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A LAX SYMMETRIC CUBICAL CATEGORY ASSOCIATED TO A DIRECTED SPACE*

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Résumé. Le domaine récent de la topologie algébrique dirigée étudie les "espaces dirigés", où chemins et homotopies peuvent être non réversibles. Les applications principales concernent la programmation parallèle.

On introduit ici, pour un espace dirigé, une catégorie fondamentale de *dimension infinie*, de type cubique *lax*: les cubes singuliers de l'espace ont une structure cubique, où les concatenations sont associatives à une re-paramétrisation invertible près, mais les dégénérescences sont seulement lax-unitaires. En outre cette structure est *symétrique*, par permutation des variables des cubes singuliers, ce qui simplifie les propriétés de cohérence.

Les "cubes de Moore" de l'espace donnent une catégorie cubique *stricte*, moyennant une construction similaire.

Abstract. The recent domain of directed algebraic topology studies 'directed spaces', where paths and homotopies need not be reversible. The main applications are concerned with concurrency.

We introduce here, for a directed space, an *infinite dimensional* fundamental category, of a *lax* cubical type: the singular cubes of the space have a cubical structure, where concatenations are associative up to invertible reparametrisation while degeneracies are only lax-unital. Moreover, this structure is *symmetric*, by permuting the variables of singular cubes; this simplifies the coherence properties.

By a similar construction, the 'Moore cubes' of the space give a *strict* symmetric cubical category.

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Introduction

Directed algebraic topology studies structures with *privileged directions*, like 'directed spaces' in some sense: for instance, ordered or locally ordered topological spaces, 'spaces with distinguished paths' (examined below), simplicial and cubical sets, etc. Such objects have *directed* paths and homotopies, which need not be reversible. The present applications deal mostly with the analysis of concurrent processes, see [FGR1, FGR2, FRGH, Ga1, Ga2, GG, GH, Go, R1, R2], but the theory aims to model non-reversible phenomena in any domain. Directed algebraic topology is the subject of a recent issue of the journal 'Applied Categorical Structures', guest-edited by the present author (vol. 15, no. 4, 2007), and of a recent book [G10]. The ideas at the basis of the present paper have been exposed at the conference 'Applied Topological Methods in Computer Sciences III', Paris 2008.

Directed spaces can be studied with homology and homotopy theories, modified to keep an account of privileged directions: namely, *preordered* homology groups [G3] and fundamental higher *categories* (in some sense) instead of the classical homology groups and fundamental higher *groupoids* of algebraic topology. Thus, directed algebraic topology is more clearly linked with higher dimensional category theory, and can also yield some geometric intuition to the latter.

Here, we want to study an infinite dimensional version of the fundamental category of a *d-space*, or *space with distinguished paths* (1.1), our main notion of directed space, which was introduced in [G2] and also studied in various works by various authors [G4-G6, G10, FhR, FjR, R2, Bu, Ga3]. While there is no problem in defining the *fundamental category* $\uparrow\Pi_1(X)$ of a d-space [G2], the construction of higher versions is complicated, even in dimension 2: see [G4] for a strict 2-categorical version and [G5, G6] for lax ones.

The present approach is cubical, rather than globular, and follows a study of weak cubical categories begun by the author in [G7-G9, G11]. We start from the standard n-dimensional cube $\mathbf{I}^n = [0, 1]^n$ and its *directed* version $\uparrow\mathbf{I}^n$ (1.2). The singular cubes of a d-space X, i.e. the maps $\uparrow\mathbf{I}^n \to X$ of d-spaces, form a 'basic symmetric pre-cubical category' $\uparrow\Box X$ (Section 1), i.e. a symmetric cubical set equipped with concatenation laws in all directions, satisfying various geometrical properties and linked by transposition symmetries; the term 'basic' means that these operations are not (yet) required to satisfy associativity, interchange and unitarity, in any sense - even weak or lax.

To take these aspects into account, we formalise in Section 2 the notion of a *u-lax symmetric cubical category*. The previous framework (a basic symmetric pre-cubical category) is enriched with *transversal maps* between n-dimensional objects; these maps include *comparisons* for associativity, interchange and unitarity, which are only assumed to be invertible in the first two cases. The new structure is thus a generalisation of a *weak symmetric cubical category* introduced in [G7, G9]; it is very similar to the 'quasi cubical' case considered in [G8] for higher cospans composed with homotopy pushouts, in relation with higher dimensional cobordism.

Then, in Sections 3 and 4, we make the previous structure $\uparrow\square X$ into the singular u-lax symmetric cubical category $\uparrow\mathbb{S}\mathrm{ng}(X)$ of the d-space X, by adding transversal maps and comparisons. Here, a transversal map $f\colon x \to y$ between two singular cubes $x, y\colon \uparrow\mathbf{I}^n \to X$ is given by a *reparametrisation mapping* $f\colon \uparrow\mathbf{I}^n \to \uparrow\mathbf{I}^n$ such that $x = yf$; the obvious *transversal composition* of such maps is strictly categorical. The operations of concatenation of the singular cubes become thus *weakly associative* (up to invertible reparametrisations) and *lax unital* (up to non-invertible reparametrisations), while interchange - here - works strictly. The non-directed structure $\mathbb{S}\mathrm{ng}(X)$ associated to a topological space X is briefly described in 4.6.

In Section 5 we outline a *strict* version of the previous framework. It is based on the *Moore* directed cubes of a d-space, defined on products of directed intervals of variable length $a_h \geq 0$

(1) $\quad I(a_1,..., a_n) \; = \; \Pi_{h=1,...,n} \uparrow[0, a_h].$

These have operations of concatenation that are *strictly associative and unital*, also because we allow these intervals to be degenerate. Transversal maps are given by 'Moore reparametrisations', but their role is less evident here, since no comparisons are needed: we get a strict symmetric cubical category $\uparrow\mathbb{M}\mathrm{Sng}(X)$.

We end, in Section 6, with a few hints to a family $\mathbb{T}(A)$ of u-lax symmetric cubical categories, depending on a topological space A, and related to higher categories of tangles, as considered in [BL, Ch]. This family is constructed starting from $\mathbb{T} = \mathbb{S}\mathrm{ng}(\mathbf{S}^0)$, the u-lax symmetric cubical category associated to the discrete space on two points $\mathbf{S}^0$, where a singular n-cube $x: \mathbf{I}^n \to \mathbf{S}^0$ can be identified to a subset of $\mathbf{I}^n$.

As a general principle of higher category theory, weak structures seem to be more important than the strict ones; this is why the strict structure of Moore cubes has here a marginal position. Let us also recall that interest is arising in category theory and algebraic topology for categorical structures (possibly higher dimensional) with lax units or 'no units' (see [MBB, Ko1, Ko2, JK, G8, G12]).

References to the rich literature on higher categories can be found in two recent books, by T. Leinster [Le] and E. Cheng, A. Lauda [CL]; but these works are mostly developed in the globular approach, rather than the cubical one. Strict cubical categories with connections (and no transversal maps) are studied in [ABS], and proved to be equivalent to the ordinary (globular) ω-categories. Weak symmetric cubical categories have been studied by the present author [G7-G9, G11]; pseudo double categories are a truncated version of the latter, studied in [GP] and three subsequent papers by the same authors.

Cubical sets have been extensively studied by R. Brown and P.J. Higgins, which introduced their connections in [BH1, BH2]. The present author began a systematic use of their symmetries in [G1]. There is a recent preprint on symmetric cubical sets, by S.B. Isaacson [Is], which investigates their non-directed homotopy theory.

For a (non-directed) topological space X, a recent preprint by R. Brown [Br] deals with a cubical structure $M_*(X)$ based on 'Moore hyperrectangles', equivalent to our Moore cubes (see a note at the end of Section 5.3). With respect to the present structure $\uparrow\mathbb{M}\mathrm{Sng}(X)$, the 'strict cubical category' $M_*(X)$ has *connections* and no *transpositions* nor *transversal maps*; it might be called a 'basic cubical category with connections', in the present terminology - where a 'cubical category' is always assumed to have transversal maps.

One dimensional reparametrisation mappings $f: \uparrow\mathbf{I} \to \uparrow\mathbf{I}$, in the same sense as here, have been studied in [G5, G6, FhR, R3].

As a matter of notation, the indices α, β take the values $0, 1$, that are more often written as $-, +$. $\mathbf{I}$ denotes the standard interval $[0, 1]$ with euclidean topology.

1. The singular cubes of a d-space and their concatenations

We briefly recall the notion of d-space, introduced in [G2]. Then we show that the singular (directed) cubes of a d-space X, with obvious concatenations in all directions, form a 'basic symmetric pre-cubical category' $\uparrow\Box X$. These operations satisfy a strict middle-four interchange; their weak associativity and lax unitarity properties will be studied in Section 4, after developing adequate structures.

1.1. Spaces with distinguished paths. A *d-space* X, or *space with distinguished paths*, is a topological space equipped with a set dX of (continuous) maps $a: \mathbf{I} \to X$, called *distinguished paths* or *directed paths* or *d-paths*, satisfying three axioms:

(i) (*constant paths*) every constant map $\mathbf{I} \to X$ is distinguished,

(ii) (*partial reparametrisation*) dX is closed under composition with every (weakly) increasing map $\mathbf{I} \to \mathbf{I}$,

(iii) (*concatenation*) dX is closed under path-concatenation: if the d-paths a, b are consecutive in X (i.e. $a(1) = b(0)$), then their ordinary concatenation $a + b$ is also a d-path.

A *directed map* $f: X \to Y$ (or *d-map*, or *map* of d-spaces) is a continuous mapping between d-spaces which preserves the directed paths: if $a \in dX$, then $fa \in dY$.

The category of d-spaces is written as d**Top**. It has all limits and colimits, constructed as in **Top** and equipped with the initial or final d-structure for the structural maps; for instance a path $\mathbf{I} \to \Pi X_j$ with values in a product is directed if and only if all its components $\mathbf{I} \to X_j$ are so. The forgetful functor U: d**Top** $\to$ **Top** preserves thus all limits and colimits; a topological space is generally viewed as a d-space by its *natural* structure, where all paths are directed (via the right adjoint to U).

Reversing d-paths, by the involution $r(t) = 1 - t$, yields the *opposite* d-space $RX = X^{op}$, where $a \in d(X^{op})$ if and only if ar is in dX. This defines the *reversor* endofunctor

(1) R: d**Top** $\to$ d**Top**, $\qquad\qquad RX = X^{op}$.

A d-space X is said to be *reversible* if it coincides with X^{op}, and *reflexive* if it is isomorphic to the latter.

1.2. Standard objects. The *directed real line*, or *d-line* $\uparrow\mathbf{R}$, is the euclidean line with directed paths given by the (weakly) increasing maps $\mathbf{I} \to \mathbf{R}$. Its cartesian power in d**Top**, the n-*dimensional real d-space* $\uparrow\mathbf{R}^n$ is similarly described (with respect to the product order of $\mathbf{R}^n$, $x \leq y$ if $x_i \leq y_i$ for all i). The *standard d-interval* $\uparrow\mathbf{I} = \uparrow[0, 1]$ has the subspace structure of the d-line; the *standard d-cube* $\uparrow\mathbf{I}^n$ is its n-th power, and a subspace of $\uparrow\mathbf{R}^n$ (with the induced structure). These d-spaces are not reversible (for $n > 0$), but they are all reflexive.

The *standard directed circle* $\uparrow\mathbf{S}^1$ will be the standard circle with the *anticlockwise structure*, where the directed paths $a: \mathbf{I} \to \mathbf{S}^1$ move this way, in the oriented plane $\mathbf{R}^2$: $a(t) = (\cos\vartheta(t), \sin\vartheta(t))$, with an increasing (continuous) argument $\vartheta: \mathbf{I} \to \mathbf{R}$.

$\uparrow\mathbf{S}^1$ can be obtained as the coequaliser in d**Top** of the following pair of maps

(1) $\partial^-, \partial^+: \{*\} \rightrightarrows \uparrow\mathbf{I}, \qquad\qquad \partial^-(*) = 0, \quad \partial^+(*) = 1.$

Indeed, the ordinary construction of this coequaliser is the quotient $\uparrow\mathbf{I}/\partial\mathbf{I}$, which identifies the endpoints; the d-structure of the quotient (*generated* by the projected paths) is the desired one precisely because of the axioms on concatenation and reparametrisation of d-paths.

The directed circle can also be described as an orbit d-space

(2) $\uparrow\mathbf{S}^1 = \uparrow\mathbf{R}/\mathbf{Z},$

with respect to the action of the group of integers on the directed line $\uparrow\mathbf{R}$, by translations; in this quotient, the distinguished paths of $\uparrow\mathbf{S}^1$ are simply the projections of the increasing paths in the line.

The *directed n-dimensional sphere* is defined, for $n > 0$, as the quotient of the directed cube $\uparrow\mathbf{I}^n$ modulo the equivalence relation which collapses its (ordinary) boundary $\partial\mathbf{I}^n$ to a single point, while $\uparrow\mathbf{S}^0$ has the discrete topology and the natural d-structure (obviously discrete)

(3) $\uparrow\mathbf{S}^n = (\uparrow\mathbf{I}^n)/(\partial\mathbf{I}^n) \quad (n > 0), \qquad\qquad \uparrow\mathbf{S}^0 = \mathbf{S}^0 = \{-1, 1\}.$

All directed spheres are reflexive.

