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REPRESENTABILITY OF THE SPLIT EXTENSION FUNCTOR FOR CATEGORIES OF GENERALIZED LIE ALGEBRAS

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Abstract

For an additive symmetric closed monoidal category $\mathbb{C}$ with equalizers, suppose M is a monoid defined with respect to the monoidal structure. In this setting we can define a *Lie algebra* with respect to M and the monoidal structure. For the category **Lie**$(M, \mathbb{C})$ of Lie algebras we show that the functor SplExt$(-, X)$: **Lie**$(M, \mathbb{C}) \to$ **Set** is representable by constructing a representation.

Pour une catégorie additive symétrique monoïdale fermée $\mathbb{C}$ avec égalisateurs, soit M un monoïde défini par rapport à la structure monoïdale. Dans ce contexte nous pouvons définir une *algèbre de Lie* par rapport à M et à la structure monoïdale. Pour la catégorie **Lie**$(M, \mathbb{C})$ d'algèbres de Lie nous montrons que le foncteur SplExt$(-, X)$: **Lie**$(M, \mathbb{C}) \to$ **Set** est représentable en construisant une représentation.

Introduction

We recall that for a Lie algebra X over a commutative ring R, a map $f : X \to X$ is called a derivation of X if f is linear and, for all x and y in X, $f(xy) = f(x)y + xf(y)$. The set Der(X) of all derivations on X can be made into a Lie algebra with Lie multiplication $fg = f \circ g - g \circ f$ and all

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other operations defined pointwise. A diagram

$$X \xrightarrow{k} A \underset{s}{\overset{p}{\rightleftarrows}} G$$

where k is the kernel of p and $ps = 1_G$ is called a split extension of G with kernel X. Any morphism between split extensions, that is, a diagram

$$\begin{array}{ccccc} X & \xrightarrow{k} & A & \underset{s}{\overset{p}{\rightleftarrows}} & G \\ \| & & \downarrow{\scriptstyle f} & & \| \\ X & \xrightarrow{k'} & A' & \underset{s'}{\overset{p'}{\rightleftarrows}} & G \end{array}$$

where the top and bottom rows are split extensions, and $fk = k'$, $p = p'f$ and $fs = s'$, is invertible since the split short five lemma holds for Lie algebras. We define an equivalence relation on the set of split extensions of G with kernel X, by requiring that extensions are equivalent if and only if there is a morphism between them. The functor SplExt$(-, X)$: $\mathbf{Lie}_R \to \mathbf{Set}$ is defined on an object G as the set of equivalence classes of split extensions of G with kernel X and on a morphism $g : G' \to G$ by pulling back. A well-known classical result can be stated as: the functor SplExt$(-, X)$ is representable with Der(X) the object of the representation, that is, there is a natural isomorphism SplExt$(-, X) \cong \mathbf{Lie}_R(-, \text{Der}(X))$. This result can be extended to any category of internal Lie algebras defined in a cartesian closed category (see Theorem 5.2 in [1]). We will generalize this result in a different direction, namely to suitably define Lie algebras over a monoid M in an additive symmetric monoidal closed category. Introducing this concept requires some auxiliary observations:

Recall that a commutative monoid in a symmetric monoidal category $(\mathbb{C}, \otimes, I, \alpha, \rho, \lambda, \sigma)$ is an object M together with two morphisms

$$\mu : M \otimes M \to M, \qquad \eta : \mathbb{Z} \to M$$

such that the diagrams

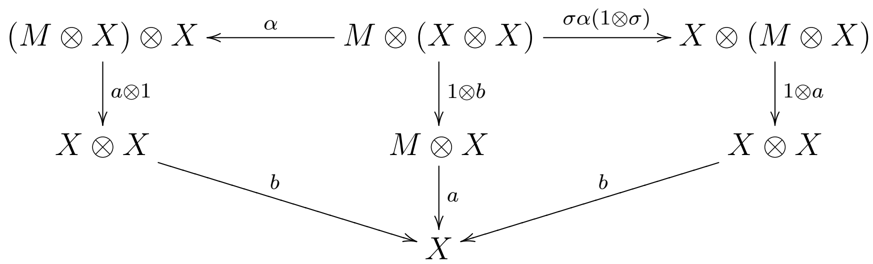
$$\begin{array}{ccccc}
M\otimes(M\otimes M) & & \xrightarrow{\alpha} & & (M\otimes M)\otimes M\\
\downarrow{\scriptstyle 1\otimes\mu} & & & & \downarrow{\scriptstyle \mu\otimes 1}\\
M\otimes M & \xrightarrow{\mu} & M & \xleftarrow{\mu} & M\otimes M\\
\downarrow{\scriptstyle\sigma} & \nearrow{\scriptstyle\mu} & & & \\
M\otimes M & & & &
\end{array}$$

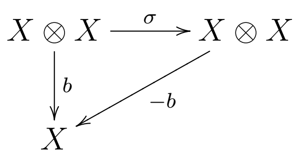
$$\begin{array}{ccccc}
I\otimes M & \xrightarrow{\eta\otimes 1} & M\otimes M & \xleftarrow{1\otimes\mu} & M\otimes I\\
 & \searrow{\scriptstyle\lambda} & \downarrow{\scriptstyle\mu} & \swarrow{\scriptstyle\rho} & \\
 & & M & &
\end{array}$$

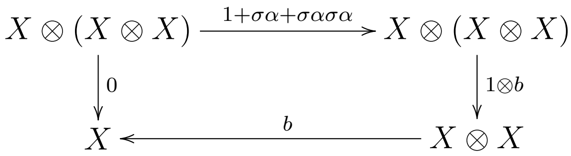
are commutative. Let us recall that when $(\mathbb{C}, \otimes, I, \alpha, \rho, \lambda, \sigma) = (\mathbf{Ab}, \otimes, \mathbb{Z}, \alpha, \rho, \lambda, \sigma)$ is the usual symmetric monoidal category of abelian groups, a commutative monoid in it is the same as a commutative ring. In this case the morphism $\mu : M \otimes M \to M$ corresponds, via the universal property of the tensor product, to a map $M \times M \to M$, call it multiplication, which is bilinear (distributive with respect to the addition of the abelian group M). The morphism $\eta : \mathbb{Z} \to M$ is determined by picking an element u in M, the image of 1. Furthermore, the commutativity of the first diagram means that multiplication is associative and commutative, while the commutativity of the second means that η makes u the identity element of M.

