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OBJECTIVE CATEGORIES AND SCHEMES

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Dedicated to B. V. M.

RESUME. Dans ce travail nous considérons les faisceaux quasi-cohérents sur un schéma comme des modules sur une catégorie "objective". On montre que la catégorie **Obj** des catégories objectives est duale de la catégorie des schémas. Nous exhibéons **Obj** comme une sous-catégorie pleine reflexive de la catégorie **POb** (catégories préobjectives) dont les objets sont des foncteurs contravariants d'un ensemble ordonné dans la catégorie des anneaux commutatifs tandis que les morphismes de **POb** sont relatifs à la structure responsable de la génération des schémas. De cette façon, la définition des morhismes des schémas prend une forme assez simple comme foncteurs entre des catégories objectives qui préservent la structure pertinente. Le résultat principal est une reconstruction des schémas plus explicite que celle due à Rosenberg (Noncommutative schemes, Compos. Math. 112 (1998), 93-125).

Abstract. Quasi-coherent sheaves over a scheme are regarded as modules over an *objective* category. The category **Obj** of objective categories is shown to be dual to the category of schemes. We exhibit **Obj** as a reflective full subcategory of a category **POb** (pre-objective categories) whose objects are contravariant functors from a poset to the category of commutative rings while the morphisms of **POb** take care of the structure responsible for the generation of schemes. In this context, morhisms of schemes just turn into functors between objective categories preserving the relevant structure. Our main result gives a more explicit version of Rosenberg's reconstruction of schemes (Noncommutative schemes, Compositio Math. 112 (1998), 93-125).

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Introduction

The most natural approach toward non-commutative algebraic geometry is based on suitable categories generalizing the abelian category $\mathbf{Qcoh}(X)$ of quasi-coherent sheaves over a scheme X. After Gabriel's reconstruction [4] of noetherian schemes X in terms of $\mathbf{Qcoh}(X)$, this approach was fully justified by Rosenberg [11] who extends Gabriel's result to arbitrary schemes.

For an affine scheme X with structure sheaf $\mathscr{O}_X$, the category of quasi-coherent sheaves coincides with the module category $\mathbf{Mod}(R)$ over the ring $R = \mathscr{O}_X(X)$ of global sections. If R is non-commutative, R can be recovered from $\mathbf{Mod}(R)$ up to Morita equivalence, i. e. instead of R itself, the category $\mathbf{proj}(R)$ of finitely generated projective R-modules can be recovered from $\mathbf{Mod}(R)$. Moreover, the objects of $\mathbf{Mod}(R)$ are additive functors $\mathscr{C}^{\mathrm{op}} \to \mathbf{Ab}$, where $\mathscr{C}$ can be chosen to be either $\mathbf{proj}(R)$ or the one-object full subcategory $\{{}_R R\}$ of $\mathbf{proj}(R)$ which can be identified with the ring R.

If the scheme X is non-affine, a reconstruction via projectives fails dramatically, even in the most simple case of a projective line X, where non-zero projectives in $\mathbf{Qcoh}(X)$ no longer exist. Nevertheless, the affine case suggests that it should be possible to associate a category $\mathscr{O}$ to any scheme X such that quasi-coherent sheaves over X become certain modules over $\mathscr{O}$. In the present article, we define a category $\mathbf{Obj}$ of such categories $\mathscr{O}$ and prove that $\mathbf{Obj}$ is dual to the category of schemes. Since objects of $\mathscr{O}$ play a particular rôle, we call the categories $\mathscr{O} \in \mathbf{Obj}$ *objective.*

More generally, we introduce *pre-objective* categories as a class of small skeletal preadditive categories $\mathscr{O}$. We stick to the classical case and assume that the endomorphism rings $\mathscr{O}(U)$ in $\mathscr{O}$ are commutative. The two axioms (O1) and (O2) for a pre-objective category $\mathscr{O}$ are based on the concept of *short monomorphism* $i \colon U \to V$, which means that every morphism $f \colon U \to V$ is of the form $f = ig$ for some $g \in \mathscr{O}(U)$. Then (O1) states that there is a short monomorphism $U \to V$ for any pair of objects U, V with $\mathrm{Hom}_{\mathscr{O}}(U, V) \neq 0$, and (O2) asserts that short monomorphisms are closed under composition. In other words, (O1) and (O2) state that the short monomorphisms form a subcategory which is fibered over a partially ordered set on $\mathrm{Ob}\,\mathscr{O}$. Using the relationship between fibered categories and pseudo-functors [6], a pre-objective category can be conceived as a functor $\rho\colon \Omega^{\mathrm{op}} \to \mathbf{CRi}$ from a partially ordered set Ω into the category $\mathbf{CRi}$ of commutative rings, together with a cohomology class $\gamma \in H^2(\Omega, \rho^{\times})$.