1.3. Directed interval and paths. A (standard) *path* in a d-space X is a d-map $a\colon \uparrow\mathbf{I} \to X$ defined on the standard d-interval. Plainly, this is the same as a structural map $a \in dX$, and will *also* be called a *directed* path when we want to stress the difference from ordinary paths in the underlying space UX.

The basic, 'first order' structure of $\uparrow\mathbf{I}$ consists of four maps, linking its 0-th cartesian power, the singleton $\uparrow\mathbf{I}^0 = \{*\}$, to $\uparrow\mathbf{I}$ or to the opposite d-space $\uparrow\mathbf{I}^{op}$

(1) $\partial^\alpha : \{*\} \rightrightarrows \uparrow\mathbf{I}, \qquad \partial^-(*) = 0, \quad \partial^+(*) = 1$ (*faces*),

$e\colon \uparrow\mathbf{I} \to \{*\}, \qquad e(t) = *$ (*degeneracy*),

$r\colon \uparrow\mathbf{I} \to \uparrow\mathbf{I}^{op}, \qquad r(t) = 1 - t$ (*reflection*).

Identifying a point x of the space X with the corresponding map $x\colon \{*\} \to X$, this basic structure determines:

(a) the endpoints of a path $a\colon \uparrow\mathbf{I} \to X$, i.e. $\partial^-(a) = a\partial^- = a(0)$, $\partial^+(a) = a\partial^+ = a(1)$,

(b) the trivial path at the point x, i.e. $0_x = e(x) = xe$,

(c) the *reflected path of* a *in* X^{op}, i.e. $r(a) = (Ra).r\colon \uparrow\mathbf{I} \to \uparrow\mathbf{I}^{op} \to X^{op}$.

Two consecutive paths $a, b\colon \uparrow\mathbf{I} \to X$ ($\partial^+(a) = \partial^-(b)$, i.e. $a(1) = b(0)$) have a concatenated path $a + b$, which is distinguished, by definition of d-structure. This amounts to saying that, in d**Top**, the *standard concatenation pushout* – pasting two copies of the d-interval, one after the other – can be realised as $\uparrow\mathbf{I}$ itself (as for spaces: pasting two copies of $\mathbf{I}$ gives $\mathbf{I}$)

(2)
$$\begin{array}{ccc} \{*\} & \xrightarrow{\partial^+} & \uparrow\mathbf{I} \\ {\scriptstyle\partial^-}\downarrow & & \downarrow{\scriptstyle c^-} \\ \uparrow\mathbf{I} & \xrightarrow[c^+]{} & \uparrow\mathbf{I} \end{array} \qquad\qquad c^-(t) = t/2, \qquad c^+(t) = (t+1)/2.$$

This pushout is preserved by cartesian product with any fixed d-space ([G2], Lemma 1.8).

Finally, there is a 'second order' structure which involves the standard directed square $\uparrow\mathbf{I}^2 = [0, 1] \times [0, 1]$ and is used to construct (directed) homotopies of (directed) paths

(3) $g^-\colon \uparrow\mathbf{I}^2 \to \uparrow\mathbf{I}$, $\quad g^-(t, t') = \max(t, t')$ (*lower connection*),

$g^+\colon \uparrow\mathbf{I}^2 \to \uparrow\mathbf{I}$, $\quad g^+(t, t') = \min(t, t')$ (*upper connection*),

$s\colon \uparrow\mathbf{I}^2 \to \uparrow\mathbf{I}^2$, $\quad s(t, t') = (t', t)$ (*transposition*).

Together with (1), these maps complete the structure of $\uparrow\mathbf{I}$ as a *lattice* in **Top** (isomorphic to the opposite lattice, via r). The choice of the superscripts of g^-, g^+ comes from the fact that the unit of g^α is $\partial^\alpha(*)$. Within homotopy theory, the importance of these binary operations has been highlighted by R. Brown and P.J. Higgins [BH1, BH2], which introduced the term of *connection*, or *higher degeneracy* (with a notation similar to the previous one for faces, degeneracy and connections: ∂^α, ε, Γ^α; notice that for simplicial sets the letter s generally denotes degeneracies).

Here, we will use the transposition symmetry s, but not the connections; the article [Br] shows as the latter can be used in the context of Moore cubes (or standard cubes, of course).

1.4. The singular symmetric cubical set of a d-space. Every d-space X has an associated *symmetric cubical set* - a notion whose general definition will be recalled below (see 1.5)

(1) $\uparrow\square X = ((\uparrow\square_n X), (\partial_i^\alpha), (e_i), (s_i))$.

Firstly, the component of $\uparrow\square X$ in degree $n \geq 0$ is the set of *singular (directed) n-cubes* of X, which will also be called *n-cubes* of X

(2) $\uparrow\square_n X = d\mathbf{Top}(\uparrow\mathbf{I}^n, X)$.

In particular, a 0-cube $x\colon \uparrow\mathbf{I}^0 \to X$ is identified with a point of X, and a 1-cube $x\colon \uparrow\mathbf{I} \to X$ is a (directed) path.

Secondly, after the basic structure recalled above, the *higher faces*, *degeneracies* and *transpositions* of the standard cubes are defined as follows

(3) $\partial_i^\alpha = \uparrow\mathbf{I}^{i-1} \times \partial^\alpha \times \uparrow\mathbf{I}^{n-i}\colon \uparrow\mathbf{I}^{n-1} \to \uparrow\mathbf{I}^n, \qquad \partial_i^\alpha(t_1,..., t_{n-1}) = (t_1,..., \alpha,..., t_{n-i}),$

$e_i = \uparrow\mathbf{I}^{i-1} \times e \times \uparrow\mathbf{I}^{n-i}\colon \uparrow\mathbf{I}^n \to \uparrow\mathbf{I}^{n-1}, \qquad e_i(t_1,..., t_n) = (t_1,..., \hat{t}_i,..., t_n),$

$s_i = \uparrow\mathbf{I}^{i-1} \times s \times \uparrow\mathbf{I}^{n-i}\colon \uparrow\mathbf{I}^{n+1} \to \uparrow\mathbf{I}^{n+1}, \qquad s_i(t_1,..., t_{n+1}) = (t_1,..., t_{i+1}, t_i,..., t_{n+1}),$

where $\alpha = 0, 1$, $i = 1,..., n$ (and, as usual, $\hat{t}_i$ means to omit the coordinate t_i).

These maps produce (contravariantly, by pre-composition) the *faces*, *degeneracies* and *transpositions* of our symmetric cubical set $\uparrow\square X$, which will be denoted by the *same* symbols

(4) $\partial_i^\alpha\colon \uparrow\square_n X \to \uparrow\square_{n-1}X, \qquad \partial_i^\alpha(x) = x.\partial_i^\alpha,$

$e_i\colon \uparrow\square_{n-1}X \to \uparrow\square_n X, \qquad e_i(x) = x.e_i,$

$s_i\colon \uparrow\square_{n+1}X \to \uparrow\square_{n+1}X \qquad s_i(x) = x.s_i \qquad (\alpha = 0, 1,\ i = 1,..., n).$

Every n-cube $x\colon \uparrow\mathbf{I}^n \to X$ has 2^n vertices: $\partial_1^\alpha\partial_2^\beta\partial_3^\gamma(x) = \partial_1^\gamma\partial_1^\beta\partial_1^\alpha(x)$, for $n = 3$.

The contravariant action of the transpositions $s_1,..., s_{n-1}$ on $\uparrow\square_n X$ can obviously be extended to a (*right*) action of the group of permutations of the coordinates of $\mathbf{I}^n$. This amounts to saying that the transpositions s_i satisfy the *Moore relations*, under which they generate the symmetric group S_n

(5) $s_i.s_i = 1, \qquad s_i.s_j.s_i = s_j.s_i.s_j \quad (i = j-1), \qquad s_i.s_j = s_j.s_i \quad (i < j-1),$

(see Coxeter-Moser [CM], 6.2; or Johnson [Jo], Section 5, Thm. 3).

Notice also that we have applied the functors

(6) $(-)_i^n = \uparrow\mathbf{I}^{i-1} \times - \times \uparrow\mathbf{I}^{n-i}\colon \mathrm{d}\mathbf{Top} \to \mathrm{d}\mathbf{Top} \qquad (i = 1,..., n),$

to deduce the higher structural maps (3) from the basic ones, ∂^α, e, s, introduced in 1.3. This procedure is usual in homotopy theory based on a standard interval, and will be repeatedly used below.

1.5. Symmetric cubical sets. Let us recall some points on the classical notion of cubical set (see [K1, K2, BH1, BH2]) and the less known notion of *symmetric* cubical set.

A cubical set $X = ((X_n), (\partial_i^\alpha), (e_i))$ is a sequence of sets $(X_n)_{n\geq 0}$ equipped with *faces* (∂_i^α) and *degeneracies* (e_i)

(1) $\partial_i^\alpha : X_n \rightleftarrows X_{n-1} : e_i$ (i = 1,..., n; $\alpha = \pm$),

satisfying the cubical relations :

(2) $\partial_i^\alpha.\partial_j^\beta = \partial_j^\beta.\partial_{i+1}^\alpha$ (j ≤ i), $e_j.e_i = e_{i+1}.e_j$ (j ≤ i),

$\partial_i^\alpha.e_j = e_j.\partial_{i-1}^\alpha$ (j < i), *or* id (j = i), *or* $e_{j-1}.\partial_i^\alpha$ (j > i).

A *morphism* $f = (f_n): X \to Y$ is a sequence of mappings $f_n: X_n \to Y_n$ commuting with faces and degeneracies. All this forms a category **Cub**, which is a category of presheaves: a cubical set can be viewed as a functor $X: \mathbb{I}^{op} \to$ **Set**, where $\mathbb{I}$ is the subcategory of **Set** consisting of the *elementary cubes* $2^n = \{0, 1\}^n$, together with the maps $\{0, 1\}^m \to \{0, 1\}^n$ which delete some coordinates and insert some 0's and 1's, without modifying the order of the remaining coordinates [GM]. Therefore, **Cub** has all limits and colimits and is cartesian closed. However, the important monoidal structure is the Kan tensor product, which is non-symmetric and biclosed [BH2] (but this is not used here).

A *symmetric cubical set* [GM, G7] is a cubical set which is further equipped with *transpositions*

(3) $s_i: X_n \to X_n$ (i = 1,..., n–1; n ≥ 2).

which satisfy the Moore relations (1.4.5) and the following coherence conditions:

		j < i	j = i	j = i+1	j > i+1
(4) $\partial_j^\alpha.s_i$	=	$s_{i-1}.\partial_j^\alpha$	∂_{i+1}^α	∂_i^α	$s_i.\partial_j^\alpha$,
$s_i.e_j$	=	$e_j.s_{i-1}$	e_{i+1}	e_i	$e_j.s_i$.

Because of the Moore relations, the symmetric group S_n operates on X_n.

A *morphism of symmetric cubical sets* $f = (f_n): X \to Y$ is a sequence of mappings $f_n: X_n \to Y_n$ commuting with faces, degeneracies and transpositions. The resulting category s**Cub** is again a category of presheaves $X: \mathbb{I}_s{}^{op} \to$ **Set**, for the *symmetric cubical site* $\mathbb{I}_s$. The latter can be defined as the subcategory of **Set** consisting of the elementary cubes $2^n = \{0, 1\}^n$ together with the maps $2^m \to 2^n$ which delete some coordinates, permute the remaining ones and insert some 0's and 1's. It is a subcategory of the *extended cubical site* $\mathbb{K}$ of [GM], which also contains the connections.

Again, s**Cub** has all limits and colimits and is cartesian closed; moreover, it inherits from **Cub** a *symmetric* monoidal closed structure [G9] and *one* internal hom (that is not used here).

1.6. An equivalent presentation. The presence of transpositions makes all faces and degeneracies determined by those belonging to a fixed direction, e.g. the 1-indexed ones, ∂_1^α and e_1. In fact, from $\partial_{i+1}^\alpha = \partial_i^\alpha.s_i$ and $e_{i+1} = s_i.e_i$, we deduce that:

(1) $\partial_{i+1}^\alpha \ = \ \partial_1^\alpha.s_1. \ldots .s_i, \qquad\qquad e_{i+1} \ = \ s_i. \ldots .s_1.e_1.$

Thus, as proved in [G8],1.2, a symmetric cubical set can be equivalently defined as a system

(2) $X \ = \ ((X_n), (\partial_1^\alpha), (e_1), (s_i)),$

$\partial_1^\alpha\colon X_n \rightleftarrows X_{n-1} :e_1, \qquad\qquad s_i\colon X_{n+1} \to X_{n+1} \qquad\qquad (n \geq 1),$

under the Moore relations for transpositions (1.4.5) and the axioms:

(3) $\partial_1^\alpha.\partial_1^\beta \ = \ \partial_1^\beta.\partial_1^\alpha.s_1, \qquad s_i.\partial_1^\alpha \ = \ \partial_1^\alpha.s_{i+1}, \qquad \partial_1^\alpha.e_1 \ = \ id,$

$e_1e_1 \ = \ s_1.e_1e_1, \qquad\qquad e_1.s_i \ = \ s_{i+1}.e_1.$

1.7. A basic symmetric pre-cubical category. The symmetric cubical set $\uparrow\Box X$ can be further equipped with partial operations of *concatenation in direction* i, or *i-concatenation*, or *i-composition* (with $i = 1,..., n$ for n-dimensional cubes); globally, we will speak of *cubical compositions* (as opposed to the transversal composition that will be introduced later).