For an ordinary Lie algebra X over a commutative ring M, the scalar multiplication $M \times X \to X$ and the Lie multiplication $X \times X \to X$ are bilinear maps, and so by the universal property of the tensor product in **Ab** they can be described as morphisms $a : M \otimes X \to X$ and $b : X \otimes X \to X$ respectively. The commutativity of the diagrams

$$\begin{array}{ccc}
M\otimes(M\otimes X) & \xrightarrow{\alpha} & (M\otimes M)\otimes X\\
\downarrow{\scriptstyle 1\otimes a} & & \downarrow{\scriptstyle \mu\otimes 1}\\
M\otimes X & & M\otimes X\\
 & \searrow{\scriptstyle a} & \downarrow{\scriptstyle a}\\
 & & X
\end{array}$$







state that

$$(mn)x = m(nx), \quad (mx)y = m(xy) = x(my), \quad xy = -yx,$$

$$x(yz) + z(xy) + y(zx) = 0$$

for all $m, n \in M$ and for all $x, y, z \in X$. These identities correspond to the axioms of a Lie algebra except that we have replaced the axiom $xx = 0$ ($x \in X$), with the axiom $xy = -yx$ ($x, y \in X$). Assuming the axiom $xx = 0$, the well known argument

$$xy = xx + xy + yx + yy - yx = (x + y)(x + y) - yx = -yx$$

shows that we have actually replaced an axiom with a formally weaker one. Assuming the axiom $xy = -yx$, the argument

$$2xx = xx + xx = xx - xx = 0$$

shows that when 2 has a multiplicative inverse in M, the two axioms are equivalent. When M is a field this corresponds to saying that M is not of characteristic 2. Since the axiom $xx = 0$ has a repeated variable in it, it is not possible to express it as the commutativity of a diagram involving tensor products. Therefore, in order to define a Lie algebra in an abstract symmetric monoidal category $(\mathbb{C}, \otimes, I, \alpha, \rho, \lambda, \sigma)$ we introduce an additional structure on $\mathbb{C}$. The structure we choose in this paper consists of a category $\mathbb{D}$, functors $U, V : \mathbb{C} \to \mathbb{D}$, and a natural transformation $\delta : U \to V(- \otimes -)$, satisfying suitable conditions (see Section 1). In Section 1 we define a generalized Lie algebra following the above as motivation. In Section 2 we define in this new setting the generalized Lie algebra of derivations and show, in Section 3, that the functor of split extensions from the category of these generalized Lie algebras to the category of sets is representable. We conclude Section 3 by remarking that the functor of split extensions of crossed modules of these generalized Lie algebras is representable.

1 Algebraic structures in monoidal categories

In this section we introduce the needed algebraic structures to define a generalized Lie algebra and construct in this context the functor which in the classical case takes associative algebras to Lie algebras. Throughout this paper we will assume that:

1. $\mathbb{C} = (\mathbb{C}, \otimes, I, \alpha, \lambda, \rho, \sigma)$ is an additive symmetric monoidal category with all finite limits; in addition we assume it to be monoidal closed, although in this section we only use the fact that the tensor is distributive with respect to finite products;

2. $(M, \mu : M \otimes M \to M, \eta : I \to M)$ is a commutative monoid in $\mathbb{C}$;

3. $\mathbb{D}$ is a category in which hom-sets are abelian groups;

4. Composition of morphisms in $\mathbb{D}$ is distributive on the right with respect to addition of morphisms, that is, for any morphisms $f, g : B \to C$ and $h : A \to B$ we have $(f + g)h = fh + gh$;

5. U and V are functors from $\mathbb{C}$ to $\mathbb{D}$ and V restricted to hom-sets is an abelian group homomorphism;

6. δ is a natural transformation from U to $V(- \otimes -)$ such that:

   Condition 1.1. *For any $C \in \mathbb{C}$ the diagram*

$$\begin{array}{ccc} UC & \xrightarrow{\delta_C} & V(C \otimes C) \\ & \searrow^{\delta_C} & \downarrow{\scriptstyle V\sigma} \\ & & V(C \otimes C) \end{array}$$

   commutes.

Example 1.2. *$(\mathbb{C}, \otimes, I, \alpha, \lambda, \rho, \sigma) = (\mathbf{Ab}, \otimes, \mathbb{Z}, \alpha, \rho, \lambda, \sigma)$ is the the usual symmetric monoidal category of abelian groups, $\mathbb{D} = \mathbf{ab}$ is the category with objects all abelian groups and morphisms all maps between their underlying sets, $U = V : \mathbf{Ab} \to \mathbf{ab}$ is the inclusion functor, and δ is defined by $\delta_C(c) = c \otimes c$ for all C in* **Ab** *and c in C. This example explains the main purpose of introducing $\mathbb{D}$, U, V, and δ: the axiom $xx = 0$ mentioned in the Introduction can now be expressed categorically as $V(b)\delta_X = 0$, where $b : X \otimes X \to X$ is a multiplication morphism on an object X (as in the Introduction).*

We recall: (i) an M-action is a pair (X, a), where X is an object in $\mathbb{C}$ and $a : M \otimes X \to X$ is a morphism in $\mathbb{C}$, such that the diagrams

$$\begin{array}{ccccc} M \otimes (M \otimes X) & \xrightarrow{\alpha} & & & (M \otimes M) \otimes X \\ \downarrow{\scriptstyle 1\otimes a} & & & & \downarrow{\scriptstyle a\otimes 1} \\ M \otimes X & \xrightarrow{a} & X & \xleftarrow{a} & M \otimes X \end{array} \qquad \begin{array}{ccc} I \otimes X & \xrightarrow{\eta\otimes 1} & M \otimes X \\ & {\scriptstyle\lambda}\searrow & \downarrow{\scriptstyle a} \\ & & X \end{array}$$

commute; (ii) a magma defined with respect to the monoidal structure in $\mathbb{C}$ is a pair (X, b), where X is an object in $\mathbb{C}$ and $b : X \otimes X \to X$ is a morphism in $\mathbb{C}$.

Definition 1.3. *A triple $(X, a : M \otimes X \to X, b : X \otimes X \to X)$ is said to be an M-magma if (X, a) is an M-action for the monoid M. For M-magmas (X, a, b) and (X', a', b'), a morphism $f : X \to X'$ in $\mathbb{C}$ is an M-magma morphism if the diagrams*

$$\begin{array}{ccc} M \otimes X & \xrightarrow{1\otimes f} & M \otimes X' \\ \downarrow{\scriptstyle a} & & \downarrow{\scriptstyle a'} \\ X & \xrightarrow{f} & X' \end{array} \qquad \begin{array}{ccc} X \otimes X & \xrightarrow{f\otimes f} & X' \otimes X' \\ \downarrow{\scriptstyle b} & & \downarrow{\scriptstyle b'} \\ X & \xrightarrow{f} & X' \end{array}$$

commute; that is, f must be a morphism of magmas and a morphism of M-actions at the same time. The category of M-magmas will be denoted M-$\mathbf{Mag}_0$.

