CAHIERS DE TOPOLOGIE ET GEOMETRIE DIFFERENTIELLE CATEGORIQUES

Vol. LIII-3 (2012)

A CONVENIENT DIFFERENTIAL CATEGORY

by Richard BLUTE, Thomas EHRHARD and Christine TASSON

Résumé. Nous montrons que les *espaces vectoriels convenables* au sens de Frölicher et Kriegl forment une *catégorie différentielle*. Ces catégories ont été introduites par Blute, Cockett et Seely en tant que modèles de la *logique linéaire différentielle* de Ehrhard et Regnier. Nous montrons que la catégorie en question rend parfaitement compte des intuitions de cette logique.
Il était déjà clair dans l'ouvrage de Frölicher et Kriegl que la catégorie des *espaces vectoriels convenables* a une structure remarquable. Nous donnons ici une interprétation catégorique à une partie importante de cette structure. Ainsi, nous montrons que cette catégorie possède une comonade dont la catégorie de coKleisli coincide avec la catégorie des fonctions infiniment différentiables et que cette comonade modélise la *modalité exponentielle* de la logique linéaire.
Le système logique suggère de nouvelles structures. Nous mettons notamment en évidence l'existence d'un morphisme de *codéréliction* qui permet d'obtenir la dérivée de n'importe quel morphisme par simple précomposition.

Abstract. We show that the category of *convenient vector spaces* in the sense of Frölicher and Kriegl is a *differential category*. Differential categories were introduced by Blute, Cockett and Seely as the categorical models of the *differential linear logic* of Ehrhard and Regnier. Indeed we claim that this category fully captures the intuition of this logic.
It was already evident in the monograph of Frölicher and Kriegl that the category of convenient vector spaces has remarkable structure. We here give much of that structure a logical interpretation. For example, this category supports a comonad for which the coKleisli category is the category of smooth maps on convenient vector spaces. We show this comonad models the *exponential modality* of linear logic.
Furthermore, we show that the logical system suggests new structure. In particular, we demonstrate the existence of a *codereliction* map. Such a map allows for the differentiation of arbitrary maps by simple precomposition.

Keywords. Linear Logic, Monoidal Categories, Topological Vector Spaces, Differentiable Structure

Mathematics Subject Classification (2010). 03F52, 18D10, 46A17

1. Introduction

Differential linear logic was introduced by Ehrhard and Regnier [5, 6] in order to describe the differentiation of higher order functionals from a syntactic or logical perspective. There are models of this logic [3, 4] with sufficient analytical structure to demonstrate that the formalism does indeed capture differentiation. But there were no models directly connected to differential geometry, which is of course where differentiation is of the highest significance. The purpose of this paper is to demonstrate that the *convenient vector spaces* of Frölicher and Kriegl [9] constitute a model of this logic.

The question of how to differentiate functions into and out of function spaces has a significant history. For instance, the importance of such structures is fundamental in the classical theory of *variational calculus*, see e.g. [7]. It is also a notoriously difficult question. This can be seen by considering the category of smooth manifolds and smooth functions between them. While products evidently exist in this category, there is no way to make the set of functions between two manifolds into a manifold. Category theory provides an appropriate framework for the analysis of function spaces, through the notion of cartesian closed categories; in particular we note that the category of smooth manifolds is not cartesian closed.

In the categorical approach to modelling logics, one typically starts with a logic presented as a sequent calculus. One then arranges equivalence classes of proofs into a category. If the equivalence relation is chosen wisely, the resulting category will be a free category with structure. For example, the conjunction-implication fragment of intuitionistic logic yields the free cartesian closed category; the tensor-implication fragment of intuitionistic linear logic yields the free symmetric monoidal closed category. Then a general model is defined as a structure preserving functor from the free category. In both these cases, the implication connective is modelled as a function space, i.e. the right adjoint to product. Any attempt to model the differential linear logic should be a category where morphisms are smooth maps for some notion of smoothness. Then, to model logical implication, the category must also be closed. This is how we will capture functional differentiation.

More precisely, a significant question raised by the work of Ehrhard and Regnier is to write down the appropriate notion of categorical model of differential linear logic. This was undertaken by Blute, Cockett and Seely in [2]. There, a notion of *differential category* is defined and several examples are given in addition to the usual one made from the syntax of the logic.

In this paper, we focus on the category of convenient vector spaces and bounded linear maps, and demonstrate that it is a differential category. Indeed, this category has a number of remarkable properties. It is symmetric monoidal closed, complete and cocomplete. But most significantly, it is equipped with a comonad, for which the resulting coKleisli category is the category of smooth maps, in an appropriate sense. It is already remarkable that the very structure of linear logic [10] appears in this category, but furthermore it is a model of the much newer theory of differential linear logic.

After describing the category of convenient spaces, we demonstrate that it is a model of intuitionistic linear logic, and that the coKleisli category corresponding to the model of the exponential modality (the comonad) is the category of smooth maps. We construct a differential operator on smooth maps, and show that it is a model of the differential inference rule of differential linear logic, i.e. a differential category.

One of the most surprising aspects of this approach to differentiation is the decomposition of the smooth maps from X to Y into a space of linear maps from $!X$ to Y, where $!X$ is the *exponential modality* of X. In fact, in the convenient setting, $!X$ is a space of distributions. It is the convenient vector space obtained by taking the Mackey closure of the linear space generated by the Dirac distributions. From this perspective, differentiation is given by precomposition with a special map called *codereliction*[1]. This may seem unusual from the functional analysis perspective, but is very natural from the linear logic viewpoint.