Indeed, acting on the concatenation pushout (1.3.2), the functors $(-)_i^n$ (1.4.6) produce the *n-dimensional i-concatenation pushout*, with embeddings $c_i^\alpha\colon \uparrow\mathbf{I}^n \to \uparrow\mathbf{I}^n$

(1)
$$\begin{array}{ccc} \uparrow\mathbf{I}^{n-1} & \xrightarrow{\ \partial_i^+\ } & \uparrow\mathbf{I}^n \\ {\scriptstyle \partial_i^-}\downarrow & & \downarrow{\scriptstyle c_i^-} \\ \uparrow\mathbf{I}^n & \xrightarrow[\ c_i^+\ ]{} & \uparrow\mathbf{I}^n \end{array} \qquad \begin{array}{l} c_i^\alpha \ = \ \uparrow\mathbf{I}^{i-1} \times c^\alpha \times \uparrow\mathbf{I}^{n-i}\colon \ \uparrow\mathbf{I}^n \to \ \uparrow\mathbf{I}^n, \\ c^-(..., t_i,...) \ = \ (..., t_i/2,...), \\ c^+(..., t_i,...) \ = \ (..., (t_i + 1)/2,...). \end{array}$$

(We have already recalled that the basic concatenation pushout is preserved by products with fixed d-spaces.) Now, given two *i-consecutive* n-cubes $x, y\colon \uparrow\mathbf{I}^n \to X$ (with $\partial_i^+ x = \partial_i^- y$), their *i-concatenation* $z = x +_i y$ is computed on the previous pushout

(2) $z\colon \uparrow\mathbf{I}^n \to \mathbf{X}, \qquad z.c_i^- = x, \quad z.c_i^+ = y.$

$\uparrow\square X$ becomes thus a *basic symmetric pre-cubical category*, i.e. a symmetric cubical set with 'geometrically consistent' cubical compositions. (This structure was called a 'reduced symmetric pre-cubical category' in [G9], Section 3.5.) More precisely, this means that $\uparrow\square X$ is a symmetric cubical set with the following additional structure.

For $1 \leq i \leq n$, the i-*concatenation* $x +_i y$ (or i-composition) of two n-cubes x, y is defined when x, y are *i-consecutive*, i.e. $\partial_i^+(x) = \partial_i^-(y)$, and satisfies the following 'geometric' relations with faces, degeneracies and transpositions:

(3) $\partial_i^-(x +_i y) = \partial_i^-(x), \qquad \partial_i^+(x +_i y) = \partial_i^+(y),$

$\partial_j^\alpha(x +_i y) = \partial_j^\alpha(x) +_{i-1} \partial_j^\alpha(y) \qquad (j < i),$

$\qquad = \partial_j^\alpha(x) +_i \partial_j^\alpha(y) \qquad (j > i),$

(4) $e_j(x +_i y) = e_j(x) +_{i+1} e_j(y) \qquad (j \leq i \leq n),$

$\qquad = e_j(x) +_i e_j(y) \qquad (i < j \leq n+1)$ (*nullary interchange*).

(5) $s_{i-1}(x +_i y) = s_{i-1}(x) +_{i-1} s_{i-1}(y), \qquad s_i(x +_i y) = s_i(x) +_{i+1} s_i(y),$

$s_j(x +_i y) = s_j(x) +_i s_j(y) \qquad (j \neq i-1, i).$

There are no other conditions: in the definition of a basic symmetric *pre*-cubical category we are not assuming that the i-compositions behave in a categorical way or satisfy the binary interchange law, in any sense - strict or weak or lax.

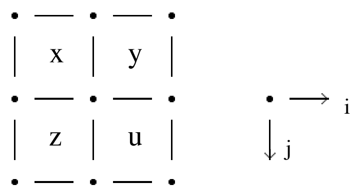
However, for the singular structure $\uparrow\square X$ which we are studying, the *binary interchange law* holds strictly. Indeed, for $1 \leq i < j \leq n$, and n-cubes x, y, z, u, we obviously have

(6) $(x +_i y) +_j (z +_i u) = (x +_j z) +_i (y +_j u)$ (*middle-four interchange*),

whenever these compositions make sense:

(7) $\partial_i^+(x) = \partial_i^-(y), \quad \partial_i^+(z) = \partial_i^-(u),$

$\partial_j^+(x) = \partial_j^-(z), \quad \partial_j^+(y) = \partial_j^-(u),$



Comparisons for associativity and unitarity of singular cubes will be introduced in Section 4.

2. Weak symmetric cubical categories with lax units

We now define a notion of cubical structure adapted to the present situation, and called a *u-lax symmetric cubical category*. It is a generalisation of the weak case introduced in [G7, G9] and is similar to the 'quasi cubical' case considered in [G8] for higher cospans composed with homotopy pushouts (the latter is even more relaxed, with weaker cubical relations for degeneracies).

Here we allow the comparisons for left and right unitarity to be *non-invertible* and *directed towards simpler expressions*, while we require that the comparisons for associativity and interchange be invertible; indeed, this is the situation that we find in our leading examples (like singular cubes, here, or cubical cospans in [G8]). One should also notice that - for associativity and interchange - there seems to be no formal reason that might distinguish a particular direction, while - for unitarity - a rewriting rule would normally point towards simplification.

2.1. Symmetric pre-cubical categories. As a first step, let us recall that a *symmetric pre-cubical category* is a *category object* $\mathbb{A}$ within the category of *basic* symmetric pre-cubical categories and their morphisms (1.7)

$$\text{(1)} \qquad \mathbb{A}^{(0)} \underset{e_0}{\overset{\partial_0^\alpha}{\leftleftarrows\!\!\rightarrow}} \mathbb{A}^{(1)} \overset{c_0}{\longleftarrow} \mathbb{A}^{(2)} \qquad\qquad (\alpha = \pm).$$

Explicitly, this means the following data and axioms.

(wcub.1) A basic symmetric pre-cubical category $\mathbb{A}^{(0)} = ((A_n), (\partial_i^\alpha), (e_i), (s_i), (+_i))$, whose entries are called *n-cubes*, or *n-dimensional objects* of $\mathbb{A}$.

(wcub.2) A basic symmetric pre-cubical category $\mathbb{A}^{(1)} = ((M_n), (\partial_i^\alpha), (e_i), (s_i), (+_i))$, whose entries are called *n-maps* of $\mathbb{A}$, or also *(n+1)-cells*.

(wcub.3) Symmetric cubical functors ∂_0^α and e_0, called 0-*faces* and 0-*degeneracy*, with $\partial_0^\alpha.e_0 = \mathrm{id}$.

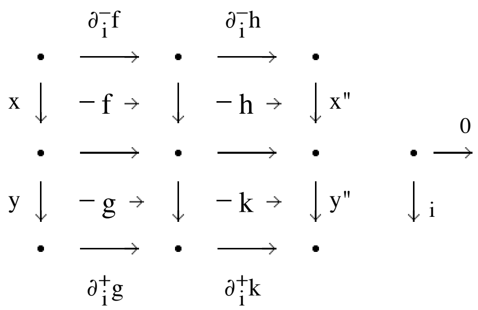
Typically, an n-map will be written as $f: x \to x'$, where $\partial_0^- f = x$, $\partial_0^+ f = x'$ are n-cubes. Every n-dimensional object x has an *identity* $e_0(x): x \to x$. Note that ∂_0^α and e_0 preserve cubical faces (∂_i^α, with $i > 0$), cubical degeneracies (e_i), transpositions (s_i) and cubical concatenations ($+_i$). In particular, given two i-consecutive n-maps f, g, their 0-faces are also i-consecutive and we have:

(2) $f +_i g: x +_i y \to x' +_i y'$ (for $f: x \to x'$, $g: y \to y'$; $\partial_i^+ f = \partial_i^- g$).

(wcub.4) A composition law c_0 which assigns to 0-consecutive n-maps $f: x \to x'$, $h: x' \to x''$ (of the same dimension), an n-map $hf: x \to x''$ (also written $h.f$). This composition law is (strictly) categorical, and forms a category $\mathbb{A}_n = (A_n, M_n, \partial_0^\alpha, e_0, c_0)$. It is also consistent with the basic symmetric pre-cubical structure, in the following sense

(3) $\partial_i^\alpha(hf) = (\partial_i^\alpha h).(\partial_i^\alpha f)$, $e_i(hf) = (e_i h)(e_i f)$, $s_i(hf) = (s_i h)(s_i f)$,

$(h +_i k).(f +_i g) = hf +_i kg$



The last condition is the (strict) middle-four interchange *between the strict composition* c_0 *and any weak one*. An n-map $f: x \to x'$ is said to be *special* if its 2^n vertices are identities

(4) $\partial^{\boldsymbol\alpha} f: \partial^{\boldsymbol\alpha} x \to \partial^{\boldsymbol\alpha} x'$ $\partial^{\boldsymbol\alpha} = \partial_1^{\alpha_1} \partial_2^{\alpha_2} \dots \partial_n^{\alpha_n}$ ($\alpha_i = \pm$).

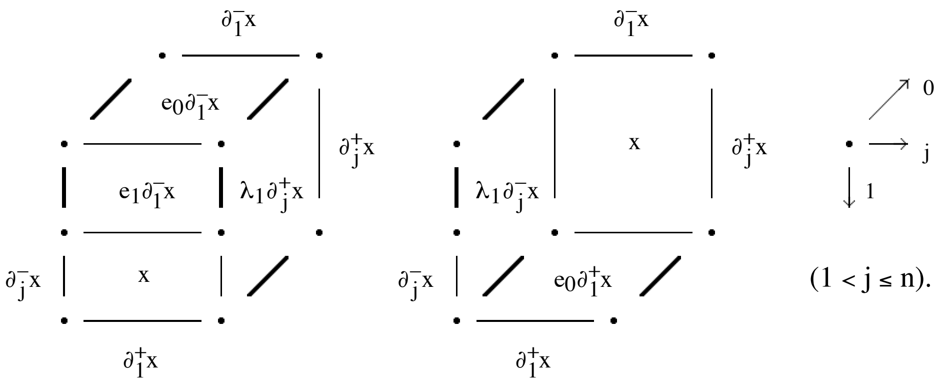
In degree 0, this just means an identity.

2.2. Comparisons. We now define a *u-lax symmetric cubical category* $\mathbb{A}$ as a symmetric pre-cubical category (2.1), which is further equipped with some *special* transversal maps, playing the role of comparisons for units, associativity and cubical interchange, as follows. (We only assign the comparisons in direction 1; all the others can be obtained with transpositions.)

(ucub.5) For every n-cube x, we have a special n-map $\lambda_1 x$, which is natural on n-maps and has the following faces (for $n > 0$)

(1) $\lambda_1 x \colon (e_1\partial_1^- x) +_1 x \ \to \ x$ (*left-unit 1-comparison*),

$\partial_1^\alpha \lambda_1 x \ = \ e_0\partial_1^\alpha x, \qquad \partial_j^\alpha \lambda_1 x \ = \ \lambda_1\partial_j^\alpha x \qquad (1 < j \leq n),$



$(1 < j \leq n).$

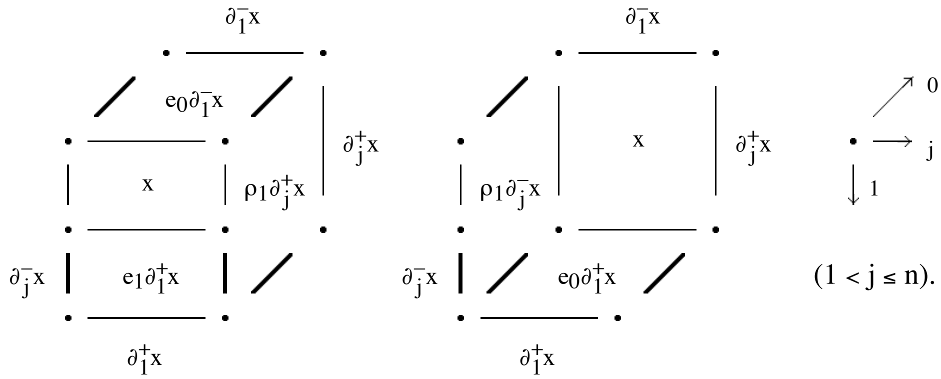
The naturality condition means that, for every n-map $f \colon x \to x'$, the following square of n-maps commutes

$$(2) \qquad \begin{array}{ccc} (e_1\partial_1^- x) +_1 x & \xrightarrow{\lambda_1 x} & x \\ {\scriptstyle (e_1\partial_1^- f) +_1 f} \downarrow & & \downarrow {\scriptstyle f} \\ (e_1\partial_1^- x') +_1 x' & \xrightarrow[\lambda_1 x']{} & x' \end{array}$$

(ucub.6) For every n-cube x, we have a special n-map $\rho_1 x$, which is natural on n-maps and has the following faces (the naturality diagram, similar to diagram (2), is not written down)

(3) $\rho_1 x\colon x +_1 (e_1\partial_1^+ x) \;\to\; x$, (*right-unit 1-comparison*),

$\partial_1^\alpha \rho_1 x \;=\; e_0\partial_1^\alpha x$, $\qquad \partial_j^\alpha \rho_1 x \;=\; \rho_1\partial_j^\alpha x$ $\qquad (1 < j \leq n)$,



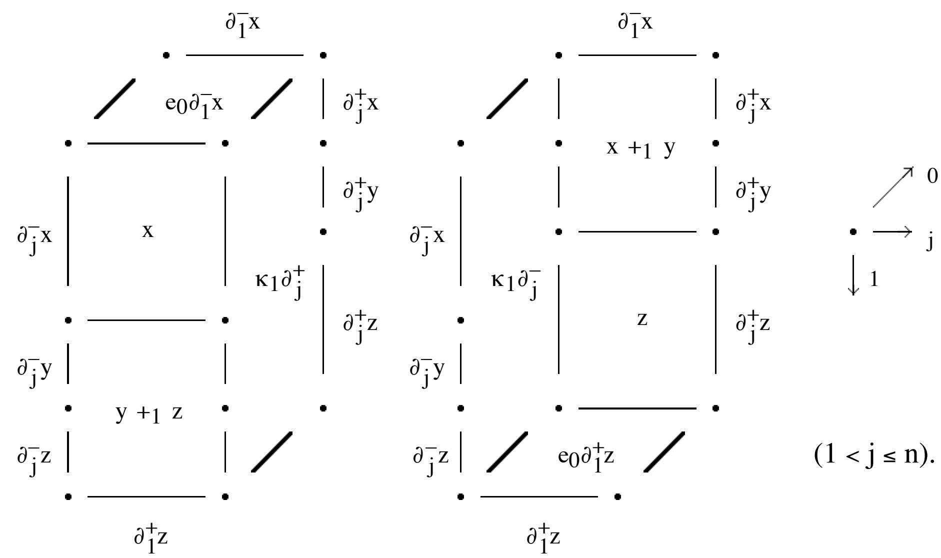
$(1 < j \leq n)$.