For an M-magma (X, a, b) consider the following condition:

Condition 1.4. *(a) The diagram*

$$\begin{array}{ccccc} M\otimes(X\otimes X) & & \xrightarrow{\quad\alpha\quad} & & (M\otimes X)\otimes X \\ \downarrow{\scriptstyle 1\otimes b} & & & & \downarrow{\scriptstyle a\otimes 1} \\ M\otimes X & \xrightarrow{\ a\ } & X & \xleftarrow{\ b\ } & X\otimes X \end{array}$$

commutes;

(b) The diagram

$$\begin{array}{ccccc} M\otimes(X\otimes X) & & \xrightarrow{\quad\sigma\alpha(1\otimes\sigma)\quad} & & X\otimes(M\otimes X) \\ \downarrow{\scriptstyle 1\otimes b} & & & & \downarrow{\scriptstyle 1\otimes a} \\ M\otimes X & \xrightarrow{\ a\ } & X & \xleftarrow{\ b\ } & X\otimes X \end{array}$$

commutes.

Let M-**Mag**$_1$ be the full subcategory of M-**Mag**$_0$ with objects all M-magmas satisfying Conditions 1.4(a) and 1.4(b). Let M-**Mag**$_2$ be the full subcategory of M-**Mag**$_1$ with objects all (X, a, b), in which the pair (X, b) is a semigroup, that is, the diagram

$$\begin{array}{ccccc} X\otimes(X\otimes X) & & \xrightarrow{\quad\alpha\quad} & & (X\otimes X)\otimes X \\ \downarrow{\scriptstyle 1\otimes b} & & & & \downarrow{\scriptstyle b\otimes 1} \\ X\otimes X & \xrightarrow{\ b\ } & X & \xleftarrow{\ b\ } & X\otimes X \end{array}$$

commutes. In the situation of Example 1.2 the categories M-**Mag**$_1$ and M-**Mag**$_2$ are the categories of non-associative and associative M-algebras respectively.

For a magma (X, b) we are also going to use the following conditions:

Condition 1.5. $V(b)\delta_X = 0$.

Condition 1.6. *(a)* $b(1 + \sigma) = 0$ *(anticommutativity);*

(b) $b(1 \otimes b)(1 + \sigma\alpha + \sigma\alpha\sigma\alpha) = 0$ *(Jacobi identity).*

Remark 1.7. *When $\mathbb{C} = \mathbb{D}$, $V = 1$, $U = (- \otimes -)$ and $\delta = 1 + \sigma$, Condition 1.6(a) becomes an instance of Condition 1.5.*

Let $\mathbf{Lie}(M, \delta)$ be the full subcategory of M-$\mathbf{Mag}_1$ with objects all (X, a, b), in which the magma (X, b) satisfies Conditions 1.5, 1.6(a) and 1.6(b). In the situation of Example 1.2, as in fact mentioned in the Introduction, Conditions 1.5, 1.6(a) and 1.6(b) correspond to the identities

$$xx = 0, \qquad xy + yx = 0, \qquad x(yz) + z(xy) + y(zx) = 0$$

respectively, and recalling that the category M-$\mathbf{Mag}_1$ is the category of non-associative algebras we see that the category $\mathbf{Lie}(M, \delta)$ is the category of Lie algebras over the commutative ring M.

Remark 1.8. *If $\mathbb{D} = \mathbb{C}$, $U = V = 1_{\mathbb{C}}$ and δ_C is the zero morphism, then Condition 1.5 is trivially satisfied by any magma (X, b). If in addition, as in Example 1.2, $(\mathbb{C}, \otimes, I, \alpha, \lambda, \rho, \sigma) = (\mathbf{Ab}, \otimes, \mathbb{Z}, \alpha, \lambda, \rho, \sigma)$ is the usual symmetric monoidal category of abelian groups, the category $\mathbf{Lie}(M, \delta)$ has as objects Lie algebras, except that the axiom $xx = 0$ has been replaced by the axiom $xy = -yx$.*

If $(X, a : R \times X \to X, b : X \times X \to X)$ is an associative algebra over a ring R and if we define $\tilde{b} : X \times X \to X$ as

$$\tilde{b}(x, y) = b(x, y) - b(y, x)$$

for all $x, y \in X$, then the triple $(X, a, \tilde{b})$ is a Lie algebra defined with respect to the ring R. This correspondence of associative algebras and Lie algebras is functorial and can been extended to our setting.

Theorem 1.9. *If $(X, a, b) \in M$-$\mathbf{Mag}_2$, then $(X, a, b(1 - \sigma)) \in \mathbf{Lie}(M, \delta)$ and the assignment $(X, a, b) \longmapsto (X, a, b(1 - \sigma))$ defines a functor $L : M$-$\mathbf{Mag}_2 \to \mathbf{Lie}(M, \delta)$ which is identity on morphisms.*

Proof. Let $\tilde{b} = b(1 - \sigma)$. It is clear that $(X, a, \tilde{b})$ is an M-magma. Condition 1.4(a) holds for $(X, a, \tilde{b})$ since

$$\begin{aligned} a(1 \otimes \tilde{b}) &= a(1 \otimes (b(1 - \sigma))) = a(1 \otimes b) - a(1 \otimes b)(1 \otimes \sigma) \\ &= b(a \otimes 1)\alpha - b(1 \otimes a)\sigma\alpha(1 \otimes \sigma)(1 \otimes \sigma) \\ &= b(1 - \sigma)(a \otimes 1)\alpha = \tilde{b}(a \otimes 1)\alpha, \end{aligned}$$

where the third equality follows by Conditions 1.4(a) and 1.4(b) for (X,a,b). Similarly, it can easily be seen that Condition 1.4(b) holds for $(X,a,\tilde{b})$. To show that the Jacobi identity, Condition 1.6(b), holds for $(X,a,\tilde{b})$, consider the equation:

$$\begin{aligned}
&\tilde{b}(1\otimes\tilde{b})(1+\sigma\alpha+\sigma\alpha\sigma\alpha)\\
&\quad= b(1-\sigma)(1\otimes b)(1-1\otimes\sigma)(1+\sigma\alpha+\sigma\alpha\sigma\alpha)\\
&\quad= b((1\otimes b)(1-1\otimes\sigma)-\sigma(1\otimes b)(1-1\otimes\sigma))(1+\sigma\alpha+\sigma\alpha\sigma\alpha)\\
&\quad= b(1\otimes b)^{(1)}+b(1\otimes b)\sigma\alpha^{(2)}+b(1\otimes b)\sigma\alpha\sigma\alpha^{(3)}\\
&\qquad -b(1\otimes b)(1\otimes\sigma)^{(4)}-b(1\otimes b)(1\otimes\sigma)\sigma\alpha^{(5)}-b(1\otimes b)(1\otimes\sigma)\sigma\alpha\sigma\alpha^{(6)}\\
&\qquad -b(b\otimes 1)\sigma^{(3)}-b(b\otimes 1)\alpha^{(1)}-b(b\otimes 1)\alpha\sigma\alpha^{(2)}\\
&\qquad +b(b\otimes 1)(\sigma\otimes 1)\sigma^{(5)}+b(b\otimes 1)(\sigma\otimes 1)\alpha^{(6)}+b(b\otimes 1)(\sigma\otimes 1)\alpha\sigma\alpha^{(4)}\\
&\quad= 0,
\end{aligned}$$

where composites labelled with the same superscript are equal. For, we only need to observe that $b(1\otimes b)=b(b\otimes 1)\alpha$ since (X,b) is a semigroup, and use that directly for (1) and (2), or together with $\alpha\sigma\alpha\sigma\alpha=\sigma$ for (3), or together with $\alpha(1\otimes\sigma)=(\sigma\otimes 1)\alpha\sigma\alpha$ for (4), or together with $\alpha(1\otimes\sigma)\sigma\alpha=(\sigma\otimes 1)\sigma$ for (5), or together with $\alpha(1\otimes\sigma)\sigma\alpha\sigma\alpha=(\sigma\otimes 1)\sigma$ for (6). From Condition 1.1 and the definition of $\tilde{b}$ it follows that Conditions 1.5 and 1.6(a) hold for $(X,\tilde{b})$. For a morphism

$$(X,a,b)\xrightarrow{f}(X',a',b')$$

let $\tilde{b}'=b'(1-\sigma)$. By calculating

$$\begin{aligned}
\tilde{b}'(f\otimes f) &= b'(1-\sigma)(f\otimes f)\\
&= b'(f\otimes f-\sigma(f\otimes f))\\
&= b'(f\otimes f-(f\otimes f)\sigma)\\
&= b'(f\otimes f)(1-\sigma)\\
&= fb(1-\sigma)\\
&= f\tilde{b},
\end{aligned}$$

we see that f is a morphism in **Lie**(M,δ). □

2 Construction of derivations

In this section we construct, for an object (X, a, b) in **Lie**(M, δ), the object Der(X), which will be shown in Section 3 to be the representing object for the functor SplExt$(-, X)$: **Lie**$(M, \delta) \to$ **Set**.

Recall that, for a Lie algebra X over a commutative ring M, the Lie algebra of derivations, Der(X), can be constructed as follows. For abelian groups A and B, let Hom(A, B) be the abelian group of homomorphisms from A to B. Defining multiplication by composition and scalar multiplication pointwise, it is easily seen that Hom(X, X) satisfies the axioms of a ring as well as those of an M-module and, moreover has scalar multiplication with the property

$$m(h_1 \circ h_2) = (mh_1) \circ h_2$$

for all $m \in M$ and $h_1, h_2 \in$ Hom(X, X). The abelian group $E(X)$ of M-module morphisms from X to X can be constructed as the equalizer of the diagram

$$\text{Hom}(X, X) \underset{f_2}{\overset{f_1}{\rightrightarrows}} \text{Hom}(M \times X, X)$$

where f_1 and f_2 are defined by

$$f_1(h)(m, x) = mh(x), \qquad f_2(h)(m, x) = h(mx)$$

for all $h \in$ Hom(X, X), $m \in M$ and $x \in X$. It is easily seen that $E(X)$ is closed under the operations defined for Hom(X, X) and has the property

$$m(h_1 \circ h_2) = h_1 \circ (mh_2)$$

for all $m \in M$ and $h_1, h_2 \in E(X)$, i.e. $E(X)$ is an associative M-algebra. As described before, any associative M-algebra $E(X)$ becomes a Lie algebra with Lie multiplication defined by

$$h_1 h_2 = h_1 \circ h_2 - h_2 \circ h_1$$

for all $h_1, h_2 \in E(L)$. Finally, the Lie algebra of derivations Der(X), can be constructed as the equalizer of the diagram

$$E(X) \underset{g_2 e}{\overset{g_1 e}{\rightrightarrows}} \text{Hom}(X \times X, X)$$

where $e : E(X) \to \mathrm{Hom}(X, X)$ is the equalizer of f_1 and f_2, and g_1 and g_2 are defined by

$$g_1(h)(x_1, x_2) = h(x_1x_2), \qquad g_2(h)(x_1, x_2) = h(x_1)x_2 + x_1h(x_2)$$

for all $h \in \mathrm{Hom}(X, X)$ and $x_1, x_2 \in L$. $\mathrm{Der}(X)$ can be seen to be closed under the operations defined for $E(X)$ and hence is a Lie algebra.

We show that this construction extends to our general context. We begin by showing that for $(X, a, b) \in \mathbf{Lie}(M, \delta)$ the internal hom-object X^X can be given a semigroup structure as well as an M-magma structure that satisfies Condition 1.4(a). We then construct the semigroup $E(X)$ as a regular sub-M-magma of the internal hom-object X^X and show that it satisfies Condition 1.4(b). We then apply the functor $L : M\text{-}\mathbf{Mag}_2 \to \mathbf{Lie}(M, \delta)$ to $E(X)$ and construct $\mathrm{Der}(X)$ as a regular subobject of $L(E(X))$.

For each object B in $\mathbb{C}$, we will denote the chosen right adjoint to the functor $- \otimes B$ by $-^B$ and denote the chosen counit of the associated adjunction by ϵ^B. For functors $F : \mathbb{X} \to \mathbb{A}$ and $G : \mathbb{A} \to \mathbb{X}$, where G is the right adjoint of F, given a morphism $h : FX \to A$, the corresponding morphism $X \to GA$ will be called the right adjunct of h (as in [6]). Similarly, given a morphism $g : X \to GA$, the corresponding morphism $FX \to A$ will be called the left adjunct of g. That is, for $g : A \to C^B$, the left adjunct of g is $\epsilon^B_C(g \otimes 1) : A \otimes B \to C$.