We note that much of the structure we describe here can be found scattered in the literature [9, 11, 12, 13], but we believe the presentation here sheds new light on both the categorical and logical structures.

Acknowledgements: The first author would like to thank NSERC for its financial support. The authors would like to especially thank Phil Scott for his helpful contributions.

2. Convenient vector spaces

In this section, we present the category of convenient spaces. They can be seen either as topological or bornological vector spaces, with the two structures satisfying a compatibility. We give a brief review of ideas related to bornology, but assume the reader is familiar with locally convex spaces. See [12] for this.

For the significance of bornology and an analysis of convergence properties, see [11]. A set is bornological if, roughly speaking, it is equipped with a notion of boundedness.

Definition 2.1. A set X is *bornological* if equipped with a *bornology*, i.e. a set of subsets $\mathcal{B}_X$, called *bounded*, such that:

- all singletons are in $\mathcal{B}_X$;
- $\mathcal{B}_X$ is downward closed with respect to inclusion;
- $\mathcal{B}_X$ is closed under finite unions.

A map between bornological spaces is *bornological* if it takes bounded sets to bounded sets. The resulting category will be denoted Born.

[1]The name arises from the fact that this is dual to the usual linear logic rule *dereliction*.

Theorem 2.2. *The category* Born *is cartesian closed.*

Proof. (**Sketch** [9, §1.2]) The product bornology is defined to be the coarsest bornology such that the projections are bornological. So a subset of $X \times Y$ is bounded if and only if its two projections are bornological.

The closedness follows from definition of the bornology on $X \Rightarrow Y$ as the set of bornological functions. A subset $B \subseteq X \Rightarrow Y$ is bounded if and only if $B(A)$ is bounded in Y, for all A bounded in X. □

As this bornology will arise in a number of different contexts, we will denote $X \Rightarrow Y$ by $\mathsf{Born}(X,Y)$. We note that the above product construction works for products of arbitrary cardinality.

Definition 2.3. A *convex bornological vector space* is a vector space E equipped with a bornology such that

1. $\mathcal{B}$ is closed under the convex hull operation.
2. If $B \in \mathcal{B}$, then $-B \in \mathcal{B}$ and $2B \in \mathcal{B}$.

The last condition ensures that addition and scalar multiplication are bornological maps, when the reals are given the usual bornology. A map of convex bornological vector spaces is just a linear, bonological map. We thus get a category that we denote CBS.

As described in [9, 12, 11], the topology and bornology of a convenient vector space are related by an adjunction, which we now describe.

Let E be a locally convex space. Say that $B \subseteq E$ is bounded if it is absorbed by every neighborhood of 0, that is to say if U is a neighborhood of 0, then there exists a positive real number λ such that $B \subseteq \lambda U$. This is called the *von Neumann bornology* associated to E. We will denote the corresponding convex bornological space by βE.

On the other hand, let E be a convex bornological space. Define a topology on E by saying that its associated topology is the finest locally convex topology compatible with the original bornology. We will denote by γE the vector space E endowed with this topology. More concretely, one says that the bornivorous disks form a neighborhood basis at 0. A *disk* is a subset A which is both convex and satisfies that $\lambda A \subseteq A$, for all λ with $|\lambda| \leq 1$. A disk A is said to be *bornivorous* when for every bounded subsets B of E, there is $\lambda \neq 0$ such that $\lambda B \subseteq A$.

Theorem 2.4. *(See Thm 2.1.10 of [9]) The functor* $\beta\colon \mathsf{LCS} \to \mathsf{CBS}$ *is right adjoint to the functor* $\gamma\colon \mathsf{CBS} \to \mathsf{LCS}$. *Moreover, if* E *is a CBS and* F *a LCS, then* $\mathsf{LCS}(\gamma E, F) = \mathsf{CBS}(E, \beta F)$.

Definition 2.5. A convex bornological space E is *topological* if $E = \beta\gamma E$. A locally convex space E is *bornological* if $E = \gamma\beta E$.

Let V be a vector space. Any subspace V' of its dual space V^* induces a bornology on V defined by: $U \subseteq V$ is bounded if and only if it is *scalarly bounded*, i.e. $\ell(U)$ is bounded in the reals, for all ℓ in V'. It follows from Lemma 2.1.23 of [9] that such bornologies are topological. Thus to specify a topological bornology, it suffices to specify such a V'. We will take advantage of this frequently in what follows.

Let tCBS denote the full subcategory of topological convex bornological vector spaces and bornological linear maps, and let bLCS denote the category of bornological locally convex spaces and continuous linear maps. We note immediately:

Corollary 2.6. *The categories* tCBS *and* bLCS *are isomorphic.*

The tCBS's that we are interested in have the desirable further properties of *separation* and *completion*. We begin with the easiest of the two notions. We note E' the space of linear bornological functionals over a tCBS E.

Definition 2.7. A convex bornological vector space E is *separated* if E' separates points, that is for any $x \neq 0 \in E$, there is $l \in E'$ such that $l(x) \neq 0$.

One can verify a number of equivalent definitions as done in [9], page 53. For example, E is separated if and only if the singleton $\{0\}$ is the only linear subspace which is bounded.

Bornological completeness is a different and weaker notion than topological completeness, so we give some details.