(ucub.7) For three 1-consecutive n-cubes x, y, z, we have an invertible special n-map $\kappa_1(x, y, z)$, which is natural on n-maps and has the following faces

(4) $\kappa_1(x, y, z)\colon\; x +_1 (y +_1 z) \;\to\; (x +_1 y) +_1 z$ (*associativity 1-comparison*),

$\partial_1^- \kappa_1(x, y, z) \;=\; e_0\partial_1^- x$, $\qquad \partial_1^+ \kappa_1(x, y, z) \;=\; e_0\partial_1^+ z$,

$\partial_j^\alpha \kappa_1(x, y, z) \;=\; \kappa_1(\partial_j^\alpha x, \partial_j^\alpha y, \partial_j^\alpha z)$ $\qquad (1 < j \leq n)$,



$(1 < j \leq n)$.

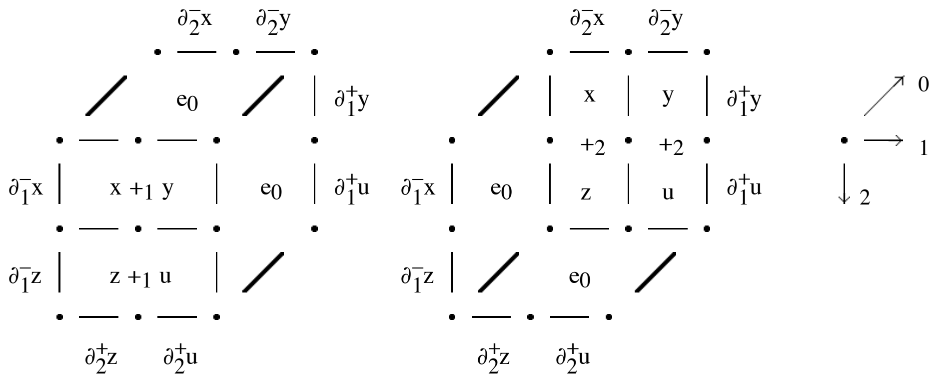
(ucub.8) Given four n-cubes x, y, z, u which satisfy the boundary conditions making the following concatenations possible, we have an invertible n-map χ_1 (*interchange 1-comparison*) which is natural on n-maps and has the following faces (partially displayed below)

(5) $\chi_1(x, y, z, u): (x +_1 y) +_2 (z +_1 u) \rightarrow (x +_2 z) +_1 (y +_2 u),$

$$\partial_1^-\chi_1(x, y, z, u) = e_0(\partial_1^- x +_2 \partial_1^- z), \qquad \partial_1^+\chi_1(x, y, z, u) = e_0(\partial_1^+ y +_2 \partial_1^+ u),$$

$$\partial_2^-\chi_1(x, y, z, u) = e_0(\partial_2^- x +_1 \partial_2^- y), \qquad \partial_2^+\chi_2(x, y, z, u) = e_0(\partial_2^+ z +_1 \partial_2^+ u),$$

$$\partial_j^\alpha\chi_1(x, y, z, u) = \chi_1(\partial_j^\alpha x, \partial_j^\alpha y, \partial_j^\alpha z, \partial_j^\alpha u) \qquad (2 < j \leq n),$$



(ucub.9) Finally, these comparisons must satisfy some conditions of coherence, listed below (2.3).

$\mathbb{A}$ is a *weak symmetric cubical category*, as defined in [G7-G9], if the unit comparisons λ, ρ are also invertible. (In this version, the axioms above are denoted as (wcub.5-9).) Among the examples studied in such papers are: the weak symmetric cubical category $\omega\mathbb{S}\mathrm{p}(\mathbf{X})$ (resp. $\omega\mathbb{C}\mathrm{osp}(\mathbf{X})$) of cubical spans (resp. cospans) on a category $\mathbf{X}$ with pullbacks (resp. pushouts); the strict symmetric cubical category $\omega\mathbb{R}\mathrm{el}$ of cubical relations; structures of 'collared cospans' related to higher cobordism.

2.3. Coherence. The coherence axiom (ucub.9) means that the following diagrams of transversal maps commute (assuming that all the cubical compositions make sense):

(i) *coherence pentagon for* $\kappa = \kappa_1$:

$$\begin{array}{ccccc} & & (x +_1 y) \; +_1 \; (z +_1 u) & & \\ & \overset{\kappa}{\nearrow} & & \overset{\kappa}{\searrow} & \\ x +_1 (y +_1 (z +_1 u)) & & & & ((x +_1 y) +_1 z) +_1 u \\ & \underset{1+\kappa}{\searrow} & & \underset{\kappa+1}{\nearrow} & \\ & x +_1 ((y +_1 z) +_1 u) & \xrightarrow{\;\kappa\;} & (x +_1 (y +_1 z)) +_1 u & \end{array} \tag{1}$$

(ii) *coherence conditions for* $\kappa = \kappa_1$, $\lambda = \lambda_1$ *and* $\rho = \rho_1$

$$\begin{array}{ccc} e_1\partial_1^-x +_1 (x +_1 y) & \xrightarrow{\;\kappa\;} & (e_1\partial_1^-x +_1 x) +_1 y \\ & {}_{\lambda}\searrow \quad \swarrow_{\lambda+1} & \\ & x +_1 y & \end{array} \tag{2}$$

$$\begin{array}{ccc} x +_1 (e_1\partial_1^-y +_1 y) & \xrightarrow{\;\kappa\;} & (x +_1 e_1\partial_1^+x) +_1 y \\ & {}_{1+\lambda}\searrow \quad \swarrow_{\rho+1} & \\ & x +_1 y & \end{array} \tag{3}$$

$$\begin{array}{ccc} x +_1 (y +_1 e_1\partial_1^+y) & \xrightarrow{\;\kappa\;} & (x +_1 y) +_1 e_1\partial_1^+y \\ & {}_{1+\rho}\searrow \quad \swarrow_{\rho} & \\ & x +_1 y & \end{array} \tag{4}$$

(iii) *coherence hexagon for* $\kappa = \kappa_1$ *and* $\chi = \chi_1$ (writing $+$ for $+_1$)

$$\begin{array}{ccc} (x + (y + z)) +_2 (x' + (y' + z')) & \xrightarrow{\;\kappa+\kappa\;} & ((x + y) + z) +_2 ((x' + y') + z') \\ \chi \downarrow & & \downarrow \chi \\ (x +_2 x') + ((y + z) +_2 (y' + z')) & & ((x + y) +_2 (x' + y')) + (z +_2 z') \\ 1{+}\chi \downarrow & & \downarrow \chi{+} \\ (x +_2 x') + ((y +_2 y') + (z +_2 z')) & \xrightarrow[\;\kappa\;]{} & ((x +_2 x') + (y +_2 y')) + (z +_2 z') \end{array} \tag{5}$$

(iv) *coherence conditions for* $\chi = \chi_1$, $\lambda = \lambda_1$ *and* $\rho = \rho_1$ (writing $+$ for $+_1$)

$$
\text{(6)}\qquad
\begin{array}{ccccc}
(e_1\partial_1^- x + x) +_2 (e_1\partial_1^- y + y) & \xrightarrow{\lambda+\lambda} & x +_2 y & \xleftarrow{\rho+\rho} & (x + e_1\partial_1^+ x) +_2 (y + e_1\partial_1^+ y) \\
\chi\downarrow & & \| & & \downarrow\chi \\
(e_1\partial_1^- x +_2 e_1\partial_1^- y) + (x +_2 y) & & \| & & (x +_2 y) + (e_1\partial_1^+ x +_2 e_1\partial_1^+ y) \\
\| & & \| & & \| \\
e_1\partial_1^-(x +_2 y) + (x +_2 y) & \xrightarrow[\lambda]{} & x +_2 y & \xleftarrow[\rho]{} & (x +_2 y) + e_1\partial_1^+(x +_2 y)
\end{array}
$$

The equality in the left (or right) column of this diagram follows from the 'geometric relations' 1.7.4 (nullary interchange) and 1.7.3. Notice also that we do not require the condition $\lambda_1 e_1 x = \rho_1 e_1 x\colon e_1 x +_1 e_1 x \to e_1 x$, which is not satisfied in our case (cf. 4.3), even though $e_1 x +_1 e_1 x$ *does coincide* with $e_1 x$.

3. Reparametrisation mappings

We study now the reparametrisation mappings $\uparrow\mathbf{I}^n \to \uparrow\mathbf{I}^n$ and their interaction with the singular cubes of a d-space, as a first step in the construction of the cubical structure $\uparrow\mathbb{S}\mathrm{ng}(X)$.

3.1. Directed reparametrisation mappings. An *n-dimensional (directed) reparametrisation mapping* $f\colon \uparrow\mathbf{I}^n \to \uparrow\mathbf{I}^n$ will be a d-map (i.e. an order-preserving continuous mapping) which sends each face of the domain to the corresponding face of the codomain, i.e. satisfies the following equivalent conditions (for $i = 1,..., n$ and $\alpha = 0,1$)

(a) $f(\partial_i^\alpha(\uparrow\mathbf{I}^n))) \subset \partial_i^\alpha(\uparrow\mathbf{I}^n)$,

(b) $f.\partial_i^\alpha = \partial_i^\alpha.e_i.f.\partial_i^\alpha$.

As a consequence, f sends each vertex of its domain to the corresponding vertex of the codomain; more generally, the 'lower' faces of any dimension are transformed into the corresponding ones. But actually, *onto them*, *and* f *itself is surjective*, as we prove below, by an Intermediate Value Theorem on the Cube (see 3.5).

Reparametrisation mappings of dimension n form a monoid $\mathcal{S}_n$, under the usual composition.

Moreover, there are faces, transpositions and degeneracies (which will be proved to form a symmetric cubical object $\mathcal{S}$ within monoids, in 3.2)

$$(1) \quad \underline{\partial}^\alpha_i\colon \mathcal{S}_n \to \mathcal{S}_{n-1}, \qquad \underline{s}_i\colon \mathcal{S}_n \to \mathcal{S}_n, \qquad \underline{e}_i\colon \mathcal{S}_n \to \mathcal{S}_{n+1},$$

$$\underline{\partial}^\alpha_i(f) = e_i.f.\partial^\alpha_i\colon {\uparrow}\mathbf{I}^{n-1} \to {\uparrow}\mathbf{I}^n \to {\uparrow}\mathbf{I}^n \to {\uparrow}\mathbf{I}^{n-1} \qquad (f \in \mathcal{S}_n),$$

$$\underline{s}_i(f) = s_i.f.s_i\colon {\uparrow}\mathbf{I}^n \to {\uparrow}\mathbf{I}^n \to {\uparrow}\mathbf{I}^n \to {\uparrow}\mathbf{I}^n \qquad (f \in \mathcal{S}_n),$$

$$\underline{e}_i(f)\colon {\uparrow}\mathbf{I}^n \to {\uparrow}\mathbf{I}^n \qquad (f \in \mathcal{S}_{n-1}),$$

$$\underline{e}_1(f)(t_1,..., t_n) = (t_1, f(t_2,..., t_n)), \qquad \underline{e}_{i+1}(f) = \underline{s}_i(....\underline{s}_2(\underline{s}_1(\underline{e}_1(f)))...).$$

We use *underlined symbols* to avoid confusing the face $\underline{\partial}^\alpha_i(f) = e_i.f.\partial^\alpha_i$ of f *as a reparametrisation mapping* with its face $\partial^\alpha_i(f) = f.\partial^\alpha_i\colon {\uparrow}\mathbf{I}^{n-1} \to {\uparrow}\mathbf{I}^n$ *as an n-cube* of its codomain; likewise for degeneracies and transpositions.

