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CLOSEDNESS PROPERTIES OF INTERNAL RELATIONS VI: APPROXIMATE OPERATIONS

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Dedicated to Francis Borceux on the occasion of his 60th birthday

Abstract

La méthode de traduction permettant de passer des «term conditions» au sens de l'algèbre universelle à des conditions purement catégoriques, qui a été présentée dans le premier article de cette série, est revisitée ici dans une nouvelle perspective, qui est basée sur l'idée de considérer les opérations approchées, introduite par D. Bourn et par l'auteur du présent article. Dans un certain sens, ces opérations approchées apparaissent comme des contreparties catégoriques des termes d'une théorie algébrique d'une variété.

The method of translating universal-algebraic term conditions into purely categorical conditions, which was presented in the first paper from this series, is now revisited with a new insight that is based on the idea of considering so called *approximate operations*, which is due to D. Bourn and the present author. In some sense, these approximate operations arise as categorical counterparts of terms of an algebraic theory of a variety.

Introduction

In [9] we saw that classes of categories, such as Mal'tsev, unital, strongly unital and subtractive categories, all can be obtained from the corresponding classes of varieties of universal algebras, using the same general method of extending classes of varieties, that are defined by suitable *term conditions*

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(i.e. a special kind of conditions on the terms of the algebraic theory of a variety), to classes of categories. These classes of categories are then defined by so called *closedness properties of internal relations*. The aim of the present paper is to show that in fact, the categorical setting also admits the presence of a special kind of “terms” — so called *approximate operations*. This idea was first put forward in [3], where only the special case of Mal’tsev categories was treated (but it was pointed out in [3] that the theory transfers to the general case). Then in [4] (see also [5]) the case of subtractive categories was considered in detail (where *approximate subtractions* were used in the construction of the abelianization functor for regular subtractive categories with binary coproducts). In this paper we present the general theory (see the Introduction in [3] for a summary of this theory in the case of Mal’tsev categories).

1 Categories with M-closed relations

In this section we recall the necessary material from [9] (omitting the proofs).

Preliminaries

By **AlgTh** we denote the category of (one-sorted and finitary) algebraic theories (see [12]). Let $\mathcal{T}$ be an object in this category and let $\mathcal{E}$ be an algebraic theory obtained from $\mathcal{T}$ by adding to it an m-ary operator p and the axioms

$$\begin{cases} p(t_{11},...,t_{1m}) = u_1, \\ \vdots \\ p(t_{n1},...,t_{nm}) = u_n, \end{cases} \tag{1}$$

where t_{ij} and u_i are certain terms in $\mathcal{T}$, having the same fixed arity k. Then we have a canonical morphism $e : \mathcal{T} \to \mathcal{E}$ of theories.

Throughout the paper we assume $n \geqslant 1$, $m \geqslant 0$ and $k \geqslant 0$.

The system of equations (1) gives rise to the extended matrix

$$M = \left(\begin{array}{ccc|c} t_{11} & \cdots & t_{1m} & u_1 \\ \vdots & & \vdots & \vdots \\ t_{n1} & \cdots & t_{nm} & u_n \end{array} \right)$$

which uniquely determines it. A special instance of this process is when in Linear Algebra we associate an extended matrix to a system of linear equations (see [9]). Similarly, the role of M in our much more general setting, is to determine if (1) is solvable in $\mathcal{T}$, i.e. whether the morphism $\mathcal{T} \to \mathcal{E}$ is a split monomorphism (which means that $\mathcal{T}$ contains an m-ary term p such that the equations in (1) are theorems in $\mathcal{T}$). More generally, we are interested in the existence of a factorization

$$\begin{array}{ccc} \mathcal{E} & \longrightarrow & \mathcal{K} \\ & \nwarrow \quad \nearrow & \\ & \mathcal{T} & \end{array}$$

for a given theory morphism $\mathcal{T} \to \mathcal{K}$. Note that once a theory morphism $\mathcal{T} \to \mathcal{K}$ is given, (1) becomes a system of term equations in $\mathcal{K}$, and when the above factorization exists we can say that (1) is *solvable in* $\mathcal{K}$. In particular, we study this problem in the case when $\mathcal{T} \to \mathcal{K}$ is a central morphism [13], i.e. every k-ary term q from $\mathcal{T}$ commutes with every l-ary term r in $\mathcal{K}$ (where k and l are arbitrary natural numbers) in the sense that the identity

$$q(r(x_{11},...,x_{1l}),...,r(x_{k1},...,x_{kl})) = r(q(x_{11},...,x_{k1}),...,q(x_{1l},...,x_{kl}))$$

is a theorem in $\mathcal{K}$. Then, it turns out that M gives rise to a condition on the category $\mathbf{Alg}_{\mathcal{K}}$ of $\mathcal{K}$-algebras, which is equivalent to solvability of (1) in $\mathcal{K}$ (see Theorem 1.2 below). To formulate this categorical condition, it is more convenient to work with the transpose of M, which we denote by the same letter

$$M = \left(\begin{array}{ccc} t_{11} & \cdots & t_{n1} \\ \vdots & & \vdots \\ t_{1m} & \cdots & t_{nm} \\ \hline u_1 & \cdots & u_n \end{array} \right).$$

Internal M-closed relations

By an *internal $\mathcal{T}$-algebra A* in a category $\mathbb{C}$ we mean an object A in $\mathbb{C}$ equipped with an ordinary internal $\mathcal{T}$-algebra structure on $Y(A)$ in the category $\mathbf{Set}^{\mathbb{C}^{\mathrm{op}}}$, where Y denotes the Yoneda embedding $Y : \mathbb{C} \to \mathbf{Set}^{\mathbb{C}^{\mathrm{op}}}$. We use standard notation: For each k-ary term v in $\mathcal{T}$, we write v^A to denote corresponding k-ary operation $v^A : Y(A)^k \to Y(A)$ of $Y(A)$. We usually omit subscripts when we write the components $v^A_X : \hom(X,A)^k \to \hom(X,A)$ of the natural transformation v^A. Sometimes we even omit the superscript and simply write v instead of v^A.