Definition 2.8. Let E be a bornological space. A net $(x_\gamma)_{\gamma\in\Gamma}$ is *Mackey-Cauchy* if there exists a bounded subset B and a net $(\mu_{\gamma,\gamma'})_{\gamma,\gamma'\in\Gamma,\Gamma'}$ of real numbers converging to 0 such that

$$x_\gamma - x_{\gamma'} \in \mu_{\gamma,\gamma'} B.$$

Contrary to what generally happens in locally convex spaces, here the convergence of Mackey-Cauchy nets is equivalent to the convergence of Mackey-Cauchy sequences.

Definition 2.9.
- A bornological space is *Mackey-complete* if every Mackey-Cauchy net converges
- A *convenient vector space (CVS)* is a Mackey-complete, separated, topological convex bornological vector space.
- The category of convenient vector spaces and bornological linear maps is denoted Con.

Later we will be considering $\mathcal{C}^\infty$, a category of convenient vector spaces and smooth maps. It will be important to distinguish the two.

We note that Kriegl and Michor in [13] denote the concept of Mackey completeness as c^∞*-completeness* and define a convenient vector space as a c^∞-complete locally convex space. If one takes the bornological maps between these as morphisms, then the result is an equivalent category.

We note that the category of convenient vector spaces is closed under several crucial operations. The following is easy to check:

Theorem 2.10 (See Theorem 2.6.5, [9], and Theorem 2.15 of [13])**.**

- *Assuming that E_j is convenient for all $j \in J$, then $\prod_{j\in J} E_j$ is convenient with respect to the product bornology, with J an arbitrary indexing set.*
- *If E is convenient, then so is $\mathsf{Born}(X, E)$ where X is an arbitrary bornological set.*

There is a standard notion of *Mackey-Cauchy completion* and *separation*. These provide an adjunction in the usual way.

Theorem 2.11 (See Section 2.6 of [9])**.** *By the process of separation and completion, we obtain a functor*

$$\omega\colon \mathsf{tCBS} \to \mathsf{Con}$$

which is left adjoint to the inclusion.

3. Monoidal structure

Theorem 3.1. *The category* Con *is symmetric monoidal closed.*

The fact that Con is a symmetric monoidal closed category is proved in the Section 3.8 of [9]. Roughly, it stems from the cartesian closedness of the category of bornological spaces and bornological maps [11]. In this paragraph, we briefly describe the main steps of the construction.

Let E and F be CVS. We will denote their algebraic tensor product by $E\hat{\otimes}F$, and define a bornology on it by specifying its dual space. Define

$$(E\hat{\otimes}F)' = \{h\colon E\hat{\otimes}F \to \mathbb{R} \mid \hat{h}\colon E \times F \to \mathbb{R} \text{ is bornological}\}$$

where $\hat{h}$ refers to the associated bilinear map, and to be bornological means with respect to the product bornology.

Now, the tensor product $E \otimes F$ in Con is the Mackey closure of the algebraic tensor product equipped with this bornology. Evidently, the tensor unit will be the base field $I = \mathbb{R}$. Let $\mathsf{Con}(E, F)$ denote the space of bornological linear maps. We endow it with the bornology induced by the dual space defined by:

$$\mathsf{Con}(E, F)' = \{h\colon \mathsf{Con}(E, F) \to \mathbb{R} \mid \text{If } U \text{ is equibounded, then } h(U) \text{ is bounded}\},$$

where a subset U of linear maps from E to F is *equibounded* if and only if for every bounded subset B of E, $U(B) = \{f(x) \mid f \in U, x \in B\}$ is bounded in F.

It follows from the cartesian closedness of the category of bornological spaces that there is an isomorphism

$$\mathsf{Con}(E_1; E_2, F) \cong \mathsf{Con}(E_1, \mathsf{Con}(E_2, F))$$

where $\mathsf{Con}(E_1; E_2, F)$ is the space of multilinear, bornological maps. Now, the algebraic tensor product, equipped with the above bornology, classifies bornological multilinear maps. Therefore, the above structure makes Con a symmetric monoidal closed category.

4. Smooth curves and maps

4.1 Smooth curves

Let E be a convenient vector space.

The notion of a smooth curve into a locally convex space E is straightforward. One simply has a curve $c\colon \mathbb{R} \to E$ and defines its derivative by:

$$c'(t) = \lim_{s \to 0} \frac{c(t+s) - c(t)}{s}.$$

Note that this limit is simply the limit in the underlying topological space of E. Then, we define a curve to be *smooth* if all iterated derivatives exist. We denote the set of smooth curves in E by $\mathcal{C}_E$.

Theorem 4.1 (See 2.14 of [13]). *Suppose E is convenient. Then:*

> *If $c\colon \mathbb{R} \to E$ is a curve such that $\ell \circ c$ is smooth for every bornological linear map $\ell\colon F \to \mathbb{R}$, then c is itself smooth.*

In order to endow $\mathcal{C}_E$ with a convenient structure, we introduce the notion of *difference quotients* which is the key idea behind the theory of finite difference methods, as described in [15]. Let $\mathbb{R}^{<i>} \subseteq \mathbb{R}^{i+1}$ consist of those $i+1$-tuples with no two elements equal. It inherits its bornological structure from $\mathbb{R}^{i+1}$. Given any function $f\colon \mathbb{R} \to E$ with E a vector space, we recursively define maps

$$\delta^i f\colon \mathbb{R}^{<i>} \to E,$$

by saying $\delta^0 f = f$, and then the prescription:

$$\delta^i f(t_0, t_1, \ldots, t_i) = \frac{i}{t_0 - t_i} [\, \delta^{i-1} f(t_0, t_1, \ldots, t_{i-1}) - \delta^{i-1} f(t_1, \ldots, t_i) \,].$$

For example,

$$\delta^1 f(t_0, t_1) = \frac{1}{t_0 - t_1} [\, f(t_0) - f(t_1) \,].$$

Notice that the extension of this map along the missing diagonal would be the derivative of f. There are similar interpretations of the higher-order formulas. So these difference formulas provide approximations to derivatives.