Notice also that we have defined all degeneracies $\underline{e}_i$ using $\underline{e}_1$ and the transpositions (according to the formula 1.6.1). Explicitly, if $f \in \mathcal{S}_{n-1}$, the reparametrisation $\underline{e}_i(f)$ operates by setting apart the i-th coordinate t_i, then applying $f \in \mathcal{S}_{n-1}$ to the remaining $n-1$ coordinates and finally reinserting t_i at the original i-th place:

$$\underline{e}_i(f)(t_1,..., t_n) = (f_1(t_1,..., \hat{t}_i,..., t_n),..., t_i,..., f_{n-1}(t_1,..., \hat{t}_i,..., t_n)).$$

In other words, $\underline{e}_i(f)$ is determined by the following two conditions

$$(2) \quad e_i.\underline{e}_i(f) = f.e_i, \qquad p_i.\underline{e}_i(f) = p_i \qquad (f \in \mathcal{S}_{n-1}),$$

where $p_i\colon {\uparrow}\mathbf{I}^n \to {\uparrow}\mathbf{I}$ denotes the i-th projection (the one omitted by $e_i\colon {\uparrow}\mathbf{I}^n \to {\uparrow}\mathbf{I}^{n-1}$). For instance, if $f\colon {\uparrow}\mathbf{I} \to {\uparrow}\mathbf{I}$ is in $\mathcal{S}_1$, its two degeneracies in $\mathcal{S}_2$ are computed by the following formulas:

$$(3) \quad \underline{e}_1(f)(t_1, t_2) = (t_1, f(t_2)), \qquad \underline{e}_2(f)(t_1, t_2) = (f(t_1), t_2) \qquad (t_1, t_2 \in \mathbf{I}).$$

Reparametrisation mappings will be used to reparametrise the singular cubes $x: \uparrow\mathbf{I}^n \to X$ of a d-space. The interactions of the two 'algebras' will be developed in 3.4.

Notice that the following squares commute (also because of (2))

$$
(4) \qquad
\begin{array}{ccc}
\uparrow\mathbf{I}^n & \xrightarrow{f} & \uparrow\mathbf{I}^n \\
\partial_i^\alpha \uparrow & & \uparrow \partial_i^\alpha \\
\uparrow\mathbf{I}^{n-1} & \xrightarrow[\underline{\partial}_i^\alpha f]{} & \uparrow\mathbf{I}^{n-1}
\end{array}
\qquad
\begin{array}{ccc}
\uparrow\mathbf{I}^{n-1} & \xrightarrow{f} & \uparrow\mathbf{I}^{n-1} \\
e_i \uparrow & & \uparrow e_i \\
\uparrow\mathbf{I}^n & \xrightarrow[\underline{e}_i f]{} & \uparrow\mathbf{I}^n
\end{array}
\qquad
\begin{array}{ccc}
\uparrow\mathbf{I}^n & \xrightarrow{f} & \uparrow\mathbf{I}^n \\
s_i \uparrow & & \uparrow s_i \\
\uparrow\mathbf{I}^n & \xrightarrow[\underline{s}_i f]{} & \uparrow\mathbf{I}^n
\end{array}
$$

3.2. Theorem (The structure of reparametrisation mappings). *Reparametrisation mappings, with the faces, degeneracies and transpositions defined above, form a symmetric cubical object in the category of monoids.*

Proof. Faces and degeneracies preserve the composition of reparametrisation mappings (and - plainly - the identity). Indeed, applying 3.1(b) and the characterisation 3.1.2 for $\underline{e}_i$, we have

$$\underline{\partial}_i^\alpha(gf) = e_i.gf.\partial_i^\alpha = e_i g.\partial_i^\alpha e_i.f\partial_i^\alpha = \underline{\partial}_i^\alpha(g).\underline{\partial}_i^\alpha(f),$$

$$e_i.\underline{e}_i(gf) = (gf)e_i, = g(e_i e_i(f)) = e_i.\underline{e}_i(g).\underline{e}_i(f),$$

$$p_i.\underline{e}_i(gf) = p_i = p_i.\underline{e}_i(g).\underline{e}_i(f),$$

$$\underline{s}_i(gf) = s_i.gf.s_i = s_i g s_i.s_i f s_i = \underline{s}_i(g).\underline{s}_i(f).$$

Finally, we verify the symmetric cubical identities, working with the simpler presentation of 1.6.3 to reduce computations, and taking into account the fact that the structural maps of cubes satisfy the follwing *dual* conditions

(1) $\partial_1^\beta\partial_1^\alpha = s_1.\partial_1^\alpha\partial_1^\beta,$ $\qquad \partial_1^\alpha.s_i = s_{i+1}.\partial_1^\alpha,$ $\qquad e_1.\partial_1^\alpha = \mathrm{id},$

$\qquad e_1e_1 = e_1e_1.s_1,$ $\qquad s_i.e_1 = e_1.s_{i+1}.$

Now, we have:

- $\underline{\partial}_1^\alpha\underline{\partial}_1^\beta(f) = (e_1e_1).f.(\partial_1^\beta\partial_1^\alpha) = (e_1e_1s_1).f.(s_1.\partial_1^\alpha.\partial_1^\beta) = \underline{\partial}_1^\beta\underline{\partial}_1^\alpha\underline{s}_1(f),$
- $\underline{s}_i.\underline{\partial}_1^\alpha(f) = s_ie_1.f.\partial_1^\alpha s_i = e_1s_{i+1}.f.s_{i+1}.\partial_1^\alpha = \underline{\partial}_1^\alpha.\underline{s}_{i+1}(f),$

- $\underline{\partial}{}_1^\alpha \underline{e}_1(f) = e_1.\underline{e}_1(f).\partial_1^\alpha = f.e_1\partial_1^\alpha = f$,
- $e_1(\underline{e}_1\underline{s}_i(f)) = (s_i.f.s_i).e_1 = s_i.f.e_1.s_{i+1} = s_i.e_1.\underline{e}_1(f).s_{i+1} = e_1.s_{i+1}.\underline{e}_1(f).s_{i+1} = e_1(\underline{s}_{i+1}\underline{e}_1(f))$,
- $p_1(\underline{e}_1\underline{s}_i(f)) = p_1 = s_{i+1}.p_1.s_{i+1} = s_{i+1}.p_1.\underline{e}_1(f).s_{i+1} = p_1.s_{i+1}.\underline{e}_1(f).s_{i+1} = p_1(\underline{s}_{i+1}\underline{e}_1(f))$.

The n-dimensional reparametrisation mapping

$$(\underline{e}_1\underline{e}_1(f))(t_1,..., t_n) = (t_1, t_2, f(t_3,..., t_n)),$$

is plainly invariant under $\underline{s}_1 = s_1.(-).s_1$. Finally, the Moore relations for the transpositions $\underline{s}_i$ follow trivially from those of the original s_i. For instance, for $i = j-1$:

- $\underline{s}_i.\underline{s}_j.\underline{s}_i(f) = (s_i.(s_j.(s_i.f.s_i).s_j).s_i) = \underline{s}_j.\underline{s}_i.\underline{s}_j(f)$. □

3.3. Concatenating reparametrisation mappings. The cubical set $\mathcal{S}$ has the following i-concatenation, or i-composition.

If $f, g \in \mathcal{S}_n$ are i-consecutive $(\underline{\partial}{}_i^+ f = \underline{\partial}{}_i^- g)$, we define:

$$(1) \quad (f +_i g)(..., t_i,...) = \begin{cases} u_i(f(t_1,..., 2t_i,..., t_n)), & \text{if } 0 \leq t_i \leq 1/2, \\ v_i(g(t_1,..., 2t_i - 1,..., t_n)), & \text{if } 1/2 \leq t_i \leq 1, \end{cases}$$

where the map $u_i\colon \uparrow\mathbf{I}^n \to \uparrow\mathbf{I}^n$ halves the i-th coordinate, while $v_i\colon \uparrow\mathbf{I}^n \to \uparrow\mathbf{I}^n$ operates on this coordinate as $t \mapsto (t + 1)/2$ (all the other coordinates staying unchanged).

Finally, it is obvious that $f +_i g$ is again a reparametrisation mapping.

3.4. Proposition (The interaction of cubes and reparametrisation mappings). *For a d-space* X, *the reparametrisations of its singular cubes agree with faces, degeneracies, transpositions and concatenations, in the following sense*

(1) $\partial_i^\alpha(xf) = \partial_i^\alpha(x).\underline{\partial}{}_i^\alpha(f), \quad e_i(xf) = e_i(x).\underline{e}_i(f), \quad s_i(xf) = s_i(x).\underline{s}_i(f)$,

(2) $(x +_i y).(f +_i g) = xf +_i yg$,

where $x, y\colon \uparrow\mathbf{I}^n \to X$ *are* i-*consecutive singular* n-*cubes and* f, g *are* i-*consecutive mappings in* $\mathbb{S}_n$.

Proof. The formulas (1) are an easy consequence of the definitions (in 3.1)

$$(3)\quad \partial_i^\alpha(xf) \ = \ (xf)\partial_i^\alpha \ = \ x.\partial_i^\alpha e_i.f\partial_i^\alpha \ = \ \partial_i^\alpha(x).\underline{\partial}_i^\alpha(f),$$
$$e_i(xf) \ = \ (xf)e_i, \ = \ xe_i.\underline{e}_i(f) \ = \ e_i(x).\underline{e}_i(f),$$
$$s_i(xf) \ = \ (xf)s_i \ = \ x.s_is_i.fs_i \ = \ s_i(x).\underline{s}_i(f).$$

The first point also proves that $xf +_i yg$ makes sense, in (2). Then, this formula is easily verified, with the definitions of concatenations of cubes and reparametrisations (in 1.7.2 and 3.3.1). □

3.5. Theorem (Intermediate Value Theorem on the Cube). *Let* $f\colon \mathbf{I}^n \to \mathbf{I}^n$ *be a continuous mapping which sends each* (n–1)-*dimensional face* to *itself. Then* f *is surjective and sends each 'lower' face (of any dimension)* onto *itself.*

Proof. Let us begin by considering the affine homotopy $h\colon f \simeq \mathrm{id}\colon \mathbf{I}^n \to \mathbf{I}^n$

$$(1)\quad h(t_1,..., t_n, t) \ = \ (1 - t).f(t_1,..., t_n) + t.(t_1,..., t_n),$$

and note that it sends each face of $\mathbf{I}^n$ into itself, because f and id both do, and each face is convex. Now, let us prove that f is surjective, by induction on n. Our thesis being trivial for $n = 0$, let us assume it holds for $n-1$ and prove it for $n > 0$.

Every restriction of f to an (n–1)-dimensional face of the cube gives a map $\mathbf{I}^{n-1} \to \mathbf{I}^{n-1}$ which satisfies the hypothesis, and is surjective; whence, the restriction $f'\colon \partial\mathbf{I}^n \to \partial\mathbf{I}^n$ to the boundary of the cube is surjective.

Collapsing the boundary $\partial\mathbf{I}^n$ to a point, we get an induced endomap of the sphere, $f''\colon \mathbf{S}^n \to \mathbf{S}^n$, which is still homotopic to the identity, by a homotopy induced by h; therefore f" is also surjective, or its image would be contained in a contractible space and f" would be homotopic to a constant map. Therefore, the image of f also contains the interior points of $\mathbf{I}^n$ and f is surjective. □

3.6. Remarks. The previous statement is trivial for $n = 0$, and amounts to the classical Intermediate Value Theorem for $n = 1$. For $n = 2$, one might describe the statement as follows: in order to cover a picture with a rectangular piece of cloth, it is sufficient to ensure that each edge of the cloth is placed on the 'corresponding' edge of the picture (so that vertices are necessarily placed at vertices and each edge covers an edge).

Notice also that, for $n \geq 2$, it is not sufficient to assume that f covers the boundary of the cube, as simple examples can show. The crucial assumption is that the restriction of f to the boundary is not homotopically trivial. This can be formulated as follows.

Intermediate Value Theorem on the Ball. Let $f \colon \mathbf{B}^n \to \mathbf{B}^n$ be a map which sends the boundary $\mathbf{S}^{n-1}$ into itself. If the restriction $f' \colon \mathbf{S}^{n-1} \to \mathbf{S}^{n-1}$ is not homotopic to a constant map (or, equivalently, if its homological degree is not null), then f is surjective.

An equivalent formulation can be found in Agoston's text [Ag], Section 7.4. (We thank Sibe Mardešić for this reference.)

4. Transversal maps and comparisons

Reparametrisation mappings are now used to define the transversal maps of the singular cubes of a d-space X. These include comparisons for the associativity and unitarity of the operations of concatenation, yielding the u-lax symmetric cubical category $\uparrow\mathbb{S}\text{ng}(X)$.

4.1. Transversal maps. For a d-space X, a *transversal map* $f \colon x \to y$ between two singular n-cubes x, y of X will be a reparametrisation mapping $f \colon \uparrow\mathbf{I}^n \to \uparrow\mathbf{I}^n$ such that $x = yf$.