Let $r = (r_i : R \to A_i)_{i\in\{1,\dots,n\}}$ and $s = (s_i : S \to A_i)_{i\in\{1,\dots,n\}}$ be two spans in $\mathbb{C}$. We say that the span s factors through the span r, if there exists a morphism $f : S \to R$ such that $r_i f = s_i$ for each $i \in \{1,\dots,n\}$.

By an internal n-ary relation r in $\mathbb{C}$ we mean a span

$$r = (r_i : R \to A_i)_{i\in\{1,\dots,n\}}$$

whose projections $r_1,\dots,r_n$ are jointly monomorphic.

Suppose A is an internal $\mathcal{T}$-algebra in $\mathbb{C}$. We say that an internal n-ary relation $r = (r_i : R \to A)_{i\in\{1,\dots,n\}}$ in $\mathbb{C}$ is *M-closed* when for any object X in $\mathbb{C}$ and for any morphisms $x_1,\dots,x_k : X \to A$, if for each $j \in \{1,\dots,m\}$ the span $(t_{ij}(x_1,\dots,x_k) : X \to A)_{i\in\{1,\dots,n\}}$ factors through the relation r, then so does the span $(u_i(x_1,\dots,x_k) : X \to A)_{i\in\{1,\dots,n\}}$. The latter property of $x_1,\dots,x_k$ can be expressed in short by saying that r is *compatible* with the following matrix:

$$\left(\begin{array}{ccc} t_{11}(x_1,\dots,x_k) & \cdots & t_{n1}(x_1,\dots,x_k) \\ \vdots & & \vdots \\ t_{1m}(x_1,\dots,x_k) & \cdots & t_{nm}(x_1,\dots,x_k) \\ \hline u_1(x_1,\dots,x_k) & \cdots & u_n(x_1,\dots,x_k) \end{array}\right)$$

Thus in general, given an extended matrix

$$\left(\begin{array}{ccc} f_{11} & \cdots & f_{n1} \\ \vdots & & \vdots \\ f_{1m} & \cdots & f_{nm} \\ \hline g_1 & \cdots & g_n \end{array}\right)$$

whose each i-th column consists of morphisms $X \to C_i$ in $\mathbb{C}$, where X is a fixed object in $\mathbb{C}$, we say that a relation $r = (r_i : R \to C_i)_{i\in\{1,\dots,n\}}$ is *compatible* with this matrix if whenever the top rows in the above matrix, regarded as spans, factor through r, so does the bottom row.

Now consider a relation $r = (r_i : R \to A_i)_{i\in\{1,\dots,n\}}$ between possibly different internal $\mathcal{T}$-algebras $A_1,\dots,A_n$ in $\mathbb{C}$. We say that r is *strictly M-closed* if for any object X in $\mathbb{C}$ and for any family of morphisms

$$(x_{ii'} : X \to A_i)_{i\in\{1,\dots,n\},i'\in\{1,\dots,k\}},$$

the relation r is compatible with respect to the following matrix:

$$\left(\begin{array}{ccc} t_{11}(x_{11},\dots,x_{1k}) & \cdots & t_{n1}(x_{n1},\dots,x_{nk}) \\ \vdots & & \vdots \\ t_{1m}(x_{11},\dots,x_{1k}) & \cdots & t_{nm}(x_{n1},\dots,x_{nk}) \\ \hline u_1(x_{11},\dots,x_{1k}) & \cdots & u_n(x_{n1},\dots,x_{nk}) \end{array}\right)$$

The categorical condition determined by M

By a *$\mathcal{T}$-enrichment* of a category $\mathbb{C}$ we mean a left inverse of the forgetful functor

$$\mathbf{Alg}_{\mathcal{T}}\mathbb{C} \to \mathbb{C},$$

where $\mathbf{Alg}_{\mathcal{T}}$ denotes the category of internal $\mathcal{T}$-algebras in $\mathbb{C}$ (see [7], [9]). Thus, a $\mathcal{T}$-enrichment equips each object C in $\mathbb{C}$ with an internal $\mathcal{T}$-algebra structure in $\mathbb{C}$, in such a way that every morphism $f : C \to D$ becomes an internal homomorphism of internal $\mathcal{T}$-algebras. A *$\mathcal{T}$-enriched category* is a category $\mathbb{C}$ given with a fixed $\mathcal{T}$-enrichment. A $\mathcal{T}$-enrichment can be also equivalently defined as a natural $\mathcal{T}$-algebra structure on the functor $\hom_{\mathbb{C}}$, i.e. an internal $\mathcal{T}$-algebra structure on $\hom_{\mathbb{C}}$ in the functor category $\mathbf{Set}^{\mathbb{C}^{op}\times\mathbb{C}}$. When we talk about a $\mathcal{T}$-enriched category $\mathbb{C}$, we will automatically regard the functor $\hom_{\mathbb{C}}$ as the corresponding internal $\mathcal{T}$-algebra in $\mathbf{Set}^{\mathbb{C}^{op}\times\mathbb{C}}$.

