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THE TOTAL EXTERIOR DIFFERENTIAL

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Abstract. The definition of mixed jets includes the finite sequences of vertical vectors tangent to jet bundles. This allows us to define differential operators on vertical forms on jet bundles by using mixed jets prolongations. The total exterior differential is a special case.

Résumé. La définition de jets mixtes inclut les suites finies de vecteurs verticaux tangents à des fibrés de jets. Cela nous permet de définir des opérateurs différentiels sur des formes verticales à un fibré de jets, en utilisant les prolongements de jets mixtes. Le différentiel exterieur total en est un cas particulier.

KEYWORDS: *Jet bundles, differential operators.*

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1. Introduction.
The total exterior differential presented in this paper is an operator that generalizes the total derivative [3][4] and its construction is strongly based on the notion of mixed jets introduced in Section 3. Our line of thought can be clarified by means of the following simple example. Let $\mathsf{T}M$ be a tangent bundle and

$$f : \mathsf{T}M \to \mathbb{R} \; : \; \mathsf{t}^1\gamma(0) \mapsto f\big(\mathsf{t}^1\gamma(0)\big) \tag{1}$$

be a differentiable function defined on $\mathsf{T}M$. The total derivative of f is the following differentiable function defined on the second tangent bundle T^2M:

$$\mathrm{d}_T f : \mathsf{T}^2M \to \mathbb{R} \; : \; \mathsf{t}^2\gamma(0) \mapsto \mathrm{D}(f \circ \mathsf{t}^1\gamma)(0), \tag{2}$$

where $\mathsf{t}^1\gamma : \mathbb{R} \to \mathsf{T}M$ is the prolongation of the curve γ to $\mathsf{T}M$, i.e. its tangent lift, and D is the usual derivative of real functions. In the definition (2) we explicitly use two basic facts: any second tangent vector is an equivalence class of curves, and any curve on the manifold M can be prolonged to a curve on the tangent bundle $\mathsf{T}M$.
If we now regard f as a 0-form on $\mathsf{T}M$, we may look for an extension of the total derivative to q-forms on $\mathsf{T}M$, i.e., to multilinear totally antisymmetric mappings

$$\Omega : \times^q_{\mathsf{T}M} \mathsf{T}(\mathsf{T}M) \to \mathbb{R}. \tag{3}$$

What we expect, as a result, is a mapping

$$\mathrm{d}_T\Omega : \times^q_{\mathsf{T}^2M} \mathsf{T}(\mathsf{T}^2M) \to \mathbb{R} \tag{4}$$

still multilinear and totally antisymmetric.
In order to follow the same pattern as above, we need to consider the elements of the fiber product $\times^q_{\mathsf{T}^2M} \mathsf{T}(\mathsf{T}^2M)$ as equivalence classes of (families of) curves on M, and we need the definition of their prolongations to $\times^q_{\mathsf{T}M} \mathsf{T}(\mathsf{T}M)$. This is made possible by the notion of mixed tangent vector.
Now notice that in (2) the derivative D acting on real functions could as well be interpreted as the exterior differential d. We will then obtain

a further extension of the total derivative, if the role played by D is taken over by the exterior differential d. In this passage, the role of the iterated tangent bundles will be played by the vertical bundles tangent to k-jets and the mixed tangent vectors will be replaced by mixed jets. The result will be the total exterior differential.
The paper is subdivided into two main parts and an appendix. The first part is devoted to the definition of mixed jets, restricted to the case of those mixed jets that can be identified with q-uples of vertical vectors tangent to k-jets. The second part deals with the total exterior differential, starting from the special case of the total derivative. In the appendix we will give the coordinate-based approach to the main constructions presented in the paper.
Remark. The total derivative appears in the Euler-Lagrange operator acting on Lagrangian forms defined on iterated tangent bundles. The total exterior differential will take over its role in the case of the Euler-Lagrange operator acting on Lagrangian forms defined on jet bundles. This is the topic of a forthcoming paper.

2. Preliminaries.

In this paper we will adopt the algebraic interpretation of jets [2], [6]. Unless otherwise specified, all mappings considered in the paper will be local and differentiable. Let M and N be differential manifolds. A mapping φ from M to N will be also denoted by $\varphi : M \to N$ without specifying its domain. The set of all mappings from M to N which are defined at $x \in M$ will be denoted by $\mathcal{D}(N|M, x)$.
Consider the following equivalence relation in $\mathcal{D}(N|M, x)$: φ and φ' are equivalent if they coincide on some open neighbourhood of x. The equivalence class of φ, denoted by $\mathsf{j}^{\mathfrak{c}}\varphi(x)$, is called the germ of φ at x. The set of all germs at x is denoted by $\mathsf{J}^{\mathfrak{c}}(N|M, x)$ and we set

$$\mathsf{J}^{\mathfrak{c}}(N|M) = \bigcup_{x \in M} \mathsf{J}^{\mathfrak{c}}(N|M, x). \tag{5}$$

Consider the special case $N = \mathbb{R}$. In this case the set of germs at x of real functions, denoted by $\mathsf{A}^{\mathfrak{c}}(M, x)$, is a commutative associative algebra with a unit element, and has a unique maximal ideal, namely

$$\mathsf{I^c}_0(M,x) = \{\mathsf{j^c}f(x) \in \mathsf{A^c}(M,x);\ f(x)=0\}\,. \tag{6}$$

In the algebra $\mathsf{A^c}(M,x)$ we have the sequence of ideals

$$\mathsf{I^c}_0(M,x), \mathsf{I^c}_1(M,x), \ldots, \mathsf{I^c}_k(M,x), \mathsf{I^c}_{k+1}(M,x), \ldots \tag{7}$$

where, for any $k \in \mathbb{N}$,

$$\mathsf{I^c}_k(M,x) = (\mathsf{I^c}_0(M,x))^{k+1}. \tag{8}$$

Inclusion relations

$$\mathsf{I^c}_k(M,x) \subset \mathsf{I^c}_{k'}(M,x) \tag{9}$$

hold for all k' and k in $\mathbb{N}$ such that $k' \leqslant k$.
In the set $\mathcal{D}(N|M,x)$ we have, for each $k \in \mathbb{N}$, another equivalence relation: φ' and φ are equivalent if

$$\mathsf{j^c}(f\circ\varphi')(x) - \mathsf{j^c}(f\circ\varphi)(x) \in \mathsf{I^c}_k(M,x) \tag{10}$$

for any function f on N for which the compositions $(f\circ\varphi')$ and $(f\circ\varphi)$ make sense. The equivalence class of φ, denoted by $\mathsf{j}^k\varphi(x)$, is called the *k-jet* of φ at x. The set of all k-jet at x is denoted by $\mathsf{J}^k(N|M,x)$ and we set

$$\mathsf{J}^k(N|M) = \bigcup_{x\in M} \mathsf{J}^k(N|M,x). \tag{11}$$

The set $\mathsf{J}^k(N|M)$ can be endowed with a structure of a differential manifold (Cf.(A18)) such that the k-jet-source projection

$$\sigma_{k(N|M)} : \mathsf{J}^k(N|M) \to M : \mathsf{j}^k\varphi(x) \mapsto x \tag{12}$$

and the k-jet-target projection

$$\tau_{k(N|M)} : \mathsf{J}^k(N|M) \to N : \mathsf{j}^k\varphi(x) \mapsto \varphi(x) \tag{13}$$

are differentiable fibrations.

The k-jet prolongation of a mapping $\varphi : M \to N$ is the mapping

$$\mathsf{j}^k\varphi : M \to \mathsf{J}^k(N|M) : x \mapsto \mathsf{j}^k\varphi(x). \tag{14}$$

The case $M = \mathbb{R}$ is of special interest: the k-tangent fibration

$$\begin{array}{c} \mathsf{T}^k N \\ {\scriptstyle \tau_{k\ N}} \Big\downarrow \\ N \end{array} \tag{15}$$

of a manifold N can be regarded as the restriction of the projection $\tau_{k\ (N|\mathbb{R})} : \mathsf{J}^k(N|\mathbb{R}) \to N$ to the fiber $\mathsf{J}^k(N|\mathbb{R}, 0)$. In view of this identification, we will always write $\mathsf{t}^k\gamma(0)$ instead of $\mathsf{j}^k\gamma(0)$ for every curve γ in N.

From the tangent fibration

$$\begin{array}{c} \mathsf{TJ}^k(N|M) \\ {\scriptstyle \tau_{\mathsf{J}^k(N|M)}} \Big\downarrow \\ \mathsf{J}^k(N|M) \end{array} \tag{16}$$

we select the subfibration

$$\begin{array}{c} \mathsf{VJ}^k(N|M) \\ {\scriptstyle \upsilon_{\mathsf{J}^k(N|M)}} \Big\downarrow \\ \mathsf{J}^k(N|M) \end{array} \tag{17}$$

vertical with respect to the fibration (12), i.e, for each $z \in \mathsf{J}^k(N|M)$,

$$\mathsf{V}_z\mathsf{J}^k(N|M) = (\upsilon_{\mathsf{J}^k(N|M)})^{-1}(z) = \ker T_z\sigma_{k(N|M)}, \tag{18}$$

where $T_z\sigma_{k(N|M)}$ is the tangent mapping of $\sigma_{k(N|M)}$ at z.
We finally recall that a q-form on a manifold M is a mapping

$$\Omega\colon \times^q_M \mathsf{T}M \longrightarrow \mathbb{R}, \tag{19}$$

multilinear and totally antisymmetric. It can be identified with a section of the fiber bundle

$$\begin{array}{c} \wedge^q\mathsf{T}^*M \\ \pi^q_M \Big\downarrow \\ M \end{array} \tag{20}$$

The space of all q-forms on the manifold M will be denoted by $\Lambda^q(M)$ and then $\Lambda(M)$ will be the exterior algebra on M.