More precisely, a transversal map should be defined as a triple $\hat{f} = (f, x, y)$, and we will use this notation when useful. Notice that y always determines x, while x determines y *if* f is bijective.

The choice of the direction of f, *from x to y,* is formal but has the advantage of agreeing with the composition of reparametrisations. In fact, the n-cubes of X and their transversal maps form a category $\uparrow\mathrm{Sng}_n(X)$, with obvious faces, identities and composition

(1) $\partial_0^-(f, x, y) = x, \quad \partial_0^+(f, x, y) = y, \qquad e_0(x) = (id, x, x),$

$c_0(f, g) = gf \colon x \to z \qquad$ (for $g \colon y \to z$, so that $x = yf = zgf$).

$\uparrow\mathrm{Sng}_0(X)$ is a discrete category: the only transversal maps between 0-cubes are the identities.

Transversal maps also form a symmetric cubical set, using the cubical structure of reparametrisation maps (defined in 3.1) and that of singular cubes

(2) $\partial_i^\alpha(f, x, y) = (\underline{\partial}_i^\alpha f, \ \partial_i^\alpha x, \ \partial_i^\alpha y), \qquad e_i(f, x, y) = (\underline{e}_i f, \ e_i x, \ e_i y),$

$s_i(f, x, y) = (\underline{s}_i f, \ s_i x, \ s_i y).$

This is legitimate, since the relation $x = yf$ implies

(3) $\partial_i^\alpha(x) = yf.\partial_i^\alpha = y.\partial_i^\alpha.e_i.f.\partial_i^\alpha = \partial_i^\alpha(y).\underline{\partial}_i^\alpha(f),$

$e_i(x) = yf.e_i = y.e_i.\underline{e}_i(f) = e_i(y).\underline{e}_i(f),$

$s_i(x) = yf.s_i = y.s_i.s_i.f.s_i = s_i(y).\underline{s}_i(f).$

Finally, we define the i-concatenation of i-consecutive transversal maps as

(4) $(f, x, y) +_i (g, z, u) = (f +_i g, \ x +_i z, \ y +_i u) \qquad (\partial_i^+(f, x, y) = \partial_i^-(g, z, u)),$

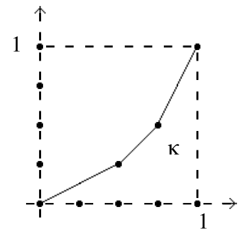
where $f +_i g \colon \mathbf{I}^n \to \mathbf{I}^n$ is the i-concatenation of reparametrisation maps (3.3.1), and the relations $x = yf$ and $z = ug$ imply that

(5) $(y +_i u)(f +_i g) = yf +_i ug = x +_i z.$

4.2. Associativity comparison. Given three consecutive paths (1-cubes) $x, y, z \colon \uparrow\mathbf{I} \to X$, the two ternary concatenations $w' = x +_1 (y +_1 z)$ and $w'' = (x +_1 y) +_1 z$

(1) $w'(t) = \begin{cases} x(2t) & (0 \leq t \leq 1/2), \\ y(4t-2) & (1/2 \leq t \leq 3/4), \\ z(4t-3) & (3/4 \leq t \leq 1), \end{cases} \qquad w''(t) = \begin{cases} x(4t) & (0 \leq t \leq 1/4), \\ y(4t-1) & (1/4 \leq t \leq 1/2), \\ z(2t-1) & (1/2 \leq t \leq 1), \end{cases}$

can be turned one into the other by a suitable *invertible* reparametrisation of the interval. Namely, we have an invertible transversal map $\kappa \colon w' \to w''$ $(w' = w''\kappa)$, where $\kappa \colon \uparrow\mathbf{I} \to \uparrow\mathbf{I}$ is the following reparametrisation function

(2) 

$$\kappa(t) = \begin{cases} t/2 & (0 \leq t \leq 1/2), \\ t - 1/4 & (1/2 \leq t \leq 3/4), \\ 2t - 1 & (3/4 \leq t \leq 1). \end{cases}$$

In degree n, we shall use the reparametrisation maps obtained from κ in the usual way

(3) $\kappa_i = \uparrow\mathbf{I}^{i-1} \times \kappa \times \uparrow\mathbf{I}^{n-i}\colon \uparrow\mathbf{I}^n \to \uparrow\mathbf{I}^n$.

It follows that the i-concatenation of singular n-cubes is associative up to the following family of *invertible* transversal maps

(4) $\kappa_i(x, y, z)\colon\ x +_i (y +_i z) \to (x +_i y) +_i z$.

This family is natural, with respect to transversal maps: given three n-maps

$f\colon x' \to x, \qquad g\colon y' \to y, \qquad h\colon z' \to z,$

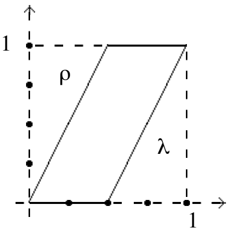
that are consecutive in direction i, we must verify that the following square commutes

(5)
$$\begin{array}{ccc} x' +_i (y' +_i z') & \xrightarrow{\kappa_i} & (x' +_i y') +_i z' \\ {\scriptstyle f +_i (g +_i h)}\downarrow & & \downarrow{\scriptstyle (f +_i g) +_i h} \\ x +_i (y +_i z) & \xrightarrow[\kappa_i']{} & (x +_i y) +_i z \end{array}$$

Plainly, it is sufficient to check this for $n = 1$ (and $i = 1$). Then, both composites transform the partition (0, 1/2, 3/4, 1) of $\mathbf{I}$ into the partition (0, 1/4, 1/2, 1), by a pasting of 'affine modifications' of f, g, h on domain and codomain. The *common result* of both compositions is thus the transversal map defined by the following reparametrisation function

$$k(t) = \begin{cases} f(2t)/4 & (0 \leq t \leq 1/2), \\ 1/4 + g(4t - 2)/4 & (1/2 \leq t \leq 3/4), \\ 1/2 + h(4t - 3)/2 & (3/4 \leq t \leq 1). \end{cases}$$

4.3. Identity comparisons. Given a path $x\colon \mathbf{I} \to X$ with endpoints $x_0 = x(0)$, $x_1 = x(1)$, the two concatenations x', x'' of x with trivial paths can be obtained from the original path x by non-invertible reparametrisations of the interval, with two piecewise affine functions λ, ρ:

(1) 

$$x' = e_1(x_0) +_1 x = x\lambda, \quad \lambda(t) = \max(0, 2t-1),$$

$$x'' = x +_1 e_1(x_1) = x\rho, \quad \rho(t) = \min(2t, 1).$$

In degree n, we shall use the reparametrisation maps

(2) $\lambda_i = \uparrow\mathbf{I}^{i-1} \times \lambda \times \uparrow\mathbf{I}^{n-i}\colon \uparrow\mathbf{I}^n \to \uparrow\mathbf{I}^n, \qquad \rho_i = \uparrow\mathbf{I}^{i-1} \times \rho \times \uparrow\mathbf{I}^{n-i}\colon \uparrow\mathbf{I}^n \to \uparrow\mathbf{I}^n.$

We have thus two natural transversal maps

(3) $\lambda_i(x)\colon x\lambda_i \to x, \qquad \rho_i(x)\colon x\rho_i \to x,$

$x\lambda_i = e_i(\partial_i^- x) +_i x, \qquad x\rho_i = x +_i e_i(\partial_i^+ x).$

We do not need other comparisons: we have already remarked, at the end of 1.7, that $\uparrow\Box X$ has a *strict* interchange of concatenations (binary and nullary).

4.4. The u-lax cubical category of a directed space. For a d-space X, we have thus defined the u-lax symmetric cubical category $\uparrow\mathbb{S}\mathrm{ng}(X)$: it consists of the basic symmetric pre-cubical category $\uparrow\Box X$ (1.7), with the addition of:

- transversal maps given by reparametrisations (4.1),
- invertible comparisons for pseudo associativity (4.2),
- comparisons for lax unitarity (4.3),
- identity comparisons for strict interchange.

The coherence conditions of 2.3 are satisfied. To verify this point for the pentagon, it is sufficient to note that the five associativity comparisons of diagram 2.3.1 are produced by the mapping

(1) $\kappa_1 = \kappa \times \uparrow\mathbf{I}^{n-1}\colon \uparrow\mathbf{I}^n \to \uparrow\mathbf{I}^n.$

Their action on the first coordinate is piecewise affine, and determined by the following partitions of the interval [0, 1]

$$
(2)\qquad
\begin{array}{ccccc}
 & & (0, 1/4, 1/2, 3/4, 1) & & \\
 & {}^{\kappa}\nearrow & & \searrow^{\kappa} & \\
(0, 1/2, 3/4, 7/8, 1) & & & & (0, 1/8/, 1/4, 1/2, 1) \\
 & {}^{1+\kappa}\searrow & & \nearrow^{\kappa+1} & \\
 & (0, 1/2, 5/8, 3/4, 1) & \xrightarrow{\ \kappa\ } & (0, 1/4, 3/8, 1/2, 1) &
\end{array}
$$

Now, both composites coincide with the piecewise affine mapping which transforms the left-hand partition into that at the right. The other axioms of coherence are verified in the same way.

Notice also that, when χ is the identity, the coherence hexagon 2.3.5 reduces to a condition of consistency of the associativity comparison $\kappa = \kappa_1$ with 2-concatenation:

(3) $\kappa(x, y, z) +_2 \kappa(x', y', z') = \kappa(x +_2 x', y +_2 y', z +_2 z')$.

4.5. Remarks. It would be interesting to quotient singular cubes up to invertible reparametrisations, but this is not easily done because - of course - we want to have induced concatenations. Now, if $x +_i y$ and $x' +_i y'$ are defined in $\uparrow\mathrm{Sng}_n(X)$, and there exist two invertible reparametrisations $f\colon x' \to x$, $g\colon y' \to y$, these need not be i-consecutive, and there are cases where there is *no* reparametrisation at all from $x' +_i y'$ to $x +_i y$.

We give such an example in dimension 2. Let us start from two 1-consecutive 2-cubes x, y of the ordinary plane $\mathbf{R}^2$, that have constant faces ∂_1^α and are injective outside of such faces, like

$$x, y\colon \uparrow\mathbf{I}^2 \to \mathbf{R}^2, \qquad x(t, t') = (t, t'.t(1-t)), \qquad y(t, t') = (t + 1, t'.t(1-t)).$$

There is precisely one transversal endomap $f\colon x \to x$, namely the identity, because its reparametrisation $f\colon \uparrow\mathbf{I}^2 \to \uparrow\mathbf{I}^2$ must be the identity on a dense subset of the standard square.

Let now $g = id \times \varphi: \uparrow\mathbf{I}^2 \to \uparrow\mathbf{I}^2$ be an invertible reparametrisation map given by a directed homeomorphism $\varphi: \uparrow\mathbf{I} \to \uparrow\mathbf{I}$ other than the identity (for instance, $\varphi(t) = t^2$). There is only one transversal endomap $yg \to y$, and is given by $g: \uparrow\mathbf{I}^2 \to \uparrow\mathbf{I}^2$.

The 2-cubes x, y and $y' = yg$ have the same (constant) 1-indexed faces, whence there are concatenated cubes $x +_1 y$ and $x +_1 y'$, with values in $\mathbf{R}^2$.

However, there is no transversal map $x +_1 y' \to x +_1 y$: indeed, its reparametrisation mapping $h: \uparrow\mathbf{I}^2 \to \uparrow\mathbf{I}^2$ should restrict to the identity on $[0, 1/2] \times \mathbf{I}$ and to $id \times \varphi$ on $[1/2, 1] \times \mathbf{I}$, which gives a contradiction on the intersection $\{1/2\} \times \mathbf{I}$ of these rectangles.

One might think of solving this problem by considering 'piecewise reparametrisations' on 'multi-partitions' of the singular cubes, but this leads to two problems.

(a) If $f: \partial_i^+(x) \to \partial_i^-(y)$ is a (global) invertible transversal map, it is easy to show that there exists a cube y' and an invertible transversal map $f': y' \to y$ such that $x +_i y'$ is defined; but $y' = yf'$ is constructed by extending the reparametrisation mapping f; a 'piecewise reparametrisation' of our faces would not allow us to construct a cube y'.

(b) The relation between cubes obtained this way is not transitive.

Extending our relation by transitivity would solve the second point but still stumble on the first. The same happens with a different approach, by *partial reparametrisation mappings* defined on convenient dense open subsets.

4.6. The non-directed case. Let now X be a (non-directed) topological space. As already said, we view it as a d-space by its natural (reversible) structure, where all paths are directed.

Now, the singular symmetric cubical set $\uparrow\square X$ (1.4) will be written as $\square X$ and equipped with *reversions* produced, contravariantly, by the reversions of the standard cube

$$(1)\quad r_i = \mathbf{I}^{i-1} \times r \times \mathbf{I}^{n-i}: \mathbf{I}^n \to \mathbf{I}^n, \qquad r_i(t_1,..., t_{n+1}) = (t_1,..., 1 - t_i,..., t_n).$$

$$r_i: \square_n X \to \square_n X \qquad r_i(x) = x.r_i \qquad (i = 1,..., n),$$

We now replace the weak symmetric cubical category $\uparrow\mathbb{S}ng(X)$ with a *larger* structure $\mathbb{S}ng(X)$, where the transversal maps $(f, x, y): x \to y$ are given by reparametrisation mappings $f: \mathbf{I}^n \to \mathbf{I}^n$ that need not preserve the natural ordering (but, of course, still have to send each face onto itself).

