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OPERADIC DEFINITION OF NON-STRICT CELLS

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Résumé

Dans [16] nous avons étendu le travail de Jacques Penon sur les ω-catégories non-strictes en définissant leurs ω-foncteurs non-stricts, leurs ω-transformations naturelles non-strictes, etc. tout ceci en utilisant des extensions de ces "étirements catégoriques" que l'on a appelés "n-étirements catégoriques" ($n \in \mathbb{N}^*$). Dans cet article nous poursuivons le travail de Michael Batanin sur les ω-catégories non-strictes [2] en définissant leurs ω-foncteurs non-stricts, leurs ω-transformations naturelles non-strictes, etc. en utilisant des extensions de son ω-opérade contractile universelle K, i.e en construisant des ω-opérades colorées contractiles universelles B^n ($n \in \mathbb{N}^*$) adaptés.

Abstract

In [16] we pursue Penon's work in higher dimensional categories by defining weak ω-functors, weak natural ω-transformations, and so on, all that with Penon's frameworks i.e with the "étirements catégoriques", where we have used an extension of this object, namely the "n-étirements catégoriques" ($n \in \mathbb{N}^*$). In this article we are pursuing Batanin's work in higher dimensional categories [2] by defining weak ω-functors, weak natural ω-transformations, and so on, using Batanin's frameworks i.e by extending his universal contractible ω-operad K, by building the adapted globular colored contractible ω-operads B^n ($n \in \mathbb{N}^*$).

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Contents

One of the fundamental but still conjectural properties of any theory of higher categories has to be the statement that n-categories as a totality have a structure of an $(n+1)$-category. Or taking the limit : there must exist an ∞-category of ∞-categories. This means that we should be able to define functors between ∞-categories, transformations between such functors, transformations between transformations etc..

A difficulty here is that these functors and transformations must be as weak as possible, meaning that they are functors, transformations etc. only up to all higher cells. There are approaches to this problem which attempt to avoid the direct construction of higher transformations using methods of homotopy theory ([8, 12, 19, 22, 23]).

Even though there are some serious advantages to such approaches I believe it is of fundamental importance to have a precise notion of n-transformation, especially for the so called algebraic model of higher category theory (see [2, 20, 21]) where an ∞-category is defined as an algebra of a special monad or algebraic theory. The very spirit of these approaches, which I believe, coincides with Grothendieck's original vision of higher category theory, requires a similar definition of higher transformations.

The first step in this direction was undertaken in [16], where I have introduced the globular complex of higher transformations for Penon ∞-categories. In this paper I construct such a complex for Batanin ∞-categories. As it was shown by Batanin [3], Penon's ∞-categories are a special case of Batanin, so this work can be considered as a generalization of my previous work. The methods of this work, apply also to Leinster's ∞-categories which is a slight variation of Batanin's original definition. I leave as an exercise for a reader interested in Leinster's n-transformations to make the necessary changes in definitions.

In my paper I use the language of the theory of T-categories invented by A.Burroni [7] and rediscovered later by Leinster and Hermida [10, 18]. I refer the reader to the book of Leinster for the main definitions. I also use the following terminology: weak ∞-Functors are called 1-Transformations, weak ∞-natural transformations are called 2-Transformations, weak ∞-modifications are called 3-Transformations, etc.

A new technique is the use of 2-colored operads. This is reminiscent to the use of 2-colored operads in the classical operad theory to define coherent maps between operadic algebras. For this purpose I develop a necessary generalisation of Batanin's techniques [2] to handle colored operads.

Batanin built his weak ∞-categories with a contractible operad equipped with a composition system. I adopt the same point of view and construct a sequence of contractible globular operads with "bicolored composition systems" (called operation systems). Like in [2], these operads are initial in an appropriate sense. This property happens to be crucial for constructing the sources and targets of the underlying graphs of the probable Weak Omega Category of Weak Omega Categories.

In more detail the construction proceeds in 4 stages: one first constructs a co-∞-graph of operation systems, followed by a co-∞-graph of globular colored operads, which will successively lead to an ∞-graph in the category of categories equipped with a monad, and finally to the ∞-graph of their algebras. These algebras will contain all Batanin's n-Transformations ($n \in \mathbb{N}^*$). This work was exposed in Calais in June 2008 in the International Category Theory Conference [15].

In "pursuing stacks" [9] Alexander Grothendieck gave his own definition of weak omega groupoids in which he saw them as models of some specific theories called "cohérateurs", and a slight modification of this definition led to a notion of weak omega category [20]. Thus in the spirit of Grothendieck, weak and higher structures should be seen as models of certain kinds of theories. Section 7 is devoted to showing, thanks to the Abstract Nerve Theorem of Mark Weber ([25]), that our approach of weak omega transformations can be seen also from the point of view of theories and their models. According to [1], our approach and that of Grothendieck seem to be very similar.

In a forthcoming paper I will show that this globular complex of higher transformations has a natural action of a globular operad. The contractibility of this operad will be studied in the third paper of this series. This will complete the proof of the hypothesis of the existence of an algebraic model of the Weak ∞-Category of the Weak ∞-Categories.

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I dedicate this work to my uncle Mohamed-Kommandar Mezouar who taught us important life lessons even while struggling with a difficult illness.

1 Pointed and Contractibles T-Graphs

From here $\mathbb{T} = (T, \mu, \eta)$ refers to the cartesian monad of strict ∞-categories. Its cartesian feature permits us to build the bigategory $\mathrm{Span}(T)$ of spans. The various concepts in this article are defined in this bicategory, which is described in Leinster [18, 4.2.1 page 138]. In all this paper if $\mathbb{C}$ is a category then $\mathbb{C}(0)$ is the class of its objects (but we often omit "0" when there is no confusion) and $\mathbb{C}(1)$ is the class of its morphisms. The symbol $:=$ means "by definition is".

1.1 T-Graphs

A T-graph (C, d, c) is a datum of a diagram of ∞-$\mathbb{G}$r such as

$$T(G) \xleftarrow{d} C \xrightarrow{c} G$$

T-graphs are endomorphisms of $\mathrm{Span}(T)$ and they form a category T-$\mathbb{G}$r (described in Leinster [18, definition 4.2.4 page 140]). If we choose $G \in \infty$-$\mathbb{G}\mathrm{r}(0)$, the endomorphisms on G (in $\mathrm{Span}(T)$) forms a subcategory of T-$\mathbb{G}$r which will be noted T-$\mathbb{G}\mathrm{r}_G$, and it is well-known that T-$\mathbb{G}\mathrm{r}_G$ is a monoidal category such as the definition of its tensor:

$$(C, d, c) \bigotimes (C', d', c') := (T(C) \times_{T(G)} C', \mu(G)T(d)\pi_0, c\pi_1),$$

and its unity object $I(G) = (G, \eta(G), 1_G)$. We can remember that $I(G)$ is also an identity morphism of $\mathrm{Span}(T)$. The ∞-graph G is called the graph of globular arities.

1.2 Pointed T-Graphs

A T-graph (C,d,c) equipped with a morphism $I(G) \xrightarrow{p} (C,d,c)$ is called a pointed T-graph. Also we note $(C,d,c;p)$ for a pointed T-graph. That also means that one has a 2-cell $I(G) \xrightarrow{p} (C,d,c)$ of $\mathrm{Span}(T)$ such as $dp = \eta(G)$ and $cp = 1_G$. We define in a natural way the category T-$\mathbb{G}\mathrm{r}_p$ of pointed T-graphs and the category T-$\mathbb{G}\mathrm{r}_{p,G}$ of G-pointed T-graphs: Their morphisms keep pointing in an obvious direction.

1.3 Contractible T-Graphs

Let (C,d,c) be a T-graph. For any $k \in \mathbb{N}$ we consider

$$D_k = \{(\alpha,\beta) \in C(k) \times C(k)/s(\alpha) = s(\beta), t(\alpha) = t(\beta) \text{ and } d(\alpha) = d(\beta)\}$$

A contraction on that T-graph, is the datum, for all $k \in \mathbb{N}$, of a map

$$D_k \xrightarrow{[,]_k} C(k+1)$$

such that

- $s([\alpha,\beta]_k) = \alpha, t([\alpha,\beta]_k) = \beta,$
- $d([\alpha,\beta]_k) = 1_{d(\alpha)=d(\beta)}.$

This maps $[,]_k$ form the bracket law (as the terminology in [16]). A T-graph which is equipped with a contraction will be called contractible and we note $(C,d,c;([,]_k)_{k\in\mathbb{N}})$ for a contractible T-graph. Nothing prevents a contractible T-graph from being equipped with several contractions. So here CT-$\mathbb{G}\mathrm{r}$ is a category of contractible T-graphs equipped with a specific contraction. The morphisms of this category preserves the contractions and one can also refer to the category CT-$\mathbb{G}\mathrm{r}_G$ where contractible T-graphs are only taken on a specific ∞-graph of globular arities G.

Remark 1 If $(\alpha,\beta)\in D_k$ then this does not lead to $c(\alpha)=c(\beta)$, but this equality will be verified for constant ∞-graphs (see below) and in particular for collections with two colours (These are the most important T-graphs in this article). We should also bear in mind CT-$\mathbb{G}\mathrm{r}_p$, the category of pointed and contractible T-graphs resulting from the previous definitions. A pointed and contractible T-graph will be noted $(C,d,c;([,]_k)_{k\in\mathbb{N}},p)$. $\square$

1.4 Constant ∞-Graphs

A constant ∞-graph is a ∞-graph G such as $\forall n,m\in\mathbb{N}$ we have $G(n)=G(m)$ and such as source and target maps are identity. We note ∞-$\mathbb{G}\mathrm{r}_c$ the corresponding category of constant ∞-graphs. Constant ∞-graph are important because it is in this context that we have an adjunction result (theorem 1) that we used to produce free colored contractibles operads of n-Transformations ($n\in\mathbb{N}^*$). We write T-$\mathbb{G}\mathrm{r}_c$ for the subcategory of T-$\mathbb{G}\mathrm{r}$ consisting of T-graphs with underlying ∞-graphs of globular arity which are constant ∞-graphs, T-$\mathbb{G}\mathrm{r}_{c,p}$ for the subcategory of T-$\mathbb{G}\mathrm{r}_p$ consisting of pointed T-graphs with underlying ∞-graphs of globular arity which are constant ∞-graphs, and we write T-$\mathbb{G}\mathrm{r}_{c,p,G}$ for the fiber subcategory in T-$\mathbb{G}\mathrm{r}_{c,p}$ (for a given G in ∞-$\mathbb{G}\mathrm{r}_c$).

2 Contractible T-Categories

2.1 T-Categories

A T-category is a monad of the bigategory $\mathrm{Span}(T)$ or in a equivalent way a monoid of the monoidal category T-$\mathbb{G}\mathrm{r}_G$ (for a specific G). The definition of T-categories are in Leinster [18, definition 4.2.2 page 140], and their category will be noted T-$\mathbb{C}\mathrm{at}$ and that of T-categories of the same ∞-graph of globular arities G will be noted T-$\mathbb{C}\mathrm{at}_G$. A T-category $(B,d,c;\gamma,u)\in T$-$\mathbb{C}\mathrm{at}$ is specifically given by the morphism of (operadic) composition $(B,d,c)\bigotimes(B,d,c)\xrightarrow{\gamma}$

(B,d,c) and the (operadic) unit $I(G) \xrightarrow{u} (B,d,c)$ fitting axioms of associativity and unity [see 18]. Note that $(B,d,c;\gamma,u)$ has $(B,d,c;u)$ as natural underlying pointed T-graph.

2.2 Contractibles T-Categories and the Theorem of Initial Objects

A T-category $(B,d,c;\gamma,u)$ will be said to be contractible if its underlying T-graph is contractible. To specify the underlying contraction of contractible T-categories we eventually noted it $(B,d,c;\gamma,u,([,]_k)_{k\in\mathbb{N}})$. The category of contractible T-categories will be noted CT-$\mathbb{C}$at, that of contractible T-categories of the same ∞-graph of globular arities G will be noted CT-$\mathbb{C}\mathrm{at}_G$. We also write CT-$\mathbb{C}\mathrm{at}_c$ for the subcategory of CT-$\mathbb{C}$at whose objects are contractible T-categories whose underlying ∞-graph of globular arities is a constant ∞-graph. Besides there is an obvious forgetful functor

$$CT\text{-}\mathbb{C}\mathrm{at}_{c,G} \xrightarrow{O} T\text{-}\mathbb{G}\mathrm{r}_{c,p,G}$$

and there is the

Theorem 1 (Theorem of Initial Objects) *O has a left adjoint F: $F \dashv O$.* $\square$

PROOF The first monad $(L,\mathfrak{m},l)$, resulting from the adjunction

$$\begin{tikzcd} T\text{-}\mathbb{C}\mathrm{at}_{c,G} \arrow[r, shift left, "U"] \arrow[r, phantom, "\top"] & T\text{-}\mathbb{G}\mathrm{r}_{c,p,G} \arrow[l, shift left, "M"] \end{tikzcd}$$

and the second monad (C,m,c), resulting from the adjunction

$$\begin{tikzcd} CT\text{-}\mathbb{G}\mathrm{r}_{c,p,G} \arrow[r, shift left, "V"] \arrow[r, phantom, "\top"] & T\text{-}\mathbb{G}\mathrm{r}_{c,p,G} \arrow[l, shift left, "H"] \end{tikzcd}$$

are built as in [2];

The hypotheses of the section 6 are satisfied because the forgetful functors U and V are monadic, T-$\mathbb{C}\mathrm{at}_{c,G}$ and CT-$\mathbb{G}\mathrm{r}_{c,p,G}$ have coequalizers and $\vec{\mathbb{N}}$-colimits and it is easy to notice that the forgetful functors U and V are faithfull and preserve $\vec{\mathbb{N}}$-colimits as well. Thus this two adjunctions are fusionable which permits, through theorem 2, to make the fusion

$$\begin{tikzcd}
T\text{-}\mathbb{C}\mathrm{at}_{c,G} \arrow[rr, shift left, "U"] \arrow[rr, phantom, "\perp"] & & T\text{-}\mathbb{G}\mathrm{r}_{c,p,G} \arrow[ll, shift left, "M"] \arrow[rr, shift left, "H"] \arrow[rr, phantom, "\perp"] \arrow[d, shift right, "F"'] \arrow[d, phantom, "\dashv"] & & CT\text{-}\mathbb{G}\mathrm{r}_{c,p,G} \arrow[ll, shift left, "V"] \\
& & CT\text{-}\mathbb{C}\mathrm{at}_{c,G} \arrow[llu, "p_1"] \arrow[u, shift right, "O"'] \arrow[rru, "p_2"'] & &
\end{tikzcd}$$

where trivially

$$CT\text{-}\mathbb{C}\mathrm{at}_{c,G} \simeq T\text{-}\mathbb{C}\mathrm{at}_{c,G} \times_{T\text{-}\mathbb{G}\mathrm{r}_{c,p,G}} CT\text{-}\mathbb{G}\mathrm{r}_{c,p,G}$$ ■

The monad of this adjunction $F \dashv O$ is noted $\mathbb{B} = (B, \rho, b)$.