3. Mixed Jets.

In this section we focus on mappings defined on a cross-product of two manifolds. The first step will be the construction of a class of ideals which describe the behaviour of the mappings on the two manifolds separately. Then we will use these ideals to define the mixed jets. For our purposes it will be sufficient to choose $\mathbb{R}^q$ as one of the two manifolds involved.
Let M be an m-dimensional differential manifold, consider the cross-product $\mathbb{R}^q \times M$ and denote by pr_1 and pr_2 the natural projections onto $\mathbb{R}^q$ and M, respectively.
Now let $(\mathbf{0}, x) \in \mathbb{R}^q \times M$ and consider the mapping

$$\mathsf{A}^\mathsf{c}(\mathbb{R}^q, \mathbf{0}) \longrightarrow \mathsf{A}^\mathsf{c}(\mathbb{R}^q \times M, (\mathbf{0}, x))\colon \mathsf{j}^\mathsf{c}f(\mathbf{0}) \mapsto \mathsf{j}^\mathsf{c}(f \circ pr_1)(\mathbf{0}, x). \tag{21}$$

We denote the image of the ideal $\mathsf{I}^{\mathsf{c}}_1(\mathbb{R}^q, \mathbf{0})$ by $\mathsf{I}^{\mathsf{c}}_1(\mathbb{R}^q, \mathbf{0}) \circ \mathsf{j}^{\mathsf{c}} pr_1(\mathbf{0}, x)$. Then we consider the ideal of $\mathsf{A}^{\mathsf{c}}(\mathbb{R}^q \times M, (\mathbf{0}, x))$ generated by this image

$$\mathsf{I}^{\mathsf{c}}_{\mathbf{1}}\big(\mathbb{R}^q; (M, x)\big)_{\mathbf{0}} := \Big(\mathsf{I}^{\mathsf{c}}_1(\mathbb{R}^q, \mathbf{0}) \circ \mathsf{j}^{\mathsf{c}} pr_1(\mathbf{0}, x)\Big). \tag{22}$$

Any element in the above ideal (22) is then the germ at $(\mathbf{0}, x)$ of a function on $\mathbb{R}^q \times M$ that is the sum of products

$$(f \circ pr_1)g, \tag{23}$$

where g is any function on $\mathbb{R}^q \times M$, and, owing to Proposition A1, in the notation introduced in the Appendix, $f : \mathbb{R}^q \to \mathbb{R}$ satisfies,

$$\partial_{\boldsymbol{\rho}} f(x) = 0 \tag{24}$$

for any q-multi-index $\boldsymbol{\rho}$ such that $|\boldsymbol{\rho}| \leqslant 1$.
We repeat the construction starting, this time, with the mapping

$$\mathsf{A}^{\mathsf{c}}(M, x) \longrightarrow \mathsf{A}^{\mathsf{c}}(\mathbb{R}^q \times M, (\mathbf{0}, x)) : \mathsf{j}^{\mathsf{c}}\varphi(x) \mapsto \mathsf{j}^{\mathsf{c}}(\varphi \circ pr_2)(\mathbf{0}, x). \tag{25}$$

We first consider the image of the ideal $\mathsf{I}^{\mathsf{c}}_k(M, x)$, which will be denoted by $\mathsf{I}^{\mathsf{c}}_k(M, x) \circ \mathsf{j}^{\mathsf{c}} pr_2(\mathbf{0}, x)$, and then the ideal of $\mathsf{A}^{\mathsf{c}}(\mathbb{R}^q \times M, (\mathbf{0}, x))$ generated by this image

$$\mathsf{I}^{\mathsf{c}}_k\big((\mathbb{R}^q, \mathbf{0}); M\big)_x = \Big(\mathsf{I}^{\mathsf{c}}_k(M, x) \circ \mathsf{j}^{\mathsf{c}} pr_2(\mathbf{0}, x)\Big). \tag{26}$$

Any element in the above ideal (26) is still the germ at $(\mathbf{0}, x)$ of a function on $\mathbb{R}^q \times M$ that is the sum of products

$$(f \circ pr_2)g, \tag{27}$$

but, this time, $f : M \to \mathbb{R}$ satisfies

$$\partial_{\boldsymbol{\mu}} f(x) = 0 \tag{28}$$

for any m-multi-index $\boldsymbol{\mu}$ such that $|\boldsymbol{\mu}|\leqslant k$.
Finally consider the ideal sum

$$\mathsf{I}^{\mathsf{c}}_{(\mathbf{1},k)}\big(\mathbb{R}^q \times M, (\mathbf{0}, x)\big) = \mathsf{I}^{\mathsf{c}}_{\mathbf{1}}\big(\mathbb{R}^q; (M, x)\big)_{\mathbf{0}} + \mathsf{I}^{\mathsf{c}}_k\big((\mathbb{R}^q, \mathbf{0}); M\big)_x. \tag{29}$$

Any element in the mixed ideal (29) is then the germ at $(\mathbf{0}, x)$ of a function on $\mathbb{R}^q \times M$ that is the sum of terms

$$(f_1 \circ pr_1)g_1 + (f_2 \circ pr_2)g_2 \tag{30}$$

with

$$\begin{aligned} \partial_{\boldsymbol{\rho}} f_1(x) = 0 \\ \partial_{\boldsymbol{\mu}} f_2(x) = 0 \end{aligned} \tag{31}$$

for any q-multi-index $\boldsymbol{\rho}$ and m-multi-index $\boldsymbol{\mu}$ such that $|\boldsymbol{\rho}|\leqslant 1$ and $|\boldsymbol{\mu}|\leqslant k$.
We have the following inclusions

$$\mathsf{I}^{\mathsf{c}}_{\mathbf{1}}\big(\mathbb{R}^q; (M, x)\big)_{\mathbf{0}} \subset \mathsf{I}^{\mathsf{c}}_1\big(\mathbb{R}^q \times M, (\mathbf{0}, x)\big) \tag{32}$$

$$\mathsf{I}^{\mathsf{c}}_k\big((\mathbb{R}^q, \mathbf{0}); M\big)_x \subset \mathsf{I}^{\mathsf{c}}_k\big(\mathbb{R}^q \times M, (\mathbf{0}, x)\big), \tag{33}$$

moreover relations

$$\mathsf{I}^{\mathsf{c}}_{(\mathbf{1},k)}\big(\mathbb{R}^q \times M, (\mathbf{0}, x)\big) \subset \mathsf{I}^{\mathsf{c}}_{(\mathbf{1},k')}\big(\mathbb{R}^q \times M, (\mathbf{0}, x)\big) \tag{34}$$

hold for all k' and k in $\mathbb{N}$ such that $k'\leqslant k$.
The mixed ideals (29) will now lead to the definition of mixed jets. Let N be an n-dimensional differential manifold. In the set $\mathcal{D}\big(N|\mathbb{R}^q \times M, (\mathbf{0}, x)\big)$ we introduce, for each $k \in \mathbb{N}$, the following equivalence relation: χ' and χ are equivalent if

$$\mathsf{j}^{\mathsf{c}}(f \circ \chi')(\mathbf{0}, x) - \mathsf{j}^{\mathsf{c}}(f \circ \chi)(\mathbf{0}, x) \in \mathsf{I}^{\mathsf{c}}_{(\mathbf{1},k)}\big(\mathbb{R}^q \times M, (\mathbf{0}, x)\big) \tag{35}$$

for any function f on N for which the compositions make sense.

The equivalence class of χ is denoted by $\mathsf{j}^{(\mathbf{1},k)}\chi(x)$ and is called the $(\mathbf{1},k)$-*jet* of χ at x. The set of all $(\mathbf{1},k)$-jets at x will be denoted by $\mathsf{J}^{(\mathbf{1},k)}(N|\mathbb{R}^q \times M, x)$ and we set

$$\mathsf{J}^{(\mathbf{1},k)}(N|\mathbb{R}^q \times M) = \bigcup_{x\in M} \mathsf{J}^{(\mathbf{1},k)}(N|\mathbb{R}^q \times M, x). \tag{36}$$

The set $\mathsf{J}^{(\mathbf{1},k)}(N|\mathbb{R}^q \times M)$ can be endowed with a differential structure (Cf.(A.31)) such that the $(\mathbf{1},k)$-jet-source projection

$$\sigma_{(\mathbf{1},k)(N|\mathbb{R}^q\times M)} : \mathsf{J}^{(\mathbf{1},k)}(N|\mathbb{R}^q \times M) \to M : \mathsf{j}^{(\mathbf{1},k)}\chi(x) \mapsto x \tag{37}$$

and the $(\mathbf{1},k)$-jet-target projection

$$\tau_{(\mathbf{1},k)(N|\mathbb{R}^q\times M)} : \mathsf{J}^{(\mathbf{1},k)}(N|\mathbb{R}^q \times M) \to N : \mathsf{j}^{(\mathbf{1},k)}\chi(x) \mapsto \chi(\mathbf{0}, x) \tag{38}$$

are differentiable fibrations.
The $(\mathbf{1},k)$-jet prolongation of a mapping $\chi : \mathbb{R}^q \times M \to N$ is the mapping

$$\mathsf{j}^{(\mathbf{1},k)}\chi : M \to \mathsf{J}^{(\mathbf{1},k)}(N|\mathbb{R}^q \times M) : x \mapsto \mathsf{j}^{(\mathbf{1},k)}\chi(x). \tag{39}$$

We conclude this section by showing how q-uples of vertical vectors tangent to jet spaces can be related to mixed jets. Let us consider the fiber product of q copies of the vertical bundle (17),

$$\begin{array}{c} \times^q_{\mathsf{J}^k(N|M)} \mathsf{VJ}^k(N|M) \\ \Big\downarrow{\scriptstyle \upsilon^q_{\mathsf{J}^k(N|M)}} \\ \mathsf{J}^k(N|M) \end{array} \tag{40}$$

Proposition 1. *The elements of* $\mathsf{J}^{(\mathbf{1},k)}(N|\mathbb{R}^q \times M)$ *are in a one-to-one correspondence with the elements of* $\times^q_{\mathsf{J}^k(N|M)} \mathsf{VJ}^k(N|M)$.