An internal n-ary relation $r = (r_i : R \to C)_{i\in\{1,\dots,n\}}$ in a $\mathcal{T}$-enriched category $\mathbb{C}$ is said to be M-closed if r is M-closed when C is regarded as an internal $\mathcal{T}$-algebra in $\mathbb{C}$ whose $\mathcal{T}$-algebra structure is the one that is associated with C by the $\mathcal{T}$-enrichment (a similar convention also applies to “strict M-closedness”).

Theorem 1.1 ([9]) *Let $\mathbb{C}$ be a finitely complete $\mathcal{T}$-enriched category. Then the following conditions are equivalent to each other:*

(a) Every internal relation $R \to C^n$ in $\mathbb{C}$ is M-closed.

(b) Every internal relation $R \to C_1 \times \dots \times C_n$ in $\mathbb{C}$ is strictly M-closed.

We say that a (finitely complete) $\mathcal{T}$-enriched category $\mathbb{C}$ *has M-closed relations* if it satisfies the equivalent conditions (a) and (b) in Theorem 1.1. Suppose $\mathbb{C} = \mathbf{Alg}_{\mathcal{K}}$. Then $\mathcal{T}$-enrichments of $\mathbb{C}$ are in one-to-one correspondence with central morphisms $\mathcal{T} \to \mathcal{K}$ (see [7]).

Theorem 1.2 ([9]) *Let $\mathcal{T} \to \mathcal{K}$ be a central morphism of theories. Then the $\mathcal{T}$-enriched category $\mathbf{Alg}_{\mathcal{K}}$ has M-closed relations if and only if the system of equations (1) corresponding to M is solvable in $\mathcal{K}$.*

By $\mathcal{T}$-$\mathbf{Cat}^{\mathrm{fc}}_M$ we denote the class of all finitely complete $\mathcal{T}$-enriched categories $\mathbb{C}$ that have M-closed relations. In [9] it was shown that the class of Mal'tsev categories, as well as several other closely related classes of categories (see Table 1), are of the form $\mathcal{T}$-$\mathbf{Cat}^{\mathrm{fc}}_M$ for suitable $\mathcal{T}$'s and M's.

It turns out that under certain restrictions on $\mathcal{T}$ and M the classes $\mathcal{T}$-$\mathbf{Cat}^{\mathrm{fc}}_M$ themselves can be characterized via a generalized form of "solvability" of (1). This was first shown in the case of the class of Mal'tsev categories in [3], where it was also observed that the same could be done more generally for the classes of the form $\mathcal{T}$-$\mathbf{Cat}^{\mathrm{fc}}_M$. The purpose of this paper is to outline this more general unified theory.

Observation 1.3 *Below we describe two procedures of producing from M another extended matrix M′ of terms in $\mathcal{T}$. It is easy to see that in both cases we have $\mathcal{T}$-$\mathbf{Cat}^{\mathrm{fc}}_M \subseteq \mathcal{T}$-$\mathbf{Cat}^{\mathrm{fc}}_{M'}$.*

"Change of variables": Let k′ be a natural number and suppose the entries in

$$M' = \left(\begin{array}{ccc} t'_{11} & \cdots & t'_{n1} \\ \vdots & & \vdots \\ t'_{1m} & \cdots & t'_{nm} \\ \hline u'_1 & \cdots & u'_n \end{array}\right)$$

are k′-ary terms. Suppose further there exist maps $f_1, \dots, f_n : \{1, \dots, k\} \to \{1, \dots, k'\}$ such that for every $i \in \{1, \dots, n\}$ the equations

$$\begin{aligned} & t'_{i1}(x_1, \dots, x_{k'}) = t_{i1}(x_{f_i(1)}, \dots, x_{f_i(k)}), \\ & \vdots \\ & t'_{im}(x_1, \dots, x_{k'}) = t_{im}(x_{f_i(1)}, \dots, x_{f_i(k)}), \\ & u'_i(x_1, \dots, x_{k'}) = u_i(x_{f_i(1)}, \dots, x_{f_i(k)}), \end{aligned}$$

are theorems in $\mathcal{T}$. Then we say that M′ is obtained from M by change of variables.

Table 1: Examples of classes of categories determined by matrices

$\mathcal{T} =$	$M =$	$\mathbb{C} \in \mathcal{T}$-$\mathbf{Cat}_M^{\mathrm{fc}}$ iff $\mathbb{C}$ is a
Th[*sets*]	$\left(\begin{array}{cc} x & y \\ y & y \\ y & x \\ \hline x & x \end{array}\right)$	Mal'tsev category [6]
Th[*pointed sets*]	$\left(\begin{array}{cc} x & y \\ 0 & y \\ 0 & x \\ \hline x & x \end{array}\right)$	Strongly unital category [2]
Th[*pointed sets*]	$\left(\begin{array}{cc} x & 0 \\ 0 & x \\ \hline x & x \end{array}\right)$	Unital category [2]
Th[*pointed sets*]	$\left(\begin{array}{cc} x & x \\ 0 & x \\ \hline x & 0 \end{array}\right)$	Subtractive category [8]

"Change of columns": Let n' be a natural number $n' \geqslant 1$ and suppose

$$M' = \left(\begin{array}{ccc} t'_{11} & \cdots & t'_{n'1} \\ \vdots & & \vdots \\ t'_{1m} & \cdots & t'_{n'm} \\ \hline u'_1 & \cdots & u'_{n'} \end{array} \right).$$

Suppose further there exists a map $f : \{1,\ldots,n'\} \to \{1,\ldots,n\}$ such that for every $i \in \{1,\ldots,n'\}$ we have:

$$\begin{array}{l} t'_{i1} = t_{f(i),1}, \\ \vdots \\ t'_{im} = t_{f(i),m}, \\ u'_i = u_{f(i)}. \end{array}$$

Then we say that M' is obtained from M by change of columns.

