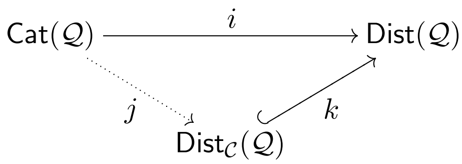
ii. for each $\phi\colon *_X \mathrel{-\!\!\circ\!\!\rightarrow} \mathbb{A}$ in $\mathcal{C}$ and each functor $G\colon \mathbb{A} \longrightarrow \mathcal{P}(\mathbb{B})$ for which each $G(a)$ is in $\mathcal{C}$, $\mathrm{colim}(\phi, G)$ is in $\mathcal{C}$ too.

There is another way of putting this. Observe first that any class $\mathcal{C}$ of presheaves on $\mathcal{Q}$-categories defines a sub-2-graph $k\colon \mathsf{Dist}_{\mathcal{C}}(\mathcal{Q}) \hookrightarrow \mathsf{Dist}(\mathcal{Q})$ by

$$\Phi\colon \mathbb{A} \mathrel{-\!\!\circ\!\!\rightarrow} \mathbb{B} \text{ is in } \mathsf{Dist}_{\mathcal{C}}(\mathcal{Q}) \stackrel{\text{def.}}{\iff} \text{ for all } a \in \mathbb{A}_0\colon \Phi(-,a) \in \mathcal{C}. \tag{9}$$

Then in fact we have:

Proposition 4.2 *A class $\mathcal{C}$ of presheaves on $\mathcal{Q}$-categories is saturated if and only if $\mathsf{Dist}_{\mathcal{C}}(\mathcal{Q})$ is a sub-2-category of $\mathsf{Dist}(\mathcal{Q})$ containing (all objects and) all identities. In this case there is an obvious factorisation*



Proof : With (9) it is trivial that $\mathcal{C}$ contains all representable presheaves if and only if $\mathsf{Dist}_{\mathcal{C}}(\mathcal{Q})$ contains all objects and all identities.

Next, assume that $\mathcal{C}$ is a saturated class of presheaves, and let $\Phi\colon \mathbb{A} \mathrel{-\!\!\circ\!\!\rightarrow} \mathbb{B}$ and $\Psi\colon \mathbb{B} \mathrel{-\!\!\circ\!\!\rightarrow} \mathbb{C}$ be arrows in $\mathsf{Dist}_{\mathcal{C}}(\mathcal{Q})$. Invoking the classifying property of $\mathcal{P}(\mathbb{C})$ and the computation of colimits in $\mathcal{P}(\mathbb{C})$, we find $\mathrm{colim}(\Phi(-,a), Y_\Psi) = \Psi \otimes \Phi(-,a)$ for each $a \in \mathbb{A}_0$. But because $\Phi(-,a) \in \mathcal{C}$ and for each $b \in \mathbb{B}_0$ also $Y_\Psi(b) = \Psi(-,b) \in \mathcal{C}$, this colimit, i.e. $\Psi \otimes \Phi(-,a)$, is an element of $\mathcal{C}$. This holds for all $a \in \mathbb{A}_0$, thus the composition $\Psi \otimes \Phi\colon \mathbb{A} \mathrel{-\!\!\circ\!\!\rightarrow} \mathbb{C}$ is an arrow in $\mathsf{Dist}_{\mathcal{C}}(\mathcal{Q})$.

Conversely, assuming $\mathsf{Dist}_{\mathcal{C}}(\mathcal{Q})$ is a sub-2-category of $\mathsf{Dist}(\mathcal{Q})$, let $\phi\colon *_A \mathrel{-\!\!\circ\!\!\rightarrow} \mathbb{B}$ be in $\mathcal{C}$ and let $F\colon \mathbb{B}\longrightarrow\mathcal{P}(\mathbb{C})$ be a functor such that, for each $b\in\mathbb{B}$, $F(b)$ is in $\mathcal{C}$. By the classifying property of $\mathcal{P}(\mathbb{C})$ we can equate the functor $F\colon\mathbb{B}\longrightarrow\mathcal{P}(\mathbb{C})$ with a distributor $\Phi_F\colon\mathbb{B}\mathrel{-\!\!\circ\!\!\rightarrow}\mathbb{C}$ and by the computation of colimits in $\mathcal{P}(\mathbb{C})$ we know that $\mathrm{colim}(\phi,F)=\Phi_F\otimes\phi$. Now $\Phi_F(-,b)=F(b)$ by definition, so $\Phi_F\colon\mathbb{B}\mathrel{-\!\!\circ\!\!\rightarrow}\mathbb{C}$ is in $\mathsf{Dist}_{\mathcal{C}}(\mathcal{Q})$; but also $\phi : *_A\mathrel{-\!\!\circ\!\!\rightarrow}\mathbb{B}$ is in $\mathsf{Dist}_{\mathcal{C}}(\mathcal{Q})$, and therefore their composite is in $\mathsf{Dist}_{\mathcal{C}}(\mathcal{Q})$, i.e. $\mathrm{colim}(\phi,F)$ is in $\mathcal{C}$, as wanted.