Remark 2 We can also prove that the forgetful functor

$$CT\text{-}\mathbb{C}\mathrm{at}_c \xrightarrow{O} T\text{-}\mathbb{G}\mathrm{r}_{c,p}$$

has a left adjoint. A way to prove it is to extend the work of [6] on "Surcatégories", and it is done in [13]. But it seems that this result is too much strong for this article where we use no more than 2 colours. However we will use this adjunction for a future paper, after the talk [17] where we need to use more than two colors. □

2.3 T-Categories equipped with a System of Operations

Consider $(B, \overline{d}, \overline{c}; \gamma, u) \in T\text{-}\mathbb{C}\mathrm{at}_G$ and $(C, d, c) \in T\text{-}\mathbb{G}\mathrm{r}_G$. If there exists a diagram of $T\text{-}\mathbb{G}\mathrm{r}_G$

$$(I(G), \eta_G, id) \xrightarrow{p} (C, d, c) \xrightarrow{k} (B, \overline{d}, \overline{c})$$

such as $k \circ p = u$, then (C, d, c) is qualified system of operations, and one can say that $(B, \overline{d}, \overline{c}; \gamma, u)$ is equipped with the system of operations (C, d, c). With this definition and the previous theorem it is clear that all pointed T-graphs $(C, d, c; p)$ induces a free contractible T-category $F(C)$, which has (C, d, c) as a system of operations. See also section 3.

3 Systems of Operations of the n-Transformations ($n \in \mathbb{N}^*$)

3.1 Preliminaries

The 2-coloured collection of the n-Transformations ($n \in \mathbb{N}^*$) are just noted C^n without specified its underlying structure, and we do the same simplification for its free contractible 2-coloured operads B^n.

From here on only the contractible 2-coloured operads of n-Transformations will be studied. All these operads are obtained applying the free functor of the theorem 1 to specific 2-coloured collections. These 2-coloured collections will be those of the n-Transformations and they count an infinite countable number of elements. Thus for each $n \in \mathbb{N}$ there is the 2-coloured collection of n-Transformations, C^n, which freely produces the free contractible 2-coloured operad B^n of n-Transformations. The pointed collection C^0 is the system of composition of Batanin's operad of weak ∞-categories, i.e. the collection gathering all the symbols of atomic operations necessary for the weak ∞-categories, plus the symbols of operadic units (the latter are given by pointing). The pointed 2-coloured collection C^1 is adapted to weak ∞-functors, i.e. it gathers all the symbols of operations of the source and target weak ∞-categories (which will be composed of different colours whether they concern the source or the target). It also brings together the unary symbols of functors as well as the symbols of operadic units. Thus as we will see, the unary symbols of functors have a domain with the same colour as the domains and codomains of the symbols of operations of source weak ∞-categories and they have a codomain with the same colour as the domains and codomains of the symbols of operations of target weak ∞-categories. However these symbols of functors have domains and codomains with different colours. The pointed 2-coloured collection C^2 is adapted to weak natural ∞-transformations, etc.

3.2 Pointed 2-Coloured Collections $C^n(n \in \mathbb{N})$

In order to clearly see the bicolour feature of these symbols of operations, we write $(1+1)(n) := \{1(n), 2(n)\}$, which enables to identify $T(1) \sqcup T(1)$ with $T(1) \cup T(2)$ and $1 \sqcup 1$ with $1 \cup 2$. So the colour 1 and the colour 2 will be referred to. Let us move to the definition of $C^n(n \in \mathbb{N})$. In the diagram

$$T(1) \cup T(2) \xleftarrow{d} C^n \xrightarrow{c} 1 \cup 2$$

C^n is a ∞-graph so that it contains both source and target maps which will be noted $C^n(m+1) \underset{t^{m+1}_m}{\overset{s^{m+1}_m}{\rightrightarrows}} C^n(m)$, $(m \in \mathbb{N})$.

3.2.1 Definition of C^0

C^0 is Batanin's system of composition, i.e. there is the following collection $T(1) \xleftarrow{d^0} C^0 \xrightarrow{c^0} 1$ such as C^0 precisely contains the symbols of the compositions of weak ∞-categories $\mu^m_p \in C^0(m)(0 \leq p < m)$, plus the operadic unary symbols $u_m \in C^0(m)$. More specifically:

$\forall m \in \mathbb{N}$, C^0 contains the m-cell u_m such as: $s^m_{m-1}(u_m) = t^m_{m-1}(u_m) = u_{m-1}$ (if $m \geq 1$); $d^0(u_m) = 1(m)(= \eta(1 \cup 2)(1(m)))$, $c^0(u_m) = 1(m)$.

$\forall m \in \mathbb{N} - \{0,1\}$, $\forall p \in \mathbb{N}$, such that $m > p$, C^0 contains the m-cell μ^m_p such as: If $p = m-1$, $s^m_{m-1}(\mu^m_{m-1}) = t^m_{m-1}(\mu^m_{m-1}) = u_{m-1}$. If $0 \leq p < m-1$, $s^m_{m-1}(\mu^m_p) = t^m_{m-1}(\mu^m_p) = \mu^{m-1}_p$. Also $d^0(\mu^m_p) = 1(m) \star^m_p 1(m)$, and inevitably $c^0(\mu^m_p) = 1(m)$.

Furthemore C^0 contains the 1-cell μ^1_0 such as $s^1_0(\mu^1_0) = t^1_0(\mu^1_0) = u_0$, $d^0(\mu^1_0) = 1(1) \star^1_0 1(1)$, also inevitably $c^0(\mu^1_0) = 1(1)$.

The system of composition C^0 has got a well-known pointing λ^0 which is defined as $\forall m \in \mathbb{N}$, $\lambda^0(1(m)) = u_m$.

3.2.2 Definition of C

Firstly we will define a collection (C,d,c) which will be useful to build the collections of n-Transformations ($n \in \mathbb{N}^*$). C contains two copies of the symbols of C^0, each having a distinct colour: The symbols formed with the letters μ and u are those of the colour 1, and those formed with the letters ν and v are those of the colour 2. Let us be more precise:

$\forall m \in \mathbb{N}$, C contains the m-cell u_m such as: $s^m_{m-1}(u_m) = t^m_{m-1}(u_m) = u_{m-1}$ (if $m \geq 1$) and $d(u_m) = 1(m)$, $c(u_m) = 1(m)$.

$\forall m \in \mathbb{N} - \{0,1\}$, $\forall p \in \mathbb{N}$, such as $m > p$, C contains the m-cell μ^m_p such as: If $p = m-1$, $s^m_{m-1}(\mu^m_{m-1}) = t^m_{m-1}(\mu^m_{m-1}) = u_{m-1}$. If $0 \leq p < m-1$, $s^m_{m-1}(\mu^m_p) = t^m_{m-1}(\mu^m_p) = \mu^{m-1}_p$. Also $d(\mu^m_p) = 1(m) \star^m_p 1(m)$, $c(\mu^m_p) = 1(m)$.

Furthemore C contains the 1-cell μ^1_0 such as $s^1_0(\mu^1_0) = t^1_0(\mu^1_0) = u_0$ and $d(\mu^1_0) = 1(1) \star^1_0 1(1)$, $c(\mu^1_0) = 1(1)$.

Besides, $\forall m \in \mathbb{N}$, C contains the m-cellule v_m such that: $s^m_{m-1}(v_m) = t^m_{m-1}(v_m) = v_{m-1}$ (if $m \geq 1$) and $d(v_m) = 2(m)$, $c(v_m) = 2(m)$.

$\forall m \in \mathbb{N} - \{0,1\}$, $\forall p \in \mathbb{N}$, such that $m > p$, C contains the m-cell ν^m_p such as: If $p = m-1$, $s^m_{m-1}(\nu^m_{m-1}) = t^m_{m-1}(\nu^m_{m-1}) = v_{m-1}$. If $0 \leq p < m-1$, $s^m_{m-1}(\nu^m_p) = t^m_{m-1}(\nu^m_p) = \nu^{m-1}_p$. Also $d(\nu^m_p) = 2(m) \star^m_p 2(m)$, $c(\nu^m_p) = 2(m)$.

Furthemore C contains the 1-cell ν^1_0 such as $s^1_0(\nu^1_0) = t^1_0(\nu^1_0) = v_0$ and $d(\nu^1_0) = 2(1) \star^1_0 2(1)$, $c(\nu^1_0) = 2(1)$.

3.2.3 Definition of $C^i (i = 1,2)$

C^1 is the system of operations of weak ∞-functors. It is built on the basis of C adding to it a single symbol of functor (for each cell level):$\forall m \in \mathbb{N}$ the F^m

m-cell is added, which is such as: If $m \geq 1$, $s^m_{m-1}(F^m) = t^m_{m-1}(F^m) = F^{m-1}$. Also $d^1(F^m) = 1(m)$ and $c^1(F^m) = 2(m)$.

C^2 is the system of operations of weak natural ∞-transformations. C^2 is built on C, adding to it two symbols of functor (for each cell level) and a symbol of natural transformation. More precisely

$\forall m \in \mathbb{N}$ we add the m-cell F^m such as: If $m \geq 1$, $s^m_{m-1}(F^m) = t^m_{m-1}(F^m) = F^{m-1}$. Also $d^2(F^m) = 1(m)$ and $c^2(F^m) = 2(m)$.

Then $\forall m \in \mathbb{N}$ we add the m-cell H^m such as: If $m \geq 1$, $s^m_{m-1}(H^m) = t^m_{m-1}(H^m) = H^{m-1}$. Also $d^2(H^m) = 1(m)$ and $c^2(H^m) = 2(m)$.

And finally we add 1-cell τ such as: $s^1_0(\tau) = F^0$ and $t^1_0(\tau) = H^0$. Also $d^2(\tau) = 1_{1(0)}$ and $c^2(\tau) = 2(1)$.

We can point out that the 2-coloured collections C^i $(i = 1,2)$ are naturally equipped with a pointing λ^i defined by $\lambda^i(1(m)) = u_m$ and $\lambda^i(2(m)) = v_m$.

3.2.4 Definition of C^n for $n \geq 3$

In order to define the general theory of n-Transformations ($n \in \mathbb{N}^*$), it is necessary to define the systems of operations C^n for the superior n-Transformations ($n \geq 3$). This paragraph can be left out in the first reading. Each collection C^n is built on C, adding to it the required cells. They contain four large groups of cells: The symbols of source and target weak ∞-categories, the symbols of operadic units (obtained on the basis of C), the symbols of functors (sources and targets), and the symbols of n-Transformations (natural transformations, modification, etc). More precisely, on the basis of C:

Symbols of Functors $\forall m \in \mathbb{N}$, C^n contains the m-cells α^m_0 and β^m_0 such as: If $m \geq 1$, $s^m_{m-1}(\alpha^m_0) = t^m_{m-1}(\alpha^m_0) = \alpha^{m-1}_0$ and $s^m_{m-1}(\beta^m_0) = t^m_{m-1}(\beta^m_0) = \beta^{m-1}_0$. Furthermore $d^n(\alpha^m_0) = d^n(\beta^m_0) = 1(m)$ and $c^n(\alpha^m_0) = c^n(\beta^m_0) = 2(m)$.

Symbols of the Higher n-Transformations $\forall p$, with $1 \leq p \leq n-1$, C^n contains the p-cells α_p and β_p which are such as: $\forall p$, with $2 \leq p \leq n-1$, $s^p_{p-1}(\alpha_p) = s^p_{p-1}(\beta_p) = \alpha_{p-1}$ and $t^p_{p-1}(\alpha_p) = t^p_{p-1}(\beta_p) = \beta_{p-1}$. If $p = 1$, $s^1_0(\alpha_1) = s^1_0(\beta_1) = \alpha^0_0$ and $t^1_0(\alpha_1) = t^1_0(\beta_1) = \beta^0_0$. What's more, $\forall p$, with $1 \leq p \leq n-1$, $d^n(\alpha_p) = d^n(\beta_p) = 1^0_p(1(0))$ and $c^n(\alpha_p) = c^n(\beta_p) = 2(p)$. Finally C^n contains the n-cell ξ_n such as $s^n_{n-1}(\xi_n) = \alpha_{n-1}, b^n_{n-1}(\xi_n) = \beta_{n-1}$ and $d^n(\xi_n) = 1^0_n(1(0))$ and $c^n(\xi_n) = 2(n)$ (Here 1^0_n is the map resulting from the reflexive structure of $T(1 \cup 2)$. See [16]).

We can see that $\forall n \in \mathbb{N}^*$, the 2-colored collection C^n is naturally equipped with the pointing $1 \cup 2 \xrightarrow{\lambda^n} (C^n, d, c)$ defined as $\forall m \in \mathbb{N}, \lambda^n(1(m)) = u_m$ and $\lambda^n(2(m)) = v_m$.