2 Approximate operations

Approximate solutions of the system of term equations corresponding to *M*

For an extended term matrix M, we use the same letter M to refer to the system of term equations corresponding to this matrix. Let A be an internal $\mathcal{T}$-algebra in a category $\mathbb{C}$ with finite products. An *approximate solution* in A, of a system of term equations M, is a morphism $p : A^m \to B$ in $\mathbb{C}$ (an "approximate operation") such that there exists a morphism $d : A \to B$ making the diagram

$$\begin{array}{ccc} A^m & \xrightarrow{p} & B \\ {\scriptstyle (t_{i1},\ldots,t_{im})}\uparrow & & \uparrow{\scriptstyle d} \\ A^k & \xrightarrow[u_i]{} & A \end{array}$$

commute for each $i \in \{1,\ldots,n\}$. The morphism d is called an *approximation* of the approximate solution p. The morphism p in the initial object in the category of all diagrams

$$A^m \xrightarrow{p} B \xleftarrow{d} A$$

with A fixed, will be called the *initial approximate solution* of M in A, and the morphism d in the same diagram will be called a *canonical approximation* of p. When $\mathbb{C}$ has finite colimits, the initial approximate solution of M in A, together with its canonical approximation d, can be obtained via the pushout

$$\begin{array}{ccc} A^m & \xrightarrow{\quad p \quad} & B \\ {\scriptstyle \begin{pmatrix} t_{11} & \cdots & t_{1m} \\ \vdots & & \vdots \\ t_{n1} & \cdots & t_{nm} \end{pmatrix}} \Big\uparrow & & \Big\uparrow {\scriptstyle d} \\ n(A^k) & \xrightarrow[{\scriptstyle \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix}}]{} & A \end{array} \tag{2}$$

By a *solution of M in A* we mean an approximate solution of M in A which has an approximation that is an identity morphism. Solutions of M in A are in one-to-one correspondence with left inverses of the canonical approximation d of the initial approximate solution of M in A (when the latter exists), and so, M is solvable in A if and only if d is a split monomorphism. This leads us to the following

Lemma 2.1 *For an internal $\mathcal{T}$-algebra A in* **Set***, whose underlying set is non-empty, the following conditions are equivalent:*

(a) M is solvable in A.

(b) The canonical approximation of the initial approximate solution of M in A is an injective map.

(c) For all $i, i' \in \{1, ..., n\}$ and $a_1, ..., a_k, b_1, ..., b_k \in A$ we have

$$\left[\forall_{j \in \{1,...,m\}} t_{ij}(a_1, ..., a_k) = t_{i'j}(b_1, ..., b_k)\right]$$
$$\implies \quad u_i(a_1, ..., a_k) = u_{i'}(b_1, ..., b_k).$$

If A is empty then (b) and (c) are always satisfied (so in this case we still have (b)⇔(c)), and (a) is satisfied if and only if $m \neq 0$.

Proof. We already know (a)⇔(b).

(b)⇒(c): Let p be the initial approximate solution of M in A and let d be its canonical approximation. Suppose

$$\forall_{j\in\{1,\dots,m\}}\, t_{ij}(a_1,\dots,a_k)=t_{i'j}(b_1,\dots,b_k).$$

Then

$$d(u_i(a_1,\dots,a_k))=p(t_{i1}(a_1,\dots,a_k),\dots,t_{im}(a_1,\dots,a_k))=$$
$$=p(t_{i'1}(b_1,\dots,b_k),\dots,t_{i'm}(b_1,\dots,b_k))=d(u_{i'}(b_1,\dots,b_k)).$$

Since d is injective, $u_i(a_1,\dots,a_k)=u_{i'}(b_1,\dots,b_k)$.

(c)⇒(a): The condition (c) insures that the following map is well-defined:

$$p:A^m\to A,\ \ p(a_1,\dots,a_m)=\begin{cases} u_i(b_1,\dots,b_k) & \text{if } \forall_{j\in\{1,\dots,m\}} a_j=t_{ij}(b_1,\dots,b_k),\\ a_1 & \text{otherwise.}\end{cases}$$

From the definition of p it is clear that p is a solution of M in A.

The last statement of the Lemma is essentially obvious. □

Uniqueness of the approximation

Proposition 2.2 *For any $\mathcal{T}$ and M, the following conditions are equivalent:*

(a) Each approximate solution of M has exactly one approximation.

(b) There exists $o\in\{1,\dots,n\}$ and there exist unary terms $v_1,\dots,v_k$ in $\mathcal{T}$ such that $u_o(v_1(x),\dots,v_k(x))=x$ is a theorem in $\mathcal{T}$.

Moreover, when (b) is satisfied, for any approximation d of an approximate solution p of M (in an internal $\mathcal{T}$-algebra A in a category $\mathbb{C}$), we have

$$d=p\circ(t^A_{o1},\dots,t^A_{om})\circ(v^A_1,\dots,v^A_k)$$

where $o,v_1,\dots,v_k$ are the same as in (b).

