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A NOTE ON DOUBLE CENTRAL EXTENSIONS IN EXACT MAL'TSEV CATEGORIES

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Dedicated to Francis Borceux on the occasion of his sixtieth birthday

Abstract

La caractérisation des extensions centrales doubles en termes de commutateurs de Janelidze (dans le cas des groupes) et de Gran et Rossi (dans le cas des variétés de Mal'tsev) est montrée d'être toujours valide dans le contexte des catégories exactes de Mal'tsev avec coégalisateurs.

The characterisation of double central extensions in terms of commutators due to Janelidze (in the case of groups) and Gran and Rossi (in the case of Mal'tsev varieties) is shown to be still valid in the context of exact Mal'tsev categories with coequalisers.

In his article [10], George Janelidze gave a characterisation of the double central extensions of groups in terms of commutators. Not only did he thus relate Galois theory to commutator theory, but he also sowed the seeds for a new approach to homological algebra, where higher-dimensional (central) extensions are used as a basic tool—see, for instance, [5, 6, 11, 16].

Expressed in terms of commutators of equivalence relations [15, 17], his result amounts to the following: a double extension

$$\begin{array}{ccc} X & \xrightarrow{c} & C \\ {\scriptstyle d}\downarrow & & \downarrow \\ D & \xrightarrow[f]{} & Z \end{array} \qquad \textbf{(A)}$$

is central if and only if $[R[d], R[c]] = \Delta_X = [R[d] \cap R[c], \nabla_X]$. Here $R[d]$ and $R[c]$ denote the kernel pairs of d and c, and Δ_X and ∇_X are the smallest and the largest

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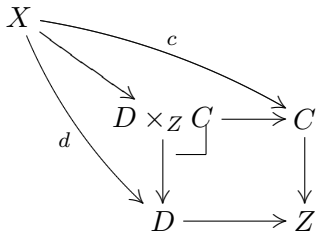
equivalence relation on X. This characterisation was generalised to the context of Mal'tsev varieties by Marino Gran and Valentina Rossi [9]. Although one of the implications (the "only if"-part) of the proof given in [9] is entirely categorical and easily seen to be valid in any Barr exact Mal'tsev category with coequalisers, the other implication is not, and makes heavy use of universal-algebraic machinery. The aim of this note is to provide a proof of the other implication which is valid in any exact Mal'tsev category with coequalisers.

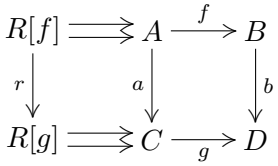
In our proof, we shall not only consider double extensions, but also three-fold and four-fold extensions. Therefore, we begin this note with a few introductory words on higher-dimensional extensions. For an in-depth discussion on this subject in the context of semi-abelian categories we refer the reader to [6] and [5].

Let $\mathcal{A}$ be a regular category, i.e., a finitely complete category with pullback-stable regular epi-mono factorisations. Given $n \geq 0$, denote by $\mathsf{Arr}^n\mathcal{A}$ the category of n-dimensional arrows in $\mathcal{A}$. (A zero-dimensional arrow is an object of $\mathcal{A}$.) n-fold extensions are defined inductively as follows. A **(one-fold) extension** is a regular epimorphism in $\mathcal{A}$. For $n \geq 1$, an $(n+1)$**-fold extension** is a commutative square **A** in $\mathsf{Arr}^{n-1}\mathcal{A}$ (an arrow in $\mathsf{Arr}^n\mathcal{A}$) such that in the induced commutative diagram



every arrow is an n-fold extension. Thus for $n = 2$ we regain the notion of double extension. Note that, since in the regular category $\mathsf{Arr}^{n-1}\mathcal{A}$ a pullback of regular epimorphisms is always a pushout, it follows that an $(n+1)$-fold extension is necessarily a pushout in $\mathsf{Arr}^{n-1}\mathcal{A}$, for any $n \geq 1$.