PROOF: Let $\mathsf{j}^{(\mathbf{1},k)}\chi(x) \in \mathsf{J}^{(\mathbf{1},k)}(N|\mathbb{R}^q \times M)$ and $j \in \{1, \ldots, q\}$. For any real number s^j in a suitable neighbourhood of $0 \in \mathbb{R}$, we can consider the mapping

$$\chi^j(s^j, \cdot\,) = \chi(0, \ldots, 0, s^j, 0, \ldots, 0, \cdot\,) \colon M \to N \tag{41}$$

and its k-jet, $\mathsf{j}^k\chi^i(s^j, \cdot\,)(x)$, at x. In this way we define q curves

$$\gamma^j_x \colon \mathbb{R} \to \mathsf{J}^k(N|M, x) \colon s^j \mapsto \mathsf{j}^k\chi^j(s^j, \cdot\,)(x) \tag{42}$$

in the fiber $\mathsf{J}^k(N|M, x)$, whose tangent vectors $\mathsf{t}\gamma^j_x(0)$ are, therefore, vertical. We set

$$\mathsf{t}\mathsf{j}^k\chi^j(0, x) := \mathsf{t}\gamma^j_x(0). \tag{43}$$

Note that, since

$$\chi^1(0, \cdot\,) = \ldots = \chi^q(0, \cdot\,) = \chi(\mathbf{0}, \cdot\,), \tag{44}$$

we have that

$$\mathsf{j}^k\chi^1(0, \cdot\,)(x) = \ldots = \mathsf{j}^k\chi^q(0, \cdot\,)(x) = \mathsf{j}^k\chi(\mathbf{0}, \cdot\,)(x) \tag{45}$$

and, as a consequence,

$$\big(\mathsf{t}\mathsf{j}^k\chi^1(0, x), \ldots, \mathsf{t}\mathsf{j}^k\chi^q(0, x)\big) \in \times^q_{\mathsf{J}^k(N|M)} \mathsf{V}\mathsf{J}^k(N|M). \tag{46}$$

We have constructed the mapping

$$\begin{aligned} \varphi^{k,q}_{(N|M)} \colon \mathsf{J}^{(\mathbf{1},k)}(N|\mathbb{R}^q \times M) &\longrightarrow \times^q_{\mathsf{J}^k(N|M)} \mathsf{V}\mathsf{J}^k(N|M) \\ \colon \mathsf{j}^{(\mathbf{1},k)}\chi(x) &\mapsto \big(\mathsf{t}\mathsf{j}^k\chi^1(0, x), \ldots, \mathsf{t}\mathsf{j}^k\chi^q(0, x)\big). \end{aligned} \tag{47}$$

We will prove that it is a bijection. Let ξ and η be charts on M and N, respectively. On the one hand $\mathsf{j}^{(\mathbf{1},k)}\chi(x)$ admits the coordinates

$$\big(\xi(x), \partial_{\boldsymbol{\rho}}\partial_{\boldsymbol{\mu}}\chi\big(\mathbf{0}, x\big)\big)_{|\boldsymbol{\rho}| \leqslant 1\ ,\ |\boldsymbol{\mu}| \leqslant k}\ , \tag{48}$$

(Cf. Appendix, (A.31)), on the other hand, the coordinate expression of the j-th curve γ_x^j in $\mathsf{VJ}^k(N|M)$ is

$$\big(\xi(x), \partial_{\boldsymbol{\mu}} \chi^j(s^j,\cdot)(x)\big)_{|\boldsymbol{\mu}|\leqslant k}\ , \tag{49}$$

(Cf. Appendix, (A.18)). From (49) it follows that the tangent vector $\mathsf{t}\gamma_x^j(0)$ has coordinates

$$\Big(\xi(x), \partial_{\boldsymbol{\mu}} \chi^j(0,\cdot)(x); \mathrm{D}\big(\partial_{\boldsymbol{\mu}} \chi^j(s^j,\cdot)(x)\big)(0)\Big)_{|\boldsymbol{\mu}|\leqslant k}, \tag{50}$$

or, owing to (44),

$$\Big(\xi(x), \partial_{\boldsymbol{\mu}} \chi(\mathbf{0},x); \mathrm{D}\big(\partial_{\boldsymbol{\mu}} \chi^j(s^j,\cdot)(x)\big)(0)\Big)_{|\boldsymbol{\mu}|\leqslant k}. \tag{51}$$

The q-uple $\big(\mathsf{t}\gamma_x^1(0),\ldots,\mathsf{t}\gamma_x^q(0)\big) \in \times^q_{\mathsf{J}^k(N|M)} \mathsf{VJ}^k(N|M)$ can then be given the coordinates

$$\begin{aligned}
&\Big(\xi(x), \partial_{\boldsymbol{\mu}} \chi(\mathbf{0},x); \mathrm{D}\big(\partial_{\boldsymbol{\mu}} \chi^1(s^1,\cdot)(x)\big)(0), .., \mathrm{D}\big(\partial_{\boldsymbol{\mu}} \chi^q(s^q,\cdot)(x)\big)(0)\Big)_{|\boldsymbol{\mu}|\leqslant k} \\
&= \Big(\big(\xi(x), \partial_{\boldsymbol{\mu}} \chi(\mathbf{0},x)\big); \partial_{\boldsymbol{\rho}} \partial_{\boldsymbol{\mu}} \chi(\mathbf{0},x)\Big)_{|\boldsymbol{\rho}|=1\,,\,|\boldsymbol{\mu}|\leqslant k} \qquad (52) \\
&= \big(\xi(x), \partial_{\boldsymbol{\rho}} \partial_{\boldsymbol{\mu}} \chi(\mathbf{0},x)\big)_{|\boldsymbol{\rho}|\leqslant 1\,,\,|\boldsymbol{\mu}|\leqslant k}
\end{aligned}$$

This together with (48) shows that the mapping (47) is injective. We prove that it is also surjective.

Let $(w^1, \ldots, w^q) \in \times^q_{\mathsf{J}^k(N|M)} \mathsf{VJ}^k(N|M)$ and let

$$\left(\bar{x}^1, \ldots, \bar{x}^m, \bar{y}^1_{\boldsymbol{\mu}}, \ldots, \bar{y}^n_{\boldsymbol{\mu}}, \bar{y}^1_{\boldsymbol{\rho\mu}}, \ldots, \bar{y}^n_{\boldsymbol{\rho\mu}}\right)_{|\boldsymbol{\rho}| \leqslant 1\,,\,|\boldsymbol{\mu}| \leqslant k} \tag{53}$$

be its coordinates. For each $A = 1, \ldots, n$, consider the following polynomial function on $\mathbb{R}^q \times \mathbb{R}^m$:

$$P^A(s^1, \ldots, s^q; x^1, \ldots, x^m) =$$

$$\sum_{|\boldsymbol{\mu}| \leq k} \frac{1}{\mu_1! \ldots \mu_m!} \, \bar{y}^A_{\boldsymbol{\mu}} \, (x^1 - \bar{x}^1)^{\mu_1} \cdots (x^m - \bar{x}^m)^{\mu_m} + \tag{54}$$

$$\sum_{\substack{|\boldsymbol{\mu}| \leq k \\ |\boldsymbol{\rho}| = 1}} \frac{1}{\rho_1! .. \rho_q! \mu_1! .. \mu_m!} \, \bar{y}^A_{\boldsymbol{\rho\mu}} \, (s^1)^{\rho_1} \cdot\cdot (s^q)^{\rho_q} (x^1 - \bar{x}^1)^{\mu_1} \cdot\cdot (x^m - \bar{x}^m)^{\mu_m}$$

The mapping

$$(P^1, \ldots, P^n) : \mathbb{R}^q \times \mathbb{R}^m \longrightarrow \mathbb{R}^n \tag{55}$$

is the coordinate expression of the mapping

$$\chi = \eta^{-1} \circ (P^1, \ldots, P^n) \circ \tilde{\xi} : \mathbb{R}^q \times M \longrightarrow N\,, \tag{56}$$

where $\tilde{\xi} = \xi \circ \mathrm{id}_{\mathbb{R}^q}$. It is easy to check that the image of the $(\mathbf{1}, k)$-jet $\mathsf{j}^{(\mathbf{1},k)}\chi(\mathbf{0}, x)$ in the mapping (47) is the assigned q-uple $(w^1, \ldots, w^q)$. ■