Proof. (a)⇒(b): Let A be the free $\mathcal{T}$-algebra in **Set** over a one-element set $\{x\}$. Consider the pushout (2). If x is the only element of A, then this means that for any unary term w in $\mathcal{T}$ we have $w(x)=x$, and so (b) is trivially satisfied. Suppose A has at least two elements. The condition (b) states nothing other than that x belongs to the image of the bottom horizontal map in (2). If this is not satisfied, then B must also have at least two elements. But this allows to define another approximation d' of p,

$$d'(a) = \begin{cases} d(a) & \text{if } a \neq x, \\ b & \text{if } a = x, \end{cases}$$

where b is any element of B such that $b \neq d(x)$.

(b)⇒(a) is a consequence of the last part of the proposition, which is easy to prove: take $o, v_1, \ldots, v_k$ as in (b); then $1_A = u_0^A \circ (v_1^A, \ldots, v_k^A)$, which gives $d = d \circ u_0^A \circ (v_1^A, \ldots, v_k^A) = p \circ (t_{o1}^A, \ldots, t_{om}^A) \circ (v_1^A, \ldots, v_k^A)$. □

Condition 2.2(b) is satisfied for each of the cases displayed in Table 1 (in fact, in each case we can take $o = 1$ and all v's to be the identity term $v(x) = x$).

When the equivalent conditions of Proposition 2.2 are satisfied, p is an approximate solution of M in an internal $\mathcal{T}$-algebra A if and only if for each $i \in \{1, \ldots, n\}$ we have

$$p \circ (t_{i1}^A, \ldots, t_{im}^A) = p \circ (t_{o1}^A, \ldots, t_{om}^A) \circ (v_1^A, \ldots, v_k^A) \circ u_i^A$$

i.e. p coequalizes the morphisms

$$A^k \underset{(t_{o1},\ldots,t_{om})\circ(v_1,\ldots,v_k)\circ u_i}{\overset{(t_{i1},\ldots,t_{im})}{\rightrightarrows}} A^m$$

where $o, v_1, \ldots, v_k$ are the same as in 2.2(c). In particular, this allows to construct the initial approximate solution of M in A as a joint coequalizer of pairs of morphisms of the above form (see e.g. Table 2).

3 The characterization theorems

The first characterization theorem

Let $\mathbb{C}$ be a $\mathcal{T}$-enriched category. Consider the following condition on a natural transformation $d : \hom_{\mathbb{C}} \to D$, where n' is any natural number $n' \geqslant 1$:

Table 2: Coequalizer constructions of some initial approximate solutions

$\mathcal{T} =$	$M =$	
Th[*sets*]	$\begin{pmatrix} x & y \\ y & y \\ y & x \\ \hline x & x \end{pmatrix}$	$A^2 \underset{(\pi_2,\pi_2,\pi_1)}{\overset{(\pi_1,\pi_2,\pi_2)}{\underset{(\pi_1,\pi_1,\pi_1)}{\rightrightarrows}}} A^3 \xrightarrow{p} B$
Th[*pointed sets*]	$\begin{pmatrix} x & y \\ 0 & y \\ 0 & x \\ \hline x & x \end{pmatrix}$	$A^2 \underset{(\pi_2,\pi_2,\pi_1)}{\overset{(\pi_1,0,0)}{\rightrightarrows}} A^3 \xrightarrow{p} B$
Th[*pointed sets*]	$\begin{pmatrix} x & 0 \\ 0 & x \\ \hline x & x \end{pmatrix}$	$A \underset{(0,1_A)}{\overset{(1_A,0)}{\rightrightarrows}} A^2 \xrightarrow{p} B$
Th[*pointed sets*]	$\begin{pmatrix} x & x \\ 0 & x \\ \hline x & 0 \end{pmatrix}$	$A \underset{(0,0)}{\overset{(1_A,1_A)}{\rightrightarrows}} A^2 \xrightarrow{p} B$

$(\mathrm{C}_{n'})$ Let $r = (r_i : R \to A_i)_{i\in\{1,\dots,n'\}}$ and $s = (s_i : S \to A_i)_{i\in\{1,\dots,n'\}}$ be arbitrary internal n'-ary relation and span, respectively, between arbitrary objects $A_1,\dots,A_{n'}$ in $\mathbb{C}$. If there exists an element $x \in D(S,R)$ such that

$$D(1_S, r_i)(x) = d_{(S,A_i)}(s_i) \quad \text{for every} \quad i \in \{1,\dots,n'\},$$

then there exists a morphism $y : S \to R$ such that

$$r_i y = s_i \quad \text{for every} \quad i \in \{1,\dots,n'\}.$$

This condition turns out to be closely related to the following condition, which is determined by M and a natural number $n' \geqslant 1$:

$(\mathrm{D}^M_{n'})$ Any internal relation $r = (r_i : R \to A_i)_{i\in\{1,\dots,n'\}}$ in $\mathbb{C}$ is strictly M'-closed, where M' is any extended matrix (of terms in $\mathcal{T}$) with n' columns, obtained from M by change of columns and variables.

It is easy to see that if $(\mathrm{C}_{n'})$ is satisfied, then $(\mathrm{C}_{n''})$ is satisfied for any $n'' \in \{1,\dots,n'\}$. A similar fact is also true for the condition $(\mathrm{D}^M_{n'})$.