Finally, if $F\colon\mathbb{A}\longrightarrow\mathbb{B}$ is any functor, then for each $a\in\mathbb{A}$ the representable $\mathbb{B}(-,Fa)\colon *_{ta}\mathrel{-\!\!\circ\!\!\rightarrow}\mathbb{B}$ is in the saturated class $\mathcal{C}$, and therefore $\mathbb{B}(-,F-)\colon\mathbb{A}\mathrel{-\!\!\circ\!\!\rightarrow}\mathbb{B}$ is in $\mathsf{Dist}_{\mathcal{C}}(\mathcal{Q})$. This accounts for the factorisation of $\mathsf{Cat}(\mathcal{Q})\longrightarrow\mathsf{Dist}(\mathcal{Q})$ over $\mathsf{Dist}_{\mathcal{C}}(\mathcal{Q})\hookrightarrow\mathsf{Dist}(\mathcal{Q})$. $\square$

We shall now characterise saturated classes of presheaves on $\mathcal{Q}$-categories in terms of KZ-doctrines on $\mathsf{Cat}(\mathcal{Q})$. (We shall indeed always deal with a saturated class of presheaves, even though certain results hold under weaker hypotheses.) We begin by pointing out a classifying property:

Proposition 4.3 *Let $\mathcal{C}$ be a saturated class of presheaves and, for a $\mathcal{Q}$-category $\mathbb{A}$, write $J_{\mathbb{A}}\colon\mathcal{C}(\mathbb{A})\longrightarrow\mathcal{P}(\mathbb{A})$ for the full subcategory of $\mathcal{P}(\mathbb{A})$ determined by those presheaves on $\mathbb{A}$ which are elements of $\mathcal{C}$. A distributor $\Phi\colon\mathbb{A}\mathrel{-\!\!\circ\!\!\rightarrow}\mathbb{B}$ belongs to $\mathsf{Dist}_{\mathcal{C}}(\mathcal{Q})$ if and only if there exists a (necessarily unique) factorisation*

$$\begin{array}{ccc} \mathbb{A} & \xrightarrow{\quad Y_\Phi \quad} & \mathcal{P}(\mathbb{B}) \\ & {\scriptstyle I_\Phi}\searrow \quad \nearrow{\scriptstyle J_{\mathbb{B}}} & \\ & \mathcal{C}(\mathbb{B}) & \end{array} \tag{10}$$

in which case Φ is cotabulated by $I_\Phi\colon\mathbb{A}\longrightarrow\mathcal{C}(\mathbb{B})$ and $I_{\mathbb{B}}\colon\mathbb{B}\longrightarrow\mathcal{C}(\mathbb{B})$ (the latter being the factorisation of $Y_{\mathbb{B}}$ through $J_{\mathbb{B}}$).

Proof : The factorisation property in (10) literally says that, for any $a\in\mathbb{A}$, the presheaf $Y_\Phi(a)$ on $\mathbb{B}$ must be an element of the class $\mathcal{C}$. But $Y_\Phi(b)=\Phi(-,b)$ hence this is trivially equivalent to the statement in (9), defining those distributors that belong to $\mathsf{Dist}_{\mathcal{C}}(\mathcal{Q})$. In particular, if $\mathcal{C}$ is saturated then $\mathsf{Dist}_{\mathcal{C}}(\mathcal{Q})$ contains all identities, hence we have factorisations $Y_{\mathbb{B}}=J_{\mathbb{B}}\circ I_{\mathbb{B}}$ of the Yoneda embeddings. Hence, whenever a factorisation as in (10) exists, we can use the fully faithful $J_{\mathbb{B}}\colon\mathcal{C}(\mathbb{B})\longrightarrow\mathcal{P}(\mathbb{B})$ to compute, starting from the classifying cotabulation of Φ, that

$$\Phi=\mathcal{P}(\mathbb{B})(Y_{\mathbb{B}}-,Y_\Phi-)=\mathcal{P}(\mathbb{B})(J_{\mathbb{B}}(I_{\mathbb{B}}(-)),J_{\mathbb{B}}(I_\Phi(-)))=\mathcal{C}(\mathbb{B})(I_{\mathbb{B}}-,I_\Phi-),$$

confirming the cotabulation of Φ by I_Φ and $I_{\mathbb{B}}$. $\square$

Any saturated class $\mathcal{C}$ thus automatically comes with the 2-functor

$$\mathcal{C}_0\colon \mathsf{Dist}_{\mathcal{C}}(\mathcal{Q})\longrightarrow \mathsf{Cat}(\mathcal{Q})\colon \Big(\Phi\colon \mathbb{A} \xrightarrow{\ \circ\ } \mathbb{B}\Big) \mapsto \Big(\Phi\otimes -\colon \mathcal{C}(\mathbb{A})\longrightarrow\mathcal{C}(\mathbb{B})\Big)$$

and the full embeddings $J_{\mathbb{A}}\colon \mathcal{C}(\mathbb{A})\longrightarrow\mathcal{P}(\mathbb{A})$ are the components of a 2-natural transformation

$$\begin{array}{ccccc} & \overset{k}{\nearrow} & \mathsf{Dist}(\mathcal{Q}) & \overset{\mathcal{P}_0}{\searrow} & \\ & & \Uparrow J & & \\ \mathsf{Dist}_{\mathcal{C}}(\mathcal{Q}) & & \xrightarrow[\mathcal{C}_0]{\qquad\qquad} & & \mathsf{Cat}(\mathcal{Q}) \end{array}$$

Composing $\mathcal{C}_0$ with $j\colon \mathsf{Cat}(\mathcal{Q})\longrightarrow\mathsf{Dist}_{\mathcal{C}}(\mathcal{Q})$ it is natural to define

$$\mathcal{C}\colon \mathsf{Cat}(\mathcal{Q})\longrightarrow\mathsf{Cat}(\mathcal{Q})\colon \Big(F\colon\mathbb{A}\longrightarrow\mathbb{B}\Big)\mapsto\Big(\mathbb{B}(-,F-)\otimes -\colon\mathcal{C}(\mathbb{A})\longrightarrow\mathcal{C}(\mathbb{B})\Big)$$

together with

$$\mathsf{Cat}(\mathcal{Q}) \underset{\mathcal{C}}{\overset{\mathcal{P}}{\rightrightarrows}}\ \Uparrow J\ \mathsf{Cat}(\mathcal{Q})$$

(slightly abusing notation). We apply previous results, particularly Proposition 2.4:

Proposition 4.4 *If $\mathcal{C}$ is a saturated class of presheaves on $\mathcal{Q}$-categories then the 2-functor $\mathcal{C}\colon\mathsf{Cat}(\mathcal{Q})\longrightarrow\mathsf{Cat}(\mathcal{Q})$ together with the transformation $J\colon\mathcal{C}\Longrightarrow\mathcal{P}$ forms a full sub-KZ-doctrine of $\mathcal{P}$. Moreover, the $\mathcal{C}$-cocomplete $\mathcal{Q}$-categories are precisely the $\mathcal{C}$-algebras, and the $\mathcal{C}$-cocontinuous functors between $\mathcal{C}$-cocomplete $\mathcal{Q}$-categories are precisely the $\mathcal{C}$-algebra homomorphisms.*

Proof : To fulfill the hypotheses in Proposition 2.4, we only need to check the factorisation of the multiplication: if we prove, for any $\mathcal{Q}$-category $\mathbb{A}$ and each $\phi\in\mathcal{C}(\mathcal{C}(\mathbb{A}))$, that the $(J*J)_{\mathbb{A}}(\phi)$-weighted colimit of $1_{\mathcal{P}(\mathbb{A})}$ is in $\mathcal{C}(\mathbb{A})$, then we obtain the required commutative diagram

$$\begin{array}{ccc} \mathcal{P}(\mathcal{P}(\mathbb{A})) & \xrightarrow{\mathrm{colim}(-,1_{\mathcal{P}(\mathbb{A})})} & \mathcal{P}(\mathbb{A}) \\ {\scriptstyle (J*J)_{\mathbb{A}}}\uparrow & & \uparrow{\scriptstyle J_{\mathbb{A}}} \\ \mathcal{C}(\mathcal{C}(\mathbb{A})) & \dashrightarrow & \mathcal{C}(\mathbb{A}) \end{array}$$

But because $(J*J)_{\mathbb{A}}=\mathcal{P}(J_{\mathbb{A}})\circ J_{\mathcal{C}(\mathbb{A})}$ we can compute that

$$\mathrm{colim}((J*J)_{\mathbb{A}}(\phi),1_{\mathcal{P}(\mathbb{A})})=\mathrm{colim}(\mathcal{P}(\mathbb{A})(-,J_{\mathbb{A}}-)\otimes\phi,1_{\mathcal{P}(\mathbb{A})})=\mathrm{colim}(\phi,J_{\mathbb{A}})$$

and this colimit indeed belongs to the saturated class $\mathcal{C}$, because both ϕ and (the objects in) the image of $J_{\mathbb{A}}$ are in $\mathcal{C}$.

A $\mathcal{Q}$-category $\mathbb{B}$ is a $\mathcal{C}$-algebra if and only if $I_{\mathbb{B}}\colon\mathbb{B}\longrightarrow\mathcal{C}(\mathbb{B})$ admits a left adjoint in $\mathsf{Cat}(\mathcal{Q})$ (because $\mathcal{C}$ is a KZ-doctrine). Suppose that $\mathbb{B}$ is indeed a $\mathcal{C}$-algebra, and write the left adjoint as $L_{\mathbb{B}}\colon\mathcal{C}(\mathbb{B})\longrightarrow\mathbb{B}$. If $\phi\colon *_X \multimap \mathbb{A}$ is a presheaf in $\mathcal{C}$ and $F\colon\mathbb{A}\longrightarrow\mathbb{B}$ is any functor, then $\mathcal{C}(F)(\phi)$ is an object of $\mathcal{C}(\mathbb{B})$, thus we can consider the object $L_{\mathbb{B}}(\mathcal{C}(F)(\phi))$ of $\mathbb{B}$. This is precisely the ϕ-weighted colimit of F, for indeed its universal property holds: for any $b\in\mathbb{B}$,

$$\begin{aligned}
\mathbb{B}(L_{\mathbb{B}}(\mathcal{C}(F)(\phi)),b) &= \mathcal{C}(\mathbb{B})(\mathcal{C}(F)(\phi),I_{\mathbb{B}}(b))\\
&= \mathcal{P}(\mathbb{B})(J_{\mathbb{B}}(\mathcal{C}(F)(\phi)),J_{\mathbb{B}}(I_{\mathbb{B}}(b)))\\
&= [\mathcal{P}(F)(J_{\mathbb{B}}(\phi)),Y_{\mathbb{B}}(b)]\\
&= [\mathbb{B}(-,F-)\otimes J_{\mathbb{B}}(\phi),\mathbb{B}(-,b)]\\
&= [J_{\mathbb{B}}(\phi),\mathbb{B}(F-,-)\otimes\mathbb{B}(-,b)]\\
&= [\phi,\mathbb{B}(F-,b)].
\end{aligned}$$

(Apart from the adjunction $L_{\mathbb{B}}\dashv I_{\mathbb{B}}$ we used the fully faithfulness of $J_{\mathbb{B}}$ and its naturality, and then made some computations with liftings and adjoints in $\mathsf{Dist}(\mathcal{Q})$.)

Conversely, suppose that $\mathbb{B}$ admits all $\mathcal{C}$-weighted colimits. In particular can we then compute, for any $\phi\in\mathcal{C}(\mathbb{B})$, the ϕ-weighted colimit of $1_{\mathbb{B}}$, and doing so gives a function $f\colon\mathcal{C}(\mathbb{B})\longrightarrow\mathbb{B}\colon\phi\mapsto\mathrm{colim}(\phi,1_{\mathbb{B}})$. But for any $\phi\in\mathcal{C}(\mathbb{B})$ and any $b\in\mathbb{B}$ it is easy to compute, from the universal property of colimits and using the fully faithfulness of $J_{\mathbb{B}}$, that

$$\begin{aligned}
\mathbb{B}(f(\phi),b) &= [\phi,\mathbb{B}(1_{\mathbb{B}}-,b)] = \mathcal{P}(\mathbb{B})(\phi,Y_{\mathbb{B}}(b))\\
&= \mathcal{P}(\mathbb{B})(J_{\mathbb{B}}(\phi),J_{\mathbb{B}}(I_{\mathbb{B}}(b))) = \mathcal{C}(\mathbb{B})(\phi,I_{\mathbb{B}}(b)).
\end{aligned}$$

This straightforwardly implies that $\phi\mapsto f(\phi)$ is in fact a functor (and not merely a function), and that it is left adjoint to $I_{\mathbb{B}}$; thus $\mathbb{B}$ is a $\mathcal{C}$-algebra.