The bijection $\varphi^{k,q}_{(N|M)}$ defined in the proof of the above proposition gives rise to the following commutative diagram

$$\begin{array}{ccccccc}
\mathsf{J}^{(\mathbf{1},k)}(N|\mathbb{R}^q \times M) & & \xrightarrow{\;\varphi^{k,q}_{(N|M)}\;} & & \times^q_{\mathsf{J}^k(N|M)}\mathsf{VJ}^k(N|M) & & \\
\swarrow{\scriptstyle \tau_{(\mathbf{1},k)(N|\mathbb{R}^q\times M)}} & & \searrow{\scriptstyle \sigma_{(\mathbf{1},k)(N|\mathbb{R}^q\times M)}} & & \swarrow{\scriptstyle \tau} & & \searrow{\scriptstyle \sigma} \\
N & = & = & = & N & & \\
 & & M & = & = & = & M
\end{array} \tag{57}$$

with

$$\tau = \tau_{k(N|M)} \circ \upsilon^q_{\mathsf{J}^k(N|M)} \tag{58}$$

and

$$\sigma = \sigma_{k(N|M)} \circ \upsilon^q_{\mathsf{J}^k(N|M)}. \tag{59}$$

A *super-representative* of a sequence $(w^1, .., w^q) \in \times^q_{\mathsf{J}^k(N|M)}\mathsf{VJ}^k(N|M)$, is any representative of the corresponding jet $\left(\varphi^{k,q}_{(N|M)}\right)^{-1}(w^1, .., w^q) \in \mathsf{J}^{(\mathbf{1},k)}(N|\mathbb{R}^q \times M)$.

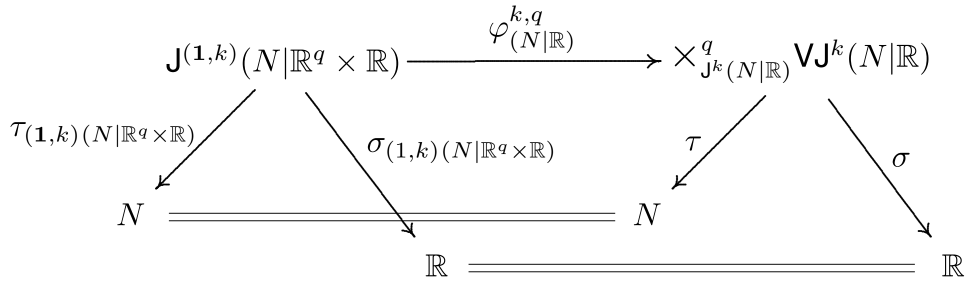
From Proposition 1 it follows that the $(\mathbf{1}, k)$-jet prolongation (39) of a mapping $\chi : \mathbb{R}^q \times M \to N$ can be identified with the mapping

$$\mathsf{j}^{(\mathbf{1},k)}\chi : M \to \times^q_{\mathsf{J}^k(N|M)}\mathsf{VJ}^k(N|M) : x \mapsto \left(\mathsf{tj}^k\chi^1(0, x), \ldots, \mathsf{tj}^k\chi^q(0, x)\right) \tag{60}$$

4. The total exterior differential.

We start the construction of the total exterior differential from a special case, i.e., the total derivative. It is a differential operator known in

the calculus of variations. An intrinsic construction of this operator, obtained as the result of a generalization of the Frölicher and Nijenhuis theory of derivations [3], was presented in [4]. We are providing here an alternative construction based on mixed jets of mappings on $\mathbb{R}^q \times \mathbb{R}$.
We will first regard q-uples in $\times^q_{\mathsf{T}^k N} \mathsf{TT}^k N$ as mixed jets. Let us consider the diagram (57) in our special case $M = \mathbb{R}$,



(61)

On the one hand the bijection $\varphi^{k,q}_{(N|\mathbb{R})}$ induces a bijection between the fibers at $0 \in \mathbb{R}$,

$$\begin{aligned} \varphi^{k,q}_{(N|\mathbb{R},0)} &: \mathsf{J}^{(\mathbf{1},k)}(N|\mathbb{R}^q \times \mathbb{R}, 0) \longrightarrow \times^q_{\mathsf{J}^k(N|\mathbb{R},0)} \mathsf{VJ}^k(N|\mathbb{R}, 0) \\ &: \mathsf{j}^{(\mathbf{1},k)}\chi(0) \mapsto (\mathsf{tj}^k\chi^1(0,0), \ldots, \mathsf{tj}^k\chi^q(0,0)), \end{aligned} \tag{62}$$

on the other hand, as we remarked in the preliminaries, we can make the identification

$$\mathsf{J}^k(N|\mathbb{R}, 0) = \mathsf{T}^k N \tag{63}$$

and then

$$\mathsf{VJ}^k(N|\mathbb{R}, 0) = \mathsf{VT}^k N = \mathsf{TT}^k N. \tag{64}$$

If, moreover, we set

$$\begin{aligned} \mathsf{T}^{(\mathbf{1},k)}N &:= \mathsf{J}^{(\mathbf{1},k)}(N|\mathbb{R}^q \times \mathbb{R}, 0) \\ \mathsf{t}^{(\mathbf{1},k)}\chi(0) &:= \mathsf{j}^{(\mathbf{1},k)}\chi(0), \end{aligned} \tag{65}$$

the mapping (62) becomes

$$\begin{aligned} \varphi_N^{k,q} : \mathsf{T}^{(\mathbf{1},k)}N &\to \times^q_{\mathsf{T}^k N}\mathsf{T}\mathsf{T}^k N \\ : \mathsf{t}^{(\mathbf{1},k)}\chi(0) &\mapsto (\mathsf{t}\mathsf{t}^k\chi^1(0,0), \ldots, \mathsf{t}\mathsf{t}^k\chi^q(0,0)). \end{aligned} \tag{66}$$

It follows that the diagram (61), when restricted to the fibers at $0 \in \mathbb{R}$, reduces to the following fiber isomorphism

$$\begin{array}{ccc} \mathsf{T}^{(\mathbf{1},k)}N & \xrightarrow{\varphi_N^{k,q}} & \times^q_{\mathsf{T}^k M}\mathsf{T}\,\mathsf{T}^k N \\ {\scriptstyle \tau_{(\mathbf{1},k)\,N}}\big\downarrow & & \big\downarrow{\scriptstyle \tau_{k\,N} \circ \tau^q_{\mathsf{T}^k N}} \\ N & = & N \end{array} \tag{67}$$

where $\tau_{(\mathbf{1},k)\,N}$ and $\tau^q_{\mathsf{T}^k N}$ are the restrictions to the fibers considered of the projections $\tau_{(\mathbf{1},k)(N|\mathbb{R}^q\times\mathbb{R})}$ and $\upsilon^q_{\mathsf{J}^k(N|\mathbb{R})}$, respectively.

Finally, as a special case of (39), the $(\mathbf{1},k)$-tangent prolongation of a local mapping $\chi : \mathbb{R}^q \times \mathbb{R} \to N$ is the mapping

$$\begin{aligned} \mathsf{t}^{(\mathbf{1},k)}\chi : \mathbb{R} &\to \times^q_{\mathsf{T}^k N}\mathsf{T}\,\mathsf{T}^k N \\ : t &\mapsto \big(\mathsf{t}\mathsf{t}^k\chi^1(0, t+\cdot)(0), \ldots, \mathsf{t}\mathsf{t}^k\chi^q(0, t+\cdot)(0)\big). \end{aligned} \tag{68}$$

The representation of the elements of $\times^q_{\mathsf{T}^k N}\mathsf{T}\,\mathsf{T}^k N$ as mixed jets makes operations on forms more efficient. We introduce an operator

$$\mathrm{d}_{T(k)} \colon \Lambda(\mathsf{T}^k N) \to \Lambda(\mathsf{T}^{k+1} N) \tag{69}$$

as follows.
Let

$$f \colon \mathsf{T}^k N \to \mathbb{R} \tag{70}$$

be a 0-form on $\mathsf{T}^k N$, then $\mathrm{d}_{T(k)} f$ is the 0-form on $\mathsf{T}^{k+1} N$ given by

$$\mathrm{d}_{T(k)}\Omega \colon \mathsf{T}^{k+1} N \to \mathbb{R} \colon \mathsf{t}^{k+1}\gamma(0) \mapsto \mathrm{D}(\Omega \circ \mathsf{t}^k\gamma)(0). \tag{71}$$

If $q > 0$ and

$$\Omega \colon \times^q_{\mathsf{T}^k N}\mathsf{T}\,\mathsf{T}^k N \to \mathbb{R} \tag{72}$$

is a q-form on $\mathsf{T}^k N$, then $\mathrm{d}_{T(k)}\Omega$ is the q-form on $\mathsf{T}^{k+1} N$ given by

$$\begin{aligned} \mathrm{d}_{T(k)}\Omega \colon & \times^q_{\mathsf{T}^{k+1} N}\mathsf{T}\,\mathsf{T}^{k+1} N \to \mathbb{R} \\ & \colon (w^1, \ldots, w^q) \mapsto \mathrm{D}(\Omega \circ \varphi_N^{k,q} \circ \mathsf{t}^{(\mathbf{1},k)}\chi)(0), \end{aligned} \tag{73}$$

where χ is any super-representative of $(w^1, \ldots, w^q)$.
The operator $\mathrm{d}_{T(k)}$ is the *total derivative*. The coordinate expression of its action is presented in the Appendix.