Theorem 3.1 *Let $\mathbb{C}$ be a $\mathcal{T}$-enriched category. For any natural number $n' \geqslant 2$ the following conditions are equivalent to each other:*

- *(a) the canonical approximation d of the initial approximate solution p of M in $\hom_{\mathbb{C}}$ satisfies $(\mathrm{C}_{n'})$;*
- *(b) there exists an approximate solution p of M in $\hom_{\mathbb{C}}$ which has an approximation d satisfying $(\mathrm{C}_{n'})$;*
- *(c) $\mathbb{C}$ satisfies $(\mathrm{D}^M_{n'})$.*

Moreover, if the above conditions are satisfied, then d in (a) necessarily has injective components.

Proof. (a)⇒(b) is obvious. (b)⇒(a) follows from the fact that if the condition $(\mathrm{C}_{n'})$ is satisfied for a composite cd then it is also satisfied for d.

(b)⇒(c): Suppose (b) is satisfied. Let X be an object and

$$r = (r_i : R \to A_i)_{i\in\{1,\dots,n'\}}$$

an internal n'-ary relation in $\mathbb{C}$. We must show that r is compatible with the matrix

$$\left(\begin{array}{ccc} t_{f(1)1}(x_{11},\dots,x_{1k}) & \cdots & t_{f(n')1}(x_{n'1},\dots,x_{n'k}) \\ \vdots & & \vdots \\ t_{f(1)m}(x_{11},\dots,x_{1k}) & \cdots & t_{f(n')m}(x_{n'1},\dots,x_{n'k}) \\ \hline u_{f(1)}(x_{11},\dots,x_{1k}) & \cdots & u_{f(n')}(x_{n'1},\dots,x_{n'k}) \end{array}\right)$$

for any map $f:\{1,\dots,n'\}\to\{1,\dots,n\}$, where each x_{ij} is an arbitrary morphism

$$x_{ij}: X \to A_i.$$

Suppose each upper j-th row, regarded as a span, factors through the relation r via a morphism $z_j : X \to R$. Consider the element $x = p_{(X,R)}(z_1,\dots,z_m) \in D(X,R)$. For each $i \in \{1,\dots,n'\}$ we have

$$D(1_X,r_i)(x) = D(1_X,r_i)(p_{(X,R)}(z_1,\dots,z_m))$$

$$= p_{(X,A_i)}(r_iz_1,\dots,r_iz_m) = p_{(X,A_i)}(t_{f(i)1}(x_{i1},\dots,x_{ik}),\dots,t_{f(i)m}(x_{i1},\dots,x_{ik}))$$
$$= d_{(X,A_i)}(u_{f(i)}(x_{i1},\dots,x_{ik})).$$

Now, applying $(\mathrm{C}_{n'})$ we get that the last row of the above matrix also factors through r.

Before proving (c)⇒(a) first we prove that (c) implies that the d in (a) has injective components. Let X and Y be any two objects in $\mathbb{C}$. Let p be the initial approximate solution of M in $\hom_{\mathbb{C}}(X,Y)$ with canonical approximation d. According to Lemma 2.1, to show that d is an injective map, it suffices to show that the condition 2.1(c) is satisfied for $A = \hom_{\mathbb{C}}(X,Y)$, which is the same as to show that for all $i,i' \in \{1,\dots,n\}$ and $a_1,\dots,a_k,b_1,\dots,b_k \in \hom_{\mathbb{C}}(X,Y)$ the relation

$$Y \xleftarrow{1_Y} Y \xrightarrow{1_Y} Y$$

is compatible with the matrix

$$\left(\begin{array}{cc} t_{i1}(a_1,\dots,a_k) & t_{i'1}(b_1,\dots,b_k) \\ \vdots & \vdots \\ t_{im}(a_1,\dots,a_k) & t_{i'm}(b_1,\dots,b_k) \\ \hline u_i(a_1,\dots,a_k) & u_{i'}(b_1,\dots,b_k) \end{array}\right).$$

But this is indeed so when (c) is satisfied, since the above matrix can be obtained from M by change of columns and variables, and because the number of columns is less than or equal to n' (this is where we use the assumption $n' \geqslant 2$).

(c)$\Rightarrow$(a): Suppose (c) is satisfied. Let

$$r = (r_i : R \to A_i)_{i\in\{1,\dots,n'\}} \quad \text{and} \quad s = (s_i : S \to A_i)_{i\in\{1,\dots,n'\}}$$

be an arbitrary internal n'-ary relation and a span, respectively, between arbitrary objects $A_1,\dots,A_{n'}$ in $\mathbb{C}$. Suppose there exists an element $x \in D(S,R)$ such that

$$D(1_S, r_i)(x) = d_{(S,A_i)}(s_i) \quad \text{for every } i \in \{1,\dots,n'\}.$$

The maps $p_{(S,R)}, d_{(S,R)}$ are jointly surjective (since $p_{(S,R)}$ is an initial approximate solution of M in $\hom_{\mathbb{C}}(S,R)$ with $d_{(S,R)}$ as its canonical approximation). This means that x falls either in the image of $d_{(S,R)}$, or of $p_{(S,R)}$. In the first case we get $x = d_{(S,R)}(y)$ for a morphism $y : S \to R$, and then naturality of d yields that

$$d_{(S,A_i)}(r_i y) = d_{(S,A_i)}(s_i)$$

for every $i \in \{1,\dots,n'\}$, which implies $r_i y = s_i$ for every $i \in \{1,\dots,n'\}$ (since each $d_{(S,A_i)}$ is an injective map). Suppose now x falls in the image of $p_{(S,R)}$, i.e. there exist morphisms $z_1,\dots,z_m : S \to R$ in $\mathbb{C}$ such that $x = p_{(S,R)}(z_1,\dots,z_m)$. Then, for each $i \in \{1,\dots,n'\}$ we have

$$d_{(S,A_i)}(s_i) = D(1_S, r_i)(x)$$
$$= D(1_S, r_i)(p_{(S,R)}(z_1,\dots,z_m)) = p_{(S,A_i)}(r_i z_1,\dots,r_i z_m).$$