Lemma 4.2 (See 1.3.22 of [9])**.** *Let $c\colon \mathbb{R} \to E$ be a function. Then c is a smooth curve if and only if for all natural numbers i, $\delta^i c$ is a bornological map.*

By Lemma 4.2, the above described difference quotients define an infinite family of maps:

$$\delta^i\colon \mathcal{C}_E \to \mathsf{Born}(\mathbb{R}^{<i>}, E).$$

Definition 4.3. Say that $U \subseteq \mathcal{C}_E$ is *bounded* if and only if its image $\delta^i(U)$ is bounded for every natural number i.

Theorem 4.4 (See 3.7 of [13])**.** *This structure makes $\mathcal{C}_E$ a convenient vector space.*

4.2 Smooth maps

We are then left with the question of how to define smoothness of a function between two locally convex spaces.

Definition 4.5. A function $f\colon E \to F$ is *smooth* if $f(\mathcal{C}_E) \subseteq \mathcal{C}_F$. Let $\mathcal{C}^\infty(E, F)$ denote the set of smooth functions from E to F.

We note the obvious fact that $\mathcal{C}_E = \mathcal{C}^\infty(\mathbb{R}, E)$, as seen by considering the identity $id\colon \mathbb{R} \to \mathbb{R}$ as a smooth curve.

Lemma 4.6 (See 2.11 of [13])**.** *A linear map between convenient vector spaces is smooth if and only if it is bornological.*

Let $\mathcal{C}^\infty$ denote the category of convenient vector spaces and smooth maps. Note that the preceding lemma implies the existence of the forgetful functor $\mathbf{U} : \mathsf{Con} \to \mathcal{C}^\infty$ which is the identity on objects and maps.

One of the crucial results of [9] and [13] is that $\mathcal{C}^\infty$ is a cartesian closed category. In fact, this category is the coKleisli category of a model of intuitionistic linear logic, from which the above follows. But this is hardly an enlightening proof! We first give a convenient vector space structure on $\mathcal{C}^\infty(E, F)$.

Now, let E and F be convenient vector spaces. If $c\colon \mathbb{R} \to E$ is a smooth curve, we get a map $c^*\colon \mathcal{C}^\infty(E, F) \to \mathcal{C}_F$ by precomposing.

Definition 4.7. Say that $U \subseteq \mathcal{C}^\infty(E,F)$ is bounded if and only if its image $c^*(U)$ is bounded in $\mathcal{C}_F$ for every smooth curve in $\mathcal{C}_E$.

The space $\mathcal{C}^\infty(E,F)$ has a natural interpretation as a projective limit:

Lemma 4.8 (See [13], p. 30)**.** *The space $\mathcal{C}^\infty(E,F)$ is the projective limit of spaces $\mathcal{C}_F$, one for each $c \in \mathcal{C}_E$. Equivalently, it consists of the Mackey-closed linear subspace of*

$$\mathcal{C}^\infty(E,F) \subseteq \prod_{c \in \mathcal{C}_E} \mathcal{C}_F$$

consisting of all collections $(f_c)_{c \in \mathcal{C}_E}$ such that $f_{c \circ g} = f_c \circ g$ for every $g \in \mathcal{C}^\infty(\mathbb{R},\mathbb{R})$.

As $\mathcal{C}^\infty(E,F)$ is equivalent to a Mackey-closed subspace of a convenient vector space:

Corollary 4.9. *The above structure makes $\mathcal{C}^\infty(E,F)$ a convenient vector space.*

As another consequence of the above Lemma, we get a characterization of smooth curves in $\mathcal{C}^\infty(E,F)$:

Corollary 4.10. *A curve $f : \mathbb{R} \to \mathcal{C}^\infty(E,F)$ is smooth if and only if $t \mapsto c^*(f(t)) : \mathbb{R} \to F$ is smooth for all smooth curves c in $\mathcal{C}_E$.*

Theorem 4.11 (See Theorem 3.12 of [13])**.** *The category $\mathcal{C}^\infty$ is cartesian closed.*

As usual, having a cartesian closed category gives us an enormous amount of structure to work with, as will be seen in what follows.