Suppose from now on that $\mathcal{A}$ is, moreover, Mal'tsev [4, 3], i.e., every (internal) reflexive relation in $\mathcal{A}$ is an equivalence relation. It was shown in [1] that, for a regular category $\mathcal{A}$, the Mal'tsev condition is equivalent to the following property: if, in a commutative diagram



f, g, a and b are extensions, then the right hand square is a double extension if and only if its kernel pair in $\mathsf{Arr}\mathcal{A}$—the morphism r in the diagram—is an extension. Since the concept of double extension is symmetric, this has the following consequences:

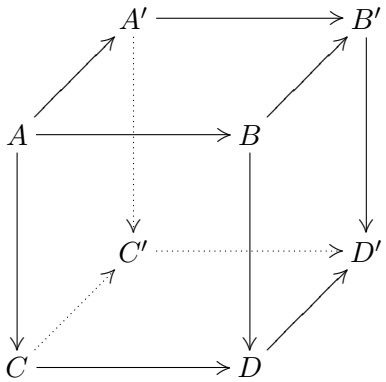
- double extensions are stable under composition;
- if a composite $g \circ f \colon A \to B \to C$ of arrows in $\mathsf{Arr}\mathcal{A}$ is a double extension and B is an extension, then $g \colon B \to C$ is a double extension;
- any split epimorphism of extensions is a double extension.

And then also the following is straightforward to prove:

- the pullback in $\mathsf{Arr}\mathcal{A}$ of a double extension $A \to B$ along a double extension $C \to B$ is a double extension.

In fact, for *any* $n \geq 2$, a commutative square in $\mathsf{Arr}^{n-2}\mathcal{A}$ consisting of $(n-1)$-fold extensions is an n-fold extension if and only if its kernel pair in $\mathsf{Arr}^{n-1}\mathcal{A}$ is an $(n-1)$-fold extension, and thus for all of the above listed properties one obtains higher dimensional versions as well. This is easily shown by induction, if one takes into account that the notion of n-fold extension (for $n \geq 3$) is symmetric in the following sense: any commutative cube in $\mathsf{Arr}^{n-3}\mathcal{A}$ can be considered in three ways as a commutative square in $\mathsf{Arr}^{n-2}\mathcal{A}$; if any of the three squares is an n-fold extension, then the same is true for the other two.

Lemma. *Let $n \geq 3$, and suppose that the following commutative cube in $\mathsf{Arr}^{n-3}\mathcal{A}$ is an n-fold extension.*



If the top square is a pullback, then so is the bottom square.

Proof. Taking pullbacks in the top and bottom squares of the cube, we obtain the comparison square

$$\begin{array}{ccc} A & \longrightarrow & A' \times_{B'} B \\ \downarrow & & \downarrow \\ C & \longrightarrow & C' \times_{D'} D. \end{array}$$

Since the cube is an n-fold extension, this square is an $(n-1)$-fold extension. In particular, it is a pushout in $\mathsf{Arr}^{n-3}\mathcal{A}$, and it follows that the lower comparison morphism is an isomorphism as soon as the upper one is so. □

From now on, we assume that the regular Mal'tsev category $\mathcal{A}$ is, moreover, (Barr) exact (every equivalence relation in $\mathcal{A}$ is effective) and that $\mathcal{A}$ admits coequalisers. This allows us to consider the commutator of equivalence relations defined in [15] (see also [14]), which is a generalisation of Jonathan Smith's definition in the context of Mal'tsev varieties [17]. Following [7] we call an object $A \in \mathcal{A}$ **abelian** if $[\nabla_A, \nabla_A] = \Delta_A$, and we write $\mathsf{Ab}\mathcal{A}$ for the full subcategory of $\mathcal{A}$ determined by all abelian objects. Then $\mathsf{Ab}\mathcal{A}$ is a reflective subcategory of $\mathcal{A}$ and the abelianisation of an object $A \in \mathcal{A}$ is given the quotient $\mathsf{ab}A = A/[\nabla_A, \nabla_A]$. It was shown in [7] that $\mathsf{Ab}\mathcal{A}$ is a **Birkhoff subcategory** of $\mathcal{A}$, which means that it is, moreover, closed in $\mathcal{A}$ under subobjects and regular quotients. Recall from [12] that the Birkhoff condition is equivalent to the following one: for any extension $f\colon A \to B$ in $\mathcal{A}$ the commutative square canonically induced by the unit η

$$\begin{array}{ccc} A & \xrightarrow{\eta_A} & \mathsf{ab}A \\ {\scriptstyle f}\downarrow & & \downarrow \\ B & \xrightarrow[\eta_B]{} & \mathsf{ab}B \end{array} \qquad \textbf{(B)}$$

is a double extension. Note that this condition, together with the lemma above for $n = 3$, implies that

- the abelianisation functor $\mathsf{ab}\colon \mathcal{A} \to \mathsf{Ab}\mathcal{A}$ preserves pullbacks of split epimorphisms along extensions.