Finally, let $G\colon\mathbb{B}\longrightarrow\mathbb{C}$ be a functor between $\mathcal{C}$-cocomplete $\mathcal{Q}$-categories. Supposing that G is $\mathcal{C}$-cocontinuous, we can compute any $\psi\in\mathcal{C}(\mathbb{B})$ that

$$G(L_{\mathbb{B}}(\psi)) = G(\mathrm{colim}(\psi),1_{\mathbb{B}}) = \mathrm{colim}(\psi),G) = L_{\mathbb{C}}(\mathcal{C}(G)(\psi)),$$

proving that G is a homomorphism between the $\mathcal{C}$-algebras $(\mathbb{B},L_{\mathbb{B}})$ and $(\mathbb{C},L_{\mathbb{C}})$. Conversely, supposing now that G is a homomorphism, we can compute for any presheaf $\phi\colon *_X\multimap\mathbb{A}$ in $\mathcal{C}$ and any functor $F\colon\mathbb{A}\longrightarrow\mathbb{B}$ that

$$G(\mathrm{colim}(\phi,F)) = G(L_{\mathbb{B}}(\mathcal{C}(F)(\phi))) = L_{\mathbb{C}}(\mathcal{C}(G)(\mathcal{C}(F)(\phi))) = \mathrm{colim}(\phi,G\circ F),$$

proving that G is $\mathcal{C}$-cocontinuous. $\square$

Also the converse of the previous Proposition is true:

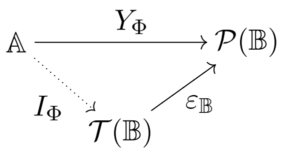
Proposition 4.5 *If $(\mathcal{T},\varepsilon)$ is a full sub-KZ-doctrine of $\mathcal{P}\colon \mathsf{Cat}(\mathcal{Q}) \longrightarrow \mathsf{Cat}(\mathcal{Q})$ then*

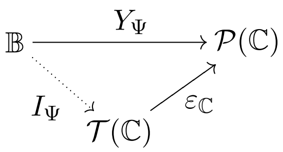
$$\mathcal{C}_{\mathcal{T}} := \{\varepsilon_{\mathbb{A}}(t) \mid \mathbb{A} \in \mathsf{Cat}(\mathcal{Q}),\ t \in \mathcal{T}(\mathbb{A})\} \tag{11}$$

is a saturated class of presheaves on $\mathcal{Q}$-categories. Moreover, the $\mathcal{T}$-algebras are precisely the $\mathcal{C}_{\mathcal{T}}$-cocomplete categories, and the $\mathcal{T}$-algebra homomorphisms are precisely the $\mathcal{C}_{\mathcal{T}}$-cocontinuous functors between the $\mathcal{C}_{\mathcal{T}}$-cocomplete categories.

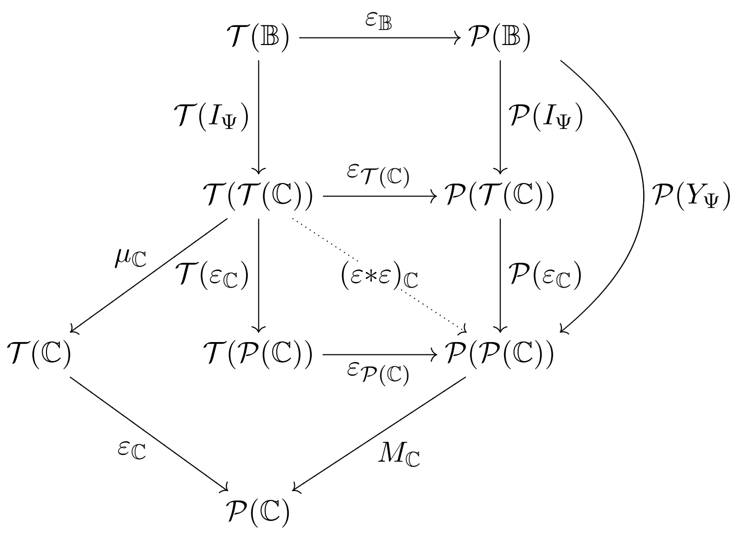
Proof : We shall write $\mathsf{Dist}_{\mathcal{T}}(\mathcal{Q})$ for the sub-2-graph of $\mathsf{Dist}(\mathcal{Q})$ determined – as prescribed in (9) – by the class $\mathcal{C}_{\mathcal{T}}$, and we shall show that it is a sub-2-category containing all (objects and) identities of $\mathsf{Dist}(\mathcal{Q})$. But a distributor $\Phi\colon \mathbb{A} \rightarrow\!\!\!\!\!\circ\; \mathbb{B}$ belongs to $\mathsf{Dist}_{\mathcal{T}}(\mathcal{Q})$ if and only if the classifying functor $Y_\Phi\colon \mathbb{A} \longrightarrow \mathcal{P}(\mathbb{B})$ factors (necessarily essentially uniquely) through the fully faithful $\varepsilon_{\mathbb{B}}\colon \mathcal{T}(\mathbb{B}) \longrightarrow \mathcal{P}(\mathbb{B})$.