For a pair $(X, a_X : M \otimes X \to X)$ where $M = (M, \mu, \eta)$ is a monoid in $\mathbb{C}$ as above, consider the following condition, which is part of the definition of an action for a monoid:

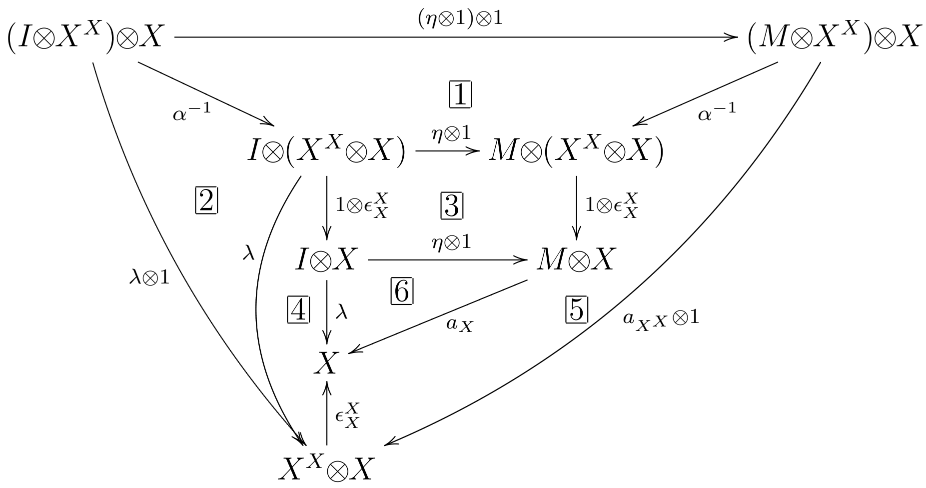
Condition 2.1. *The diagram*

$$\begin{array}{ccc} I\otimes X & \xrightarrow{\eta\otimes 1} & M\otimes X \\ & {\scriptstyle\lambda}\searrow \quad \swarrow{\scriptstyle a_X} & \\ & X & \end{array}$$

commutes.

Proposition 2.2. *If (X, a_X) satisfies Condition 2.1 and if $a_{X^X} : M \otimes X^X \to X^X$ is the right adjunct of $a_X(1 \otimes \epsilon^X_X)\alpha^{-1} : (M \otimes X^X) \otimes X \to X$ then (X^X, a_{X^X}) satisfies Condition 2.1.*

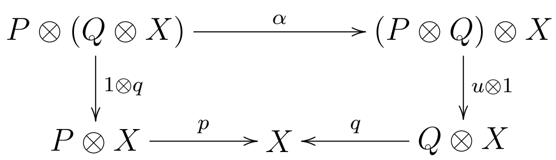
Proof. In the diagram



1 commutes since α is a natural transformation; 2 commutes as an immediate consequence of the axioms of a monoidal category; 3 commutes since $\otimes$ is a bifunctor; 4 commutes since λ is a natural transformation; 5 commutes by definition of $a_{X^X} : M \otimes X^X \to X^X$; 6 commutes by assumption on (X, a_X). That is, $\lambda\otimes 1 = (a_{X^X}\otimes 1)((\eta\otimes 1)\otimes 1) = (a_{X^X}(\eta\otimes 1))\otimes 1$, which tells us that the left adjuncts of the morphims $\lambda, a_{X^X}(\eta\otimes 1) : I\otimes X^X \to X^X$ are equal to each other. Therefore these two morphisms are equal to each other themselves, as desired. $\square$

For a sextuple $(P, Q, X, u : P \otimes Q \to Q, p : P \otimes X \to X, q : Q \otimes X \to X)$ we consider the following condition:

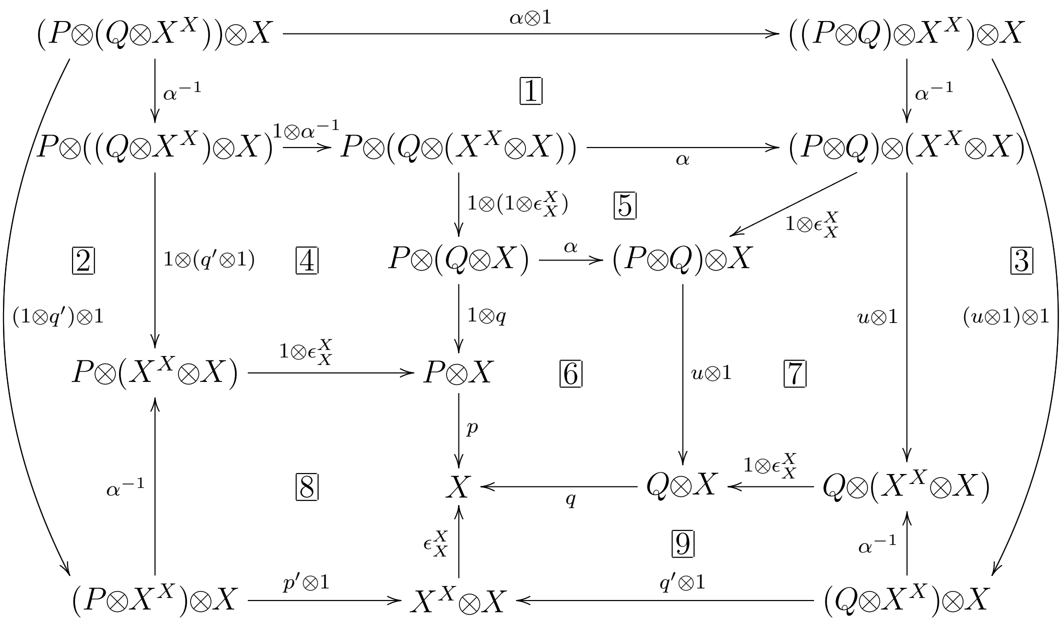
Condition 2.3. *The diagram*



commutes.

Lemma 2.4. *Suppose* $(P, Q, X, u : P \otimes Q \to Q, p : P \otimes X \to X, q : Q\otimes X \to X)$ *satisfies Condition 2.3,* $p' : P\otimes X^X \to X^X$ *is the right adjunct of* $p(1\otimes\epsilon^X_X)\alpha^{-1} : (P\otimes X^X)\otimes X \to X$ *and* $q' : Q\otimes X^X \to X^X$ *is the right* *adjunct of* $q(1 \otimes \epsilon_X^X)\alpha^{-1} : (Q \otimes X^X) \otimes X \to X$ *then* (P, Q, X^X, u, p', q') *satisfies Condition 2.3.*

Proof. In the diagram



[1] commutes by the axioms of a monoidal category; [2], [3] and [5] commute since α is natural transformation; [4] and [9] commute from the definition of q'; [8] commutes by the definition of p'; [7] commutes since $\otimes$ is a bifunctor; [6] commutes by assumption on u, p and q (Condition 2.3). That is, $(q' \otimes 1)((u \otimes 1) \otimes 1)(\alpha \otimes 1) = (p' \otimes 1)((1 \otimes q') \otimes 1)$, or, equivalently, $(q'(u \otimes 1)\alpha) \otimes 1 = (p'(1 \otimes q')) \otimes 1$ – which means that the left adjuncts of the morphisms $p'(1 \otimes q'), q'(u \otimes 1)\alpha : P \otimes (Q \otimes X^X) \to X^X$ are equal to each other. Therefore these two morphisms are equal to each other themselves, as desired. $\square$