3.3 The Co-∞-Graph of Coloured Operads of the n-Transformations ($n \in \mathbb{N}^*$)

In order not to make heavy notations we can write with the same notation δ^n_{n+1} and κ^n_{n+1}, sources and targets of the co-∞-graph of coloured collections, the co-∞-graph of coloured operads, and the ∞-graph in $\mathbb{M}nd$ below. There is no risk of confusion. The set $\{C^n / n \in \mathbb{N}\}$ has got a natural structure of co-∞-graph. This co-∞-graph is generated by diagrams

$$C^n \underset{\kappa^n_{n+1}}{\overset{\delta^n_{n+1}}{\rightrightarrows}} C^{n+1}$$

of pointed 2-coloured collections. For $n \geq 2$, these diagrams are defined as follows: First the $(n+1)$-colored collection contains the same symbols of operations as C^n for the j-cells, $0 \leq j \leq n-1$ or $n+2 \leq j < \infty$. For the n-cells and the $(n+1)$-cells the symbols of operations will change: C^n contains the n-cell ξ_n whereas C^{n+1} contains the n-cells α_n and β_n, in addition contains the $(n+1)$-cell ξ_{n+1}. If one notes $C^n - \xi_n$ the n-coloured collection obtained on the basis of C^n by taking from it the n-cell ξ_n, then δ^n_{n+1} is defined as follows: $\delta^n_{n+1}|_{C^n-\xi_n}$ (i.e the restriction of δ^n_{n+1} to $C^n-\xi_n$) is the canonical injection $C^n-\xi_n \hookrightarrow C^{n+1}$ and $\delta^n_{n+1}(\xi_n)=\alpha_n$. In a similar way κ^n_{n+1} is defined as follows: $\kappa^n_{n+1}|_{C^n-\xi_n}=\delta^n_{n+1}|_{C^n-\xi_n}$ and $\kappa^n_{n+1}(\xi_n)=\beta_n$. We can notice that δ^n_{n+1} and κ^n_{n+1} keeps pointing, i.e we have for all $n\geq 1$ the equalities $\delta^n_{n+1}\lambda^n=\lambda^{n+1}$ and $\kappa^n_{n+1}\lambda^n=\lambda^{n+1}$.

The morphisms of 2-colored pointing collections of the diagram

$$C^O \underset{\kappa^0_1}{\overset{\delta^0_1}{\rightrightarrows}} C^1 \underset{\kappa^1_2}{\overset{\delta^1_2}{\rightrightarrows}} C^2 \underset{\kappa^2_3}{\overset{\delta^2_3}{\rightrightarrows}} C^3$$

have a similar definition:

By considering notation of section 3.2, we have for all integer $0\leq p<n$ and for all $\forall m\in\mathbb{N}$:

$\delta^0_1(\mu^n_p)=\mu^n_p$; $\delta^0_1(u_m)=u_m$; $\kappa^0_1(\mu^n_p)=\nu^n_p$; $\kappa^0_1(u_m)=v_m$.

Also: $\delta^1_2(\mu^n_p)=\mu^n_p$; $\delta^1_2(u_m)=u_m$; $\delta^1_2(\nu^n_p)=\nu^n_p$; $\delta^1_2(v_m)=v_m$; $\delta^1_2(F^m)=F^m$. And $\kappa^1_2(\mu^n_p)=\mu^n_p$; $\kappa^1_2(u_m)=u_m$; $\kappa^1_2(\nu^n_p)=\nu^n_p$; $\kappa^1_2(v_m)=v_m$; $\kappa^1_2(F^m)=H^m$.

Finally: $\delta^2_3(\mu^n_p)=\mu^n_p$; $\delta^2_3(u_m)=u_m$; $\delta^2_3(\nu^n_p)=\nu^n_p$; $\delta^2_3(v_m)=v_m$; $\delta^2_3(F^m)=\alpha^m_0$; $\delta^2_3(H^m)=\beta^m_0$; $\delta^2_3(\tau)=\alpha_1$. And $\kappa^2_3(\mu^n_p)=\mu^n_p$; $\kappa^2_3(u_m)=u_m$; $\kappa^2_3(\nu^n_p)=\nu^n_p$; $\kappa^2_3(v_m)=v_m$; $\kappa^2_3(F^m)=\alpha^m_0$; $\kappa^2_3(H^m)=\beta^m_0$; $\kappa^2_3(\tau)=\beta_1$.

The pointed 2-coloured collections C^n $(n\in\mathbb{N}^*)$ are the sytems of operations of the n-Transformations. Each of them freely produces the contractible 2-colored operads B^n $(n\in\mathbb{N}^*)$. Each of these contractible operads is equipped with a system of operations given by the pointed 2-coloured collection C^n. These operads B^n are the operads of the n-Transformations $(n\in\mathbb{N}^*)$ and are the most important objects in this article. They produce the monads $\underline{B}^n$ whose algebras are the sought-after n-Transformations (see section 4 below). Due to the universal property of the unit b of the monad $\mathbb{B}$, $C^n \xrightarrow{b(C^n)} B^n = B(C^n)$, one obtains the co-$\infty$-graph $B^\bullet$ of the coloured operads of the n-Transformations.

$$\begin{tikzcd} B^0 \arrow[r, shift left, "\delta_1^0"] \arrow[r, shift right, "\kappa_1^0"'] & B^1 \arrow[r, shift left, "\delta_2^1"] \arrow[r, shift right, "\kappa_2^1"'] & B^2 \arrow[r, dotted, shift left] \arrow[r, dotted, shift right] & B^{n-1} \arrow[r, shift left, "\delta_n^{n-1}"] \arrow[r, shift right, "\kappa_n^{n-1}"'] & B^n \arrow[r, dotted, dash] & {} \\ C^0 \arrow[u, "b(C^0)"] \arrow[r, shift left, "\delta_1^0"] \arrow[r, shift right, "\kappa_1^0"'] & C^1 \arrow[u, "b(C^1)"'] \arrow[r, shift left, "\delta_2^1"] \arrow[r, shift right, "\kappa_2^1"'] & C^2 \arrow[u, "b(C^2)"'] \arrow[r, dotted, shift left] \arrow[r, dotted, shift right] & C^{n-1} \arrow[u, "b(C^{n-1})"'] \arrow[r, shift left, "\delta_n^{n-1}"] \arrow[r, shift right, "\kappa_n^{n-1}"'] & C^n \arrow[u, "b(C^n)"'] \arrow[r, dotted, dash] & {} \end{tikzcd}$$

The commutativity property of these diagrams is important for the consistence of algebras (see section 4.5). In particular morphisms

$$\begin{tikzcd} B^0 \arrow[r, shift left, "\delta_1^0"] \arrow[r, shift right, "\kappa_1^0"'] & B^1 \end{tikzcd}$$

are obtain with the following way:

First we consider "morphisms of colors" (in the category of ω-graphs)

$$\begin{tikzcd} 1 \arrow[r, shift left, "i_1"] \arrow[r, shift right, "i_2"'] & 1 \cup 2 \end{tikzcd}$$

such as $\forall n \in \mathbb{N}, i_1(1(n)) = 1(n)$ and $i_2(1(n)) = 2(n)$

Then we build for each $j \in \{1,2\}$ the following diagram

$$\begin{tikzcd} B^0 \arrow[r, dotted, "u_j"] \arrow[dr, "(d^0,c^0)"'] & (T(i_j) \times i_j)^*(B^1) \arrow[r, "v_j"] \arrow[d] & B^1 \arrow[d, "(d^1,c^1)"] \\ & T(1) \times 1 \arrow[r, "T(i_j)\times i_j"'] & T(1 \cup 2) \times (1 \cup 2) \end{tikzcd}$$

where the right square is cartesian (we change the color of the operad B^1 by pullback) and where the new operads $(T(i_j) \times i_j)^*(B^1)$ has a composition system and is contractible as well. So by universality, for each $j \in \{1,2\}$, we get the unique morphism u_j and we write $v_1 \circ u_1 = \delta_1^0$ and $v_2 \circ u_2 = \kappa_1^0$. Also it is not difficult to see the co-globularity property of the diagram

$$\begin{tikzcd} B^0 \arrow[r, shift left, "\delta_1^0"] \arrow[r, shift right, "\kappa_1^0"'] & B^1 \arrow[r, shift left, "\delta_2^1"] \arrow[r, shift right, "\kappa_2^1"'] & B^2 \end{tikzcd}$$

4 Monads and Algebras of the n-Transformations ($n \in \mathbb{N}^*$)

$\mathbb{M}nd$ is the category of the categories equipped with a monad, and $\mathbb{A}dj$ is the category of the adjunction pairs. These categories are defined in [16].

4.1 Monads $\underline{B}^n$ of the n-Transformations ($n \in \mathbb{N}^*$).

If $\mathscr{C}$ is a topos we shall note $\mathscr{C}\diagup B \xrightarrow{f^*} \mathscr{C}\diagup A$ the pullback functor associated with an arrow $A \xrightarrow{f} B$ of $\mathscr{C}$, and $\mathscr{C}\diagup A \xrightarrow{\Sigma_f} \mathscr{C}\diagup B$ the composition functor. We have the usual adjunctions: $\Sigma_f \dashv f^* \dashv \pi_f$, where π_f is the internal product functor.

Each T-category produces a monad which is described in [18, 4.3 page 150]. Hence $\forall n \in \mathbb{N}^*$, the operad B^n of the n-Transformations produce a monad $\underline{B}^n$ on ∞-$\mathbb{G}\mathrm{r}/1\cup 2$. More precisely, if we note (B^n, d^n, c^n) its underlying T-graph we have: $\underline{B}^n := \Sigma_{c^n}(d^n)^*\widehat{T}$ (where we put $\widehat{T}(C,d,c) := (T(C), T(d), T(c))$). A bicolour ∞-graph $G \xrightarrow{g} 1\cup 2$ is often noted G because there is no risk of confounding. We can therefore write $\underline{B}^n(G)$ instead of $\underline{B}^n(g)$, and it will be the same for the natural transformations δ_n^{n-1} and κ_n^{n-1} (see below) and we write $\underline{B}^n(G) := T(G) \times_{T(1\cup 2)} B^n$ (implied $\underline{B}^n(g) = c^n \circ \pi_1$) and the definition of $\underline{B}^n$ on morphisms is as easy. Projection on $T(G) \times_{T(1\cup 2)} B^n$ are noted π_0 and π_1. The definition of $\underline{B}^0$ is similar.

4.2 The ∞-graph of $\mathbb{M}nd$ of Monads of n-Transformations ($n \in \mathbb{N}^*$)

Considering $G \xrightarrow{g} 1\cup 2$, a bicolour ∞-graph. If we apply to it the monads $\underline{B}^n$ and $\underline{B}^{n-1}$ we obtain the equalities $d^n\pi_1 = T(g)\pi_0$, $d^{n-1}\pi_1 = T(g)\pi_0$. We also have $d^{n-1} = d^n\delta_n^{n-1}$ (To remove any confusion on our abuses of notations, the reader is encouraged to draw corresponding diagram). Thus we have

$d^n \circ \delta_n^{n-1} \circ \pi_1 = 1_{T(1\cup 2)} \circ d^{n-1} \circ \pi_1 = 1_{T(1\cup 2)} \circ T(g) \circ \pi_0 = T(g) \circ 1_{T(G)} \circ \pi_0$.
Hence the existence of a single morphism of ∞-graph

$$T(G) \times_{T(1\cup 2)} B^{n-1} \xrightarrow{\delta_n^{n-1}(G)} T(G) \times_{T(1\cup 2)} B^n$$

such as $\delta_n^{n-1}\pi_1 = \pi_1 \delta_n^{n-1}(G)$ and $\pi_0 = \pi_0 \delta_n^{n-1}(G)$. In particular we obtain the equality $c^n \pi_1 \delta_n^{n-1}(G) = c^{n-1}\pi_1$. It is then easy to see that to each bicolour ∞-graph is associated the morphism (of ∞-$\mathbb{G}/1\cup 2$): $\underline{B}^{n-1}(G) \xrightarrow{\delta_n^{n-1}(G)} \underline{B}^n(G)$ (these morphisms are still simply called $\delta_n^{n-1}(G)$). It is very easy to see that the set of these morphisms produce a natural transformation $\underline{B}^{n-1} \xrightarrow{\delta_n^{n-1}} \underline{B}^n$. It is shown that δ_n^{n-1} fits the axioms $\mathbb{M}nd1$ and $\mathbb{M}nd2$ of the morphisms of monads (these axioms are in [16]; particularly because $B^{n-1} \xrightarrow{\delta_n^{n-1}} B^n$ is a morphism of operads). Hence we get the morphism of $\mathbb{M}nd$

$$(\infty\text{-}\mathbb{G}\mathrm{r}/1\cup 2, \underline{B}^n) \xrightarrow{\delta_n^{n-1}} (\infty\text{-}\mathbb{G}\mathrm{r}/1\cup 2, \underline{B}^{n-1})$$

Thus the morphisms of coloured operads $B^{n-1} \underset{\kappa_n^{n1}}{\overset{\delta_n^{n1}}{\rightrightarrows}} B^n$ $(n \geq 2)$, create natural transformations $\underline{B}^{n-1} \underset{\kappa_n^{n1}}{\overset{\delta_n^{n-1}}{\rightrightarrows}} \underline{B}^n$ which fits into the axioms $\mathbb{M}nd1$ and $\mathbb{M}nd2$ of morphisms of monads. So we get the diagrams of $\mathbb{M}nd(n \geq 2)$

$$(\infty\text{-}\mathbb{G}\mathrm{r}/1\cup 2, \underline{B}^n) \underset{\kappa_n^{n-1}}{\overset{\delta_n^{n-1}}{\rightrightarrows}} (\infty\text{-}\mathbb{G}\mathrm{r}/1\cup 2, \underline{B}^{n-1})$$

Similarly the morphisms $B^0 \underset{\kappa_1^0}{\overset{\delta_1^0}{\rightrightarrows}} B^1$ produce two natural transformations $\underline{B}^0 \circ i_1^* \xrightarrow{\delta_1^0} i_1^* \circ \underline{B}^1$, $\underline{B}^0 \circ i_2^* \xrightarrow{\kappa_1^0} i_2^* \circ \underline{B}^1$ (i_1^* and i_2^* are the colour functors) which also fits $\mathbb{M}nd1$ and $\mathbb{M}nd2$, which leads to the diagram of $\mathbb{M}nd$

$$(\infty\text{-}\mathbb{G}\mathrm{r}/1\cup 2, \underline{B}^1) \underset{\kappa_1^0}{\overset{\delta_1^0}{\rightrightarrows}} (\infty\text{-}\mathbb{G}\mathrm{r}/1\cup 2, \underline{B}^0)$$

It is generally appeared that the building of the monad associated to a T-category is functorial, so the diagram of $\mathbb{M}nd$

$$\cdots \rightrightarrows (\infty\text{-}\mathbb{G}\mathrm{r}/1\cup 2,\underline{B}^n) \cdots \rightrightarrows (\infty\text{-}\mathbb{G}\mathrm{r}/1\cup 2,\underline{B}^1) \rightrightarrows (\infty\text{-}\mathbb{G}\mathrm{r}/1\cup 2,\underline{B}^0)$$

is a ∞-graph: The ∞-graph $\underline{B}^\bullet$ in $\mathbb{M}nd$ of the monads of the n-Transformations ($n\in\mathbb{N}^*$).