This cohomology class γ vanishes for pre-objective categories $\mathscr{O}$ with a greatest object (Proposition 3), which are just our concern here. To make such pre-objective categories $\mathscr{O}$ into a suitable category $\mathbf{POb}$, we introduce the concept of *short limit* $S(f)$ of an endomorphism $f \in \mathscr{O}(U)$ in $\mathscr{O}$ and call an $\mathscr{O}$-module M is *quasi-coherent* if $M\colon \mathscr{O}^{\mathrm{op}} \to \mathbf{Ab}$ respects short limits of arbitrary endomorphisms in $\mathscr{O}$. On the other hand, we say that an object $U \in \mathrm{Ob}\,\mathscr{O}$ is *affine* if the representable functor $\mathrm{Hom}_{\mathscr{O}}(-, U)$ respects short limits of endomorphisms of U and if U satisfies two other properties related to the partially ordered structure of $\mathrm{Ob}\,\mathscr{O}$. Now a *morphism* between pre-objective categories is just a functor $F\colon \mathscr{O} \to \mathscr{O}'$ which respects the relevant structure of $\mathscr{O}$, namely, short monomorphisms, short limits, finite meets, and joins of affine objects - as far as they exist.

We call a pre-objective category $\mathscr{O}$ *objective* if joins of objects exist and the full subcategory $\mathscr{O}_{\mathrm{aff}}$ of affine objects is dense [9] in $\mathscr{O}$. The latter categorical property is closely related to the recollement of schemes. With the benefit of hindsight, our categorification of schemes appears to be quite natural and almost inevitable. As already indicated above, we prove that the full subcategory $\mathbf{Obj} \subset \mathbf{POb}$ of objective categories is dual to the category of schemes (Theorem 2). In this context, the awkward definition of morphisms between schemes takes a more pleasant form. Recall that such a morphism consists of a continuous map $\varphi\colon X \to Y$ between the base spaces together with a morphism $\vartheta\colon \mathscr{O}_Y \to \varphi_*\mathscr{O}_X$ of sheaves into the opposite direction which induces a local ring homomorphism $\vartheta_x^{\#}\colon \mathscr{O}_{Y,\varphi(x)} \to \mathscr{O}_{X,x}$ at the stalks. By contrast, morphisms between objective categories are just functors which respect the structure that ought to be respected.

Note that as a (dual) objective category, a scheme is nothing else than a pre-objective category with joins and enough affines, while a pre-objective category is given by a fibering of unit groups of commutative rings over a poset. Conversely, we associate a scheme to any pre-objective category (Theorem 1) and show that the full subcategory **Obj** of **POb** is reflective (Theorem 3). Thus in a sense, schemes are related to pre-objective categories like sheaves are related to presheaves.

Within this framework, a quasi-coherent sheaf over a scheme with corresponding objective category $\mathscr{O}$ becomes an $\mathscr{O}_{\mathrm{aff}}$-module which respects short limits. An application to flat covers (see [3]) will be reserved to a subsequent publication. Here we just give a brief discussion of Rosenberg's result [11] and show how to reconstruct an objective category $\mathscr{O}$ from the abelian category $\mathbf{Qcoh}(\mathscr{O})$ of quasi-coherent $\mathscr{O}_{\mathrm{aff}}$-modules in a quite explicit way. More generally, we associate an objective category $\mathscr{O}_{\mathscr{A}}$ to any abelian category $\mathscr{A}$ such that $\mathscr{O}_{\mathscr{A}} \cong \mathscr{O}$ in the special case $\mathscr{A} = \mathbf{Qcoh}(\mathscr{O})$. To this end, a *point* of an abelian category $\mathscr{A}$ is defined to be a quasi-injective object P with $\mathrm{End}_{\mathscr{A}}(P)$ a field such that every non-zero subobject of P generates P. In contrast to Gabriel's reconstruction of schemes which makes use of injective objects, we confine ourselves to quasi-injectives. If R is a commutative ring, the points of $\mathbf{Mod}(R)$ correspond to the prime ideals of R. Using points of $\mathscr{A}$, we introduce objects E of *finite type*, and to any such E, we associate a subset U_E of the set $\mathrm{Spec}\,\mathscr{A}$ of points. Then the U_E define a topology on $\mathrm{Spec}\,\mathscr{A}$, and every open set U in $\mathrm{Spec}\,\mathscr{A}$ gives rise to a Serre subcategory $\mathscr{T}_U$ of $\mathscr{A}$. If $\mathscr{O}_{\mathscr{A}}(U)$ denotes the center of the abelian quotient category $\mathscr{A}/\mathscr{T}_U$, we obtain a pre-objective category $\mathscr{O}_{\mathscr{A}}$ which is objective and isomorphic to $\mathscr{O}$ whenever $\mathscr{A} = \mathbf{Qcoh}(\mathscr{O})$ for a given $\mathscr{O} \in \mathbf{Obj}$.