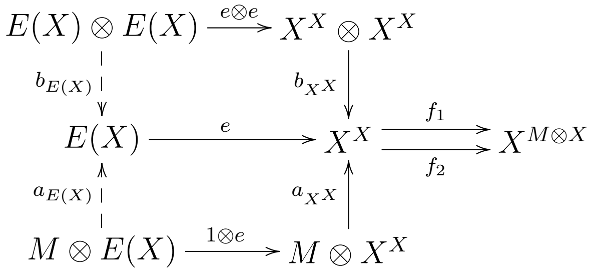
Proposition 2.5. *Let* (X, a_X) *be an* M*-action and, let* $a_{X^X} : M \otimes X^X \to X^X$ *and* $b_{X^X} : X^X \otimes X^X \to X^X$ *be the right adjuncts of* $a(1 \otimes \epsilon_X^X)\alpha^{-1} : (M \otimes X^X) \otimes X \to X$ *and* $\epsilon_X^X(1 \otimes \epsilon_X^X)\alpha^{-1} : (X^X \otimes X^X) \otimes X \to X$ *respectively. Then* (X^X, a_{X^X}) *is an* M*-action,* (X^X, b_{X^X}) *is a semigroup, and Condition 1.4(a) is satisfied.*

Proof. It is clear that since (X, a_X) is an M-action, the sextuple (M, M, X, μ, a_X, a_X) satisfies Condition 2.3. From Lemma 2.4 it follows that $(M, M, X^X, \mu, a_{X^X}, a_{X^X})$ satisfies Condition 2.3. This together with Proposition 2.2 applied to (X, a_X) shows that (X^X, a_{X^X}) is an M-action. From the definition of b_{X^X} we see that $(X^X, X^X, X, b_{X^X}, \epsilon^X_X, \epsilon^X_X)$ satisfies Condition 2.3 and by Lemma 2.4 $(X^X, X^X, X^X, b_{X^X}, b_{X^X}, b_{X^X})$ satisfies Condition 2.3 and therefore (X^X, b_{X^X}) is a semigroup. From the definition of a_{X^X} the sextuple $(M, X^X, X, a_{X^X}, a_X, \epsilon^X_X)$ satisfies Condition 2.3 and by Lemma 2.4 the sextuple $(M, X^X, X^X, a_{X^X}, a_{X^X}, b_{X^X})$ satisfies Condition 2.3 and therefore (X^X, a_{X^X}, b_{X^X}) satisfies Condition 1.4(a). □

Let $f_1 : X^X \to X^{M\otimes X}$ and $f_2 : X^X \to X^{M\otimes X}$ be the right adjuncts of $\epsilon^X_X(1 \otimes a_X) : X^X \otimes (M \otimes X) \to X$ and $a_X(1 \otimes \epsilon^X_X)\sigma\alpha(1 \otimes \sigma) : X^X \otimes (M \otimes X) \to X$ respectively, and let $e : E(X) \to X^X$ be the equalizer of f_1 and f_2.

Proposition 2.6. *For the object $E(X)$ there exist unique morphisms $b_{E(X)} : E(X) \otimes E(X) \to E(X)$ and $a_{E(X)} : M \otimes E(X) \to E(X)$ for which e becomes an M-magma morphism and $(E(X), a_{E(X)}, b_{E(X)})$ is in M-**Mag**$_2$.*

Proof. In the diagram



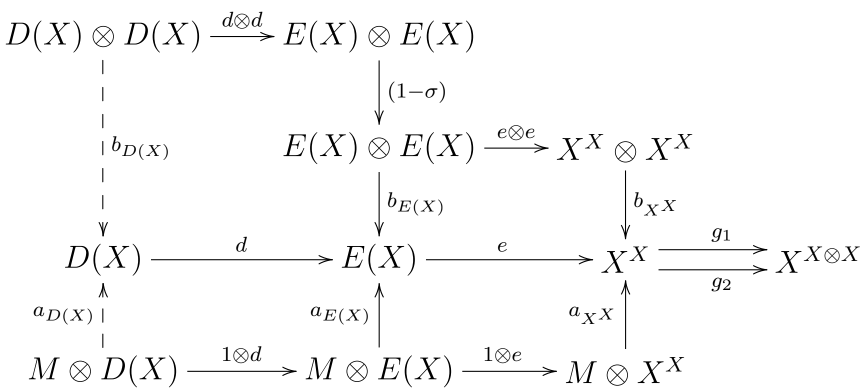
it can be seen, by considering the left adjuncts of $f_1 b_{X^X}(e\otimes e)$ and $f_2 b_{X^X}(e\otimes e)$ and the left adjuncts of $f_1 a_{X^X}(1 \otimes e)$ and $f_1 a_{X^X}(1 \otimes e)$, that the arrows $b_{X^X}(e \otimes e)$ and $a_{X^X}(1 \otimes e)$ equalize f_1 and f_2 and so, by the universal property of the equalizer e, there exist unique arrows $b_{E(X)}$ and $a_{E(X)}$ making the diagram commute. The left adjuncts of the morphisms $e a_{E(X)}(1 \otimes b_{E(X)})$ and $e b_{E(X)}(1 \otimes a_{E(X)})\sigma\alpha(1 \otimes \sigma)$ can been seen to be equal and since e is a monomorphism this shows that $(E(X), a_E(X), b_E(X))$ satisfies Condition 1.4(b). On the other hand, according to our construction of

a_{X^X} and b_{X^X}, the monomorphism e becomes an M-magma morphism from $(E(X), a_E(X), b_E(X))$ to (X^X, a_{X^X}, b_{X^X}), which implies that $(E(X), a_E(X), b_E(X))$ satisfies Condition 1.4(a) and that $(E(X), b_E(X))$ is a semigroup. This completes the proof. $\square$

By Theorem 1.9 we have that $L(E(X), b_{E(X)}, a_{E(X)}) = (E(X), \tilde{b}_{E(X)} = b_{E(X)}(1-\sigma), a_{E(X)})$ is in **Lie**(M, δ). For $(X, a_X, b_X) \in$ **Lie**(M, δ) let $g_1 : X^X \to X^{X \otimes X}$ be the right adjunct of $\epsilon_X^X(1 \otimes b_X) : X^X \otimes (X \otimes X) \to X$, let $g_2 : X^X \to X^{X\otimes X}$ be the right adjunct of the sum of the morphisms $b_X(\epsilon_X^X \otimes 1)\alpha : X^X \otimes (X \otimes X) \to X$ and $b_X(1 \otimes \epsilon_X^X)\sigma\alpha(1 \otimes \sigma) : X^X \otimes (X \otimes X) \to X$, and let $d : D(X) \to E(X)$ be the equalizer of $g_1 e$ and $g_2 e$.