4.3 The ∞-Graph of $\mathbb{C}AT$ of Batanin's Algebras of n-Transformations ($n\in\mathbb{N}^*$)

As in [16, § 4.3] we know that we have the functors

$$\mathbb{M}nd \xrightarrow{A} \mathbb{A}dj \xrightarrow{D} \mathbb{C}AT$$

where A is the functor, which is linked with any monad, its pair of adjunction functors and where D is the projection functor which associates X with any adjunction $X \underset{F}{\overset{G}{\rightleftarrows}} Y$. So it is easy to see that $D\circ A$ associates its category of Eilenberg-Moore algebras to any monads. Particularly the functor $\mathbb{M}nd \xrightarrow{D\circ A} \mathbb{C}AT$ produces the following ∞-graph of $\mathbb{C}AT$

$$\cdots \rightrightarrows \mathbb{A}lg(\underline{B}^n) \underset{\beta^n_{n-1}}{\overset{\sigma^n_{n-1}}{\rightrightarrows}} \mathbb{A}lg(\underline{B}^{n-1}) \cdots \rightrightarrows \mathbb{A}lg(\underline{B}^1) \underset{\beta^1_0}{\overset{\sigma^1_0}{\rightrightarrows}} \mathbb{A}lg(\underline{B}^0)$$

which is the ∞-graph $\mathbb{A}lg(\underline{B}^\bullet)$ of algebras of n-Transformations ($n\in\mathbb{N}$). It is the most important ∞-graph of this article since it contains all Batanin's n-Transformations ($n\in\mathbb{N}$).

4.4 Domains and Codomains of Algebras

Let us remember the morphisms of $\mathbb{M}nd$: $(C,T) \xrightarrow{(Q,t)} (C',T')$ are given by functors $C \xrightarrow{Q} C'$ and natural transformations $T'\circ Q \xrightarrow{t} Q\circ T$ whose fits $\mathbb{M}nd1$ and $\mathbb{M}nd2$. If we apply the functor $\mathbb{M}nd \xrightarrow{D\circ A} \mathbb{C}AT$ to these morphisms, one can get the functor, $\mathbb{A}lg(T) \to \mathbb{A}lg(T')$, defined on the objects as $(G,v) \longmapsto (Q(G), Q(v) \circ t(G))$. We can now describe the functors σ_{n-1}^n and β_{n-1}^n ($n \geq 1$):

- If $n \geq 2$ then $\mathbb{A}lg(\underline{B}^n) \xrightarrow{\sigma_{n-1}^n} \mathbb{A}lg(\underline{B}^{n-1})$, $(G,v) \longmapsto (G, v \circ \delta_n^{n-1}(G))$ and $\mathbb{A}lg(\underline{B}^n) \xrightarrow{\beta_{n-1}^n} \mathbb{A}lg(\underline{B}^{n-1})$, $(G,v) \longmapsto (G, v \circ \kappa_n^{n-1}(G))$.
- If $n = 1$ then $\mathbb{A}lg(\underline{B}^1) \xrightarrow{\sigma_0^1} \mathbb{A}lg(\underline{B}^0)$, $(G,v) \longmapsto (i_1^*(G), i_1^*(v) \circ \delta_1^0(G))$ and $\mathbb{A}lg(\underline{B}^1) \xrightarrow{\beta_0^1} \mathbb{A}lg(\underline{B}^0)$, $(G,v) \longmapsto (i_2^*(G), i_2^*(v) \circ \kappa_1^0(G))$.

4.5 Consistence of Algebras

As Penon's [16], Batanin's n-Transformations ($n \in \mathbb{N}^*$) are particular in that they describe the hole semantics of their domain and codomain algebras as follows: If we have an algebra (G,v) of n-Transformations, then a symbol of operation of the operad B^n which has its counterpart in the operad B^p ($0 \leq p < n$) will be semantically interpreted similarly via this algebra (G,v) or via the algebra $\sigma_p^n(G,v)$ or the algebra $\beta_p^n(G,v)$.

Remark 3 This terminology is taken from measure theory where different coverings of a measurable subset are measured with the same value by a determined measure, which makes sense to that measure. $\square$

This is the simple consequence of the commutative property of diagrams in section 3.3 applied to a bicolour ∞-graph.

So as to illustrate this property of consistence, let us take for example the symbol of operation H^m of the operad B^2 (identified with $b(C^2)(H^m)$). It will be semantically interpreted by an algebra $(G,v) \in \mathbb{A}lg(\underline{B}^2)$ on a m-cell $a \in G(m)$ (of colour 1), similarly to how the F^m symbol of the B^1 operad is interpreted by the target algebra $\beta_1^2(G,v) \in \mathbb{A}lg(\underline{B}^1)$. Indeed the equalities $\kappa_2^1\pi_1 = \pi_1\kappa_2^1(G)$ and $\kappa_2^1 b(C^1) = b(C^2)\kappa_2^1$ immediately suggests that: $(a, F^m) \xmapsto{\kappa_2^1(G)} (a, H^m)$, then $v(a, H^m) = (v \circ \kappa_2^1(G))(a, F^m) =$

$\beta_1^2(G,v)(a,F^m)$, which expresses consistence. In short, we will say that Batanin's algebras (as Penon's algebras) are consistent.

5 Dimension 2

5.1 Dimension of Algebras

The dimension of Penon's algebras is defined in [21] and in [16]. The dimension of Batanin's algebras is totally similar, but we must precisely define the structures of the underlying ∞-magmas of these algebras so as to have a reflexive structure. So we can note $B^n \times_{T(1\cup 2)} T(G) \xrightarrow{v} G$ a $\underline{B}^n$-algebra i.e a weak n-transformation ($n \geq 1$). The two ∞-magmas ([16]) of this algebra are defined as follows: $\alpha \circ_p^n \beta := v(\mu_p^n; \eta(\alpha) \star_p^n \eta(\beta))$ and $1_\alpha := v([u_n, u_n]; 1_{\eta(\alpha)})$, if $\alpha, \beta \in G(n)$ and are with colour 1. Furthemore $\alpha \circ_p^n \beta := v(\nu_p^n; \eta(\alpha) \star_p^n \eta(\beta))$ and $1_\alpha := v([v_n, v_n]; 1_{\eta(\alpha)})$, if $\alpha, \beta \in G(n)$ and are with colour 2. Then (G,v) has dimension 2 if its two underlying ∞-magmas has dimension 2. We have the same definition for $\underline{B}^0$-algebras (i.e weak ∞-categories).

5.2 The $\underline{B}^1$-Algebras of dimension 2 are Pseudo-2-Functors

Let (G,v) be a $\underline{B}^1$-algebra of dimension 2. The $\underline{B}^0$-algebra's source of (G,v): $\sigma_0^1(G,v) = (i_1^*(G), i_1^*(v) \circ \delta_1^0(G))$ put on $i_1^*(G)$ a bicategory structure which coincides with the one produced by (G,v) on $i_1^*(G)$. In the same way, the $\underline{B}^0$-algebra target of (G,v): $\beta_0^1(G,v) = (i_2^*(G), i_2^*(v) \circ \kappa_1^0(G))$ put on $i_2^*(G)$ a bicategory structure which coincides with that one produced by (G,v) on $i_2^*(G)$. All these coincidences come from the consistence of algebras, and so we can therefore make all our calculations merely with the $\underline{B}^1$-algebra (G,v) to show the given below axiom of associativity-distributivity (that we call *AD*-axiom) of pseudo-2-functors. For other axioms of the pseudo-2-

functors, which are easier, we proceed in the same way. Let $F^m (m \in \mathbb{N})$ be the unary operations symbols of functors of the operad B^1. The $\underline{B}^1$-algebra of dimension 2 interprets these symbols into pseudo-2-functors. Indeed if $B^1 \times_{T(1\cup 2)} T(G) \xrightarrow{v} G$ is a $\underline{B}^1$-algebra of dimension 2 then we get: $\forall m \in \mathbb{N}$, $F(a) := v(F^m ; \eta(a))$ if $a \in G(m)$ (a has the colour 1), which defines a morphism of ∞-graphs $i_1^*(G) \xrightarrow{F} i_2^*(G)$ where $i_1^*(G)$ and $i_2^*(G)$ are bicategories. So we will show that this morphism F fits the AD-axiom of pseudo-2-functors. Let $x \xrightarrow{a} y \xrightarrow{b} z \xrightarrow{c} t$ be a 1-cellules diagram of $i_1^*(G)$, we are going to check that we get the following commutativity

$$\begin{tikzcd}
F(a)\circ_0^1 (F(b)\circ_0^1 F(c)) \arrow[rr, Rightarrow, "1_{F(a)}\circ_0^2 d(b,c)"] & & F(a)\circ_0^1 F(b\circ_0^1 c) \arrow[d, Rightarrow, "d(a,b\circ_0^1 c)"] \\
(F(a)\circ_0^1 F(b))\circ_0^1 F(c) \arrow[u, Rightarrow, "a(F(a),F(b),F(c))"] \arrow[d, Rightarrow, "d(a,b)\circ_0^2 1_{F(c)}"'] & \square & F(a\circ_0^1 (b\circ_0^1 c)) \\
F(a\circ_0^1 b)\circ_0^1 F(c) \arrow[rr, Rightarrow, "d(a\circ_0^1 b,c)"] & & F((a\circ_0^1 b)\circ_0^1 c) \arrow[u, Rightarrow, "F(a(a,b,c))"']
\end{tikzcd}$$

where $a\circ_0^1 (b\circ_0^1 c) \xRightarrow{a(a,b,c)} (a\circ_0^1 b)\circ_0^1 c$ is an associativity coherence cell and $F(a)\circ_0^1 F(b) \xRightarrow{d(a,b)} F(a\circ_0^1 b)$ is a distributivity coherence cell (particular to pseudo-2-functors). The strategy to demonstrate the AD-axiom is simple: We build a diagram of 3-cells of B^1 which will be semantically interpreted by the $\underline{B}^1$-algebras of dimension 2 as the AD-axiom. To be clearer, the operadic multiplication of the coloured operad B^1

$$B^1 \times_{T(1\cup 2)} T(B^1) \xrightarrow{\gamma} B^1$$

will be noted γ_i for each i-cellular level. Let the following 2-cells in B^1:

$$d := [\gamma_1(v_0^1 ; \eta(F^1)\star_0^1 \eta(F^1)); \gamma_1(F^1 ; \eta(\mu_0^1))];$$

$$a_1 := [\gamma_1(\mu_0^1 ; \eta(\mu_0^1)\star_0^1 \eta(u_1)); \gamma_1(\mu_0^1 ; \eta(u_1)\star_0^1 \eta(\mu_0^1))];$$

$a_2 := [\gamma_1(\nu_0^1;\eta(\nu_0^1)\star_0^1\eta(v_1));\, \gamma_1(\nu_0^1;\eta(v_1)\star_0^1\eta(\nu_0^1))]$.

Remark 4 The operation symbol d is interpreted by the algebra as the distributivity coherence cells of the pseudo-2-functors. The symbols a_1 and a_2 are interpreted as the associativity coherence cells, the first one for the weak ∞-category source the second one for the weak ∞-category target. $\square$

Then we can consider the following 2-cells of B^1:

$\rho_1 = \gamma_2(\nu_0^2;\eta([F^1;F^1])\star_0^2\eta(d))$;

$\rho_2 = \gamma_2(d;1_{\eta(u_1)}\star_0^2 1_{\eta(\mu_0^1)})$;

$\rho_3 = \gamma_2(F^2;\eta(a_1))$;

$\rho_4 = \gamma_2(d;1_{\eta(\mu_0^1)}\star_0^2 1_{\eta(u_1)})$;

$\rho_5 = \gamma_2(\nu_0^2;\eta(d)\star_0^2\eta([F^1;F^1]))$

$\rho_6 = \gamma_2(a_2;1_{\eta(F^1)}\star_0^2 1_{\eta(F^1)}\star_0^2 1_{\eta(F^1)})$.