5. Convenient vector spaces as a differential category

5.1 Differential categories

Differential categories were introduced as the categorical models of differential linear logic. We assume a symmetric, monoidal closed category with biproducts[2]. The biproducts induce an additive structure on Hom-sets, which is necessary for the equations described below. We also assume the existence of a symmetric monoidal comonad called the *exponential modality* and denoted $\boldsymbol{!}$. Such a functor has structure maps of the following form:

$$\rho\colon \boldsymbol{!} \to \boldsymbol{!}\,\boldsymbol{!},\ \ \epsilon\colon \boldsymbol{!} \to id,\ \ \varphi\colon \boldsymbol{!}A \otimes \boldsymbol{!}B \to \boldsymbol{!}(A \otimes B),\ \ \varphi\colon I \to \boldsymbol{!}I,$$

satisfying a standard set of properties. See [16] for an excellent overview of the topic. In the presence of biproducts, the functor $\boldsymbol{!}$ determines a *bialgebra modality*, i.e. for each object A, the object $\boldsymbol{!}A$ naturally has the structure of a bialgebra:

$$\Delta\colon \boldsymbol{!}A \to \boldsymbol{!}A \otimes \boldsymbol{!}A, \qquad e\colon \boldsymbol{!}A \to I,$$
$$\nabla\colon \boldsymbol{!}A \otimes \boldsymbol{!}A \to \boldsymbol{!}A, \qquad \nu\colon I \to \boldsymbol{!}A.$$

The bialgebra structure on $\boldsymbol{!}A$ is obtained via the exponential isomorphism:

$$\boldsymbol{!}(A \oplus B) \cong \boldsymbol{!}A \otimes \boldsymbol{!}B$$

Then, for example, the comultiplication is obtained by applying the functor $\boldsymbol{!}$ to the biproduct map $A \to A \oplus A$, and then composing with the above isomorphism.

To model the remaining differential structure, we need to have a *deriving transformation*, i.e. a natural transformation of the form:

$$d_A\colon A \otimes \boldsymbol{!}A \longrightarrow \boldsymbol{!}A$$

satisfying equations corresponding to the standard rules of calculus:

- The derivative of a constant is 0.
- Leibniz rule.
- The derivative of a linear function is a constant.
- Chain rule.

[2]Actually, weaker axioms suffice [2].

In fact, it suffices to have a natural transformation called *codereliction* [6, 2]:

$$\mathsf{coder}_A\colon A \longrightarrow \,!\,A,$$

satisfying certain equations which are analogues of the above listed equations:

[dC.1] $\mathsf{coder}; e = 0,$

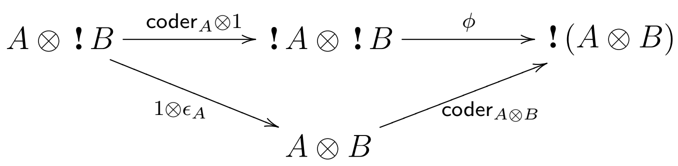
[dC.2] $\mathsf{coder}; \Delta = \mathsf{coder} \otimes \nu + \nu \otimes \mathsf{coder},$

[dC.3] $\mathsf{coder}; \epsilon = 1,$

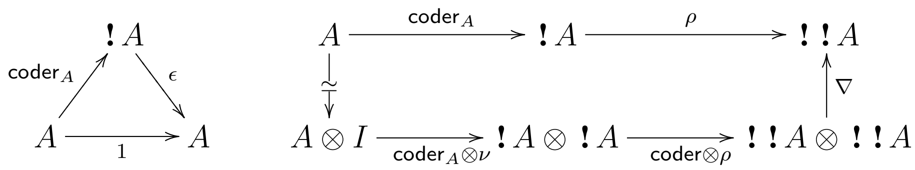
[dC.4] $(\mathsf{coder} \otimes 1); \nabla; \rho = (\mathsf{coder} \otimes \Delta); ((\nabla; \mathsf{coder}) \otimes \rho)); \nabla.$

As shown by Fiore [8] these equations are equivalent to the diagrams below:

1. Strength



2. Comonad



We can finally recover the deriving transformation from the codereliction:

$$d_A : A \otimes\, !\,A \xrightarrow{\mathsf{coder}\otimes 1} \,!\,A \otimes\, !\,A \xrightarrow{\ \nabla\ } \,!\,A.$$

Thanks to the conditions satisfied by the codereliction, we deduce the rules of the deriving transformation: the strength condition entails that the derivative of a constant is zero and the Leibniz rule; the first comonad condition induces the linearity rule; and the second the chain rule.

5.2 The exponential modality on convenient vector spaces

In the category of convenient vector spaces, the comonad described in Theorem 5.1.1 of [9] precisely demonstrates the relationship between linear maps and smooth maps which was envisioned by the differential linear logic.

We begin by noting that if E is a convenient vector space and $x \in E$, there is a canonical morphism of the form $\delta_x \colon \mathcal{C}^\infty(E,\mathbb{R}) \to \mathbb{R}$, defined by $\delta_x(f) = f(x)$. This is of course the *Dirac delta distribution*.

Lemma 5.1. *The Dirac distribution map $\delta \colon E \to \mathcal{C}^\infty(E,\mathbb{R})'$ is smooth.*

Proof. First, we show the map is well-defined. Let $x \in E$, it is easy to see that δ_x is linear. Let us check it is bornological. Let U be a bounded subset of $\mathcal{C}^\infty(E,\mathbb{R})$, that is $c^*(U)$ is bounded in $\mathbb{R}$ for every smooth curve $c \in \mathcal{C}_E$. In particular, $\delta_x(U) = U(x) = \mathrm{const}_x^*(U)$ is bounded. Here, const_x is the constant curve at x.

Now, let us show that δ is smooth. Let c and f be smooth curves into E and $\mathcal{C}^\infty(E,\mathbb{R})$ respectively. The map $t \mapsto \delta_{c(t)} f(t) = f(t)(c(t))$ is smooth. We conclude by cartesian closedness. □

Definition 5.2. The *exponential modality* $!E$ is the Mackey-closure of the linear span of the set $\delta(E)$ in $\mathcal{C}^\infty(E,\mathbb{R})'$. It obtains its bornology as a subspace of $\mathcal{C}^\infty(E,\mathbb{R})'$.