(To see this, keep in mind that a split epimorphism of double extensions is always a three-fold extension.) This important property was first discovered by Marino Gran in [8], and we shall need it in the proof of our theorem.

Recall from [12] that an extension $f\colon A \to B$ is **trivial** (with respect to the Birkhoff subcategory $\mathsf{Ab}\mathcal{A}$) if the induced square **B** is a pullback; it is **central** if there exists an extension $p\colon E \to B$ such that the pullback $p^*(f)\colon E \times_B A \to E$ of f along

p is a trivial extension; it is **normal** when the projections of its kernel pair $R[f]$ are trivial. Let us denote by $\mathsf{Ext}\mathcal{A}$ and $\mathsf{CExt}\mathcal{A}$ the full subcategories of $\mathsf{Arr}\mathcal{A}$ determined by all extensions and all central extensions, respectively. It was shown in [8] (see also [2, 13]) that the central extensions (with respect to $\mathsf{Ab}\mathcal{A}$) are precisely those extensions $f\colon A \to B$ with $[R[f], \nabla_A] = \Delta_A$. As explained in [13] (in the case of Mal'tsev varieties—but the argument remains valid), this implies in particular that the category $\mathsf{CExt}\mathcal{A}$ is reflective in $\mathsf{Ext}\mathcal{A}$ and that the centralisation of an extension $f\colon A \to B$ is given by the induced quotient $\mathsf{centr}f = A/[R[f], \nabla_A] \to B$. The centralisation functor $\mathsf{centr}\colon \mathsf{Ext}\mathcal{A} \to \mathsf{CExt}\mathcal{A}$ has the following property, which is a consequence of the fact that the commutator of equivalence relations is preserved by regular images [15]: for any double extension $f\colon A \to B$, the square in $\mathsf{Arr}\mathcal{A}$ canonically induced by the unit η^1

$$\begin{array}{ccc} A & \xrightarrow{\eta^1_A} & \mathsf{centr}A \\ {\scriptstyle f}\downarrow & & \downarrow \\ B & \xrightarrow[\eta^1_B]{} & \mathsf{centr}B \end{array} \qquad (\mathbf{C})$$

is a three-fold extension. Using the terminology of [5, 6] this means that $\mathsf{CExt}\mathcal{A}$ is a strongly $\mathcal{E}^1$-Birkhoff subcategory of $\mathsf{Ext}\mathcal{A}$, where $\mathcal{E}^1$ denotes the class of all double extensions. Applying the lemma for $n = 4$, it follows that

- the centralisation functor $\mathsf{centr}\colon \mathsf{Ext}\mathcal{A} \to \mathsf{CExt}\mathcal{A}$ preserves pullbacks of split epimorphisms of extensions along double extensions.

(To see this, keep in mind that a split epimorphism of three-fold extensions is always a four-fold extension.) Taking this into account, one is then able to prove also the following consequences of the strong $\mathcal{E}^1$-Birkhoff property of $\mathsf{CExt}\mathcal{A}$, all of which are well-known in the case of one-fold extensions [12]. Analogous to the one-dimensional case, a *double* extension $f\colon A \to B$ is **trivial** when the induced square **C** is a pullback; it is **central** if there exists a double extension $p\colon E \to B$ such that the pullback $p^*(f)\colon E \times_B A \to E$ of f along p is a trivial double extension; it is **normal** when the projections of its kernel pair $R[f]$ are trivial.

- The pullback in $\mathsf{Arr}\mathcal{A}$ of a trivial double extension along a double extension is a trivial double extension;

- the pullback in $\mathsf{Arr}\mathcal{A}$ of a double central extension along a double extension is a double central extension;

- a double central extension that is a split epimorphism in $\mathsf{Arr}\mathcal{A}$ is necessarily trivial.

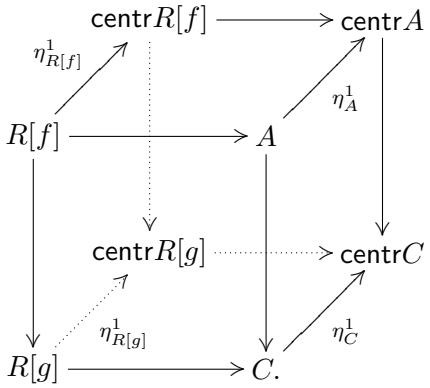
And it follows that

- the concepts of central and normal double extension coincide.