This 2-cells are the conglomerations of operation symbols that are interpreted by algebras as the coherence 2-cells of the diagram of the AD-axiom of pseudo-2-functors

$$\begin{tikzcd}
\bullet \arrow[r, Rightarrow, "\rho_1"] & \bullet \arrow[d, Rightarrow, "\rho_2"] \\
\bullet \arrow[u, Rightarrow, "\rho_6"] \arrow[d, Rightarrow, "\rho_5"'] \arrow[r, phantom, "\square"] & \bullet \\
\bullet \arrow[r, Rightarrow, "\rho_4"] & \bullet \arrow[u, Rightarrow, "\rho_3"']
\end{tikzcd}$$

Then we consider the following 2-cells of B^1

$\Lambda_1 = \gamma_2(\nu_1^2;\eta(\gamma_2(\nu_1^2;\eta(\rho_2)\star_1^2\eta(\rho_1)))\star_1^2\eta(\rho_6))$;

$\Lambda_1' = \gamma_2(\nu_1^2;\eta(\rho_2)\star_1^2\eta(\gamma_2(\nu_1^2;\eta(\rho_1)\star_1^2\eta(\rho_6))))$;

$\Lambda_2 = \gamma_2(\nu_1^2; \eta(\gamma_2(\nu_1^2; \eta(\rho_3) \star_1^2 \eta(\rho_4))) \star_1^2 \eta(\rho_5));$

$\Lambda'_2 = \gamma_2(\nu_1^2; \eta(\rho_3) \star_1^2 \eta(\gamma_2(\nu_1^2; \eta(\rho_4) \star_1^2 \eta(\rho_5)))).$

We can show that these 2-cells are parallels and with the same domain, so they are connected with coherences 3-cells

$\Theta_1 = [\Lambda_1, \Lambda'_1]$, $\Theta_2 = [\Lambda'_1, \Lambda_2]$, $\Theta_3 = [\Lambda_2, \Lambda'_2]$,

and the interpretation by $\underline{B}^1$-algebras of dimension 2 of this 3-cells gives the *AD*-axiom of pseudo-2-functors.

5.3 The $\underline{B}^2$-Algebras of dimensions 2 are Natural Pseudo-2-Transformations

Let (G,v) be a $\underline{B}^2$-algebra of dimension 2. The $\underline{B}^0$-algebra source of (G,v): $\sigma_1^2(\sigma_0^1(G,v)) = (i_1^*(G), i_1^*(v \circ \delta_2^1(G)) \circ \delta_1^0(G))$ put in $i_1^*(G)$ a bicategory structure which coincides with the one produced by (G,v) on $i_1^*(G)$. In the same way, the $\underline{B}^0$-algebra target of (G,v): $\beta_1^2(\beta_0^1(G,v)) = (i_2^*(G), i_2^*(v \circ \kappa_2^1(G)) \circ \kappa_1^0(G))$ put in $i_2^*(G)$ a bicategory structure which coincides with the one produced by (G,v) on $i_2^*(G)$. And the $\underline{B}^1$-algebra source of (G,v): $\sigma_1^2(G,v) = (G, v \circ \delta_2^1(G))$ produces a pseudo-2-functor F_1 (see above) which coincides with the one produced by (G,v) i.e the one built with the ∞-graph morphism $i_1^*(G) \xrightarrow{F_1} i_2^*(G)$ defined as: $F_1(a) := v(F^m; \eta(a))$ if $a \in i_1^*(G)(m)$. Besides the $\underline{B}^1$-algebra target of (G,v): $\beta_1^2(G,v) = (G, v \circ \kappa_2^1(G))$ produces a pseudo-2-functor H_1 which coincides with the one produced by (G,v) i.e the one built with the ∞-graph morphism $i_1^*(G) \xrightarrow{H_1} i_2^*(G)$ defined as: $H_1(a) := v(H^m; \eta(a))$ if $a \in i_1^*(G)(m)$. All these coincidences come from the consistence of algebras, and we can therefore make all our calculations merely with the $\underline{B}^2$-algebra (G,v) (without using its source algebra or its target algebra) to show the axiom below of compatibility with associativity-distributivity of natural pseudo-2-transformations (that we call *CAD*-axiom). Then let τ be the unary operation symbol of natural transformation of the operad B^2. This symbol is interpreted by the $\underline{B}^2$-algebras of dimension 2 as natural pseudo-2-transformations. Indeed if $B^2 \times_{T(1\cup 2)} T(G) \xrightarrow{v} G$ is an $\underline{B}^2$-algebra of dimension 2 then we write

$$\tau_1(a) := v(\tau; 1_{\eta(a)}), \text{ if } a \in G(0)(a \text{ has colour} 1),$$

We can see that it defines a 1-cells family τ_1 in $i_2^*(G)$ indexed by $i_1^*(G)(0)$

$$\begin{tikzcd} i_1^*(G) \arrow[r, bend left, "F_1"] \arrow[r, bend right, "H_1"'] \arrow[r, phantom, "\Downarrow\tau_1"] & i_2^*(G) \end{tikzcd}$$

We are going to show that the previous family τ_1 fits the *CAD*-axiom of natural pseudo-2-transformations. For other axioms of natural pseudo-2-transformations, which are easier, we proceed in the same way. Let $x \xrightarrow{a} y \xrightarrow{b} z$ be an 1-cells diagram of $i_1^*(G)$, we are going to prove that we have the following commutativity

$$\begin{tikzcd} H_1(b)\circ_0^1(H_1(a)\circ_0^1\tau_1(x)) \arrow[rr, Rightarrow, "1_{H_1(b)}\circ_0^2\omega(a)"] \arrow[d, Rightarrow, "a(H_1(b),H_1(a),\tau_1(x))"'] & & H_1(b)\circ_0^1(\tau_1(y)\circ_0^1F_1(a)) \arrow[d, Rightarrow, "a(H_1(b),\tau_1(y),F_1(a))"] \\ (H_1(b)\circ_0^1H_1(a))\circ_0^1\tau_1(x) \arrow[d, Rightarrow, "d_1(a,b)\circ_0^2 1_{\tau_1(x)}"'] & & (H_1(b)\circ_0^1\tau_1(y))\circ_0^1F_1(a) \arrow[d, Rightarrow, "\omega(b)\circ_0^2 1_{F_1(a)}"] \\ H_1(b\circ_0^1 a)\circ_0^1\tau_1(x) \arrow[d, Rightarrow, "\omega(b\circ_0^1 a)"'] & \square & (\tau_1(z)\circ_0^1F_1(b))\circ_0^1F_1(a) \arrow[d, Rightarrow, "a(\tau_1(z),F_1(b),F_1(a))"] \\ \tau_1(z)\circ_0^1F_1(b\circ_0^1 a) & & \tau_1(z)\circ_0^1(F_1(b)\circ_0^1F_1(a)). \arrow[ll, Rightarrow, "1_{\tau_1(z)}\circ_0^2 d_0(b,a)"] \end{tikzcd}$$

where in particular $H_1(a)\circ_0^1\tau_1(x) \xrightarrow{\omega(a)} \tau_1(y)\circ_0^1F_1(a)$ is a coherence cell specific to natural pseudo-2-transformations. The strategy to demonstrate the *CAD*-axiom is similar to the previous demonstration (for the *AD*-axiom of pseudo-2-functors): We build a diagram of 3-cells of B^2 that will be semantically interpreted by the $\underline{B}^2$-algebras of dimension 2 as the CAD-axiom. Like before operadic composition is

$$B^2 \times_{T(1\cup 2)} T(B^2) \xrightarrow{\gamma} B^2$$

will be noted γ_i for each i-cellular level. So we can consider the following 2-cells of B^2

$$\omega := [\gamma_1(v_0^1; \eta(H^1) \star_0^1 \eta(\tau)); \gamma_1(v_0^1; \eta(\tau) \star_0^1 \eta(F^1))];$$

$$d^F := [\gamma_1(v_0^1; \eta(F^1) \star_0^1 \eta(F^1)); \gamma_1(F^1; \eta(\mu_0^1))];$$

$$d^H := [\gamma_1(v_0^1; \eta(H^1) \star_0^1 \eta(H^1)); \gamma_1(H^1; \eta(\mu_0^1))];$$

$$a := [\gamma_1(v_0^1; \eta(v_1) \star_0^1 \eta(v_0^1)); \gamma_1(v_0^1; \eta(v_0^1) \star_0^1 \eta(v_1))];$$

$$b := [\gamma_1(v_0^1; \eta(v_0^1) \star_0^1 \eta(v_1)); \gamma_1(v_0^1; \eta(v_1) \star_0^1 \eta(v_0^1))].$$

Then we consider the following 2-cells

$$\rho_1 = \gamma_2(v_0^2; \eta([H^1; H^1]) \star_0^2 \eta(\omega));$$

$$\rho_2 = \gamma_2(a; 1_{\eta(H^1)} \star_0^2 1_{\eta(\tau)} \star_0^2 1_{\eta(F^1)});$$

$$\rho_3 = \gamma_2(v_0^2; \eta(\omega) \star_0^2 \eta([F^1; F^1]));$$

$$\rho_4 = \gamma_2(b; 1_{\eta(\tau)} \star_0^2 1_{\eta(F^1)} \star_0^2 1_{\eta(F^1)});$$

$$\rho_5 = \gamma_2(v_0^2; \eta([\tau; \tau]) \star_0^2 \eta(d^F));$$

$$\rho_6 = \gamma_2(\omega; 1_{\eta(\mu_0^1)});$$

$$\rho_7 = \gamma_2(v_0^2; \eta(d) \star_0^2 \eta([\tau; \tau]));$$

$$\rho_8 = \gamma_2(a; 1_{\eta(H^1)} \star_0^2 1_{\eta(H^1)} \star_0^2 1_{\eta(\tau)}).$$

We also consider one 2-cell ρ'_5 built as follows:

$$\delta^F := [\gamma_1(F^1;\eta(\mu^1_0));\gamma_1(v^1_0;\eta(F^1)\star^1_0\eta(F^1))].$$

In that case we define

$$\rho'_5 = \gamma_2(v^2_0;\eta([\tau;\tau])\star^2_0\eta(\delta^F)).$$

These 2-cells are the conglomeration of operation symbols that are interpreted by algebras as the coherence 2-cells of the diagram of the *CAD*-axiom of natural pseudo-2-transformations

$$\begin{tikzcd}
\bullet \arrow[r, Rightarrow, "\rho_1"] \arrow[d, Rightarrow, "\rho_8"'] & \bullet \arrow[d, Rightarrow, "\rho_2"] \\
\bullet \arrow[d, Rightarrow, "\rho_7"'] \arrow[dr, phantom, "\square"] & \bullet \arrow[d, Rightarrow, "\rho_3"] \\
\bullet \arrow[d, Rightarrow, "\rho_6"'] & \bullet \arrow[d, Rightarrow, "\rho_4"] \\
\bullet & \bullet \arrow[l, Rightarrow, "\rho_5"]
\end{tikzcd}$$

To built the ten coherence 2-cells Λ_i $(1 \leq i \leq 10)$ below, which enables to conclude, we need the following additional 2-cells

$$\Theta_1 = \gamma_2(v^2_1;\eta(\gamma_2(v^2_1;\eta(v^2_1)\star^2_1\eta(v_2)))\star^2_1\eta(v_2));$$

$$\Theta_2 = \gamma_2(v^2_1;\eta(\gamma_2(\mu^2_1;\eta(v_2)\star^2_1\eta(v^2_1)))\star^2_1\eta(v_2));$$

$$\Theta_3 = \gamma_2(v^2_1;\eta(v_2)\star^2_1\eta(\gamma_2(v^2_1;\eta(v^2_1)\star^2_1\eta(v_2))));$$

$$\Theta_4 = \gamma_2(v^2_1;\eta(v_2)\star^2_1\eta(\gamma_2(v^2_1;\eta(v_2)\star^2_1\eta(v^2_1))));$$

$$\Theta_5 = \gamma_2(v^2_1;\eta(v^2_1)\star^2_1\eta(v^2_1)).$$

The 2-cells $\Lambda_i(1 \leq i \leq 10)$ are then defined in the following way

$$\Lambda_1 = \gamma_2(\Theta_1;\eta(\rho_4)\star^2_1\eta(\rho_3)\star^2_1\eta(\rho_2)\star^2_1\eta(\rho_1));$$

$\Lambda_2 = \gamma_2(\Theta_2; \eta(\rho_4) \star_1^2 \eta(\rho_3) \star_1^2 \eta(\rho_2) \star_1^2 \eta(\rho_1));$

$\Lambda_3 = \gamma_2(\Theta_3; \eta(\rho_4) \star_1^2 \eta(\rho_3) \star_1^2 \eta(\rho_2) \star_1^2 \eta(\rho_1));$

$\Lambda_4 = \gamma_2(\Theta_4; \eta(\rho_4) \star_1^2 \eta(\rho_3) \star_1^2 \eta(\rho_2) \star_1^2 \eta(\rho_1));$

$\Lambda_5 = \gamma_2(\Theta_5; \eta(\rho_4) \star_1^2 \eta(\rho_3) \star_1^2 \eta(\rho_2) \star_1^2 \eta(\rho_1)).$

We can note as well $\lambda = \eta(\rho_5') \star_1^2 \eta(\rho_6) \star_1^2 \eta(\rho_7) \star_1^2 \eta(\rho_8)$. And consider $\Lambda_6 = \gamma_2(\Theta_1; \lambda)$, $\Lambda_7 = \gamma_2(\Theta_2; \lambda)$; $\Lambda_8 = \gamma_2(\Theta_3; \lambda)$; $\Lambda_9 = \gamma_2(\Theta_4; \lambda)$; $\Lambda_{10} = \gamma_2(\Theta_5; \lambda)$.

We can prove that these 2-cells are parallels and with the same domain, so they are connected with coherences 3-cells: $\zeta_i := [\Lambda_i; \Lambda_{i+1}]$ $(1 \leq i \leq 9)$. And the interpretation by $\underline{B}^2$-algebras of dimension 2 of these 3-cells gives the *CAD*-axiom of natural pseudo-2-transformations.

6 Fusion of Adjunctions

As we saw in theorem 1 we need to do the "fusion" of two monads to obtain a new monad, which inherits at the same time properties of these two monads. This monad is the contractible monoids monad $\mathbb{B} = (B, \rho, b)$ of the theorem 1 which permits us to build the operads of n-Transformations ($n \in \mathbb{N}$). The fusion between adjunctions require some hypotheses (see below) and naturally we shall see that our two adjunctions fill these hypotheses.

The following "fusion theorem" is a generalization of techniques used by Batanin in [2]. This theorem is going to be shown especially powerful because the required hypotheses are so simple. As a result the fusion product of two adjunctions is possible under conditions that we can often run into.