We will now introduce a more general operator, the total exterior differential d_H, where the role played in $\mathrm{d}_{T(k)}$ by the derivation will be played by the exterior differential.
Let

$$f \colon \mathsf{J}^k(N|M) \to \mathbb{R} \tag{74}$$

be a 0-form on $\mathsf{J}^k(N|M)$, then $\mathrm{d}_H f$ is the 0-form on $\mathsf{J}^{k+1}(N|M)$ given by

$$\mathrm{d}_H f \colon \mathsf{J}^{k+1}(N|M) \to \mathsf{T}^* M \colon \mathsf{j}^{k+1}\varphi(x) \mapsto \mathrm{d}(f \circ \mathsf{j}^k\varphi)(x). \tag{75}$$

More generally, we can consider a 0-form on $\mathsf{J}^k(N|M)$ with values in $\wedge^p \mathsf{T}^* M$, i.e., a bundle morphism

$$\begin{array}{ccc} \mathsf{J}^k(N|M) & \xrightarrow{\quad \Omega \quad} & \wedge^p \mathsf{T}^* M \\ \Big\| & & \Big\downarrow \pi^p_M \\ \mathsf{J}^k(N|M) & \xrightarrow{\quad \sigma_{k(N|M)} \quad} & M \end{array} \tag{76}$$

We obtain a 0-form on $\mathsf{J}^{k+1}(N|M)$ with values in $\wedge^{p+1} \mathsf{T}^* M$

$$\begin{array}{ccc} \mathsf{J}^{k+1}(N|M) & \xrightarrow{\quad \mathrm{d}_H \Omega \quad} & \wedge^{p+1} \mathsf{T}^* M \\ \Big\| & & \Big\downarrow \pi^{p+1}_M \\ J^{k+1}(N|M) & \xrightarrow{\quad \sigma_{k+1(N|M)} \quad} & M \end{array} \tag{77}$$

with the mapping $\mathrm{d}_H \Omega$ defined by

$$\mathrm{d}_H \Omega \left(\mathsf{j}^{k+1} \varphi(x) \right) = \mathrm{d}(\Omega \circ \mathsf{j}^k \varphi)(x). \tag{78}$$

We finally consider a vertical q-form on $\mathsf{J}^k(N|M)$ with values in the set of p-forms on M, i.e., a bundle morphism

$$\begin{array}{ccc} \times^q_{\mathsf{J}^k(N|M)} \mathsf{V}\mathsf{J}^k(N|M) & \xrightarrow{\quad \Omega \quad} & \wedge^p \mathsf{T}^* M \\ \upsilon^q{}_{\mathsf{J}^k(N|M)} \Big\downarrow & & \Big\downarrow \pi^p_M \\ J^k(N|M) & \xrightarrow{\quad \sigma_{k(N|M)} \quad} & M \end{array} \tag{79}$$

Applying d_H to it we obtain

$$\begin{array}{ccc} \times^q_{\mathsf{J}^{k+1}(N|M)} \mathsf{VJ}^{k+1}(N|M) & \xrightarrow{\mathrm{d}_H\Omega} & \wedge^{p+1}\mathsf{T}^*M \\ \Big\downarrow {\scriptstyle \upsilon^q{}_{\mathsf{J}^{k+1}(N|M)}} & & \Big\downarrow {\scriptstyle \pi^{p+1}_M} \\ J^{k+1}(N|M) & \xrightarrow{\sigma_{k+1(N|M)}} & M \end{array} \tag{80}$$

where the mapping $\mathrm{d}_H\Omega$ is defined by

$$\begin{aligned} \mathrm{d}_H\Omega \colon & \times^q_{\mathsf{J}^{k+1}(N|M)} \mathsf{VJ}^{k+1}(N|M) \to \wedge^{p+1}\mathsf{T}^*M \\ & \colon (w^1, \dots, w^q) \mapsto \mathrm{d}(\omega \circ \varphi^{k,q}_{(N|M)} \circ \mathsf{j}^{(\mathbf{1},k)}\chi)(x), \end{aligned} \tag{81}$$

and χ is a super-representative of $(w^1, \dots, w^q)$.
The coordinate expression of the action of d_H is presented in the Appendix. We conclude this presentation by remarking that, as is evident from its construction, d_H is only one of the possible operators that can be defined by using mixed jets. Our choice is due to its future application in the definition of the Euler-Lagrange operator.

5. Appendix.

In this section we give the coordinate-based approach to the main constructions presented in the paper. First we give the differential characterizations of the ideals we have introduced, from these we derive differentiable atlases for the spaces of jets, and then we provide the coordinate expressions of the actions of the operators $\mathrm{d}_{T(k)}$ and d_H.
In the sequel all the charts on manifolds will be arbitrarily chosen within those which are compatible with the compositions involved.
We will adopt the following abridged notation.

Let $f : M \to \mathbb{R}$. We set, for any chart $\xi = (x^1, \dots, x^m)$ on M and any $i = 1, \dots, m$,

$$\partial_i f = \frac{\partial(f \circ \xi^{-1})}{\partial x^i} \circ \xi, \tag{A.1}$$

and for any m-multi-index $\boldsymbol{\mu} = (\mu_1, \dots, \mu_m)$,

$$\partial_{\boldsymbol{\mu}} f = \frac{\partial^{|\boldsymbol{\mu}|}(f \circ \xi^{-1})}{(\partial x^1)^{\mu_1} \cdots (\partial x^m)^{\mu_m}} \circ \xi, \tag{A.2}$$

where $|\boldsymbol{\mu}| = \mu_1 + \cdots + \mu_m$.
Similarly, for any $\varphi : M \to N$, we set

$$\partial_{\boldsymbol{\mu}} \varphi = \frac{\partial^{|\boldsymbol{\mu}|}(\eta \circ \varphi \circ \xi^{-1})}{(\partial x^1)^{\mu_1} \cdots (\partial x^m)^{\mu_m}} \circ \xi, \tag{A.3}$$

where $\eta = (y^1, \dots, y^n)$ is any chart on N.
Finally, let $\chi : \mathbb{R}^q \times M \to N$, then for any other q-multi-index $\boldsymbol{\rho} = (\rho_1, \dots, \rho_q)$, we set

$$\partial_{\boldsymbol{\rho}} \partial_{\boldsymbol{\mu}} \chi = \frac{\partial^{|\boldsymbol{\rho}|+|\boldsymbol{\mu}|}(\eta \circ \chi \circ \tilde{\xi}^{-1})}{(\partial s^1)^{\rho_1} \cdots (\partial s^q)^{\rho_q} (\partial x^1)^{\mu_1} \cdots (\partial x^m)^{\mu_m}} \circ \tilde{\xi} \tag{A.4}$$

where, this time $\tilde{\xi} = \mathrm{id}_{\mathbb{R}^q} \times \xi$ and $\mathrm{id}_{\mathbb{R}^q} = (s^1, \dots, s^q)$.
In particular, for a mapping $\chi : \mathbb{R}^q \times \mathbb{R} \to N$, and any $h \in \mathbb{N}$, we set

$$\partial_{\boldsymbol{\rho}} \partial^h \chi = \frac{\partial^{|\boldsymbol{\rho}|+h}(\eta \circ \chi)}{(\partial s^1)^{\rho_1} \cdots (\partial s^q)^{\rho_q} (\partial t)^h} \tag{A.5}$$

with $(s^1, \dots, s^q, t)$ coordinates in $\mathbb{R}^q \times \mathbb{R}$.

The symbols we adopted for partial derivatives of mappings do not contain any reference to the charts on M and N used in their definition. This is because all the claims which follow are independent of the choice of these charts, so that there is no need to mention them explicitly. The following propositions establish a link between the definition of jet based on ideals of local algebras and the standard definition of jet utilizing partial derivatives of mappings [5].

Proposition A1. *Let $f \in \mathcal{D}(\mathbb{R}|M,x)$. Then, for each $k \in \mathbb{N}$, the following conditions are equivalent.*

(i) $\mathsf{j}^{\mathsf{c}} f(x) \in \mathsf{I}^{\mathsf{c}}_k(M,x)$*;*

(ii) $\partial_{\boldsymbol{\mu}} f(x) = 0$*, for any m-multi-index $\boldsymbol{\mu}$ such that* $|\boldsymbol{\mu}|\leqslant k$.