By hypothesis there is a factorisation $Y_{\mathbb{A}} = \varepsilon_{\mathbb{A}} \circ \eta_{\mathbb{A}}$ for any $\mathbb{A} \in \mathsf{Cat}(\mathcal{Q})$, so $\mathsf{Dist}_{\mathcal{T}}(\mathcal{Q})$ contains all identities. Secondly, suppose that $\Phi\colon \mathbb{A} \rightarrow\!\!\!\!\!\circ\; \mathbb{B}$ and $\Psi\colon \mathbb{B} \rightarrow\!\!\!\!\!\circ\; \mathbb{C}$ are in $\mathsf{Dist}_{\mathcal{T}}(\mathcal{Q})$, meaning that there exist factorisations





The following diagram is then easily seen to commute:



But we can compute, for any $\phi \in \mathcal{P}(\mathbb{B})$, that

$$(M_{\mathbb{C}} \circ \mathcal{P}(Y_\Psi))(\phi) \quad = \quad \mathrm{colim}(\mathcal{P}(\mathbb{A})(-, Y_\Psi -) \otimes \phi, 1_{\mathcal{P}(\mathbb{A})})$$

$$\begin{aligned} &= \operatorname{colim}(\phi, Y_\Psi) \\ &= \Psi \otimes \phi \\ &= \mathcal{P}_0(\Psi)(\phi) \end{aligned}$$

and therefore $Y_{\Psi\otimes\Psi} = \mathcal{P}_0(\Psi) \circ Y_\Phi = M_{\mathbb{C}} \circ \mathcal{P}(Y_\Psi) \circ \varepsilon_{\mathbb{B}} \circ I_\Phi = \varepsilon_{\mathbb{C}} \circ \mu_{\mathbb{C}} \circ \mathcal{T}(I_\Psi) \circ I_\Phi$, giving a factorisation of $Y_{\Psi\otimes\Phi}$ through $\varepsilon_{\mathbb{C}}$, as wanted.

The arguments to prove that a $\mathcal{Q}$-category $\mathbb{B}$ is a $\mathcal{T}$-algebra if and only if it is $\mathcal{C}_{\mathcal{T}}$-cocomplete, and that a $\mathcal{T}$-algebra homomorphism is precisely a $\mathcal{C}_{\mathcal{T}}$-cocontinuous functor between $\mathcal{C}_{\mathcal{T}}$-cocomplete $\mathcal{Q}$-categories, are much like those in the proof of Proposition 4.4. Omitting the calculations, let us just indicate that for a $\mathcal{T}$-algebra $\mathbb{B}$, thus with a left adjoint $L_{\mathbb{B}}: \mathcal{T}(\mathbb{B}) \longrightarrow \mathbb{B}$ to $\eta_{\mathbb{B}}$, for any weight $\phi: *_X \xrightarrow{\circ} \mathbb{A}$ in $\mathcal{C}_{\mathcal{T}}$ – i.e. $\phi = \varepsilon_{\mathbb{A}}(t)$ for some $t \in \mathcal{T}(\mathbb{A})$ – and any functor $F: \mathbb{A} \longrightarrow \mathbb{B}$, the object $L_{\mathbb{B}}(\mathcal{T}(F))(t)$ is the ϕ-weighted colimit of F. And conversely, if $\mathbb{B}$ is a $\mathcal{C}_{\mathcal{T}}$-cocomplete $\mathcal{Q}$-category, then $\mathcal{T}(\mathbb{B}) \longrightarrow \mathbb{B}: t \mapsto \operatorname{colim}(\varepsilon_{\mathbb{B}}(t), 1_{\mathbb{B}})$ is the left adjoint to $\eta_{\mathbb{B}}$, making $\mathbb{B}$ a $\mathcal{T}$-algebra. $\square$

If $\mathcal{C}$ is a saturated class of presheaves and we apply Proposition 4.4 to obtain a full sub-KZ-doctrine $(\mathcal{C}, J)$ of $\mathcal{P}: \mathsf{Cat}(\mathcal{Q}) \longrightarrow \mathsf{Cat}(\mathcal{Q})$, then the application of Proposition 4.5 gives us back precisely that same class $\mathcal{C}$ that we started from. The other way round is slightly more subtle: if $(\mathcal{T}, \varepsilon)$ is a full sub-KZ-doctrine of $\mathcal{P}$ then Proposition 4.5 gives us a saturated class $\mathcal{C}_{\mathcal{T}}$ of presheaves, and this class in turn determines by Proposition 4.4 a full KZ-doctrine of $\mathcal{P}$, let us write it as $(\mathcal{T}', \varepsilon')$, which is *equivalent* to $\mathcal{T}$. More exactly, each (fully faithful) $\varepsilon_{\mathbb{A}}: \mathcal{T}(\mathbb{A}) \longrightarrow \mathcal{P}(\mathbb{A})$ factors over the fully faithful and injective $\varepsilon'_{\mathbb{A}}: \mathcal{T}'(\mathbb{A}) \longrightarrow \mathcal{P}(\mathbb{A})$, and this factorisation is fully faithful and surjective, thus an equivalence. These equivalences are the components of a 2-natural transformation $\delta: \mathcal{T} \Longrightarrow \mathcal{T}'$ which commutes with ε and ε'.