In general, $!E$ is smaller than $\mathcal{C}^\infty(E,\mathbb{R})'$, but in the case where E is finite-dimensional, the two coincide; this is the content of Corollary 5.1.8 of [9]. Furthermore, in this case, the elements of $!E$ correspond to the distributions of compact support, as demonstrated in Proposition 5.1.5 of [9]. See also Théorème XXV, p.89 of [17].

Thanks to the following lemma, the Dirac delta distributions are linearly independant. In the sequel, we will define bornological (multi)linear functions over the exponential of convenient vector spaces by their values over the Dirac delta distributions and extending them by linearity to their linear span and then by Mackey-completion thanks to Theorem 2.11.

Lemma 5.3. *Let $v_1, v_2, \dots, v_n$ be a set of pairwise distinct vectors in E. Then the corresponding δ-functionals are linearly independent in $\mathcal{C}^\infty(E,\mathbb{R})'$.*

Proof. Suppose we have

$$r_1\delta_{v_1} + r_2\delta_{v_2} + \ldots + r_n\delta_{v_n} = 0.$$

We will show that $r_1 = 0$. Since E is separated, there exist bounded linear functionals on E, denoted $\ell_2, \ell_3, \ldots, \ell_n$, such that for all i, $\ell_i(v_1) \neq \ell_i(v_i)$.

Consider the smooth function on E defined by $f = \Pi_{i=2}^{n}(\ell_i - c_{\ell_i(v_i)})$. Here c_r denotes the constant function at r. The result follows from applying the the above equation to f. □

Proposition 5.4. *Endowed with the bornological linear maps $\phi_I : I \to\ !I$ defined by $\phi_I(1) = \delta_1$ and $\phi :\ !E \otimes\ !F \to\ !(E \otimes F)$ defined on basis elements by $\phi(\delta_x \otimes \delta_y) = \delta_{x\otimes y}$ and then extending linearly and completing, the endofunctor $!$ is symmetric monoidal.*

We will now demonstrate that this determines a comonad on Con.

Theorem 5.5. *[See [9], Theorem 5.1.1] We have the following canonical adjunction:*

$$\mathcal{C}^\infty(E, \mathbf{U}F) \cong \mathsf{Con}(\,!E, F)$$

Proof. We establish the bijection, leaving the straightforward calculation of naturality to the reader. So let $\varphi\colon\ !E \to F$ be a bornological linear map. Define a smooth map from E to F by $\hat{\varphi}(e) = \varphi(\delta_e)$. Note that $\hat{\varphi}$ is smooth because it is the composite of φ and δ; φ is smooth since it is bornological and linear.

Conversely, suppose $f\colon E \to F$ is a smooth map. Define a linear map $\tilde{f}$ from the linear span of $\delta(E)$ to F by defining $\tilde{f}(\delta_e) = f(e)$, and extending linearly. Let us show that $\tilde{f}$ is bornological. Let U be bounded in the linear span of $\delta(E)$. The image $\tilde{f}(U)$ is equal to $U(\{f\})$ which is bounded as the image of a singleton set.

We can then extend f to the Mackey completion of the span of $\delta(E)$, using the adjunction of Theorem 2.11. We get a bornological linear function $\bar{f} :\ !E \to F$.

It is clear that this determines a bijection and hence an adjunction. □

We now describe the structure that comes out of this adjunction:

- The counit is the linear map $\epsilon\colon\ !E \to E$, defined by $\epsilon(\delta_x) = x$, and then extending linearly and applying the adjunction of Theorem 2.11.
- The unit is the smooth map $\iota\colon E \to\ !E$, defined by $\iota(x) = \delta_x$.
- The associated comonad has comultiplication $\rho\colon\ !E \to\ !!E$ given by $\rho(\delta_x) = \delta_{\delta_x}$.

Proposition 5.6. *The category* Con *has finite biproducts which are compatible with the monoidal structure:*

$$!(E \oplus F) \cong\ !E \otimes\ !F$$

Proof (See Lemma 5.2.4 of [9]). The existence of finite biproducts is straightforward, as in the usual vector space setting.

The trick in establishing the isomorphism, as usual, is to verify that $!(E \times F)$ satisfies the universal property of the tensor product.

First we note that there is a bilinear map $m_{E,F}\colon\ !E \times\ !F \to\ !(E \times F)$. Consider the smooth map $\iota_{E\times F}\colon E \times F \to\ !(E \times F)$. By cartesian closedness, we get a smooth map $E \to \mathcal{C}^\infty(F, !(E \times F))$, which extends to a linear map $!E \to \mathcal{C}^\infty(F, !(E \times F)) \cong \mathsf{Con}(!F, !(E \times F))$. The transpose is the desired bilinear map. It satisfies $m_{E,F} \circ (\iota_E \times \iota_F) = \iota_{E\times F}$. Note that the map $m_{E,F}$ is in fact determined by this equation, since $!E$ is the Mackey closure of the linear span of the image of ι_E. In particular, we have

$$!\sigma \circ m_{E,F} \circ \sigma = m_{F,E}$$

where σ is the symmetry.