Since in the pushout

$$\begin{array}{ccc}
\hom(S,A_i)^m & \xrightarrow{\;p_{(S,A_i)}\;} & D(S,A_i) \\
{\scriptsize\begin{pmatrix} t_{11} & \cdots & t_{1m} \\ \vdots & & \vdots \\ t_{n1} & \cdots & t_{nm} \end{pmatrix}} \Big\uparrow & & \Big\uparrow {\scriptsize d_{(S,A_i)}} \\
n(\hom(S,A_i)^k) & \xrightarrow[{\scriptsize\begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix}}]{} & \hom(S,A_i)
\end{array}$$

$d_{(S,A_i)}$ is an injective map, the equality

$$d_{(S,A_i)}(s_i) = p_{(S,A_i)}(r_i z_1, ..., r_i z_m)$$

implies the existence of an element

$$e_i \in n(\hom(S,A_i)^k)$$

such that

$$\begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix}(e_i) = s_i \quad \text{and} \quad \begin{pmatrix} t_{11} & \cdots & t_{1m} \\ \vdots & & \vdots \\ t_{n1} & \cdots & t_{nm} \end{pmatrix}(e_i) = (r_i z_1, ..., r_i z_m).$$

Thus we obtain that for each $i \in \{1, ..., n'\}$ there exists $i^* \in \{1, ..., n\}$ and $f_{i1}, ..., f_{ik} : S \to A_i$ such that

$$u_{i^*}(f_{i1}, ..., f_{ik}) = s_i$$

and

$$t_{i^* j}(f_{i1}, ..., f_{ik}) = r_i z_j \quad \text{for every} \quad j \in \{1, ..., m\}.$$

Now, consider the extended matrix

$$\left(\begin{array}{ccc} t_{1^*1}(f_{11}, ..., f_{1k}) & \cdots & t_{n'^*1}(f_{n'1}, ..., f_{n'k}) \\ \vdots & & \vdots \\ t_{1^*m}(f_{11}, ..., f_{1k}) & \cdots & t_{n'^*m}(f_{n'1}, ..., f_{n'k}) \\ \hline u_{1^*}(f_{11}, ..., f_{1k}) & \cdots & u_{n'^*}(f_{n'1}, ..., f_{n'k}) \end{array}\right).$$

The top rows of this matrix, considered as spans between $A_1, ..., A_n$, factor through the relation r (each j-th row factors through r via z_j). This matrix is obtained from M by change of columns and change of variables. Hence, (c) implies that the relation r is compatible with the above matrix, and so its bottom row also factors through r, i.e. there exists a morphism $y : S \to R$ such that $r_i y = s_i$ for every $i \in \{1, ..., n'\}$. □

Remark 3.2 *In the case when $n' = 2$ and M is the matrix from Table 1, which corresponds to the notion of a Mal'tsev category, Theorem 3.1 becomes Theorem 3.2 of [3].*

Remark 3.3 *Suppose $n' = 1$. Then, in Theorem 3.1, we still have (a)⇒(b)⇒(c), which is evident from the proof of Theorem 3.1. The question whether we also have (c)⇒(a) or not remains open. However, if either $\mathcal{T} = \mathrm{Th}[sets]$ or $\mathcal{T} = \mathrm{Th}[pointed\ sets]$, then it is easy to show that we do have (c)⇒(a) (but then d in (a) does not necessarily have injective components).*

Recall that n denotes the number of columns of M. Remark 3.3, Theorem 3.1 and Observation 1.3 together imply:

Theorem 3.4 *For a finitely complete $\mathcal{T}$-enriched category $\mathbb{C}$ the following conditions are equivalent:*

- *(a) the canonical approximation d of the initial approximate solution p of M in $\hom_{\mathbb{C}}$ satisfies (C_n);*
- *(b) there exists an approximate solution p of M in $\hom_{\mathbb{C}}$ which has an approximation d satisfying (C_n);*
- *(c) $\mathbb{C}$ has M-closed relations.*

Moreover, if the above conditions are satisfied then 3.1(a) and 3.1(b) are satisfied for any natural number $n' \geqslant 1$.

The second characterization theorem

Let $\mathbb{C}$ be a $\mathcal{T}$-enriched category with finite coproducts. Then every object X in $\mathbb{C}$ has an internal $\mathcal{T}$-coalgebra structure; for each k-ary term t in $\mathcal{T}$ the co-operation $t_X : X \to kX$ of X is given by $t_X = t(\iota_1, ..., \iota_k)$, where $\iota_1, ..., \iota_k$ denote the coproduct injections $X \to kX$. Further, this way the $\mathcal{T}$-enrichment of $\mathbb{C}$ gives rise to an internal $\mathcal{T}$-coalgebra structure on $1_{\mathbb{C}}$ in the functor category $\mathbb{C}^{\mathbb{C}}$.