We summarise all the above in the following:

Theorem 4.6 *Propositions 4.4 and 4.5 determine an essentially bijective correspondence between, on the one hand, saturated classes $\mathcal{C}$ of presheaves on $\mathcal{Q}$-categories, and on the other hand, full sub-KZ-doctrines $(\mathcal{T}, \varepsilon)$ of the free cocompletion KZ-doctrine $\mathcal{P}: \mathsf{Cat}(\mathcal{Q}) \longrightarrow \mathsf{Cat}(\mathcal{Q})$; a class $\mathcal{C}$ and a doctrine $\mathcal{T}$ correspond with each other if and only if the $\mathcal{T}$-algebras and their homomorphisms are precisely the $\mathcal{C}$-cocomplete $\mathcal{Q}$-categories and the $\mathcal{C}$-cocontinuous functors between them. Proposition 3.3 implies that, in this case, there is a normal lax* Sup*-functor $\mathcal{T}': \mathsf{Dist}(\mathcal{Q}) \longrightarrow \mathsf{Dist}(\mathcal{Q})$, sending a distributor $\Phi: \mathbb{A} \xrightarrow{\circ} \mathbb{B}$ to the distributor $\mathcal{T}'(\Phi): \mathcal{T}(\mathbb{A}) \xrightarrow{\circ} \mathcal{T}(\mathbb{B})$ with elements*

$$\mathcal{T}'(\Phi)(t, s) = \mathcal{P}(\mathbb{B})(\varepsilon_{\mathbb{B}}(t), \Phi \otimes \varepsilon_{\mathbb{A}}(s)), \quad \textit{for } s \in \mathcal{T}(\mathbb{A}),\ t \in \mathcal{T}(\mathbb{B}),$$

which makes the following diagram commute:

$$\begin{array}{ccc}
\mathsf{Dist}(\mathcal{Q}) & \xrightarrow{\ \mathcal{T}'\ } & \mathsf{Dist}(\mathcal{Q}) \\
{\scriptstyle i}\uparrow & & \uparrow{\scriptstyle i} \\
\mathsf{Cat}(\mathcal{Q}) & \xrightarrow[\ \mathcal{T}\ ]{} & \mathsf{Cat}(\mathcal{Q})
\end{array}$$

5. Conical cocompletion and the Hausdorff doctrine

5.1 Conical colimits

Let $\mathbb{A}$ be a $\mathcal{Q}$-category. Putting, for any $a, a' \in \mathbb{A}$,

$$a \leq a' \stackrel{\text{def.}}{\iff} ta = ta' \text{ and } 1_{ta} \leq \mathbb{A}(a, a')$$

defines an order relation on the objects of $\mathbb{A}$. (There are equivalent conditions in terms of representable presheaves.) For a given $\mathcal{Q}$-category $\mathbb{A}$ and a given object $X \in \mathcal{Q}_0$, we shall write $(\mathbb{A}_X, \leq_X)$ for the ordered set of objects of $\mathbb{A}$ of type X. Because elements of different type in $\mathbb{A}$ can never have a supremum in $(\mathbb{A}_0, \leq)$, it would be very restrictive to require this order to admit arbitrary suprema; instead, experience shows that it makes good sense to require each $(\mathbb{A}_X, \leq_X)$ to be a sup-lattice: we then say that $\mathbb{A}$ is **order-cocomplete** [Stubbe, 2006]. As spelled out in that reference, we have:

Proposition 5.1 *For a family $(a_i)_{i \in I}$ in $\mathbb{A}_X$, the following are equivalent:*

i. $\bigvee_i a_i$ *exists in* $\mathbb{A}_X$ *and* $\mathbb{A}(\bigvee_i a_i, -) = \bigwedge_i \mathbb{A}(a_i, -)$ *holds in* $\mathsf{Dist}(\mathcal{Q})(\mathbb{A}, *_X)$,

ii. $\bigvee_i a_i$ *exists in* $\mathbb{A}_X$ *and* $\mathbb{A}(-, \bigvee_i a_i) = \bigvee_i \mathbb{A}(-, a_i)$ *holds in* $\mathsf{Dist}(\mathcal{Q})(*_X, \mathbb{A})$,

iii. if we write $(I, \leq)$ for the ordered set in which $i \leq j$ precisely when $a_i \leq_X a_j$ and $\mathbb{I}$ for the free $\mathcal{Q}(X,X)$-category on the poset $(I, \leq)$, $F\colon \mathbb{I} \longrightarrow \mathbb{A}$ for the functor $i \mapsto a_i$ and $\gamma\colon *_X \multimap \mathbb{I}$ for the presheaf with values $\gamma(i) = 1_X$ for all $i \in \mathbb{I}$, then the γ-weighted colimit of F exists.

In this case, $\mathrm{colim}(\gamma, F) = \bigvee_i a_i$ *and it is the* **conical colimit** *of* $(a_i)_{i \in I}$ *in* $\mathbb{A}$.

It is important to realise that such conical colimits – which are enriched colimits! – can be characterised by a property of weights:

Proposition 5.2 *For a presheaf $\phi\colon *_X \multimap \mathbb{A}$, the following conditions are equivalent:*

i. there exists a family $(a_i)_{i\in I}$ *in* $\mathbb{A}_X$ *such that for any functor* $G:\mathbb{A}\longrightarrow\mathbb{B}$*, if the* ϕ*-weighted colimit of* G *exists, then it is the conical colimit of the family* $(G(a_i))_i$*,*

ii. there exists a family $(a_i)_{i\in I}$ *in* $\mathbb{A}_X$ *for which* $\phi = \bigvee_i \mathbb{A}(-,a_i)$ *holds in* $\mathsf{Dist}(\mathcal{Q})(*_X,\mathbb{A})$*,*

iii. there exist an ordered set $(I,\leq)$ *and a functor* $F:\mathbb{I}\longrightarrow\mathbb{A}$ *with domain the free* $\mathcal{Q}(X,X)$*-category on* $(I,\leq)$ *such that, if we write* $\gamma:*_X \multimap \mathbb{I}$ *for the presheaf with values* $\gamma(i)=1_X$ *for all* $i\in\mathbb{I}$*, then* $\phi=\mathbb{A}(-,F-)\otimes\gamma$*.*

In this case, we call ϕ *a* **conical presheaf***.*

Proof : (i⇒ii) Applying the hypothesis to the functor $Y_{\mathbb{A}}:\mathbb{A}\longrightarrow\mathcal{P}(\mathbb{A})$ – indeed $\mathrm{colim}(\phi,Y_{\mathbb{A}})$ exists, and is equal to ϕ by the Yoneda Lemma – we find a family $(a_i)_{i\in I}$ such that ϕ is the conical colimit in $\mathcal{P}(\mathbb{A})$ of the family $(Y_{\mathbb{A}}(a))_i$. This implies in particular that $\phi=\bigvee_i\mathbb{A}(-,a_i)$.