We need one last consequence of the strong $\mathcal{E}^1$-Birkhoff property of $\mathsf{CExt}\mathcal{A}$. For this, consider a three-fold extension, pictured as the right hand square in the following diagram in $\mathsf{Arr}\mathcal{A}$.

$$\begin{array}{ccccc} R[f] & \rightrightarrows & A & \xrightarrow{f} & B \\ \downarrow & & \downarrow & & \downarrow \\ R[g] & \rightrightarrows & C & \xrightarrow[g]{} & D \end{array}$$

By applying the centralisation functor $\mathsf{centr}\colon \mathsf{Ext}\mathcal{A} \to \mathsf{CExt}\mathcal{A}$ to the left hand commutative square of (say) first projections, we obtain a commutative cube in $\mathsf{Arr}\mathcal{A}$ which is a four-fold extension as a split epimorphism of three-fold extensions:



It follows from the lemma that the bottom square in this cube is a pullback as soon as the top square is a pullback, i.e., if f is a normal extension, then so is g. Since the concepts of central and normal double extension coincide, it follows that

- a quotient of a double central extension by a three-fold extension is again a double central extension.

We are now in a position to prove the characterisation of double central extensions. As mentioned before, we only need to consider one implication: for the other, we refer the reader to [9].

Let **A** be a double extension such that $[R[d], R[c]] = \Delta_X = [R[d] \cap R[c], \nabla_X]$. The first condition $[R[d], R[c]] = \Delta_X$ says that there exists a partial Mal'tsev operation $p\colon R[c] \times_X R[d] \to X$, i.e., a morphism p that satisfies the conditions $p(\alpha, \gamma, \gamma) = \alpha$ and $p(\alpha, \alpha, \gamma) = \gamma$. Recall from [4] that such a p, in a regular Mal'tsev category, necessarily satisfies the conditions $dp(\alpha, \beta, \gamma) = d(\gamma)$ and $cp(\alpha, \beta, \gamma) = c(\alpha)$. We use the notation $R[d]\square R[c]$ for the largest double equivalence relation on $R[d]$ and $R[c]$, which "consists" of all quadruples $(\alpha, \beta, \delta, \gamma)$ of "elements" of X that satisfy $c(\alpha) = c(\beta)$, $c(\delta) = c(\gamma)$, $d(\alpha) = d(\delta)$ and $d(\beta) = d(\gamma)$. Such a quadruple may be pictured as

$$\begin{pmatrix} \alpha & c & \beta \\ d & & d \\ \delta & c & \gamma \end{pmatrix}. \qquad \textbf{(D)}$$

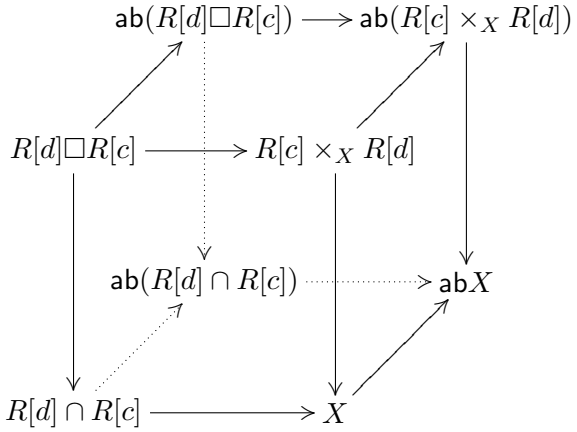
Writing

$$\pi\colon R[d]\square R[c] \to R[c] \times_X R[d]$$

for the canonical comparison map (π sends a quadruple **D** in $R[d]\square R[c]$ to the triple (α, β, γ)) and $q\colon R[d]\square R[c] \to R[d] \cap R[c]$ for the map which sends a quadruple **D** to the couple $(p(\alpha, \beta, \gamma), \delta)$ in $R[d] \cap R[c]$, we obtain the pullback of split epimorphisms

$$\begin{array}{ccc} R[d]\square R[c] & \xrightarrow{\pi} & R[c] \times_X R[d] \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle p} \\ R[d] \cap R[c] & \xrightarrow[p_1]{} & X. \end{array}$$

Applying the abelianisation functor gives us the following commutative cube, in which the slanted arrows are components of the unit η.