PROOF: We prove that (i) implies (ii). Each element of $\mathsf{I}^{\mathsf{c}}_k(M,x)$ is the finite sum of germs of functions of the form

$$f = g_0 g_1 \cdot \ldots \cdot g_k \tag{A.6}$$

such that

$$\mathsf{j}^{\mathsf{c}} g_h(x) \in \mathsf{I}^{\mathsf{c}}_0(M,x)\ ,\ 0\leqslant h\leqslant k. \tag{A.7}$$

It will then suffice to prove the claim for this kind of products.
We have $\partial_i f = \sum_{h=0}^{k} g_0 g_1 \cdot \ldots \cdot g_{h-1}\,(\partial_i g_h)\, g_{h+1} \cdot \ldots \cdot g_k$, for each $i = 1, \ldots, m$, hence

$$\mathsf{j}^{\mathsf{c}} \partial_i f(x) \in \mathsf{I}^{\mathsf{c}}_{k-1}(M,x). \tag{A.8}$$

By finite iteration, we obtain

$$\mathsf{j}^{\mathsf{c}} f(x) \in \mathsf{I}^{\mathsf{c}}_k(M,x) \Rightarrow \mathsf{j}^{\mathsf{c}} \partial_{\boldsymbol{\mu}} f(x) \in \mathsf{I}^{\mathsf{c}}_{k-|\boldsymbol{\mu}|}(M,x), \tag{A.9}$$

for $|\boldsymbol{\mu}|\leqslant k$. From the inclusion relations (9) we deduce that

$$\mathsf{j}^{\mathsf{c}} \partial_{\boldsymbol{\mu}} f(x) \in \mathsf{I}^{\mathsf{c}}_0(M,x), \tag{A.10}$$

whence (ii) immediately follows.

We now prove that (ii) implies (i). Suppose that f fulfils (ii). Then consider its Taylor expansion at x, with Lagrange remainder:

$$f = f(x) + \sum_{|\boldsymbol{\mu}|=1}^{k} \frac{1}{\mu_1!..\mu_m!} \partial_{\boldsymbol{\mu}} f(x) \big(x^1 - x^1(x)\big)^{\mu_1} \cdot \ldots \cdot \big(x^m - x^m(x)\big)^{\mu_m} + R \tag{A.11}$$

Owing to our hypothesis and the properties of the Lagrange remainder,

$$\mathsf{j}^{\mathfrak{c}} f(x) = \mathsf{j}^{\mathfrak{c}} R(x) \in \mathsf{I}^{\mathfrak{c}}_k(M, x). \tag{A.12}$$

■

Proposition A2. *Let $\varphi', \varphi \in \mathcal{D}(N|M, x)$. Then, for each $k \in \mathbb{N}$, the following conditions are equivalent.*

(i) $\mathsf{j}^k \varphi'(x) = \mathsf{j}^k \varphi(x)$;

(ii) $\partial_{\boldsymbol{\mu}} \varphi'(x) = \partial_{\boldsymbol{\mu}} \varphi(x)$, *for any m-multi-index $\boldsymbol{\mu}$ such that $|\boldsymbol{\mu}| \leqslant k$.*

PROOF: If (i) holds then, by definition, we have that, for any function f on N,

$$\mathsf{j}^{\mathfrak{c}}(f \circ \varphi' - f \circ \varphi)(x) \in \mathsf{I}^{\mathfrak{c}}_k(M, x), \tag{A.13}$$

and then Proposition A1 implies that

$$\partial_{\boldsymbol{\mu}}(f \circ \varphi')(x) = \partial_{\boldsymbol{\mu}}(f \circ \varphi)(x), \tag{A.14}$$

for $|\boldsymbol{\mu}| \leqslant k$. If we apply (A.14) to coordinate functions y^A on N, we have

$$\partial_{\boldsymbol{\mu}}(y^A \circ \varphi')(x) = \partial_{\boldsymbol{\mu}}(y^A \circ \varphi)(x), \qquad A = 1, \ldots, n, \tag{A.15}$$

whence (ii) follows.

We now prove that (ii) implies (i). Let $f : N \to \mathbb{R}$ be any function on N. By virtue of the chain rule for derivatives (cf. [1]), our hypothesis implies that

$$\begin{aligned}\partial_{\boldsymbol{\mu}}(f\circ\varphi') &= \partial_{\boldsymbol{\mu}}(f\circ\eta^{-1}\circ\eta\circ\varphi') \\ &= \frac{\partial^{|\boldsymbol{\mu}|}\big((f\circ\eta^{-1})\circ(\eta\circ\varphi'\circ\xi^{-1})\big)}{(\partial x^1)^{\mu_1}\cdots(\partial x^m)^{\mu_m}}\circ\xi \\ &= \frac{\partial^{|\boldsymbol{\mu}|}\big((f\circ\eta^{-1})\circ(\eta\circ\varphi\circ\xi^{-1})\big)}{(\partial x^1)^{\mu_1}\cdots(\partial x^m)^{\mu_m}}\circ\xi \\ &= \partial_{\boldsymbol{\mu}}(f\circ\varphi)\end{aligned} \tag{A.16}$$

and then, from Proposition A1 we have that

$$\mathsf{j}^{\mathfrak{c}}(f\circ\varphi'-f\circ\varphi)(x)\in\mathsf{I}^{\mathfrak{c}}_k(M,x), \tag{A.17}$$

whence (i) follows from the very definition of jets. ■

Proposition A2 indicates how to construct a differential structure on $\mathsf{J}^k(N|M)$: let ξ and η be charts on M and N, respectively, then $\mathsf{j}^k\varphi(x)$ can be given the coordinates

$$\big(\xi(x),\partial_{\boldsymbol{\mu}}\varphi(x)\big)_{|\boldsymbol{\mu}|\leqslant k}\,. \tag{A.18}$$

Proposition A3. *Let $f\in\mathcal{D}(\mathbb{R}|\mathbb{R}^q\times M,(\mathbf{0},x))$. Then, for each $k\in\mathbb{N}$, the following conditions are equivalent.*

(i) $\mathsf{j}^{\mathfrak{c}}f(\mathbf{0},x)\in\mathsf{I}^{\mathfrak{c}}_{(1,k)}\big(\mathbb{R}^q\times M,(\mathbf{0},x)\big)$*;*

(ii) $\partial_{\boldsymbol{\rho}}\partial_{\boldsymbol{\mu}}f(\mathbf{0},x)=0$*, for any q-multi-index $\boldsymbol{\rho}$ and any m-multi-index $\boldsymbol{\mu}$ such that $|\boldsymbol{\rho}|\leqslant 1$ and $|\boldsymbol{\mu}|\leqslant k$.*

PROOF: We prove that (i) implies (ii). Each element of $\mathsf{I}^{\mathfrak{c}}_{(1,k)}\big(\mathbb{R}^q\times M,(\mathbf{0},x)\big)$ is the finite sum of germ at $(\mathbf{0},x)$ of functions of the form

$$(f_1\circ pr_1)g_1+(f_2\circ pr_2)g_2, \tag{A.19}$$

where $\mathsf{j}^{\mathsf{c}}f_1(\mathbf{0}) \in \mathsf{I}^{\mathsf{c}}_1(\mathbb{R}^q, \mathbf{0})$, $\mathsf{j}^{\mathsf{c}}f_2(x) \in \mathsf{I}^{\mathsf{c}}_k(M, x)$ and the projections pr_1, pr_2 refer to the cross-product $\mathbb{R}^q \times M$. It will then suffice to prove the claim for this kind of functions. We first consider the derivatives of

$$F_1 = (f_1 \circ pr_1)g_1. \tag{A.20}$$

For all $\boldsymbol{\mu}$ and $|\boldsymbol{\rho}| \leqslant 1$, we have

$$\partial_{\boldsymbol{\rho}}\partial_{\boldsymbol{\mu}}F_1(\mathbf{0}, x) = \partial_{\boldsymbol{\rho}}\big((f_1 \circ pr_1)\partial_{\boldsymbol{\mu}}g_1\big)(\mathbf{0}, x). \tag{A.21}$$

Since $\mathsf{j}^{\mathsf{c}}f_1(\mathbf{0}) \in \mathsf{I}^{\mathsf{c}}_1(\mathbb{R}^q, \mathbf{0})$, it follows that $\mathsf{j}^{\mathsf{c}}(f_1 \circ pr_1)(\mathbf{0}, x) \in \mathsf{I}^{\mathsf{c}}_1(\mathbb{R}^q \times M, (\mathbf{0}, x))$ by (22) and inclusion (32). As a consequence,

$$\mathsf{j}^{\mathsf{c}}(\partial_{\boldsymbol{\rho}}\partial_{\boldsymbol{\mu}}F_1)(\mathbf{0}, x) \in \mathsf{I}^{\mathsf{c}}_1(\mathbb{R}^q \times M, (\mathbf{0}, x)), \tag{A.22}$$

and Proposition A1 proves that

$$\partial_{\boldsymbol{\rho}}\partial_{\boldsymbol{\mu}}F_1(\mathbf{0}, x) = 0. \tag{A.23}$$

We then consider

$$F_2 = (f_2 \circ pr_2)g_2. \tag{A.24}$$

We have, for all $\boldsymbol{\rho}$,

$$\partial_{\boldsymbol{\rho}}F_2 = (f_2 \circ pr_2)\partial_{\boldsymbol{\rho}}g_2. \tag{A.25}$$