(ii⇒iii) For $\phi=\bigvee_i\mathbb{A}(-,a_i)$ it is always the case that $[\bigvee_i\mathbb{A}(-,a_i),-]=\bigwedge_i[\mathbb{A}(-,a_i),-]$, i.e. $\mathcal{P}(\mathbb{A})(\bigvee_i\mathbb{A}(-,a_i),-)=\bigwedge_i\mathcal{P}(\mathbb{A})(\mathbb{A}(-,a_i),-)$. Thus ϕ is the conical colimit in $\mathcal{P}(\mathbb{A})$ of the family $(\mathbb{A}(-,a_i))_i$, and Proposition 5.1 allows for the conclusion.

(iii⇒i) If, for some functor $G:\mathbb{A}\longrightarrow\mathbb{B}$, $\mathrm{colim}(\phi,G)$ exists, then, by the hypothesis that $\phi=\mathbb{A}(-,F-)\otimes\gamma$, it is equal to $\mathrm{colim}(\mathbb{A}(-,F-)\otimes\gamma,G)=\mathrm{colim}(\gamma,G\circ F)$. The latter is the conical colimit of the family $(G(F(i)))_{i\in I}$; thus the family $(F(i))_i$ fulfills the requirement. □

A warning is in order. Proposition 5.2 attests that the conical presheaves on a $\mathcal{Q}$-category $\mathbb{A}$ are those which are a supremum of some family of representable presheaves on $\mathbb{A}$. Of course, neither that family of representables, nor the family of representing objects in $\mathbb{A}$, need to be unique.

Now comes the most important observation concerning conical presheaves.

Proposition 5.3 *The class of conical presheaves is saturated.*

Proof : We shall check both conditions in Proposition 4.1. All representable presheaves are clearly conical, so the first condition is fulfilled. As for the second condition, consider a conical presheaf $\phi:*_X\multimap\mathbb{A}$ and a functor $G:\mathbb{A}\longrightarrow\mathcal{P}(\mathbb{B})$ such that each $G(a):*_{ta}\multimap\mathbb{B}$ is a conical presheaf too. The ϕ-weighted colimit of G certainly exists, hence the first statement in Proposition 5.2 applies: it says that $\mathrm{colim}(\phi,G)$ is the conical colimit of a family of conical presheaves. In other words, $\mathrm{colim}(\phi,G)$ is a supremum of a family of suprema of representables, and is therefore a supremum of representables too, hence a conical presheaf. □

5.2 The Hausdorff doctrine

Applying Theorem 4.6 to the class of conical presheaves we get:

Definition 5.4 *We write $\mathcal{H}\colon \mathsf{Cat}(\mathcal{Q})\longrightarrow\mathsf{Cat}(\mathcal{Q})$ for the KZ-doctrine associated with the class of conical presheaves. We call it the* **Hausdorff docrine on** $\mathsf{Cat}(\mathcal{Q})$*, and we say that $\mathcal{H}(\mathbb{A})$ is the* **Hausdorff $\mathcal{Q}$-category** *associated to a $\mathcal{Q}$-category $\mathbb{A}$. We write $\mathcal{H}'\colon \mathsf{Dist}(\mathcal{Q})\longrightarrow\mathsf{Dist}(\mathcal{Q})$ for the normal lax* Sup*-functor which extends $\mathcal{H}$ from* $\mathsf{Cat}(\mathcal{Q})$ *to* $\mathsf{Dist}(\mathcal{Q})$.

To justify this terminology, and underline the concordance with [Akhvlediani *et al.*, 2009], we shall make this more explicit. According to Proposition 4.4, $\mathcal{H}(\mathbb{A})$ is the full subcategory of $\mathcal{P}(\mathbb{A})$ determined by the conical presheaves on $\mathbb{A}$. By Proposition 5.2 however, the objects of $\mathcal{H}(\mathbb{A})$ can be equated with suprema of representables; so suppose that

$$\phi=\bigvee_{a\in A}\mathbb{A}(-,a)\quad\text{and}\quad\phi'=\bigvee_{a'\in A'}\mathbb{A}(-,a')$$

for subsets $A\subseteq\mathbb{A}_X$ and $A'\subseteq\mathbb{A}_Y$. Then we can compute that

$$\begin{aligned}
\mathcal{H}(\mathbb{A})(\phi',\phi) &= \mathcal{P}(\mathbb{A})(\phi',\phi)\\
&= [\phi',\phi]\\
&= [\bigvee_{a'}\mathbb{A}(-,a'),\bigvee_{a}\mathbb{A}(-,a]\\
&= \bigwedge_{a'}[\mathbb{A}(-,a'),\bigvee_{a}\mathbb{A}(-,a)]\\
&= \bigwedge_{a'}\bigvee_{a}[\mathbb{A}(-,a'),\mathbb{A}(-,a)]\\
&= \bigwedge_{a'}\bigvee_{a}\mathbb{A}(a',a).
\end{aligned}$$

(The penultimate equality is due to the fact that each $\mathbb{A}(-,a')\colon *_Y \,{-\!\!\circ\!\!\rightarrow}\, \mathbb{A}$ is a left adjoint in the quantaloid $\mathsf{Dist}(\mathcal{Q})$, and the last equality is due to the Yoneda lemma.) This is precisely the expected formula for the "Hausdorff distance between (the conical presheaves determined by) the subsets A and A' of $\mathbb{A}$". It must be noted that [Schmitt, 2006, Proposition 3.42] describes a very similar situation particularly for symmetric categories enriched in the commutative quantale of postive real numbers.