The weak symmetric cubical category $\mathbb{S}ng(X)$ is also equipped with *reversions*, extending those of its cubes: if $(f, x, y): x \to y$ is a transversal map between singular n-cubes of X, we let

(2) $\underline{r}_i(f) = r_i.f.r_i: \mathbf{I}^n \to \mathbf{I}^n, \qquad r_i(f, x, y) = (\underline{r}_if, r_ix, r_iy).$

The theory of *reversible symmetric cubical sets* is sketched in [GM], Section 9. (The site $!\mathbb{K}$ considered there also contains the connections, that can be discarded.)

The definition of a *weak reversible symmetric cubical category* is not difficult to set up, blending the theory mentioned above with the non-reversible notion studied above. Of course, new consistency and coherence conditions must be added, like:

(3) $r_1(x +_1 y) = r_1(y) +_1 r_1(x),$

$r_1(\lambda_1 x) = \rho_1(r_1(x)), \qquad r_1(\kappa_1(x, y, z)) = \kappa_1(r_1(z), r_1(y), r_1(x)).$

5. The Moore symmetric cubical category of a d-space

In this section we briefly consider a strict version of the previous construction. It is based on the *Moore* directed cubes of a d-space, defined on 'multi-intervals'. Their cubical compositions are *strictly* associative and unital. Reparametrisation mappings of multi-intervals provide transversal maps and the (extended) Moore symmetric cubical category $\uparrow\mathbb{M}\mathbb{S}ng(X)$ of a d-space.

5.1. Multi-intervals. A (*directed*) *multi-interval* will be a product of directed intervals, possibly degenerate, of variable length $a \geq 0$ (or $a_h \geq 0$)

(1) $I(a) = \uparrow[0, a], \qquad I(a_1,..., a_n) = \prod_{h=1,...,n} \uparrow[0, a_h].$

The topological dimension of this parallelepiped can be any integer between 0 and n, but we say that it has *formal dimension* n and *span* $(a_1,..., a_n) \in [0, \infty[^n$.

There is precisely one directed multi-interval of formal dimension 0, i.e. the empty product $\{*\}$; its span is the empty family.

The *faces*, *degeneracies* and *transpositions* of multi-intervals are defined as follows

(2) $\partial_i^\alpha\colon \prod_{h\neq i} I(a_h) \to \prod_h I(a_h), \qquad \partial_i^\alpha(t_1,..., t_{n-1}) = (t_1,..., \alpha a_i,..., t_{n-1}),$

$e_i\colon \prod_h I(a_h) \to \prod_{h\neq i} I(a_h), \qquad e_i(t_1,..., t_n) = (t_1,..., \hat{t}_i,..., t_n),$

$s_i\colon \prod_h I(a_{\sigma_i(h)}) \to \prod_h I(a_h), \qquad s_i(t_1,..., t_n) = (t_1,..., t_{i+1}, t_i,..., .., t_n) \qquad (i < n).$

Here, $\sigma_i\colon \{1,..., n\} \to \{1,..., n\}$ denotes the involution that interchanges i and i+1. Notice also that - as a consequence of using multi-intervals instead of standard cubes:

- the upper face ∂_i^+ is determined by its codomain, or equivalently by its domain *and the number* a_i,

- the degeneracy e_i is determined by its domain, or equivalently by its codomain *and* a_i,

- ∂_i^- and s_i are determined by their domain, or equivalently by their codomain.

In order to reparametrise cubes, we will need degeneracies whose *codomain* is known; in such a case, the degeneracy e_i in (2) can be written as e_i^a, to mean that it is determined by its codomain $\prod_{h\neq i} I(a_h)$ *together with* $a_i = a \geq 0$.

Working with the singular cubes, in the previous sections, we have used the *standard degeneracy* e_i^1, which preserves them. Below, working with Moore cubes, we will use the *strict degeneracy* e_i^0, that has the advantage of giving strict identities for i-concatenation

(3) $e_i^0\colon \prod_h I(a_h) \to \prod_{h\neq i} I(a_h) \qquad (a_i = 0).$

5.2. The cocubical relations. The faces and degeneracies of multi-intervals satisfy cocubical relations analogous to those of a cocubical set. We display them on diagrams because the symbols ∂_i^α, e_i, s_i are far from containing the whole information

(1) $\partial^\beta_j.\partial^\alpha_i = \partial^\alpha_{i+1}\partial^\beta_j \quad (j \leq i),$

$$\begin{array}{ccc} \prod_{h\neq i+1,j} I(a_h) & \xrightarrow{\partial^\alpha_i} & \prod_{h\neq j} I(a_h) \\ \partial^\beta_j \downarrow & & \downarrow \partial^\beta_j \\ \prod_{h\neq i+1} I(a_h) & \xrightarrow[\partial^\alpha_{i+1}]{} & \prod_h I(a_h) \end{array}$$

(2) $e_i.e_j = e_j.e_{i+1} \quad (j \leq i),$

$$\begin{array}{ccc} \prod_h I(a_h) & \xrightarrow{e_i} & \prod_{h\neq j} I(a_h) \\ e_{i+1} \downarrow & & \downarrow e_i \\ \prod_{h\neq i+1} I(a_h) & \xrightarrow[e_j]{} & \prod_{h\neq i+1,j} I(a_h) \end{array}$$

(3) $e_i.\partial^\alpha_i = id$

$$\prod_{h\neq i} I(a_h) \longrightarrow \prod_h I(a_h) \longrightarrow \prod_{h\neq i} I(a_h)$$

(4) $e_j.\partial^\alpha_i = \partial^\alpha_{i-1}.e_j \quad (j < i),$

$$\begin{array}{ccc} \prod_{h\neq i} I(a_h) & \xrightarrow{\partial^\alpha_i} & \prod_h I(a_h) \\ e_j \downarrow & & \downarrow e_j \\ \prod_{h\neq j,i-1} I(a_h) & \xrightarrow[\partial^\alpha_{i-1}]{} & \prod_{h\neq j} I(a_h) \end{array}$$

(5) $e_j.\partial^\alpha_i = \partial^\alpha_i.\, e_{j-1} \quad (j > i),$

$$\begin{array}{ccc} \prod_{h\neq i} I(a_h) & \xrightarrow{\partial^\alpha_i} & \prod_h I(a_h) \\ e_{j-1} \downarrow & & \downarrow e_j \\ \prod_{h\neq i,j-1} I(a_h) & \xrightarrow[\partial^\alpha_i]{} & \prod_{h\neq j} I(a_h) \end{array}$$

This structure is thus a sort of 'cocubical aggregate' of directed spaces, more general than a cocubical directed space: in dimension n there are various objects (all the multi-intervals of formal dimension n), instead of a single one. The cocubical set of the standard cubes $\uparrow\mathbf{I}^n$ is a 'substructure' of the present structure.

Furthermore, the transpositions $s_i\colon \prod_h I(a_{\sigma(h)}) \to \prod_h I(a_h)$ satisfy the Moore relations (1.4.5) and the coherence conditions with faces and degeneracies already stated above (in the contravariant form of *cubical* relations, see 1.5.4):

		j < i	j = i	j = i+1	j > i+1
(6)	$s_i.\partial^\alpha_j$ =	$\partial^\alpha_j.s_{i-1}$	∂^α_{i+1}	∂^α_i	$\partial^\alpha_j.s_i$,
	$e_j.s_i$ =	$s_{i-1}.e_j$	e_{i+1}	e_i	$s_i.e_j$.

For instance, the first two rewriting rules on $s_i.\partial^\alpha_j$ (for $j < i$ and $j = i$, respectively) amount to the commutativity of the following diagrams:

(7)
$$\begin{array}{ccc} \Pi_{h\neq j}\, I(a_{\sigma_i(h)}) & \xrightarrow{\partial^\alpha_j} & \Pi_h\, I(a_{\sigma_i(h)}) \\ s_{i-1}\downarrow & & \downarrow s_i \\ \Pi_{h\neq j}\, I(a_h) & \xrightarrow[\partial^\alpha_j]{} & \Pi_h\, I(a_h) \end{array} \qquad \begin{array}{ccc} \Pi_{h\neq i+1}\, I(a_{\sigma_i(h)}) & \xrightarrow{\partial^\alpha_i} & \Pi_h\, I(a_{\sigma_i(h)}) \\ & \searrow^{\partial^\alpha_{i+1}} & \downarrow s_i \\ & & \Pi_h\, I(a_h) \end{array}$$

5.3. Moore cubes. We now introduce the *Moore symmetric cubical set* of a d-space X

(1) $\mathbb{M}(X) = ((\mathbb{M}_nX), (\partial^\alpha_i), (e_i), (s_i))$.

A *Moore (directed) n-cube* of X will be a map with values in X and defined on a multi-interval of formal dimension n

(2) $x: \Pi_{h=1,...,n}\, I(a_h) \to X, \qquad sp(x) = (a_1,..., a_n)$.

Faces, *degeneracies* and *transpositions* of $\mathbb{M}(X)$ are obtained by pre-composing with those of multi-intervals, and will be denoted by the same symbols

(3) $\partial^\alpha_i: \mathbb{M}_nX \to \mathbb{M}_{n-1}X, \qquad \partial^\alpha_i(x) = x.\partial^\alpha_i: \Pi_{h\neq i}\, I(a_h) \to X$,

$e_i: \mathbb{M}_{n-1}X \to \mathbb{M}_nX, \qquad e_i(x) = x.e^0_i: \Pi_{h<i}\, I(a_h) \times \{0\} \times \Pi_{h\geq i}\, I(a_h) \to X$,

$s_i: \mathbb{M}_nX \to \mathbb{M}_nX \qquad s_i(x) = x.s_i: \Pi_h\, I(a_{\sigma_i(h)}) \to X \qquad (i \leq n-1)$.

Notice, again, that the *(strict)* degeneracy $e_i(x) = x.e^0_i$ of a singular cube $x: \uparrow\mathbf{I}^{n-1} \to X$ is *not* a singular cube of X: the *standard* degeneracy used in the previous section is $x.e^1_i: \uparrow\mathbf{I}^n \to X$. We use the same notation $e_i(x)$, since the context is generally sufficient to specify whether we are working within Moore or singular cubes of X.

These maps satisfy cubical relations, dual to those considered above, so that $\mathbb{M}(X)$ is indeed a symmetric cubical set, as defined in 1.5. Again, the contravariant action of the transpositions $s_1,..., s_{n-1}$ on $\mathbb{M}_nX$ can be extended to a (*right*) action of the symmetric group S_n.

As we have already seen in 1.6, the presence of transpositions makes all faces and degeneracies determined by (say) the 1-indexed ones, ∂_1^α and e_1:

(4) $\partial_{i+1}^\alpha = \partial_1^\alpha.s_1. ... .s_i, \qquad e_{i+1} = s_i. ... .s_1.e_1.$

For a (non-directed) topological space X, R. Brown [Br] has recently given a cubical construction $M_*(X)$ similar to the present $\mathbb{M}(X)$. His n-cubes, called *Moore hyperrectangles*, are pairs (x, a), where $x: [0, \infty[^n \to X$ is a map, $a = (a_1,..., a_n) \in [0, \infty[^n$, and $x(t_1,..., t_n)$ is independent of the coordinate t_i for $t_i \geq a_i$. This is obviously equivalent to the present definition of n-cubes (letting all paths of X be distinguished), but the cubical structure considered in [Br] is different from the present one, as already mentioned in the Introduction.

5.4. A basic symmetric cubical category. The symmetric cubical set $\mathbb{M}(X)$ can be further equipped with partial operations, called *cubical compositions*, the *concatenation in direction i*, or *i-concatenation*, or *i-composition.*

In dimension n, and for $i = 1,..., n$, it is based on the following *i-concatenation pushout*, with embeddings c_i^α

$$\begin{array}{ccc} \prod_{h\neq i} I(a_h) & \xrightarrow{\partial_i^+} & \prod_h I(a_h) \\ \partial_i^- \downarrow & & \downarrow c_i^- \\ \prod_h I(b_h) & \xrightarrow[c_i^+]{} & \prod_h I(d_h) \end{array} \qquad (1) \qquad \begin{array}{l} a_h = b_h = d_h \ \text{ for } h \neq i, \\ d_i = a_i + b_i, \end{array}$$

$$c_i^-(t_1,..., t_n) = (t_1,..., t_n), \qquad c_i^+(t_1,..., t_n) = (t_1,..., a_i + t_i,..., t_n).$$

Now, given two *i-consecutive* Moore n-cubes $x: \prod_h I(a_h) \to X$ and $y: \prod_h I(b_h) \to X$ (with $\partial_i^+x = \partial_i^-y$), their i-concatenation $z = x +_i y$ is computed on the previous pushout

(2) $z: \prod_h I(d_h) \to \mathbf{X}, \qquad z.c_i^- = x, \quad z.c_i^+ = y,$

$sp(z) = (a_1,..., a_i + b_i,..., a_n) = (b_1,..., a_i + b_i,..., b_n),$

$z(t_1,..., t_n) = x(t_1,..., t_n)$ for $t_i \leq a_i,$

$z(t_1,..., t_n) = y(t_1,..., t_i - a_i,..., t_n)$ for $t_i \geq a_i.$

$\mathbb{M}(X)$ becomes thus a *basic symmetric cubical category*, i.e. a symmetric cubical set with 'geometrically consistent' cubical compositions, that satisfy a strict interchange law, *are strictly associative and have strict identities given by degeneracies*. (Again, the term 'basic' refers to the fact that transversal maps have not yet been added.)