Proposition 2.7. *For the object $D(X)$ there exist unique morphisms $b_{D(X)} : D(X) \otimes D(X) \to D(X)$ and $a_{D(X)} : M \otimes D(X) \to D(X)$ for which d is an M-magma morphism from $(D(X), a_{D(X)}, b_{D(X)})$ to $L(E(X), a_E(X), b_E(X))$ and $(D(X), a_{D(X)}, b_{D(X)})$ is in* **Lie**(M, δ).

Proof. In the diagram



it can be seen, by considering the left adjuncts of $g_1 b_{X^X}(e \otimes e)(1-\sigma)(d \otimes d)$ and $g_2 b_{X^X}(e\otimes e)(1-\sigma)(d \otimes d)$ and the left adjuncts of $g_1 a_{X^X}(1 \otimes e)(1 \otimes d)$ and $g_2 a_{X^X}(1 \otimes e)(1 \otimes d)$, that the morphisms $b_{X^X}(e \otimes e)(1-\sigma)(d \otimes d)$ and $a_{X^X}(1 \otimes e)(1 \otimes d)$ equalize g_1 and g_2 and so, by the universal property of the equalizer d, there exist unique arrows $b_{D(X)}$ and $a_{D(X)}$ making the diagram commute. Since d is a monomorphism we see that $(D(X), a_{D(X)}, b_{D(X)})$ is in **Lie**(M, δ). $\square$

We now define the object Der(X) of a derivation of $X = (X, a_X, b_X)$ as Der$(X) = D(X) = (D(X), a_D(X), b_D(X))$.

3 Representability of split extension functor for the category Lie(M, δ)

In this section we show that the functor SplExt$(-, X)$ can be defined for the category **Lie**(M, δ) and prove that it is representable by showing that Der$(X) = D(X)$ is the representing object.

To define the functor SplExt$(-, X)$ it is sufficient to show that the split short five lemma holds for **Lie**(M, δ) and that the category **Lie**(M, δ) has pullbacks of all split epimorphisms along arbitrary morphisms.

It is easily seen that the category **Lie**(M, δ) is pointed and finitely complete. Since $\mathbb{C}$ is additive the split short five lemma holds in $\mathbb{C}$ and since the forgetful functor $W : \textbf{Lie}(M, \delta) \to \mathbb{C}$ preserves limits and reflects isomorphisms, the split short five lemma holds also in **Lie**(M, δ).

Consider the diagram

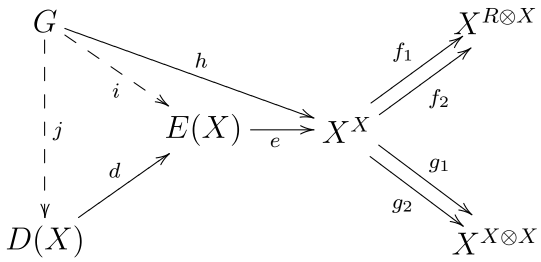
$$\begin{array}{ccccc} X & \underset{k}{\overset{l}{\leftrightarrows}} & A & \underset{s}{\overset{p}{\rightleftarrows}} & G \\ \| & & \downarrow f & & \| \\ X & \underset{k'}{\overset{l'}{\leftrightarrows}} & A' & \underset{s'}{\overset{p'}{\rightleftarrows}} & G \end{array}$$

where f is a morphism (hence an isomorphism) of split extensions in **Lie**(M, δ), and l and l' are the unique M-action morphisms with $kl = 1_A - sp$ and $k'l' = 1_{A'} - s'p'$; we shall write $A = (A, a, b)$ and $A' = (A', a', b')$. Since k' is a monomorphism and $k'l'f = (1_{A'} - s'p')f = f - s'p'f = f - fsp = f(1_A - sp) = fkl = k'l$ we have $l'f = l$; therefore

$$lb(s \otimes k) = l'fb(s \otimes k) = l'b'(f \otimes f)(s \otimes k) = l'b'(s' \otimes k').$$

Consequently, if we define $h : G \to X^X$ as the right adjunct of the composite $lb(s \otimes k)$, we see that h depends only on the isomorphism class of the split extensions.

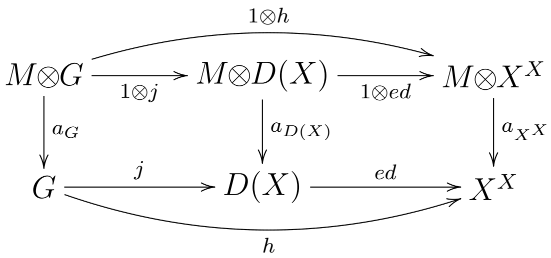
In the diagram

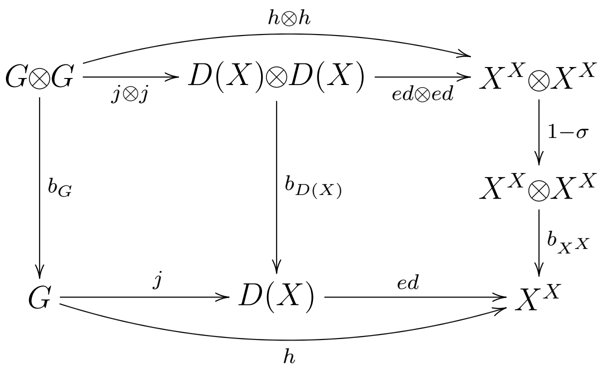


where the solid arrows are defined as before, it can be seen, by considering the left adjuncts of f_1h and f_2h and the left adjuncts of g_1h and g_2h, that h equalizes f_1 and f_2 as well as g_1 and g_2, and so by the universal properties of the equalizers e and d, there exist arrows i and j making the diagram commute.

Proposition 3.1. *The morphism* $j : G \to D(X)$ *is a morphism in* **Lie**(M, δ).