Lemma 1 *Let us consider the adjunction* $\mathscr{C} \underset{F}{\overset{U}{\underset{\leftarrow}{\overset{\rightarrow}{\top}}}} \mathscr{B}$ *such as* $\mathscr{C}$ *has a co-equalizer and* U *is faithful. Let the diagram* $B \underset{d_1}{\overset{d_0}{\rightrightarrows}} U(C)$ *in* $\mathscr{B}$*, then there* *is a unique morphism $C \xrightarrow{q} Q$ of $\mathscr{C}$ verifying $U(q)d_0 = U(q)d_1$ and which is universal for this property, i.e if we give ourselves another morphism $C \xrightarrow{q'} Q'$ of $\mathscr{C}$ such as $U(q')d_0 = U(q')d_1$, then there is a unique morphism $Q \xrightarrow{h} Q'$ of $\mathscr{C}$ such as $U(h)U(q) = U(q')$.* □

PROOF Given $\overline{d}_0$, $\overline{d}_1$ the morphisms of $\mathscr{C}$ which are the extensions of d_0 and d_1, and let us put $\widehat{d_0} = U(\overline{d}_0)$ and $\widehat{d_1} = U(\overline{d}_1)$. Let us note $C \xrightarrow{q} Q$ the coequalizer of $\overline{d}_0$ and $\overline{d}_1$. We get $U(q)d_0 = U(q)U(\overline{d}_0)\eta_X = U(q)U(\overline{d}_1)\eta_X = U(q)d_1$. We can show that q is universal for this property. Let $C \xrightarrow{q'} Q'$ another morphism of $\mathscr{C}$ verifying $U(q')d_0 = U(q')d_1$. So $U(q')U(\overline{d}_0)\eta_X = U(q')U(\overline{d}_1)\eta_X$, i.e $U(q'\overline{d}_0)\eta_X = U(q'\overline{d}_1)\eta_X$, Therefore we have $q'\overline{d}_0 = q'\overline{d}_1$ with $q = coker(\overline{d}_0, \overline{d}_1)$, which shows that there is a unique morphism $Q \xrightarrow{h} Q'$ of $\mathscr{C}$ such as $hq = q'$ and also this morphism is unique such as $U(h)U(q) = U(q')$, because U is faithful. ■

Let the following adjunction be: $(\mathscr{C},\mathtt{A}) \underset{F}{\overset{U}{\underset{\leftarrow}{\overset{\rightarrow}{\top}}}} (\mathscr{B},\mathtt{A})$. It is fusionnable if the following properties are verified:

- $\mathscr{C}$ has coequalizers and $\overrightarrow{\mathbb{N}}$-colimits.
- $\mathscr{B}$ have $\overrightarrow{\mathbb{N}}$-colimits.
- U is faithful and preserves $\overrightarrow{\mathbb{N}}$-colimits.

Remark 5 Here $\overrightarrow{\mathbb{N}}$-colimits is the notation used in [6] for directed colimits. □

Let us go to the fusion theorem.

Theorem 2 *Let us consider the adjunction $\mathscr{C} \underset{M}{\overset{U}{\underset{\leftarrow}{\overset{\rightarrow}{\top}}}} \mathscr{B}$ with monad $(L, \mathfrak{m}, l)$, and the adjunction $\mathscr{D} \underset{H}{\overset{V}{\underset{\leftarrow}{\overset{\rightarrow}{\top}}}} \mathscr{B}$ with monad (C, m, c). We suppose that these* *adjunctions are fusionnable. In this case, if we consider the cartesian square of categories*

$$\begin{tikzcd} \mathscr{C}\times_{\mathscr{B}}\mathscr{D} \arrow[r, "p_2"] \arrow[d, "p_1"'] & \mathscr{D} \arrow[d, "V"] \\ \mathscr{C} \arrow[r, "U"'] & \mathscr{B} \end{tikzcd}$$

then the forgetful functor $\mathscr{C}\times_{\mathscr{B}}\mathscr{D}\xrightarrow{O}\mathscr{B}$ has a left adjoint: $F\dashv O$. □

PROOF • Let $X\in\mathscr{B}(0)$. At first, we are going to build by induction an object $B(X)$ of $\mathscr{B}$ and secondly we shall reveal that $B(X)$ has got the expected universal property.

- If $n=0$ we give ourselves the following diagram of $\mathscr{B}$:

$$C_0=X\xrightarrow{l_0=l(C_0)}L(C_0)\xrightarrow{\phi_0=1}L_0\xrightarrow{c_0=c(L_0)}C(L_0)\xrightarrow{\psi_0=1}C_1\xrightarrow{l_1=l(C_1)}L(C_1)$$

Thanks to the lemma, we obtain the morphism ϕ_1 with the diagram

$$\begin{tikzcd} L(C_0) \arrow[r, shift left, "d_0=l_1\psi_0c_0\phi_0"] \arrow[r, shift right, "d_1=L(j_0)=L(\psi_0c_0\phi_0l_0)"'] & L(C_1) \arrow[r, "\phi_1"] & L_1 \end{tikzcd}$$

What allows to extend the previous diagram

$$C_1\xrightarrow{l_1}L(C_1)\xrightarrow{\phi_1}L_1\xrightarrow{c_1=c(L_1)}C(L_1)$$

And it allows again to obtain the morphism ψ_1

$$\begin{tikzcd} C(L_0) \arrow[r, shift left, "\delta_0=c_1\phi_1l_1\psi_0"] \arrow[r, shift right, "\delta_1=C(k_0)=C(\phi_1l_1\psi_0c_0)"'] & C(L_1) \arrow[r, "\psi_1"] & C_2 \end{tikzcd}$$

and thus to prolong once more the previous diagram

$$C_1\xrightarrow{l_1}L(C_1)\xrightarrow{\phi_1}L_1\xrightarrow{c_1}C(L_1)\xrightarrow{\psi_1}C_2\xrightarrow{l_2}L(C_2)$$

We do an induction. We can suppose that up to the rank n we can build these diagrams. In particular we give ourselves the following diagram

$$C_n \xrightarrow{l_n} L(C_n) \xrightarrow{\phi_n} L_n \xrightarrow{c_n} C(L_n) \xrightarrow{\psi_n} C_{n+1} \xrightarrow{l_{n+1}} L(C_{n+1})$$

where we especially note $j_n = \psi_n c_n \phi_n l_n$. We are going to show that we can prolong this type of diagram in the rank $n+1$. Thanks to the Lemma, we consider the morphism ϕ_{n+1}

$$\begin{tikzcd} L(C_n) \arrow[r, shift left, "d_0 = l_{n+1}\psi_n c_n \phi_n"] \arrow[r, shift right, "d_1 = L(j_n) = L(\psi_n c_n \phi_n l_n)"'] & L(C_{n+1}) \arrow[r, "\phi_{n+1}"] & L_{n+1} \end{tikzcd}$$

what allows to prolong the previous diagram

$$C_{n+1} \xrightarrow{l_{n+1}} L(C_{n+1}) \xrightarrow{\phi_{n+1}} L_{n+1} \xrightarrow{c_{n+1} = c(L_{n+1})} C(L_{n+1})$$

where we can particularly note $k_n = \phi_{n+1} l_{n+1} \psi_n c_n$. Then we consider, due to to the lemma, the morphism ψ_{n+1}

$$\begin{tikzcd} C(L_n) \arrow[r, shift left, "\delta_0 = c_{n+1}\phi_{n+1} l_{n+1}\psi_n"] \arrow[r, shift right, "\delta_1 = C(k_n) = C(\phi_{n+1} l_{n+1} \psi_n c_n)"'] & C(L_{n+1}) \arrow[r, "\psi_{n+1}"] & C_{n+2} \end{tikzcd}$$

and thus to prolong still the previous diagram

$$C_{n+1} \xrightarrow{l_{n+1}} L(C_{n+1}) \xrightarrow{\phi_{n+1}} L_{n+1} \xrightarrow{c_{n+1}} C(L_{n+1}) \xrightarrow{\psi_{n+1}} C_{n+2} \xrightarrow{l_{n+2}} L(C_{n+2})$$

Thus for all $n \in \mathbb{N}$ we have this construction, what brings to light the filtered diagram built with these diagrams. This filtered diagram is noted B_*. In particular the diagrams

$$\begin{tikzcd} L(C_{n-1}) \arrow[r, shift left, "d_0 = l_n \psi_{n-1} c_{n-1} \phi_{n-1}"] \arrow[r, shift right, "d_1 = L(\psi_{n-1} c_{n-1} \phi_{n-1} l_{n-1})"'] & L(C_n) \arrow[d, "\phi_n"'] \arrow[r, shift left, "d_0 = l_{n+1}\psi_n c_n \phi_n"] \arrow[r, shift right, "d_1 = L(\psi_n c_n \phi_n l_n)"'] & L(C_{n+1}) \arrow[d, "\phi_{n+1}"] \\ & L_n \arrow[r, dotted, "\lambda_n"'] & L_{n+1} \end{tikzcd}$$

show that

$\phi_{n+1}l_{n+1}\psi_n c_n\phi_n l_n\psi_{n-1}c_{n-1}\phi_{n-1} = \phi_{n+1}l_{n+1}\psi_n c_n\phi_n L(\psi_{n-1}c_{n-1}\phi_{n-1}$
$l_{n-1})$.

Thus according to the lemma, there is a unique morphism $L_n \xrightarrow{\lambda_n} L_{n+1}$, which is the forgetting of a morphism of $\mathscr{C}$, returning commutative these diagrams. Thus we obtain the filtered diagram L_* of $\mathscr{B}$ which is the forgetting of a diagram filtered of $\mathscr{C}$

$$L_0 \xrightarrow{\lambda_0} L_1 \xrightarrow{\lambda_1} \cdots\cdots \to L_n \xrightarrow{\lambda_n} L_{n+1} \xrightarrow{\lambda_{n+1}} \cdots\cdots$$

where B_* is an expanded diagram of L_* i.e we have

$$\overbrace{C_0 \xrightarrow{l_0} L(C_0) \xrightarrow{\phi_0} L_*}^{B_*}$$

We also have the diagram

$$\begin{tikzcd}
C(L_{n-2}) \arrow[r, shift left, "\delta_0=c_{n-1}\phi_{n-1}l_{n-1}\psi_{n-2}"] \arrow[r, shift right, "\delta_1=C(\phi_{n-1}l_{n-1}\psi_{n-2}c_{n-2})"'] & C(L_{n-1}) \arrow[r, shift left, "\delta_0=c_n\phi_n l_n\psi_{n-1}"] \arrow[r, shift right, "\delta_1=C(\phi_n l_n\psi_{n-1}c_{n-1})"'] \arrow[d, "\psi_{n-1}"'] & C(L_n) \arrow[d, "\psi_n"] \\
& C_n \arrow[r, dotted, "\kappa_n"] & C_{n+1}
\end{tikzcd}$$

which shows that

$\psi_n c_n\phi_n l_n\psi_{n-1}c_{n-1}\phi_{n-1}l_{n-1}\psi_{n-2} = \psi_n c_n\phi_n l_n\psi_{n-1}C(\phi_{n-1}l_{n-1}\psi_{n-2}$
$c_{n-2})$.

Thus according to the lemma, there is a unique morphism $C_n \xrightarrow{\kappa_n} C_{n+1}$ which is the forgetting of a morphism of $\mathscr{D}$ returning commutative these diagrams. Therefore we obtain the filtered diagram C_* of $\mathscr{B}$ which is the forgetting of a diagram filtered of $\mathscr{D}$

$$C_1 \xrightarrow{\kappa_1} C_2 \xrightarrow{\kappa_2} \cdots\cdots \to C_n \xrightarrow{\kappa_n} C_{n+1} \xrightarrow{\kappa_{n+1}} \cdots\cdots$$

where B_* is an expanded diagram of C_*, i.e we have

$$\overbrace{C_0 \xrightarrow{c_0\phi_0 l_0} C(L_0) \xrightarrow{\psi_0} C_*}^{B_*}$$

Thus these diagrams B_*, L_* and C_* have the same colimit $B(X)$ in $\mathscr{B}$. We put $L_* = U(M_*)$ and $M_* \to \Delta M_X$ its colimit (in $\mathscr{C}$), $C_* = V(H_*)$ and $H_* \to \Delta H_X$ its colimit (in $\mathscr{D}$). The functors U and V preserving $\overrightarrow{\mathbb{N}}$-colimits, therefore $B(X)$ is the forgetting of the pair (M_X, H_X) which is an object of $\mathscr{C} \times_{\mathscr{B}} \mathscr{D}$: $B(X) = O((M_X, H_X)) = U(M_X) = V(H_X)$. We put $F(X) = (M_X, H_X)$ which gives, as we are going to see, the desired left adjoint of the forgetful functor O, and where (B, ρ, b) is the associated monad. $B(X)$ inherits at the same time the structure of the object M_X (which lives in $\mathscr{C}$) and the structure of the object H_X (which lives in $\mathscr{D}$). It is the reason why the monad (B, ρ, b) can be called "fusion" of monads $(L, \mathfrak{m}, l)$ and (C, m, c). We note b_X the produced arrow $X \xrightarrow{b_X} B(X)$ The continuation consists in showing the universal character of b_X. We are going to show that if we give ourselves a morphism $X \xrightarrow{f} B_0$ of $\mathscr{B}$ such as B_0 is the forgetting of an object (M_0, H_0) of $\mathscr{C} \times_{\mathscr{B}} \mathscr{D}$, then there is a unique morphism $(M_X, H_X) \xrightarrow{(h,k)} (M_0, H_0)$ of $\mathscr{C} \times_{\mathscr{B}} \mathscr{D}$ such as $O(h,k)b_X = f$. For that, we are going to use the filtered diagram B_* with which we are going to build by induction a cocone $B_* \to \Delta B_0$, and it will display the existence of the pair (h,k).