5.5. Moore reparametrisation mappings. An *n-dimensional (directed) reparametrisation mapping* with *domain-span* $(a_1,..., a_n)$ and *codomain-span* $(b_1,..., b_n)$.

(1) $f: I(a_1,..., a_n) \rightarrow I(b_1,..., b_n),$

will be a d-map (i.e. an order-preserving continuous mapping) which sends each face of the domain multi-interval to the corresponding face of the codomain, i.e. satisfies the following equivalent conditions (for $i = 1,..., n$ and $\alpha = 0,1$)

(a) $f(\partial_i^\alpha(\Pi_h I(a_h))) \subset \partial_i^\alpha(\Pi_h I(b_h)),$

(b) $f.\partial_i^\alpha = \partial_i^\alpha.e_i.f.\partial_i^\alpha,$

(in the second formula, notice that e_i is determined by its domain, which is the codomain of f).

Again, the faces of any dimension are transformed *onto* the corresponding ones and f *itself is surjective* (by the obvious extension of Theorem 3.5). In particular, $a_h = 0$ implies $b_h = 0$. The topological dimension of the domain is thus greater than or equal to that of the codomain, while the formal dimensions are the same.

Multi-intervals of formal dimension n and their reparametrisation mappings form a category $\mathcal{R}_n$, under the usual composition. There are faces, transpositions and degeneracies:

(2) $\underline{\partial}_i^\alpha: \mathcal{R}_n \rightleftarrows \mathcal{R}_{n-1}, \qquad \underline{s}_i: \mathcal{R}_n \rightarrow \mathcal{R}_n, \qquad \underline{e}_i: \mathcal{R}_n \rightarrow \mathcal{R}_{n+1},$

$\underline{\partial}_i^\alpha(f) = e_i.f.\partial_i^\alpha: \Pi_{h\neq i} I(a_h) \rightarrow \Pi_h I(a_h) \rightarrow \Pi_h I(b_h) \rightarrow \Pi_{h\neq i} I(b_h),$

$\underline{s}_i(f) = s_i.f.s_i\colon \Pi_h\, I(a_{\sigma_i(h)}) \to \Pi_h\, I(a_h) \to \Pi_h\, I(b_h) \to \Pi_h\, I(b_{\sigma_i(h)}),$

$\underline{e}_i(f)\colon \Pi_{h<i}\, I(a_h) \times \{0\} \times \Pi_{h>i}\, I(a_h) \to \Pi_{h<i}\, I(b_h) \times \{0\} \times \Pi_{h>i}\, I(b_h),$

$\underline{e}_1(f)(0, t_2,..., t_n) = (0, f(t_2,..., t_n)), \qquad \underline{e}_{i+1}(f) = \underline{s}_i(....\underline{s}_2(\underline{s}_1(\underline{e}_1(f)))...).$

Moore reparametrisation mappings, with the faces, degeneracies and transpositions defined above, form *a symmetric cubical object in the category of small categories*. (The proof is similar to that of Theorem 3.2, for standard reparametrisations.)

Notice that these faces and transpositions extend those of standard reparametrisations, while *degeneracies do not*. Here $\underline{e}_i(f)$ is determined by the following condition

(3) $e^0_i.\underline{e}_i(f) = f.e^0_i.$

5.6. Concatenating reparametrisation mappings. The cubical object $\mathcal{R}$ has the following i-directed concatenation. Take two reparametrisation mappings f, g that are i-consecutive, i.e. $\underline{\partial}^+_i f = \underline{\partial}^-_i g$

(1) $f\colon \Pi_h\, I(a_h) \to \Pi_h\, I(b_h), \qquad g\colon \Pi_h\, I(c_h) \to \Pi_h\, I(d_h),$

$(a_h = c_h,\ b_h = d_h, \quad \text{for } h \neq i).$

Their i-concatenation has spans $(a_1,..., a_i + c_i,..., a_n)$ and $(b_1,..., b_i + d_i,..., b_n)$

(2) $$(f +_i g)(t_1,.., t_n) = \begin{cases} f(t_1,..., t_i,..., t_n), & \text{for } 0 \leq t_i \leq a_i, \\ (0,..., b_i,..., 0) + g(t_1,..., t_i - a_i,..., t_n), & \text{for } a_i \leq t_i \leq a_i + b_i. \end{cases}$$

For a d-space X, reparametrisation of its Moore cubes agrees with faces, degeneracies, transpositions and i-indexed compositions, in the following sense

(3) $\partial^\alpha_i(xf) = \partial^\alpha_i(x).\underline{\partial}^\alpha_i(f), \qquad e_i(xf) = e_i(x).\underline{e}_i(f), \qquad s_i(xf) = s_i(x).\underline{s}_i(f),$

(4) $(x +_i y).(f +_i g) = xf +_i yg,$

where f, g are i-consecutive mappings in $\mathcal{R}_n$, as in 5.6.1, and x, y are i-consecutive Moore n-cubes (such that xf and yg are defined). The proof is similar to that of Proposition 3.4, for standard reparametrisations.

5.7. Moore transversal maps. For a d-space X, a *transversal map* f: x → y between two Moore n-cubes x, y of X will be a reparametrisation mapping such that x = yf

(1) $f\colon \Pi_h I(a_h) \to \Pi_h I(b_h), \qquad y\colon \Pi_h I(b_h) \to X, \qquad x = yf\colon \Pi_h I(a_h) \to X.$

Also here, a transversal map should be defined as a triple $\hat{f} = (f, x, y)$, and we will use this notation when useful. Again, y determines x (and conversely if f is bijective).

The Moore n-cubes of X and their transversal maps form a category $\uparrow\mathbb{M}\mathrm{Sng}_n(X)$, with obvious faces, identities and composition

(2) $\partial_0^-(f, x, y) = x, \quad \partial_0^+(f, x, y) = y, \qquad e_0(x) = (\mathrm{id}, x, x),$

$c_0(f, g) = gf\colon x \to z \qquad$ (for $g\colon y \to z$, so that $x = yf = zgf$).

This category is discrete in degree 0: the only transversal maps between 0-cubes are the identities.

Transversal maps also form a symmetric cubical set, using the cubical structure of Moore cubes (5.3) and of their reparametrisation maps (5.5)

(3) $\partial_i^\alpha(f, x, y) = (\underline{\partial}_i^\alpha f,\ \partial_i^\alpha x,\ \partial_i^\alpha y), \qquad e_i(f, x, y) = (\underline{e}_i f,\ e_i x,\ e_i y),$

$s_i(f, x, y) = (\underline{s}_i f,\ s_i x,\ s_i y).$

Finally, one defines the i-concatenation of i-consecutive transversal maps as

(4) $(f, x, y) +_i (g, z, u) = (f +_i g,\ x +_i z,\ y +_i u) \qquad (\partial_i^+(f, x, y) = \partial_i^-(g, z, u)),$

where $f +_i g$ is the i-concatenation of reparametrisation maps (5.6).

This completes the definition of the *Moore symmetric cubical category* $\uparrow\mathbb{M}\mathrm{Sng}(X)$ of the d-space X.

6. Some hints to lax cubical structures of tangles

We end with a few hints to a family $\mathbb{T}(A)$ of u-lax symmetric cubical categories, depending on a topological space A, and related to higher categories of tangles, as considered in [BL, Ch].

In this section the faces, degeneracies and transpositions of the standard cubes $\mathbf{I}^n$ are written as $\delta_i^\alpha: \mathbf{I}^{n-1} \to \mathbf{I}^n$, $\varepsilon_i: \mathbf{I}^n \to \mathbf{I}^{n-1}$ and $\sigma_i: \mathbf{I}^n \to \mathbf{I}^n$.

6.1. A preparatory structure. Let us come back to the u-lax symmetric cubical category $\mathbb{S}\mathrm{ng}(S)$ associated to a topological space S (4.6).

It is interesting to note that, if we start from the discrete space on two points $\mathbf{S}^0 = \{-1, 1\}$, a singular n-cube $x: \mathbf{I}^n \to \mathbf{S}^0$ can be identified with a subset $X \subset \mathbf{I}^n$, namely the counterimage $x^{-1}\{1\}$. We obtain thus a particular u-lax symmetric cubical category $\mathbb{T} = \mathbb{S}\mathrm{ng}(\mathbf{S}^0)$, which can be viewed as a starting point to define a u-lax symmetric cubical category of tangles.

Concretely, $\mathbb{T}$ can be described in the following terms.

(a) An n-cube is a subset $X \subset \mathbf{I}^n$.

(b) Faces, degeneracies and transpositions of n-cubes are obtained as counterimages of the corresponding maps δ_i^α, ε_i and σ_i between standard cubes

(1) $\partial_i^\alpha(X) = (\delta_i^\alpha)^{-1}(X), \qquad e_i(X) = (\varepsilon_i)^{-1}(X), \qquad s_i(X) = (\sigma_i)^{-1}(X) = \sigma_i(X).$

(c) The i-concatenation $X +_i Y$ of i-consecutive n-cubes is defined by the union of their images in two i-consecutive halves of $\mathbf{I}^n$:

(2) $X +_i Y = \varphi_i(X) \cup \psi_i(Y),$

$\varphi_i(t_1,..., t_n) = (t_1,..., t_i/2,..., t_n), \qquad \psi_i(t_1,..., t_n) = (t_1,..., (t_i+1)/2,..., t_n).$

(d) A transversal map $(f, X, Y): X \to Y$ is given by a reparametrisation mapping $f: \mathbf{I}^n \to \mathbf{I}^n$ (see 4.6) such that $X = f^{-1}(Y)$ (which *implies* $f(X) = Y$, because f is surjective).

(e) Their faces are (again) defined by the faces of reparametrisation mappings *and* of n-cubes; similarly for degeneracies, transpositions and concatenations

(3) $\partial_i^\alpha(f, x, y) = (\underline{\partial}_i^\alpha f,\ \partial_i^\alpha x,\ \partial_i^\alpha y), \qquad e_i(f, x, y) = (\underline{e}_i f,\ e_i x,\ e_i y),$

$s_i(f, x, y) = (\underline{s}_i f,\ s_i x,\ s_i y),$

$(f, x, y) +_i (g, z, u) = (f +_i g,\ x +_i z,\ y +_i u) \qquad (\partial_i^+(f, x, y) = \partial_i^-(g, z, u)).$

(f) Interchange is strict. Invertible comparisons for associativity and non-invertible ones for unitarity are given by the reparametrisation mappings κ_i, λ_i, ρ_i: $\mathbf{I}^n \to \mathbf{I}^n$ defined in 4.2. 4.3.

Replacing the discrete topology on $\{-1, 1\}$ with the *Sierpinski topology*, where the point 1 is open (resp. closed), we obtain the substructure $\mathbb{T}'$ (resp. $\mathbb{T}''$) whose n-cubes are the open (resp. closed) subsets of $\mathbf{I}^n$. Replacing $\mathbf{S}^0$ with the discrete space $S_p = \{0, 1,..., p\}$ on p+1 points, we obtain a u-lax symmetric cubical category $\mathbb{T}_p = \mathbb{S}\mathrm{ng}(S_p)$ where an n-cube amounts to an (ordered!) family $(X_1,..., X_p)$ of p disjoint subsets of $\mathbf{I}^n$.

6.2. Tangles. Finally, to approach the theory of tangles, we modify the previous construction obtaining a u-lax symmetric cubical category $\mathbb{T}(A)$, depending on a fixed topological space A, and giving back $\mathbb{T}$ when A is the singleton. (A standard case would be to choose the k-dimensional cube $\mathbf{I}^k$.)

$\mathbb{T}(A)$ is defined as follows.

(a) An n-cube is a subset $X \subset A \times \mathbf{I}^n$.

(b) Faces, degeneracies and transpositions of n-cubes are obtained as counterimages

(1) $\partial_i^\alpha(X) = (A \times \delta_i^\alpha)^{-1}(X)$, $\quad e_i(X) = (A \times \varepsilon_i)^{-1}(X)$,

$s_i(X) = (A \times \sigma_i)^{-1}(X) = (A \times \sigma_i)(X)$.

(c) The i-concatenation $X +_i Y$ of (i-consecutive) n-cubes is defined by the union of their images in two i-consecutive halves of $A \times \mathbf{I}^n$:

(2) $X +_i Y = (A \times \varphi_i)(X) \cup (A \times \psi_i)(Y)$.

(d) A transversal map $(f, X, Y): X \to Y$ is given by a reparametrisation mapping $f: \mathbf{I}^n \to \mathbf{I}^n$ (see 4.6) such that $X = (A \times f)^{-1}(Y)$ (which implies $(A \times f)(X) = Y$).

(e) Their faces, degeneracies, transpositions and concatenations are defined as in 6.1.3. Comparisons as in 6.1(f).

More generally, the u-lax symmetric cubical category $\mathbb{T}_p = \mathbb{S}\mathrm{ng}(S_p)$ considered above yields a structure $\mathbb{T}_p(A)$, of interest for *p-colored tangles*.

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