Proof. Consider the diagrams





where $G = (G, a_G, b_G)$. Considering the left adjuncts of $a_{X^X}(1 \otimes h)$ and ha_G (in the first diagram), and considering the left adjuncts of $b_{X^X}(1-\sigma)(h \otimes h)$

and hb_G (in the second diagram), the diagram formed by the outer arrows can be seen to commute. Therefore, since e and d are monomorphisms and the right hand square in each diagram commutes, the left hand squares also commute. □

For each G in **Lie**(M, δ), using the above construction we define the map τ_G : SplExt$(G, X) \to$ **Lie**$(M, \delta)(G, \mathrm{Der}(X))$ as follows:

$$\tau_G([\ X \xrightarrow{k} A \underset{s}{\overset{p}{\rightleftarrows}} G\ ]) = j.$$

Proposition 3.2. *The maps τ_G form a natural transformation.*

Proof. Let $A = (A, a, b)$ and $A' = (A', a', b')$ be objects in **Lie**(M, δ) and let $f : G' \to G$ be any morphism in **Lie**(M, δ), such that in the diagram

$$\begin{array}{ccccc} X & \underset{k'}{\overset{l'}{\leftrightarrows}} & A' & \underset{s'}{\overset{p'}{\rightleftarrows}} & G' \\ \downarrow{\scriptstyle 1_X} & & \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ X & \underset{k}{\overset{l}{\leftrightarrows}} & A & \underset{s}{\overset{p}{\rightleftarrows}} & G \end{array}$$

(A', f', p') is the pullback of f and p in **Lie**(M, δ), l and l' are the unique M-action morphisms with $kl = 1_A - sp$ and $k'l' = 1'_A - s'p'$, and the top and bottom rows excluding l and l' are split extensions. Let h' be the right adjunct of $l'b'(s' \otimes k')$ and j' be the unique morphism with $edj' = h'$, that is,

$$\tau'_G([\ X \xrightarrow{k'} A' \underset{s'}{\overset{p'}{\rightleftarrows}} G'\ ]) = j'.$$

Since $lb(s \otimes k)(f \otimes 1) = lb(sf \otimes k) = lb(f's' \otimes f'k') = lfb'(s' \otimes k') = l'b'(s' \otimes k')$ and h and h' are the right adjuncts of $lb(s \otimes k)$ and $l'b'(s' \otimes k')$ respectively, it follows that $hf = h'$. Therefore we have $edjf = hf = h' = edj'$ and since ed is monomorphism we conclude that $jf = j'$ and that the diagram

$$\begin{array}{ccc} \mathrm{SplExt}(G, X) & \xrightarrow{\tau_G} & \mathbf{Lie}(M, \delta)(G, \mathrm{Der}(X)) \\ \downarrow{\scriptstyle \mathrm{SplExt}(f,X)} & & \downarrow{\scriptstyle \mathbf{Lie}(M,\delta)(f,\mathrm{Der}(X))} \\ \mathrm{SplExt}(G', X) & \xrightarrow{\tau_{G'}} & \mathbf{Lie}(M, \delta)(G', \mathrm{Der}(X)) \end{array}$$

commutes. □

Theorem 3.3. *The functor* $\mathrm{SplExt}(-, X) : \mathbf{Lie}(M, \delta) \to \mathbf{Set}$ *is representable with representation* $(\tau, \mathrm{Der}(X))$.

Proof. We show that the natural transformation $\tau : \mathrm{SplExt}(-, X) \to \mathbf{Lie}(M, \delta)(-, \mathrm{Der}(X))$ is a natural isomorphism. For an arrow $z : G \to \mathrm{Der}(X)$ in $\mathbf{Lie}(M, \delta)$ let $r : G \otimes X \to X$ be the left adjunct of edz, and let $X \rtimes_z G = (X \oplus G, a, b)$, where

$$a = \iota_1 a_X (1 \otimes \pi_1) + \iota_2 a_G (1 \otimes \pi_2)$$

and

$$b = \iota_1 (b_X (\pi_1 \otimes \pi_1) + r(\pi_2 \otimes \pi_1)(1 - \sigma)) + \iota_2 b_G (\pi_2 \otimes \pi_2),$$

in obvious notation. It can been seen that $X \rtimes_z G$ is in $\mathbf{Lie}(M, \delta)$ and that the diagram

$$X \xrightarrow{\iota_1} X \rtimes_r G \underset{\iota_2}{\overset{\pi_2}{\rightleftarrows}} G$$

is a split extension in $\mathbf{Lie}(M, \delta)$. Let $\hat{\tau}_G : \mathbf{Lie}(M, \delta)(G, \mathrm{Der}(X)) \to \mathrm{SplExt}(G, X)$ be the map defined as follows:

$$\hat{\tau}_G(z) = [\, X \xrightarrow{\iota_1} X \rtimes_r G \underset{\iota_2}{\overset{\pi_2}{\rightleftarrows}} G \,]$$

It can be seen that $\hat{\tau_G} = {\tau_G}^{-1}$ and hence $(\tau, \mathrm{Der}(X))$ is a representation of $\mathrm{SpltExt}(-, X)$. $\square$

Remark 3.4. *Since the category* $\mathbf{Cat}(\mathbf{Lie}(M, \delta))$ *of internal categories in* $\mathbf{Lie}(M, \delta)$ *can be presented as* $\mathbf{Lie}(M', \delta')$ *for suitable* M' *and* δ' *(it essentially follows from the results of [5]), by Theorem 3.3 the functor* SplExt: $\mathbf{Cat}(\mathbf{Lie}(M, \delta)) \to \mathbf{Set}$ *is representable.*

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REPRESENTATION OF METRIC JETS

by Elisabeth BURRONI and Jacques PENON

We dedicate this article to Francis Borceux

Abstract

Guided by the heuristic example of the tangential $\mathrm{T}f_a$ of a map f differentiable at a which can be canonically represented by the unique continuous affine map it contains, we extend, in this article, into a specific metric context, this property of representation of a metric jet. This yields a lot of relevant examples of such representations.

L'application affine continue qui est tangente, en un point a fixé, à une application f différentiable en ce point, peut être très naturellement considérée comme un représentant de la tangentielle Tf_a de f en a. Cet exemple sera notre guide heuristique pour trouver un context métrique spécifique dans lequel cette propriété de représentation d'un jet métrique soit possible. Au passage, on fournit de nombreux exemples pertinents de telles représentations.

Key words : differential calculus, Gateaux differentials, fractal maps, jets, metric spaces, categories

AMS classification : 58C25, 58C20, 28A80, 58A20, 54E35, 18D20

INTRODUCTION

This article is the sequel of a paper published in TAC [6] ; most of the proofs of the statements given here can be found in the second chapter of a paper published in arXiv [5].

We recall that maps $f, g : M \longrightarrow M'$ (where M,M' are metric spaces) are tangent at a (not isolated in M), what we denote $f \succ\!\!\prec_a g$, if $f(a) = g(a)$ and $\lim_{a\neq x\to a} \frac{d(f(x),g(x))}{d(x,a)} = 0$; a metric jet (in short, a jet) is an equivalence class for this relation $\succ\!\!\prec_a$, restricted to the set of the