We check that $m_{E,F}$ satisfies the appropriate universality. Assume $f\colon\ !E \times\ !F \to G$ is a bornological bilinear map. Let us show that f is smooth. Let $(c_1, c_2)\colon \mathbb{R} \to\ !E \times\ !F$ be a smooth curve. We want to show that $t \mapsto f(c_1(t), c_2(t))$ is a smooth curve into G. Thanks to Theorem 4.1, it is sufficient to show that for every linear bornological functional l over G, the real function $l \circ f \circ (c_1, c_2)\colon \mathbb{R} \to \mathbb{R}$ is smooth. Now, notice that, from simple calculations of difference quotients, we get

$$\delta^1(l \circ f \circ (c_1, c_2)) = l \circ f \circ (\delta^1(c_1), c_2) + l \circ f \circ (c_1, \delta^1(c_2))$$

and hence $\delta^1(l \circ f \circ (c_1, c_2))$ is bornological. More generally, every difference quotient of $l \circ f \circ (c_1, c_2)$ is bornological. From Lemma 4.2, we get that it is smooth. Then, in turn, $f \circ (\iota \times \iota)$ is smooth. By Theorem 5.5, f lifts to a linear map $\bar{f} \colon \mathbf{!}\,(E \times F) \to G$. By definition, $\bar{f} \circ \delta_{(x_1,x_2)} = f(x_1, x_2)$. Hence f factors through m and $\bar{f}$.

Therefore, the universal property is satisfied by $\mathbf{!}\,(E \times F)$ which is hence isomorphic to $\mathbf{!}\,E \otimes \mathbf{!}\,F$. □

Theorem 5.7. *The category* Con *is a model of intuitionistic linear logic.*

From the biproduct structure we deduce the bialgebra structure:

- $\Delta \colon \mathbf{!}\,E \to \mathbf{!}\,E \otimes \mathbf{!}\,E$ is $\Delta(\delta_x) = \delta_x \otimes \delta_x$, and then extending linearly and using the functor ω to extend to the completion.
- $e \colon \mathbf{!}\,E \to I$ is $e(\delta_x) = 1$.
- $\nabla \colon \mathbf{!}\,E \otimes \mathbf{!}\,E \to \mathbf{!}\,E$ is $\nabla(\delta_x \otimes \delta_y) = \delta_{x+y}$.
- $\nu \colon I \to \mathbf{!}\,E$ is $\nu(1) = \delta_0$.

Thus it remains to establish a *codereliction map* of the form:

$$\mathsf{coder} \colon E \to \mathbf{!}\,E$$

Theorem 5.8. *The category* Con *is a differential category, with codereliction given by*

$$\mathsf{coder}(v) = \lim_{t \to 0} \frac{\delta_{tv} - \delta_0}{t}$$

The first part of the proof, that coder is a bornological linear map from E to $\mathbf{!}\,E$, is an adaptation of the proof by Michor and Kriegl of Theorem 5.9 below. As we will see, their more general result then follows.

Proof. Let us first recall that δ is smooth, hence $t \mapsto \delta_{tv}$ is a smooth curve and the limit is well defined. We now prove that coder $: E \to \mathbf{!}\,E$ is smooth. Let c be a smooth curve in $\mathcal{C}_E$. Then, for any real t, $c^*(coder)(t) = \lim_{s \to 0} \frac{\delta_{sc(t)} - \delta_0}{s}$. Consider the smooth map $h : \mathbb{R} \times \mathbb{R} \to \mathbf{!}\,E$ defined by $h(s,t) = \delta_{sc(t)}$. Its partial derivative at 0 with respect to the second argument is smooth and gives us the partial derivative at 0: $\partial_2 h(t,0) = c^*(\mathsf{coder})(t)$. Hence, $c^*(\mathsf{coder})$ is a smooth curve. And we have proved that $coder$ is smooth.

We now check that the codereliction is linear. It is obviously homogeneous. Then, for any $v, w \in E$, we consider the smooth map $g : \mathbb{R} \times \mathbb{R} \to\ !E$, defined by $g(t,s) = \delta_{tv+sw}$. By computation of the derivative of the smooth map $t \mapsto g(t,t)$, we get: $(t \mapsto g(t,t))'(0) = \partial_1 g(0,0) + \partial_2 g(0,0)$, that is $\mathsf{coder}(v+w) = coder(v) + coder(w)$.

We have proved that coder is linear and smooth, thus it is bornological thanks to Lemma 4.6. It remains only to check the two codereliction equations:

1. Strength: an element $v \otimes \delta_y \in E \otimes\ !F$ is sent to $\lim_{t \to 0} \frac{(\delta_{t(v \otimes y)} - \delta_0)}{t}$ under both legs of the diagram.
2. Comonad: the first comonad law follows from the continuity of ϵ; for the second one, the clockwise chase of $v \in E$ gives us $\lim_{t \to 0} \frac{\delta_{\delta_{tv}} - \delta_{\delta_0}}{t}$ and the counterclockwise gives us $\lim_{s,t \to 0} \frac{\delta_{[\frac{s}{t}(\delta_{tv} - \delta_0) + \delta_0]} - \delta_{\delta_0}}{s}$. To prove the two are equal, it is sufficient to consider the limit on the diagonal $s = t \to 0$.

