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A CHARACTERIZATION OF FINITE COCOMPLETE HOMOLOGICAL AND OF SEMI-ABELIAN CATEGORIES

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Abstract

Semi-abelian and finitely cocomplete homological categories are characterized in terms of four resp. three simple axioms, in terms of the basic categorical notions introduced in the first few chapters of MacLane's classical book.

Résumé

Les catégories semi-abéliennes et homologiques finiment cocomplètes sont définies en quatre (respectivement trois) axiomes simples exprimés en termes de notions catégoriques de base introduites dans les premiers chapitres du livre classique de MacLane.

Key words: Homological categories, exact categories, semi-abelian categories
MSC: 18E10,18A30

After a long development, the notion of semi-abelian category as introduced in [5] turns out to be kind of a best-possible generalization of the classical notion of abelian category, in that it both includes a maximum of interesting examples and provides a convenient general framework for many concepts, in particular such arising from group theory, homological and homotopical algebra, see for instance [1], [3], [4], [8]. But also the weaker notion of homological category already allows for many interesting properties [2], in particular when combined with finite cocompleteness where it supports an alternative approach to actions, commutators, modules and crossed modules as is shown in [6] and in forthcoming joint work with Van der Linden.

So the aim of this note is to provide a short and very elementary definition of the notions of semi-abelian and finitely cocomplete homological category, only in terms of the most basic concepts found in

MacLane's classical book [7]. We emphasize that this does not mean an attempt of relapsing into "old-style" category theory but originates from purely pedagogical motivations: namely, to advertize semi-abelian category theory to non-category theorists, in particular to group theorists and algebraic topologists (such as the first author) who may definitely find strong interest in these and related further developments once the presentation of the basic notions appears sufficiently familiar to be easily understood.

Consider the following axioms about a category $\mathbb{C}$, where we denote by kernel or cokernel the corresponding injection or projection arrow, resp.

A1. $\mathbb{C}$ is pointed, finitely complete and cocomplete;

A2. For any split epimorphism $p : X \to Y$ with section $s : Y \to X$ and with kernel $k : K \hookrightarrow X$, the arrow $\langle k, s\rangle : K + Y \to X$ is a cokernel;

A3. The pullback of a cokernel is a cokernel;

A4. The image of a kernel by a cokernel is a kernel.

Note that when $\mathbb{C}$ is the category of groups, axiom A2 says that a semi-direct product group $G = N \rtimes T$ is generated by N and T. Note also that the mere formulation of Axiom A4 supposes that the category is regular, but this follows from Axioms A1, A2 and A3, as will be shown.

Theorem. Let $\mathbb{C}$ be a category. Then:

(1) $\mathbb{C}$ is finitely cocomplete homological if and only if it satisfies the three axioms A1-A3, or equivalently A'1-A2-A3, where A'1 is the axiom
A'1. *$\mathbb{C}$ is pointed, finitely complete, has binary sums and has coequalizers of equivalence relations;*

(2) $\mathbb{C}$ is semi-abelian iff it satisfies all the four axioms A1-A4.

Proof. (1): First note that condition A2 is stronger than condition PM_0^+ of [5] (which asserts that the factorization is an extremal epimorphism). Moreover, for a finitely complete pointed category with finite sums, 3.1 and 3.2 of [5] show that the latter condition is equivalent to protomodularity. Since moreover in any finitely complete pointed protomodular category any regular epimorphism is the cokernel of its kernel (see e.g.

[2, Proposition 3.1.23]), the axiom A3 is equivalent to stability by pullbacks of regular epimorphisms, and A1 of course implies existence of coequalizers of kernel pairs. So any finitely cocomplete homological category satisfies A1-A3, and any category which satisfies those axioms is regular, pointed and homological, and it just remains to show that it is finitely cocomplete, i.e. that any parallel pair of morphisms has a coequalizer since by A1 $\mathbb{C}$ has finite coproducts. But the analysis of the proof given in [2, Proposition 5.1.3] of the fact that this holds in any semi-abelian category, shows that the effectiveness hypothesis for equivalence relations is only used to show that some equivalence relation has a coequalizer, hence it can be replaced by the hypothesis that any equivalence relation has one.
(2): In [5, 3.7], one finds a proof that semi-abelianness is equivalent to the following axiom system:

- (SA'1=SA1) $\mathbb{C}$ has binary products and sums and a zero object;
- (SA'2=SA2) $\mathbb{C}$ has pullbacks of (split) monomorphisms;
- (SA'3) $\mathbb{C}$ has cokernels of kernels, and every morphism with zero kernel is a monomorphism;
- (SA'4=SA4) The (split) Short Five Lemma holds true in $\mathbb{C}$;
- (SA'5) = our A3;
- (SA'6)= our A4.

By (1) above, Axioms A1-A3 hold in any semi-abelian category, and so does A4 since it is SA'6. Conversely, Axioms A1-A4 obviously imply Axioms SA'1 and SA'2, and it follows again from [5, Paragraphs 3.1 and 3.2] that they imply SA4 (at least in its weaker form), i.e. imply the protomodularity of the category. A1 obviously implies the first part of SA'3. Finally, it remains to show that A1-A4 imply its second part, i.e. that every morphism with zero kernel is a monomorphism. But this is true in any pointed protomodular category with finite limits [2, Proposition 3.1.21].

□

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