Since $\mathsf{j}^{\mathsf{c}}f_2(x) \in \mathsf{I}^{\mathsf{c}}_k(M, x)$, it follows, from (26) and inclusion (33), that $\mathsf{j}^{\mathsf{c}}(f_2 \circ pr_2)(\mathbf{0}, x) \in \mathsf{I}^{\mathsf{c}}_k(\mathbb{R}^q \times M, (\mathbf{0}, x))$. Then, $\mathsf{j}^{\mathsf{c}}\big(\partial_{\boldsymbol{\rho}}F_2\big)(\mathbf{0}, x) \in \mathsf{I}^{\mathsf{c}}_k(\mathbb{R}^q \times M, (\mathbf{0}, x))$, and Proposition A1 ensures that

$$\partial_{\boldsymbol{\mu}}\partial_{\boldsymbol{\rho}}F_2\big(\mathbf{0}, x\big) = 0 \tag{A.26}$$

for all $|\boldsymbol{\mu}| \leqslant k$. This, together with (A.23), shows that (i) implies (ii). We now show that (ii) implies (i). Let $f \in \mathcal{D}(\mathbb{R}|\mathbb{R}^q \times M, (\mathbf{0}, x))$ and consider its Taylor polynomial at $(\mathbf{0}, x)$ of order $k+1$ with Lagrange remainder. Up to a multiplicative constant, its generic term is

$$\partial_{\boldsymbol{\rho}}\partial_{\boldsymbol{\mu}}f(\mathbf{0}, x)(s^1)^{\rho_1} \cdots (s^q)^{\rho_q}(x^1 - x^1(x))^{\mu_1} \cdots (x^m - x^m(x))^{\mu_m}, \tag{A.27}$$

where $|\boldsymbol{\rho}| + |\boldsymbol{\mu}| \leqslant k+1$. We have

$$|\boldsymbol{\rho}| \geqslant 2 \Rightarrow \mathsf{j}^{\mathsf{c}}\big((s^1)^{\rho_1} \cdots (s^q)^{\rho_q}\big)(\mathbf{0}) \in \mathsf{I}^{\mathsf{c}}_1(\mathbb{R}^q, \mathbf{0}), \tag{A.28}$$

$$|\boldsymbol{\mu}| \geqslant k+1 \Rightarrow \mathsf{j}^{\mathsf{c}}\big((x^1 - x^1(x))^{\mu_1} \cdots (x^m - x^m(x))^{\mu_m}\big)(x) \in \mathsf{I}^{\mathsf{c}}_k(M, x). \tag{A.29}$$

We distinguish three possible cases.
If $|\boldsymbol{\rho}| \geqslant 2$, by (A.28), the term (A.27) belongs to $\mathsf{I}^{\mathsf{c}}_{\mathbf{1}}\big(\mathbb{R}^q; (M, x)\big)_{\mathbf{0}}$.

If $|\boldsymbol{\mu}| \geqslant k+1$, by (A.29), the term (A.27) belongs to $\mathsf{I}^{\mathsf{c}}_k\big((\mathbb{R}^q, \mathbf{0}); M\big)_x$.
If $|\boldsymbol{\rho}| \leqslant 1$ and $|\boldsymbol{\mu}| \leqslant k$, then, by assumption, the term (A.27) vanishes.
It follows that the above Taylor polynomial belongs to $\mathsf{I}^{\mathsf{c}}_{(\mathbf{1},k)}\big(\mathbb{R}^q \times M, (\mathbf{0}, x)\big)$, and then f fulfils (i), provided that this is the case for the Lagrange remainder. Note that this is the sum of terms of the following form:

$$C\,(s^1)^{\rho_1} \cdots (s^q)^{\rho_q} \big(x^1 - x^1(x)\big)^{\mu_1} \cdots \big(x^m - x^m(x)\big)^{\mu_m}, \tag{A.30}$$

where $C \in \mathcal{D}(\mathbb{R}|\mathbb{R}^q \times M, (\mathbf{0}, x))$ and $|\boldsymbol{\rho}| + |\boldsymbol{\mu}| = k+2$, which implies that $|\boldsymbol{\rho}| \geqslant 2$ or $|\boldsymbol{\mu}| \geqslant k+1$. Hence, applying again (A.28) and (A.29) we deduce that the germ at $(\mathbf{0}, x)$ of the remainder belongs to $\mathsf{I}^{\mathsf{c}}_{(\mathbf{1},k)}\big(\mathbb{R}^q \times M, (\mathbf{0}, x)\big)$. This completes the proof. ■

In the same way as we have derived Proposition A2 from Proposition A1 one can derive the following proposition from Proposition A3.

Proposition A4. *Let $\chi', \chi \in \mathcal{D}\big(N|\mathbb{R}^q \times M, (\mathbf{0}, x)\big)$. Then, for each $k \in \mathbb{N}$, the following conditions are equivalent.*

(i) $\mathsf{j}^{(\mathbf{1},k)}\chi'(x) = \mathsf{j}^{(\mathbf{1},k)}\chi(x)$;

(ii) $\partial_{\boldsymbol{\rho}}\partial_{\boldsymbol{\mu}}\chi'\big(\mathbf{0}, x\big) = \partial_{\boldsymbol{\rho}}\partial_{\boldsymbol{\mu}}\chi\big(\mathbf{0}, x\big)$, *for any q-multi-index $\boldsymbol{\rho}$ and any m-multi-index $\boldsymbol{\mu}$ such that $|\boldsymbol{\rho}| \leqslant 1$ and $|\boldsymbol{\mu}| \leqslant k$.*

Proposition A4 indicates how to construct a differential structure on $\mathsf{J}^{(\mathbf{1},k)}(N|\mathbb{R}^q \times M)$: let ξ and η be charts on M and N, respectively, then $\mathsf{j}^{(\mathbf{1},k)}\chi(x)$ can be given the coordinates

$$\big(\xi(x), \partial_{\boldsymbol{\rho}}\partial_{\boldsymbol{\mu}}\chi(\mathbf{0},x)\big)_{|\boldsymbol{\rho}|\leqslant 1\ ,\ |\boldsymbol{\mu}|\leqslant k}. \tag{A.31}$$

We now give the coordinate expression of the action of the total derivative $\mathrm{d}_{T(k)}$. In the sequel we will use Einstein's convention on repeated indices. We will first consider 0-forms and coordinate 1-forms and then we will derive from these the action of $\mathrm{d}_{T(k)}$ on arbitrary q-forms. For

the sake of semplicity, we will subdivide the coordinates on $\mathsf{T}^k N$ into blocks $\big(y^{(h)}\big)_{0\leqslant h\leqslant k}$ and treat each block $y^{(h)} = (y^{(h)1}, \dots, y^{(h)n})$ as a

single coordinate.
Let $f : \mathsf{T}^k N \to \mathbb{R}$. We have, owing to (71),

$$\mathrm{d}_{T(k)} f : \mathsf{T}^{k+1} N \longrightarrow \mathbb{R}\,, \tag{A.32}$$

$$\begin{aligned} \mathrm{d}_{T(k)} f\big(\mathsf{t}^{k+1}\gamma(0)\big) &= D(f \circ \mathsf{t}^k\gamma)(0) \\ &= \sum_{h=0}^{k} \left.\frac{\partial f}{\partial y^{(h)}}\right|_{\mathsf{t}^k\gamma(0)} \partial^{h+1}\gamma(0). \end{aligned} \tag{A.33}$$

Let us now consider blocks of coordinate 1-forms and treat them as single coordinate 1-forms, in particular we set $\mathrm{d}y^{(h)} : \mathsf{T}\mathsf{T}^k N \to \mathbb{R}^n \colon u^{(k)j}\frac{\partial}{\partial y^{(k)j}} \mapsto u^{(h)}$.

We have, owing to (71),

$$\mathrm{d}_{T(k)}\mathrm{d}y^{(h)} : \mathsf{T}\mathsf{T}^{k+1} N \longrightarrow \mathbb{R}^n\,, \tag{A.34}$$

$$\mathrm{d}_{T(k)}\mathrm{d}y^{(h)}(w) = D\big(\mathrm{d}y^{(h)} \circ \varphi_N^{k,1} \circ \mathsf{t}^{(1,k)}\chi\big)(0)\,, \tag{A.35}$$

where $\chi : \mathbb{R} \times \mathbb{R} \to N$ is a super-representative of w.
The coordinate expression of the prolongation $\mathsf{t}^{(1,k)}\chi : \mathbb{R} \to \mathsf{T}^{(1,k)}N$ is, owing to (A.31),

$$\left(\partial^h \chi(0,\cdot), \partial_1 \partial^h \chi(0,\cdot)\right)_{0 \leqslant h \leqslant k} \tag{A.36}$$

and it is also the coordinate expression of the prolongation

$$\varphi_N^{k,1} \circ \mathsf{t}^{(1,k)}\chi : \mathbb{R} \to \mathsf{TT}^k N \tag{A.37}$$

So we have

$$\begin{aligned} \mathrm{d}_{T(k)}(\mathrm{d}y^{(h)})(w) &= \mathrm{D}\left(\partial_1 \partial_h \chi(0,\cdot)\right)(0) \\ &= \partial_1 \partial_{h+1} \chi(0,0) \\ &= \mathrm{d}y^{(h+1)}(w). \end{aligned} \tag{A.38}$$