Similarly for functors: given a functor $F\colon\mathbb{A}\longrightarrow\mathbb{B}$ between $\mathcal{Q}$-categories, the functor $\mathcal{H}(F)\colon\mathcal{H}(\mathbb{A})\longrightarrow\mathcal{H}(\mathbb{B})$ sends a conical presheaf ϕ on $\mathbb{A}$ to the conical presheaf $\mathbb{B}(-,F-)\otimes\phi$ on $\mathbb{B}$. Supposing that $\phi=\bigvee_{a\in A}\mathbb{A}(-,a)$ for some $A\subseteq\mathbb{A}_X$, it is straightforward to check that

$$\begin{aligned}\mathbb{B}(-,F-)\otimes\phi &= \bigvee_{x\in\mathbb{A}}\left(\mathbb{B}(-,Fx)\circ\bigvee_{a\in A}\mathbb{A}(x,a)\right)\\ &= \bigvee_{a\in A}\left(\bigvee_{x\in\mathbb{A}}\mathbb{B}(-,Fx)\circ\mathbb{A}(x,a)\right)\\ &= \bigvee_{a\in A}\mathbb{B}(-,Fa).\end{aligned}$$

That is to say, "$\mathcal{H}(F)$ sends (the conical presheaf determined by) $A\subseteq\mathbb{A}$ to (the conical presheaf determined by) $F(A)\subseteq\mathbb{B}$".

Finally, by Proposition 3.3, the action of $\mathcal{H}'$ on a distributor $\Phi\colon\mathbb{A}\multimap\mathbb{B}$ gives a distributor $\mathcal{H}'(\Phi)\colon\mathcal{H}(\mathbb{A})\multimap\mathcal{H}'(\mathbb{B})$ whose value in $\phi\in\mathcal{H}(\mathbb{A})$ and $\psi\in\mathcal{H}(\mathbb{B})$ is $\mathcal{P}(\mathbb{B})(\psi,\Phi\otimes\phi)$. Assuming that

$$\phi=\bigvee_{a\in A}\mathbb{A}(-,a)\text{ and }\psi=\bigvee_{b\in B}\mathbb{B}(-,b)$$

for some $A\subseteq\mathbb{A}_X$ and $B\subseteq\mathbb{B}_Y$, a similar computation as above shows that

$$\mathcal{H}'(\Phi)(\psi,\phi)=\bigwedge_{b\in B}\bigvee_{a\in A}\Phi(b,a).$$

This is the expected generalisation of the previous formula, to measure the "Hausdorff distance between (the conical presheaves determined by) $A\subseteq\mathbb{A}$ and $B\subseteq\mathbb{B}$ through $\Phi\colon\mathbb{A}\multimap\mathbb{B}$".

5.3 Other examples

The following examples of saturated classes of presheaves have been considered by [Kelly and Schmitt, 2005] in the case of categories enriched in symmetric monoidal categories.

Example 5.5 (Minimal and maximal class) The smallest saturated class of presheaves on $\mathcal{Q}$-categories is, of course, that containing only representable presheaves. It is straightforward that the KZ-doctrine on $\mathsf{Cat}(\mathcal{Q})$ corresponding with this class is the identity functor. On the other hand, the class of all presheaves on $\mathcal{Q}$-categories corresponds with the free cocompletion KZ-doctrine on $\mathsf{Cat}(\mathcal{Q})$.

Example 5.6 (Cauchy completion) The class of all left adjoint presheaves, also known as **Cauchy presheaves**, on $\mathcal{Q}$-categories is saturated. Indeed, all representable presheaves are left adjoints. And suppose that $\Phi: \mathbb{A} \multimap \mathbb{B}$ and $\Psi: \mathbb{B} \multimap \mathbb{C}$ are distributors such that, for all $a \in \mathbb{A}$ and all $b \in \mathbb{B}$, $\Phi(-, a): *_{ta} \multimap \mathbb{B}$ and $\Psi(-, b): *_{tb} \multimap \mathbb{C}$ are left adjoints. Writing $\rho_b: \mathbb{C} \multimap *_{tb}$ for the right adjoint to $\Psi(-, b)$, it is easily verified that Ψ is left adjoint to $\bigvee_{b \in \mathbb{B}} \mathbb{B}(-, b) \otimes \rho_b$. This makes sure that $(\Psi \otimes \Phi)(-, a) = \Psi \otimes \Phi(-, a)$ is a left adjoint too, and by Proposition 4.2 we can conclude that the class of Cauchy presheaves is saturated. The KZ-doctrine on $\mathsf{Cat}(\mathcal{Q})$ which corresponds to this saturated class of presheaves, sends a $\mathcal{Q}$-category $\mathbb{A}$ to its **Cauchy completion** [Lawvere, 1973; Walters, 1981; Street, 1983].

Inspired by the examples in [Lawvere, 1973] and the general theory in [Kelly and Schmitt, 2005], Vincent Schmitt [2006] has studied several other classes of presheaves for ordered sets (viewed as categories enriched in the 2-element Boolean algebra) and for generalised metric spaces (viewed as categories enriched in the quantale of positive real numbers). He constructs *saturated* classes of presheaves by requiring that each element of the class "commutes" (in a suitable way) with all elements of a given (*not-necessarily saturated*) class of presheaves. These interesting examples do not seem to generalise straightforwardly to general quantaloid-enriched categories, so we shall not survey them here, but refer instead to [Schmitt, 2006] for more details.

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Isar Stubbe
Laboratoire de Mathématiques Pures et Appliquées
Université du Littoral-Côte d'Opale
50, rue Ferdinand Buisson
62228 Calais (France)
isar.stubbe@lmpa.univ-littoral.fr