□

Using this codereliction map, we can build a more general differentiation operator by precomposition:

Consider $f\colon\ !E \to F$ then define $df\colon E \otimes\ !E \to F$ as the composite:

$$E \otimes\ !E \xrightarrow{\mathsf{coder} \otimes 1}\ !E \otimes\ !E \xrightarrow{\nabla}\ !E \xrightarrow{f} F$$

$$v \otimes \delta_x \longmapsto \lim_{t \to 0} \frac{\delta_{tv} - \delta_0}{t} \otimes \delta_x \longmapsto \lim_{t \to 0} \frac{\delta_{tv+x} - \delta_x}{t} \longmapsto \lim_{t \to 0} \frac{f(tv+x) - f(x)}{t}$$

We then obtain the following result of Kriegl and Michor as a corollary:

Theorem 5.9 (See [13], Theorem 1.3.18)**.** *Let E and F be convenient vector spaces. The differentiation operator*

$$d\colon \mathcal{C}^\infty(E,F) \to \mathcal{C}^\infty(E, \mathsf{Con}(E,F))$$

defined as

$$df(x)(v) = \lim_{t \to 0} \frac{f(x+tv) - f(x)}{t}$$

is linear and bounded. In particular, this limit exists and is linear in the variable v.

Conversely, if we start with the general differentiation operator, we can recover codereliction as the differential at 0 of ι, that is:

$$\mathsf{coder}(v) = d\iota(0)(v) = \lim_{t \to 0} \frac{\delta_{tv} - \delta_0}{t}$$

6. Conclusion

Fundamental to understanding the structure of convenient vector spaces is the duality between bornology and topology in the definition of convenient vector spaces. Another place where there is such duality is the notion of a finiteness space, introduced in [4]. But there, the duality is between bornology and the *linear topology* of Lefschetz [14]. The advantage of the present setting is that the topology takes place in the more familiar world of locally convex spaces. However, it remains an interesting question to work out a similar structure in the Lefschetz setting. This program was initiated in the thesis of the third author [19].

Evidently, a next fundamental question is the logical/syntactic structure of integration. One would like an *integral linear logic*, which would again treat integration as an inference rule. It should not be a surprise at this point that convenient vector spaces are extremely well-behaved with respect to integration. The category Con will likely provide an excellent indicator of the appropriate structure.

One can also ask about other classes of functions beside the smooth ones. Chapter 3 of [13] is devoted to the calculus of holomorphic and real-analytic functions on convenient vector spaces. It is an important question as to whether there is an analogous comonad to be found, inducing the category of holomorphic maps as its coKleisli category. Then one can investigate whether the corresponding logic is in any way changed.

Of course, once one has a good notion of structured vector spaces, it is always a good question to ask whether one can build manifolds from such spaces. Manifolds based on convenient vector spaces is the subject of the latter half of [13], and it seems an excellent idea to view these structures from the logical perspective developed here.

Convenient vector spaces and similar structures are under active consideration today, see [1, 18]. We hope the logical perspective introduced here gives new insights in this domain.

References

[1] J. Baez, A. Hoffnung. Convenient Categories of Smooth Spaces, *to appear in* Transactions of the American Mathematics Society, (2010).

[2] R. Blute, R, Cockett, R. Seely. Differential categories, Mathematical Structures in Computer Science 16, pp. 1049-1083, (2006).

[3] T. Ehrhard. On Köthe sequence spaces and linear logic. Mathematical Structures in Computer Science 12, pp. 579-623, (2002).

[4] T. Ehrhard. Finiteness spaces. Mathematical Structures in Computer Science 15, pp. 615-646, (2005)

[5] T. Ehrhard, L. Regnier. The differential λ-calculus. Theoretical Computer Science 309, pp. 1-41, (2003).

[6] T. Ehrhard, L. Regnier. Differential interaction nets. Theoretical Computer Science 364, pp. 166–195, (2006).

[7] I. Gelfand and S. Fomin, *Calculus of Variations*, Dover Publishing, (2000).

[8] M. Fiore, Differential structure in models of multiplicative biadditive intuitionistic linear logic. Proceedings of TLCA, pp 163–177, (2007).

[9] A. Frölicher, A. Kriegl, *Linear Spaces and Differentiation Theory*, Wiley, (1988).

[10] J.-Y. Girard. Linear logic. Theoretical Computer Science 50, pp. 1–102, (1987).

[11] H. Hogbe-Nlend. *Bornologies and Functional Analysis*, North-Holland, (1977).

[12] H. Jarchow. *Locally Convex Spaces*, Teubner, (1981).

[13] A. Kriegl, P. Michor. *The Convenient Setting of Global Analysis*. American Mathematical Society, (1997).

[14] S. Lefschetz. *Algebraic Topology*, American Mathematical Society, (1942).

[15] A. Levin. *Difference Algebra*, Springer-Verlag, (2008).

[16] P.-A. Melliès. *Categorical semantics of linear logic*, Société Mathématique de France, (2008).

[17] L. Schwartz. *Théorie des distributions*, Hermann, (1966).

[18] A. Stacey. Comparative smootheology, Theory and Applications of Categories 25, pp. 64-117, (2011).

[19] C. Tasson. *Sémantiques et syntaxes vectorielles de la logique linéaire*, Thesis, (2009).

Richard Blute
Department of Mathematics
University of Ottawa
Ottawa, Ontario, K1N 6N5, CANADA
rblute@uottawa.ca

Thomas Ehrhard
CNRS, PPS, UMR 7126
Univ Paris Diderot, Sorbonne Paris Cit
F-75205 Paris, France
thomas.ehrhard@pps.jussieu.fr

Christine Tasson
Univ Paris Diderot, Sorbonne Paris Cit
PPS, UMR 7126, CNRS
F-75205 Paris, France
christine.tasson@pps.jussieu.fr