We conclude that

$$\mathrm{d}_{T(k)} \mathrm{d}y^{(h)} = \mathrm{d}y^{(h+1)}. \tag{A.39}$$

Let now

$$\Omega = \Omega_{h_1 \ldots h_q} \mathrm{d}y^{(h_1)} \wedge \ldots \wedge \mathrm{d}y^{(h_q)}, \quad 0 \leqslant h_\ell \leqslant k\,. \tag{A.40}$$

be a q-form on $\mathsf{T}^k M$. Its total derivative

$$\mathrm{d}_{T(k)}\Omega = (\mathrm{d}_{T(k)}\Omega)_{h_1 \ldots h_q} \mathrm{d}x^{(h_1)} \wedge \ldots \wedge \mathrm{d}x^{(h_q)}, \quad 0 \leqslant h_\ell \leqslant k+1, \tag{A.41}$$

is the q-form on $\mathsf{T}^{k+1}M$ whose components are

$$(\mathrm{d}_{T(k)}\Omega)_{h_1 \ldots h_q} = \mathrm{d}_{T(k)}\Omega_{h_1 \ldots h_q} + \sum_{\ell=1}^{q} \Omega_{h_1 \ldots h_{\ell-1}\, h_\ell - 1\, h_{\ell+1} \ldots h_q}, \tag{A.42}$$

where

$$\Omega_{h_1 \ldots h_q} = 0 \qquad \text{if } h_\ell = k+1 \text{ or } h_\ell < 0 \text{ for some } \ell. \tag{A.43}$$

We now give the coordinate expression of the action of the differential operator d_H. We adopt the following convention. We denote the coordinates on $\mathsf{J}^k(N|M)$ by $(x^i, y^A_{\boldsymbol{\mu}})$, where $1 \leqslant i \leqslant m$, $1 \leqslant A \leqslant n$, $\boldsymbol{\mu} \in \mathbb{N}^m$ and $|\boldsymbol{\mu}| \leqslant k$.
Let $f : \mathsf{J}^k(N|M) \to \mathbb{R}$. We have, owing to (75),

$$\mathrm{d}_H f : \mathsf{J}^{k+1}(N|M) \to \mathsf{T}^* M \tag{A.44}$$

$$\mathrm{d}_H f(\mathsf{j}^{k+1}\varphi(x)) = \mathrm{d}(f \circ \mathsf{j}^k \varphi)(x) = \left(\frac{\partial f}{\partial x^i} + \frac{\partial f}{\partial y^A_{\boldsymbol{\mu}}} \partial_{\boldsymbol{\mu}+\boldsymbol{\delta}_i} \varphi^A \right) \mathrm{d}x^i, \tag{A.45}$$

where $\boldsymbol{\delta}_i$ is the ith element of the canonical basis of $\mathbb{R}^m$, and we have used the coordinate expression of the prolongation $\mathsf{j}^k\varphi : M \to \mathsf{J}^k(N|M)$, which owing to (A.18), is $(x^i, \partial_{\boldsymbol{\mu}} \varphi^A)$, $1 \leqslant i \leqslant m$, $1 \leqslant A \leqslant n$, $\boldsymbol{\mu} \in \mathbb{N}^m$, and $|\boldsymbol{\mu}| \leqslant k$. Let us now consider the action of d_H on the exact vertical 1-form $\mathrm{d}y^A_{\boldsymbol{\mu}}$. To this end let $W \in \mathsf{VJ}^{k+1}(N|M)$ and set

$W = \mathsf{j}^{(1,k+1)}\chi(0,x)$. We have

$$\mathrm{d}_H \mathrm{d}y^A_{\boldsymbol{\mu}}(W) = \mathrm{d}\big(\mathrm{d}y^A_{\boldsymbol{\mu}} \circ \varphi^{k,1}_{(N|M)} \circ \mathsf{j}^{(1,k)}\chi\big)(x), \tag{A.46}$$

where χ is a super-representative of W. The coordinate expression of the prolongation $\mathsf{j}^{(1,k)}\chi : M \to \mathsf{J}^{(1,k)}(N|\mathbb{R} \times M)$ is, owing to (A.31),

$$(\partial_1 \partial_{\boldsymbol{\mu}} \chi^A)_{|\boldsymbol{\mu}| \leqslant k} \tag{A.47}$$

and it is also the coordinate expression of the prolongation

$$\varphi^{k,1}_{(N|M)} \circ \mathsf{j}^{(1,k)}\chi : M \to \mathsf{VJ}^k(N|M). \tag{A.48}$$

So we have

$$
\begin{aligned}
\mathrm{d}_H \mathrm{d}y^A_{\boldsymbol{\mu}}(W) &= \mathrm{d}\left(\partial_1 \partial_{\boldsymbol{\mu}} \chi^A\right)(x) \\
&= \left(\partial_1 \partial_{\boldsymbol{\mu}+\boldsymbol{\delta}_i} \chi^A \mathrm{d}x^i\right)(x) \\
&= W^A_{\boldsymbol{\mu}+\boldsymbol{\delta}_i} \mathrm{d}x^i(x) \\
&= \mathrm{d}y^A_{\boldsymbol{\mu}+\boldsymbol{\delta}_i}(W) \mathrm{d}x^i(x)
\end{aligned}
\tag{A.49}
$$

This shows that

$$
\mathrm{d}_H \mathrm{d}y^A_{\boldsymbol{\mu}} = \mathrm{d}y^A_{\boldsymbol{\mu}+\boldsymbol{\delta}_i} \otimes \mathrm{d}x^i. \tag{A.50}
$$

Let now

$$
\begin{array}{ccc}
\times^q_{\mathsf{J}^k(N|M)} \mathsf{VJ}^k(N|M) & \xrightarrow{\quad \Theta \quad} & \wedge^p \mathsf{T}^* M \\
\Big\downarrow {\scriptstyle \upsilon^q{}_{\mathsf{J}^k(N|M)}} & & \Big\downarrow {\scriptstyle \pi^p_M} \\
J^k(N|M) & \xrightarrow{\quad \sigma_{k(N|M)} \quad} & M
\end{array}
\tag{A.51}
$$

and set

$$
\Theta = \Theta^{\boldsymbol{\mu}_1 \ldots \boldsymbol{\mu}_q}_{A_1 \ldots A_q; i_1 \ldots i_p} \, \mathrm{d}y^{A_1}_{\boldsymbol{\mu}_1} \wedge \ldots \wedge \mathrm{d}y^{A_q}_{\boldsymbol{\mu}_q} \otimes \mathrm{d}x^{i_1} \wedge \ldots \wedge \mathrm{d}x^{i_p}. \tag{A.52}
$$

If we apply (A.45) and (A.50) to (A.52) we obtain that

$$
\mathrm{d}_H \Theta = (\mathrm{d}_H \Theta)^{\boldsymbol{\mu}_1 \ldots \boldsymbol{\mu}_q}_{A_1 \ldots A_q; i_1 \ldots i_{p+1}} \, \mathrm{d}y^{A_1}_{\boldsymbol{\mu}_1} \wedge \ldots \wedge \mathrm{d}y^{A_q}_{\boldsymbol{\mu}_q} \otimes \mathrm{d}x^{i_1} \wedge \ldots \wedge \mathrm{d}x^{i_{p+1}}, \tag{A.53}
$$

where

$$(\mathrm{d}_H\Theta)^{\boldsymbol{\mu}_1\cdots\boldsymbol{\mu}_q}_{A_1\ldots A_q;i_1\ldots i_{p+1}} = \left(\mathrm{d}_H\Theta^{\boldsymbol{\mu}_1\cdots\boldsymbol{\mu}_q}_{A_1\ldots A_q;i_1\ldots i_p}\right)_{i_{p+1}} + \sum_{\ell=1}^{q}\Theta^{\boldsymbol{\mu}_1\cdots\boldsymbol{\mu}_{\ell-1}\boldsymbol{\mu}_\ell-\boldsymbol{\delta}_{i_{p+1}}\boldsymbol{\mu}_{\ell+1}\cdots\boldsymbol{\mu}_q}_{A_1\ldots A_q;i_1\ldots i_p} \tag{A.54}$$

and

$$\Theta^{\boldsymbol{\mu}_1\cdots\boldsymbol{\mu}_q}_{A_1\ldots A_q;i_1\ldots i_p} = 0 \tag{A.55}$$

if $|\boldsymbol{\mu}_\ell| = k+1$ or $\boldsymbol{\mu}_\ell$ has a negative component for some ℓ.

References

[1] M. Barile, F. Barone and W.M.Tulczyjew, *Polarizations and differential calculus in affine spaces*, Linear Multilinear Algebra **55**, (2007), pp. 121–146.

[2] M. Barile, F. Barone and W.M.Tulczyjew, *Local algebra bundles and α-jets of mappings*, Bull. Math. Soc. Sci. Math. Roumanie (N.S.) **50**, (2007), pp. 207–222.

[3] A. Frölicher and A. Nijenhuis, *Theory of vector-valued differential forms. I. Derivations of the graded ring of differential forms*, Nederl. Akad. Wetensch. Proc. Ser. A. **59**, (1956), pp. 338–359.

[4] G. Pidello and W.M. Tulczyjew, *Derivations of differential forms on jet bundles*, Ann. Mat. Pura Appl. **147** (1987), pp. 249–265.

[5] D.J. Saunders, *The Geometry of Jet Bundles*, London Math. Soc., Lecture Note Series 142, Cambridge University Press (1989).

[6] A. Weil, *Théorie des points proches sur les variétés différentielles.* Géométrie différentielle. Colloques Internationaux du Centre National de la Recherche Scientifique, Strasbourg, 1953, pp. 111–117. Centre National de la Recherche Scientifique, Paris, (1953).

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