– Let $g_0 = f$ and f_0 which is the extension of f from $L_0 = L(X)$:

$$\begin{tikzcd}
C_0 = X \arrow[r, "f=g_0"] \arrow[d, "l_0"'] & B_0 \\
L(C_0) \arrow[ur, "x_0"] \arrow[d, "\phi_0=1"'] & \\
L_0 \arrow[uur, "f_0=x_0"'] &
\end{tikzcd}$$

– We can suppose that this construction is up to the rank n. Thus in particular we have the following diagram

$$\begin{tikzcd}
C_0 \arrow[r, "f"] \arrow[d, dotted] & B_0 \\
C_n \arrow[ur, "g_n"] \arrow[d, "l_n"'] & \\
L(C_n) \arrow[uur, "x_n"] \arrow[d, "\phi_n"'] & \\
L_n \arrow[uuur, "f_n"] \arrow[d, "c_n"'] & \\
C(L_n) \arrow[uuuur, "y_n"'] &
\end{tikzcd}$$

Also the natural transformation $1_{\mathscr{B}} \xrightarrow{c} C$ applied to

$$C(L_{n-1}) \xrightarrow{\phi_n l_n \psi_{n-1}} L_n$$

gives the equality

$$C(\phi_n l_n \psi_{n-1})c(C(L_{n-1})) = c_n \phi_n l_n \psi_{n-1} = \delta_0$$

thus $y_n\delta_0 = y_n C(\phi_n l_n \psi_{n-1})c(C(L_{n-1}))$. On the other hand

$$y_n\delta_0 = y_n\delta_0 m(L_{n-1})c(C(L_{n-1}))$$

(unity axiom of monads), which leads to the equality

$$y_nC(\phi_n l_n \psi_{n-1}) = y_n\delta_0 m(L_{n-1})$$

(do not forget that $y_n\delta_0$ is the forgetting of a morphism of $\mathscr{D}$ because $y_n\delta_0 = y_{n-1}$). What allows to write

$$\begin{aligned} y_n\delta_1 &= y_nC(k_{n-1}) = y_nC(\phi_n l_n \psi_{n-1} c_{n-1}) \\ &= y_nC(\phi_n l_n \psi_{n-1})C(c_{n-1}) = y_n\delta_0 m(L_{n-1})C(c_{n-1}) \\ &= y_n\delta_0 \text{ (unity axiom of monads)} \end{aligned}$$

So the universality of ψ_n implies the existence of a unique morphism of $\mathscr{D}$ that the forgetting g_{n+1} is such as $g_{n+1}\psi_n = y_n$. We also have the extension x_{n+1} of g_{n+1} from $L(C_{n+1})$. Then the natural transformation $1_{\mathscr{B}} \xrightarrow{l} L$ applied to $L(C_n) \xrightarrow{\psi_n c_n \phi_n} C_{n+1}$ gives the equality

$$L(\psi_n c_n \phi_n) l(L(C_n)) = l_{n+1}\psi_n c_n \phi_n = d_0$$

thus $x_{n+1}d_0 = x_{n+1}L(\psi_n c_n \phi_n) l(L(C_n))$, and

$$x_{n+1}d_0 = x_{n+1}d_0\mathfrak{m}(C_n) l(L(C_n)) \quad \text{(unity axiom of monads)}$$

which leads to the equality

$$x_{n+1}L(\psi_n c_n \phi_n) = x_{n+1}d_0\mathfrak{m}(C_n)$$

(do not forget that $x_{n+1}d_0$ is the forgetting of a morphism of $\mathscr{C}$ because $x_{n+1}d_0 = x_n$). What allows to write

$$\begin{aligned} x_{n+1}d_1 &= x_{n+1}L(j_n) = x_{n+1}L(\psi_n c_n \phi_n l_n) \\ &= x_{n+1}L(\psi_n c_n \phi_n)L(l_n) = x_{n+1}d_0\mathfrak{m}(C_n)L(l_n) \\ &= x_{n+1}d_0 \quad \text{(unity axiom of monads)} \end{aligned}$$

Then the universality of ϕ_{n+1} implies the existence of a unique morphism of $\mathscr{C}$ which the forgetting f_{n+1} is such as $f_{n+1}\phi_{n+1} = x_{n+1}$. We also have the extension y_{n+1} of f_{n+1} from $C(L_{n+1})$.

– Thus we obtain a cone $B_* \to \Delta B_0$, with $B_0 = O(M_0, H_0) = U(M_0) = V(H_0)$. We have the two cocones as well $L_* \to \Delta U(M_0)$ and $C_* \to \Delta V(H_0)$. The functor U preserving the $\overrightarrow{\mathbb{N}}$-colimits, the diagram of $\mathscr{B}$

$$\begin{tikzcd} L_* \arrow[r] \arrow[dr] & \Delta U(M_0) \\ & \Delta U(M_X) \end{tikzcd}$$

results of the diagram of $\mathscr{C}$

$$\begin{tikzcd} M_* \arrow[r] \arrow[dr] & \Delta M_0 \\ & \Delta M_X \end{tikzcd}$$

such as $M_* \to \Delta M_X$ is a colimit. There is consequently a unique morphism h of $\mathscr{C}$ such as the triangle commutes

$$\begin{tikzcd} M_* \arrow[r] \arrow[dr] & \Delta M_0 \\ & \Delta M_X \arrow[u, dotted, "\Delta h"'] \end{tikzcd} .$$

In the same way the functor V preserves $\overrightarrow{\mathbb{N}}$-colimits, so the diagram of $\mathscr{B}$

$$\begin{tikzcd} C_* \arrow[r] \arrow[dr] & \Delta V(H_0) \\ & \Delta V(H_X) \end{tikzcd}$$

results of the diagram of $\mathscr{D}$

$$\begin{tikzcd} H_* \arrow[r] \arrow[dr] & \Delta H_0 \\ & \Delta H_X \end{tikzcd}$$

such as $H_* \to \Delta H_X$ is a colimit. Therefore there is a unique morphism k of $\mathscr{D}$ such as the following triangle commutes

$$\begin{tikzcd} H_* \arrow[r] \arrow[dr] & \Delta H_0 \\ & \Delta H_X \arrow[u, dotted, "\Delta k"'] \end{tikzcd} .$$

It shows the existence of the unique morphism (h,k) of $\mathscr{C} \times_{\mathscr{B}} \mathscr{D}$ such as

$$\begin{tikzcd} B_* \arrow[r] \arrow[dr] & \Delta B_0 \\ & \Delta B(X) \arrow[u, dotted, "O(h,k)"'] \end{tikzcd} .$$

In consequence we obtain the morphism (h,k) of $\mathscr{C} \times_{\mathscr{B}} \mathscr{D}$ such as $O(h,k)b_X = f$. Let (h',k') another morphism of $\mathscr{C} \times_{\mathscr{B}} \mathscr{D}$ making the following triangle commute

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "b_X"'] & B_0 = O(M_0,H_0) \\ B(X) = O(M_X,H_X) \arrow[ur, "O(h',k')"'] & \end{tikzcd} .$$

We are going to prove by induction that it makes commutative the following triangle of natural transformations

$$\begin{tikzcd} B_* \arrow[r] \arrow[dr] & \Delta B_0 \\ & \Delta B(X) \arrow[u, dotted, "O(h',k')"'] \end{tikzcd} .$$

then it will immediatly prove the unicity of (h,k).

The cocone $B_* \to \Delta B(X)$ is explicitly given by the following diagram

$$\begin{tikzcd}
C_0 \arrow[r, "b_X"] \arrow[d, dotted] & B(X) \\
C_n \arrow[ur, "g_n^X"] \arrow[d, "l_n"'] & \\
L(C_n) \arrow[uur, "x_n^X"] \arrow[d, "\phi_n"'] & \\
L_n \arrow[uuur, "f_n^X"] \arrow[d, "c_n"'] & \\
C(L_n) \arrow[uuuur, "y_n^X"'] &
\end{tikzcd}$$

We need to prove that $\forall n \in \mathbb{N}$ we have the equalities: $O(h',k')g_n^X = g_n$, $O(h',k')f_n^X = f_n$, $O(h',k')x_n^X = x_n$, $O(h',k')y_n^X = y_n$.

- If $n = 0$ we have $V(k')x_0^X l_0 = V(k')b_X = f = x_0 l_0$ (don't forget that $V(k') = U(h') = O(h',k')$) thus $V(k')x_0^X = x_0$. We trivially have $V(k')f_0^X = f_0$ because $f_0 = x_0$ and $f_0^X = x_0^X$. Also, $V(k')y_0^X c_0$ $= V(k')f_0^X = f_0 = y_0 c_0$, so $V(k')y_0^X = y_0$. And g_1 is unique such as $g_1 \psi_0 = y_0$. However $V(k')g_1^X \psi_0 = V(k')y_0^X = y_0$, thus $V(k')g_1^X = g_1$.
- We can suppose that until $n \geq 1$, we have these equalities; g_{n+1} is unique such as $g_{n+1}\psi_n = y_n$. But $V(k')g_{n+1}^X \psi_n = V(k')y_n^X =$ y_n, thus $V(k')g_{n+1}^X = g_{n+1}$. Also $V(k')x_{n+1}^X l_{n+1} = V(k')g_{n+1}^X =$ $g_{n+1} = x_{n+1} l_{n+1}$. Thus $V(k')x_{n+1}^X = x_{n+1}$. And f_{n+1} is unique such as $f_{n+1}\phi_{n+1} = x_{n+1}$. Nevertheless $V(k')f_{n+1}^X \phi_{n+1} = V(k')x_{n+1}$ $= x_{n+1}$, thus $V(k')f_{n+1}^X = f_{n+1}$. So we have $V(k')y_{n+1}^X c_{n+1} =$ $V(k')f_{n+1}^X = f_{n+1} = y_{n+1}c_{n+1}$, which proves that $V(k')y_{n+1}^X =$ y_{n+1}.

Finally we obtain the following fusion diagram

$$\begin{tikzcd}
\mathscr{C} \arrow[rr, shift left=1.5, "U"] \arrow[rr, phantom, "\top"] & & \mathscr{B} \arrow[ll, shift left=1.5, "L"] \arrow[rr, shift left=1.5, "C"] \arrow[rr, phantom, "\perp"] \arrow[d, shift left=1.5, "F"] \arrow[d, phantom, "\vdash"] & & \mathscr{D} \arrow[ll, shift left=1.5, "V"] \\
& & \mathscr{C} \times_{\mathscr{B}} \mathscr{D} \arrow[llu, "p_1"] \arrow[rru, "p_2"'] \arrow[u, shift left=1.5, "O"] & &
\end{tikzcd}$$

■

7 Theories of the n-Transformations ($n \in \mathbb{N}^*$) and their Models.

The goal of this section is to build, thanks to the Nerve Theorem ([25]), the equivalence in $\mathbb{G}lob(\mathbb{C}AT)$ of 7.3 which shows that n-Transformations can be seen as models for some very elegant theories which are colored in a precise sense (see 7.2). We refer to the papers [14], [5] for materials that we are going to use here. Here $\mathbb{A}$r is the category of categories with arities, $\mathbb{A}$r$\mathbb{M}$nd is the category of categories with arities equipped with monads, and $\mathbb{M}$nd$\mathbb{A}$r is the category of monads with arities. More specifically objects of $\mathbb{A}$r are noted $(\Theta_0, i_0, \mathscr{A})$ where $\Theta_0 \xrightarrow{i_0} \mathscr{A}$ is a fully faithfull functor, and objects of $\mathbb{A}$r$\mathbb{M}$nd and of $\mathbb{M}$nd$\mathbb{A}$r are noted $((\Theta_0, i_0, \mathscr{A}), (T, \eta, \mu))$ or $(\Theta_0, i_0, \mathscr{A})$ when there is no confusion about monads T which act on $\mathscr{A}$. Strongly cartesian monads [5] are the most important example of monads with arities for our purpose, because all monads arising from operads of the n-transformations are strongly cartesians (see proposition 2). But before this easy but important proposition 2, we are going to show some interesting objects of $co\mathbb{G}lob(\mathbb{C}AT)$ (in 7.1 and 7.2), the category of coglobular objects in $\mathbb{C}AT$.

7.1 Coglobular Complex of Kleisli of the n-Transformations ($n \in \mathbb{N}^*$).

Here categories $\mathbb{M}$nd and $\mathbb{A}$dj are slightly different from those which were defined in 4 (see [14, 24] for their definitions) and are adapted for building the coglobular complex of Kleisli of the n-Transformations ($n \in \mathbb{N}^*$). Consider the functor $\mathbb{M}\mathrm{nd} \xrightarrow{\mathbb{K}} \mathbb{A}\mathrm{dj}$ which send the monad $(\mathcal{G},(T,\eta,\mu))$ to the adjunction $(Kl(T),\mathcal{G},L_T,U_T,\eta_T,\varepsilon_T)$ coming from the Kleisli construction. Objects of $Kl(T)$ are objects of $\mathcal{G}$ and morphisms $G \xrightarrow{f} G'$ of $Kl(T)$ are given by morphisms $G \xrightarrow{f} T(G')$ of $\mathcal{G}$. Also if $G \xrightarrow{g} G'$ lives in $\mathcal{G}$ then $L_T(g) = \eta(G') \circ g$ and if $G \xrightarrow{f} G'$ lives in $Kl(T)$ then $U_T(f) = \mu(G') \circ T(f)$. Finally $\mathbb{K}$ send the morphism (Q,q) of $\mathbb{M}$nd to the morphism (P,Q) of $\mathbb{A}$dj such that if $G \xrightarrow{f} G'$ is a morphism of $Kl(T)$ then $P(f) = q(G')Q(f)$. Then consider the coglobular complex of CT-$\mathbb{C}\mathrm{at}_c$ of the globular contractible colored operads of the n-Transformations 3.3

$$B^0 \underset{\kappa_1^0}{\overset{\delta_1^0}{\rightrightarrows}} B^1 \underset{\kappa_2^1}{\overset{\delta_2^1}{\rightrightarrows}} B^2 \cdots\cdots B^{n-1} \underset{\kappa_n^{n-1}}{\overset{\delta_n^{n-1}}{\rightrightarrows}} B^n \cdots\cdots$$