Theorem 3.5 *A finitely complete $\mathcal{T}$-enriched category $\mathbb{C}$ with finite coproducts has M-closed relations if and only if for every object X in $\mathbb{C}$, every n-ary internal relation $r : R \to (kX)^n$ in $\mathbb{C}$ is compatible with*

$$M_X = \left(\begin{array}{ccc} (t_{11})_X & \cdots & (t_{n1})_X \\ \vdots & & \vdots \\ (t_{1m})_X & \cdots & (t_{nm})_X \\ \hline (u_1)_X & \cdots & (u_n)_X \end{array} \right).$$

Proof. The "only if" part is obvious. The proof of the "if" part relies on the observation that for any morphisms $f_1, \ldots, f_k : X \to C$, composing each entry of M_X with the morphism

$$\begin{pmatrix} f_1 \\ \vdots \\ f_k \end{pmatrix} : kX \to C$$

yields the matrix

$$\left(\begin{array}{ccc} t_{11}(f_1, \ldots, f_k) & \cdots & t_{n1}(f_1, \ldots, f_k) \\ \vdots & & \vdots \\ t_{1m}(f_1, \ldots, f_k) & \cdots & t_{nm}(f_1, \ldots, f_k) \\ \hline u_1(f_1, \ldots, f_k) & \cdots & u_n(f_1, \ldots, f_k) \end{array}\right).$$

It is a simple observation that each row of this matrix factors through $r : R \to C^n$ if and only if the corresponding row of M_X factors through the pullback of r along the morphism

$$\begin{pmatrix} f_1 \\ \vdots \\ f_k \end{pmatrix}^n : (kX)^n \to C^n.$$

So every n-ary relation on C is M-closed when every n-ary relation on kX is compatible with M_X. □

If $\mathbb{C}$ is a regular category [1], then for each object X in $\mathbb{C}$ the following conditions are equivalent to each other:

- every relation $r : R \to (kX)^n$ in $\mathbb{C}$ is compatible with M_X;
- there exists an *approximate co-solution* $p : W \to mX$ of M in X whose approximation $d : W \to X$ is a regular epimorphism;
- the approximation of the *terminal approximate co-solution* of M in X is a regular epimorphism.

The equivalence of these three conditions follows from the fact that in a regular category, for a pair of morphisms

$$\begin{array}{ccc} & & Y \\ & & \downarrow{\scriptstyle g} \\ X & \xrightarrow[f]{} & C \end{array}$$

the following conditions are equivalent to each other:

- f factors through every monomorphism through which g factors;
- the above diagram can be filled up to a commutative square

  $$\begin{array}{ccc} W & \longrightarrow & Y \\ {\scriptstyle g'}\downarrow & & \downarrow{\scriptstyle g} \\ X & \xrightarrow[f]{} & C \end{array}$$

  where g' is a regular epimorphism;
- the pullback of g along f is a regular epimorphism (that is to say, we can take the above square to be a pullback).

So from Theorem 3.5 we deduce:

Theorem 3.6 *For a regular $\mathcal{T}$-enriched category $\mathbb{C}$ with finite coproducts, the following conditions are equivalent to each other:*

(a) $\mathbb{C}$ *has M-closed relations.*

(b) For every object X in $\mathbb{C}$ there exists an approximate co-solution of M in X having an approximation that is a regular epimorphism.

(c) For every object X in $\mathbb{C}$ the canonical approximation of the terminal approximate co-solution of M in X is a regular epimorphism.

Remark 3.7 *There is a close connection between the condition on the approximation considered in Theorem 3.1 and the condition on the approximation considered in Theorem 3.6 — see [3].*

Remark 3.8 *Theorems 3.4 and 3.6 were obtained in the special cases of Mal'tsev and subtractive categories in [3] and [4], respectively. As it was emphasized in [4] and in [5] (see also [11]), Theorem 3.6 provides a convenient tool for working in a category $\mathbb{C}$ with M-closed relations, since the behavior of approximate co-solutions of M mimic up to a great degree the behavior of term solutions of M in varieties with M-closed relations; in particular, this allows in many cases to translate a universal-algebraic argument involving terms into a purely categorical argument — thus giving a straightforward method of lifting certain results from Universal Algebra to Category Theory (see e.g. [4] and [11]).*

4 Final remarks

A natural next step is to try to extend the results of this paper to the case of the more general type of matrices considered in [10] (which allow to replace the system of term equations (1) in Theorem 1.2 with a more general kind of system of term equations). However, before that one should first probably try to understand Theorem 3.4 better, which perhaps leads to trying to obtain an analogous result where in 3.4(a) and in 3.4(b), the natural transformation $d : \hom_{\mathbb{C}} \to D$, instead of being an approximation of an approximate solution of M in $\hom_{\mathbb{C}}$, would have the following stronger property: D is equipped with an internal $\mathcal{T}$-algebra structure where M is solvable, and d is a homomorphism of internal $\mathcal{T}$-algebras. For instance, it is easy to show that a finitely complete pointed category $\mathbb{C}$ is unital if and only if there is an internal magma D in $\mathbf{Set}^{\mathbb{C}^{\mathrm{op}}\times\mathbb{C}}$ and a homomorphism $d : \hom_{\mathbb{C}} \to D$ of internal pointed sets, which satisfies (C_2) (and specifically, we can take D to be the free internal magma over $\hom_{\mathbb{C}}$, with unit the base point of $\hom_{\mathbb{C}}$). By the way, the same result remains to be true when we replace "magma" with "monoid" or "commutative monoid".

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