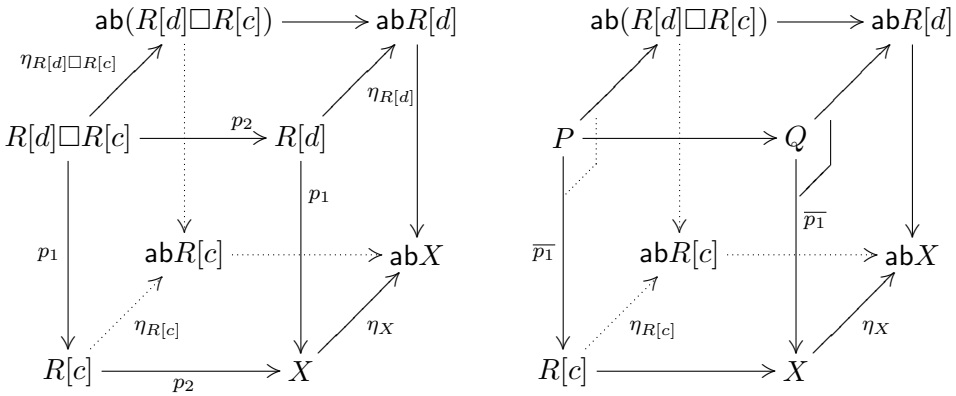


Since the reflector ab preserves pullbacks of extensions along split epimorphisms, the back square of this cube is a pullback.

The second condition $[R[d] \cap R[c], \nabla_X] = \Delta_X$ tells us that the extension $(d, c)\colon X \to D \times_X C$ is central. This is equivalent to the kernel pair projection $p_1\colon R[d] \cap R[c] \to X$ being a trivial extension, which is another way to say that the bottom square in the above cube is a pullback. Hence the two conditions together imply that so is its top square

$$\begin{array}{ccc} R[d]\square R[c] & \xrightarrow{\pi} & R[c] \times_X R[d] \\ {\scriptstyle \eta_{R[d]\square R[c]}}\downarrow & & \downarrow{\scriptstyle \eta_{R[c]\times_X R[d]}} \\ \mathsf{ab}(R[d]\square R[c]) & \xrightarrow[\mathsf{ab}\pi]{} & \mathsf{ab}(R[c] \times_X R[d]). \end{array}$$

Now consider the left hand side cube and the induced right hand side cube of pullbacks.

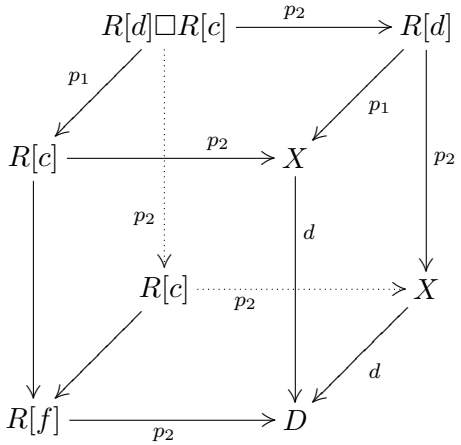


Taking into account that, since $R[c] \times_X R[d]$ is a pullback of a split epimorphism along a split epimorphism, $\mathsf{ab}(R[c] \times_X R[d]) = \mathsf{ab}R[c] \times_{\mathsf{ab}X} \mathsf{ab}R[d]$, the foregoing results imply that the left hand side cube is a limit diagram. Hence the comparison square

$$\begin{array}{ccc} R[d]\square R[c] & \longrightarrow & R[d] \\ \downarrow & & \downarrow \\ P & \longrightarrow & Q \end{array}$$

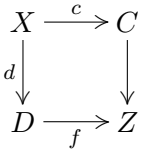
between the two cubes is a pullback, which means that the front square (considered as a horizontal arrow) of the left hand side cube is a trivial double extension. (The vertical arrows p_1 in this double extension are split epimorphisms, so their centralisation is their trivialisation—the two arrows $\overline{p_1}$ on the right hand side.) A fortiori,

it is a double central extension. Now consider the commutative cube below. Considered as a horizontal arrow, it is a split epimorphism between pullbacks of regular epimorphisms; consequently it is a three-fold extension.



We have just seen that this cube's top square, considered as a horizontal arrow, is a double central extension. It follows that the bottom square, also considered as a horizontal arrow, is a double central extension as well, being a quotient of a double extension along a three-fold extension. But this bottom square is one of the projections of the kernel pair of the double extension **A**, so that also **A** is central, and we obtain:

Theorem. *In a Barr exact Mal'tsev category with finite colimits, a double extension*



is central if and only if $[R[d], R[c]] = \Delta_X = [R[d] \cap R[c], \nabla_X]$. □

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