For each $j \in \mathbb{N}$ we note $(\underline{B}^j,\mu^j,\eta^j)$ the corresponding monads (see 4). Given the following functors "choice of a color" $\omega-\mathbb{G}r \xrightarrow{i_{j_*}} \omega-\mathbb{G}r/1\cup 2$ for each $j \in \{1,2\}$ which send the ω-graph G to the bicolored ω-graph $i_j \circ !_G$ and which send a morphism f to f. It result from the morphisms of color $1 \xrightarrow{i_j} 1\cup 2$ (see 3.3). By definition of the monads $\underline{B}^0$ and $\underline{B}^1$ we have the following natural transformations $i_{1*}\underline{B}^0 \xrightarrow{\delta_1^0} \underline{B}^1 i_{1*}$ and $i_{2*}\underline{B}^0 \xrightarrow{\kappa_1^0} \underline{B}^1 i_{2*}$ and furthemore we have for each $j \geqslant 1$ the following natural transformations $\underline{B}^j \xrightarrow{\delta_{j+1}^j} \underline{B}^{j+1}$ and $\underline{B}^j \xrightarrow{\kappa_{j+1}^j} \underline{B}^{j+1}$ and it is easy to see that these natural transformations fit well the axioms of morphisms of $\mathbb{M}$nd (and it is similar to the construction in [16]). The functoriality of the building a monad from a

$\mathbb{T}$-Category implied that we can build the corresponding coglobular complex of $\mathbb{M}$nd (similar to 4.2)

$$\underline{B}^0 \underset{\kappa_1^0}{\overset{\delta_1^0}{\rightrightarrows}} \underline{B}^1 \underset{\kappa_2^1}{\overset{\delta_2^1}{\rightrightarrows}} \underline{B}^2 \cdots\cdots \rightrightarrows \underline{B}^n \underset{\kappa_{n+1}^n}{\overset{\delta_{n+1}^n}{\rightrightarrows}} \cdots$$

If $\mathbb{A}\mathrm{dj} \xrightarrow{\mathbb{P}} \mathbb{C}AT$ is the projection functor, then the functor

$$\mathbb{M}\mathrm{nd} \xrightarrow{\mathbb{K}} \mathbb{A}\mathrm{dj} \xrightarrow{\mathbb{P}} \mathbb{C}AT$$

brings to light the following coglobular complex of Kleisli of the n-Transformations ($n \in \mathbb{N}^*$)

$$Kl(\underline{B}^0) \underset{\kappa_1^0}{\overset{\delta_1^0}{\rightrightarrows}} Kl(\underline{B}^1) \underset{\kappa_2^1}{\overset{\delta_2^1}{\rightrightarrows}} Kl(\underline{B}^2) \cdots\cdots \rightrightarrows Kl(\underline{B}^{n-1}) \underset{\kappa_n^{n-1}}{\overset{\delta_n^{n-1}}{\rightrightarrows}} Kl(\underline{B}^n) \cdots\cdots$$

7.2 Coglobular Complex of the Theories of the n-Transformations ($n \in \mathbb{N}^*$).

We are going to exhibit the categories of arities for the n-Transformations where we can immediately see their colored nature. Then we construct the theories of the n-Transformations where in particular we can see again their bicolored features and then we describe these colored theories as full subcategories of their Kleisli categories. Finally we exhibit the coglobular complex of the theories of the n-Transformations.

Given Θ_0 the category of graphic trees (see [2], [11], [4]). Theories build with sums $\Theta_0 \sqcup \ldots \sqcup \Theta_0$ are called n-colored if the sum use Θ_0 n times.

We have the following easy proposition

Proposition 1 *For all $n \in \mathbb{N}^*$ the following canonical inclusion functors*

$$\Theta_0 \sqcup \ldots \sqcup \Theta_0 \overset{i_0}{\hookrightarrow} \omega - \mathbb{G}r/1 \cup 2 \cup \ldots \cup n$$

produce categories with arities. □

For the n-Transformations the following morphisms of $\mathbb{A}$r are important

$$\begin{tikzcd}
\Theta_0 \arrow[r, shift left, "i_{1*}"] \arrow[r, shift right, "i_{2*}"'] \arrow[d, hook, "i_0"'] & \Theta_0 \sqcup \Theta_0 \arrow[d, hook, "i_0"] \\
\omega-\mathbb{G}r \arrow[r, shift left, "i_{1*}"] \arrow[r, shift right, "i_{2*}"'] & \omega-\mathbb{G}r/1\cup 2
\end{tikzcd}$$

where i_{1*} and i_{2*} are the functors "choice of a color" (see section 7.1).

Let us consider the case of the categories with arities equipped with monads of the n-Transformations $((\Theta_0, i_0, \omega-\mathbb{G}r), (\underline{B}^0, \eta^0, \mu^0))$ and $((\Theta_0 \sqcup \Theta_0, i_0, \omega-\mathbb{G}r/1\cup 2), (\underline{B}^i, \eta^i, \mu^i))$ if $i \geqslant 1$

We have the following factorisation

$$\begin{tikzcd}
\Theta_0 \arrow[r, "i_0"] \arrow[dr, "j"'] & \omega-\mathbb{G}r \arrow[r, "L^0"] & \underline{B}^0-\mathbb{A}lg \\
& \Theta_{\underline{B}^0} \arrow[ur, "i"'] &
\end{tikzcd}$$

and for each $i \geqslant 1$ we have the following factorisations

$$\begin{tikzcd}
\Theta_0 \sqcup \Theta_0 \arrow[r, "i_0"] \arrow[dr, "j"'] & \omega-\mathbb{G}r/1\cup 2 \arrow[r, "L^i"] & \underline{B}^i-\mathbb{A}lg \\
& \Theta_{\underline{B}^i} \arrow[ur, "i"'] &
\end{tikzcd}$$

where the functors j are identity on the objects and the functors i are fully faithfull (see [14, 25]). The categories $\Theta_{\underline{B}^0}$, $\Theta_{\underline{B}^1}$, ...,$\Theta_{\underline{B}^i}$, ...etc. are the theories of the n-Transformations (by abuse we call $\Theta_{\underline{B}^0}$ the theory of the 0-Transformations, which is actually the theory built by Clemens Berger in [4]). We can also give to them the following alternative definition: Each $\Theta_{\underline{B}^i}$ can be seen as the full subcategory of the Kleisli category $Kl(\Theta_{\underline{B}^i})$ (see the paragraph section 7.1) which objects are the bicolored trees if $i \geqslant 1$ (i.e belong in $\Theta_0 \sqcup \Theta_0$), and which objects are the trees if $i = 0$. With this description we obtain the coglobular complex of the theories of the n-Transformations which is seen as a subcomplex of the coglobular complex of the Kleisli categories of the n-Transformations

$$\begin{tikzcd}
\Theta_{\underline{B}^0} \arrow[r, shift left, "\delta_1^0"] \arrow[r, shift right, "\kappa_1^0"'] \arrow[d, hook] & \Theta_{\underline{B}^1} \arrow[r, shift left, "\delta_2^1"] \arrow[r, shift right, "\kappa_2^1"'] \arrow[d, hook] & \Theta_{\underline{B}^2} \arrow[r, dotted] \arrow[d, hook] & \Theta_{\underline{B}^{n-1}} \arrow[r, shift left, "\delta_n^{n-1}"] \arrow[r, shift right, "\kappa_n^{n-1}"'] \arrow[d, hook] & \Theta_{\underline{B}^n} \arrow[r, dotted, no head] \arrow[d, hook] & {} \\
Kl(\underline{B}^0) \arrow[r, shift left, "\delta_1^0"] \arrow[r, shift right, "\kappa_1^0"'] & Kl(\underline{B}^1) \arrow[r, shift left, "\delta_2^1"] \arrow[r, shift right, "\kappa_2^1"'] & Kl(\underline{B}^2) \arrow[r, dotted] & Kl(\underline{B}^{n-1}) \arrow[r, shift left, "\delta_n^{n-1}"] \arrow[r, shift right, "\kappa_n^{n-1}"'] & Kl(\underline{B}^n) \arrow[r, dotted, no head] & {}
\end{tikzcd}$$

7.3 An application of the Nerve Theorem.

Given $\mathscr{A}$ a category with a final object 1, and a functor $\mathscr{A} \xrightarrow{F} \mathscr{B}$

We have the following factorisation:

$$\begin{tikzcd}
\mathscr{A} \arrow[rr, "F"] \arrow[dr, "F_1"'] & & \mathscr{B} \\
& \mathscr{B}/F(1) \arrow[ur, "cod"'] &
\end{tikzcd}$$

where $F_1(a) := F(!_a)$. In that case we have the following important definition

Definition 1 (Street 2001) The last F is qualified as Parametric Right Adjoint (p.r.a for short) if F_1 has a left adjoint. $\square$

Definition 2 A monad $(\mathscr{G}, (T, \eta, \mu))$ is a strongly cartesian monad if T is p.r.a. and if its unit and multiplication are cartesian. $\square$

Remark 6 In 2001 Ross Street has called them p.r.a monads, Mark Weber in [25] has called them locally right adjoint monads (l.r.a monads), but we adopt here the terminology of the paper [5]. $\square$

Monads of the n-Transformations are in fact strongly cartesian monads (see the proposition 2 below, where the proof is left to the reader) which allow us to exhibit the coglobular complex in $\mathbb{M}\mathrm{nd}\,\mathbb{A}\mathrm{r}$ of the n-Transformations and thus, thanks to the Nerve Theorem [25] we get the globular complex of nerves of the n-Transformations and finally the equivalence in $\mathbb{G}lob(\mathbb{C}AT)$, which express the definition of the n-Transformations as models for theories, that is the outcome of this section. It is well known that $(\omega-\mathbb{G}r,(\underline{B}^0,\eta^0,\mu^0))$ is a strongly cartesian monad [25]. In fact all monads of the n-Transformations ($n\in\mathbb{N}^*$) have this property

Proposition 2 *For all* $i\geqslant 1$ *the monad* $(\omega-\mathbb{G}r/1\bigcup 2,(\underline{B}^i,\eta^i,\mu^i))$ *is strongly cartesian. Furthermore* $(\Theta_0\sqcup\Theta_0,i_0,\omega-\mathbb{G}r/1\cup 2)$ *is their canonical arities (see remark 2.10 in [5]).*

So we obtain the coglobular complex in $\mathbb{M}\mathrm{nd}\,\mathbb{A}\mathrm{r}$ of the n-Transformations

$$(\Theta_0,i_0,\omega-\mathbb{G}r)\underset{\kappa_1^0}{\overset{\delta_1^0}{\rightrightarrows}}(\Theta_0\sqcup\Theta_0,i_0,\omega-\mathbb{G}r/1\cup 2)\underset{\kappa_2^1}{\overset{\delta_2^1}{\rightrightarrows}}\cdots$$

$$(\Theta_0\sqcup\Theta_0,i_0,\omega-\mathbb{G}r/1\cup 2)\underset{\kappa_{i+1}^i}{\overset{\delta_{i+1}^i}{\rightrightarrows}}\cdots$$

which brings to light the globular complex of nerves of the n-Transformations

$$\begin{tikzcd}
\cdots \arrow[r, dotted] & \underline{B}^n-\mathbb{A}lg \arrow[r, shift left, "\sigma_{n-1}^n"] \arrow[r, shift right, "\beta_{n-1}^n"'] \arrow[d, "N_{\underline{B}^n}"'] & \underline{B}^{n-1}-\mathbb{A}lg \arrow[r, dotted] \arrow[d, "N_{\underline{B}^{n-1}}"] & \underline{B}^1-\mathbb{A}lg \arrow[r, shift left, "\sigma_0^1"] \arrow[r, shift right, "\beta_0^1"'] \arrow[d, "N_{\underline{B}^1}"'] & \underline{B}^0-\mathbb{A}lg \arrow[d, "N_{\underline{B}^0}"] \\
\cdots \arrow[r, dotted] & \widehat{\Theta}_{\underline{B}^n} \arrow[r, shift left, "\sigma_{n-1}^n"] \arrow[r, shift right, "\beta_{n-1}^n"'] & \widehat{\Theta}_{\underline{B}^{n-1}} \arrow[r, dotted] & \widehat{\Theta}_{\underline{B}^1} \arrow[r, shift left, "\sigma_0^1"] \arrow[r, shift right, "\beta_0^1"'] & \widehat{\Theta}_{\underline{B}^0}
\end{tikzcd}$$

which finally achieve the goal of this section by showing the following equivalence in $\mathbb{G}lob(\mathbb{C}AT)$ given by the nerves functors, i.e each nerve functor $N_{\underline{B}^n}$ of the commutative diagram below is an equivalence of categories

$$
\begin{tikzcd}
\cdots \arrow[r, dotted] & \underline{B}^n - \mathbb{A}lg \arrow[r, shift left, "\sigma^n_{n-1}"] \arrow[r, shift right, "\beta^n_{n-1}"'] \arrow[d, "N_{\underline{B}^n}"'] & \underline{B}^{n-1} - \mathbb{A}lg \arrow[r, dotted] \arrow[d, "N_{\underline{B}^{n-1}}"] & \underline{B}^1 - \mathbb{A}lg \arrow[r, shift left, "\sigma^1_0"] \arrow[r, shift right, "\beta^1_0"'] \arrow[d, "N_{\underline{B}^1}"'] & \underline{B}^0 - \mathbb{A}lg \arrow[d, "N_{\underline{B}^0}"] \\
\cdots \arrow[r, dotted] & \mathbb{M}\mathrm{od}(\Theta_{\underline{B}^n}) \arrow[r, shift left, "\sigma^n_{n-1}"] \arrow[r, shift right, "\beta^n_{n-1}"'] & \mathbb{M}\mathrm{od}(\Theta_{\underline{B}^{n-1}}) \arrow[r, dotted] & \mathbb{M}\mathrm{od}(\Theta_{\underline{B}^1}) \arrow[r, shift left, "\sigma^1_0"] \arrow[r, shift right, "\beta^1_0"'] & \mathbb{M}\mathrm{od}(\Theta_{\underline{B}^0})
\end{tikzcd}
$$

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