1 Short limits

Let $\mathscr{C}$ be a category. We call a monomorphism $i \colon X \to Y$ *short* if every morphism $f \colon X \to Y$ is of the form $f = ig$ for some endomorphism

g of X. Clearly, a short monomorphism $X \to Y$ is unique up to an automorphism of X. So we could speak of a *short subobject* X of Y.

Consider a functor $C\colon \mathscr{I} \to \mathscr{C}$ with $\mathscr{I}$ small. We define a *short cone* c over C to be a collection of short monomorphisms $c_i\colon X \to C_i$ (with $i \in \operatorname{Ob}\mathscr{I}$) such that for every $\alpha\colon i \to j$ in $\mathscr{I}$ there is a commutative square

$$\begin{array}{ccc} X & = & X \\ \downarrow{\scriptstyle c_i} & & \downarrow{\scriptstyle c_j} \\ C_i & \xrightarrow{C_\alpha} & C_j \end{array} \tag{1}$$

in $\mathscr{C}$. We call X (together with c) a *short limit* of C if every short cone $c'\colon X' \to C$ factors uniquely through c, i. e. there is a unique $f\colon X' \to X$ with $c'_i = c_i f$ for all $i \in \operatorname{Ob}\mathscr{I}$. We denote it by $\underleftarrow{\text{shlim}}\, C_i$. For $\mathscr{I} = \varnothing$, a short limit $\underleftarrow{\text{shlim}}\, \varnothing$ is just a terminal object.

Example 1. For a ring R, the short submodules of an R-module M are fully invariant, and every sum of short submodules of M is again a short submodule. So the short submodules form a complete lattice. The short limit $\underleftarrow{\text{shlim}}\, M_i$ of a non-empty family of submodules $M_i \subset M$ is given by the largest submodule L of $\bigcap M_i$ such that every inclusion $L \hookrightarrow M_i$ is a short monomorphism.

Definition 1. We call a preadditive category $\mathscr{C}$ *commutative* if the ring $\operatorname{End}_{\mathscr{C}}(X)$ is commutative for each $X \in \operatorname{Ob}\mathscr{C}$.

As usual, we regard a partially ordered set Ω as a small skeletal category with at most one morphism $a \to b$ between any pair of objects. If such a morphism exists, we write $a \leqslant b$. An ordinal λ will be regarded as a well-ordered set

$$\lambda = \{\alpha \in \mathbf{Ord} \mid \alpha < \lambda\},$$

i. e. a full subcategory of the category **Ord** of all ordinals.

For a preadditive category $\mathscr{C}$, every morphism $f\colon X_0 \to X_1$ in $\mathscr{C}$ gives rise to a functor $2 \to \mathscr{C}$, also denoted by f. The short limit $S(f)$

of an endomorphism $f\colon X \to X$ (viewed as a functor $2 \to \mathscr{C}$) is given by a commutative diagram

$$\begin{array}{ccc} S(f) & & \\ \Big\downarrow{\scriptstyle i} & \searrow^{j} & \\ X & \xrightarrow{\;f\;} & X \end{array} \tag{2}$$

with short monomorphisms i, j. Thus $j = if^{\times}$ with an automorphism $f^{\times}$ of $S(f)$. This gives a commutative square

$$\begin{array}{ccc} S(f) & \xrightarrow{\;f^{\times}\;} & S(f) \\ \Big\downarrow{\scriptstyle i} & & \Big\downarrow{\scriptstyle i} \\ X & \xrightarrow{\;f\;} & X, \end{array} \tag{3}$$

and $S(f) = \underleftarrow{\mathrm{shlim}}\, f$ means that every short monomorphism $i'\colon Y \to X$ with $fi' = i'e$ for some automorphism e of Y factors through i. We call $S(f)$ the *support* of f. If $\mathscr{C}$ is commutative and skeletal, the automorphism $f^{\times}$ in (3) is unique. In fact, if we replace i by ie with an isomorphism $e\colon Y \xrightarrow{\sim} S(f)$, then $Y = S(f)$, and thus $f \cdot ie = if^{\times}e = ie \cdot f^{\times}$.

Regarding $\mathbb{Z}$ as a partially ordered set, let $f^{\mathbb{Z}}\colon \mathbb{Z} \to \mathscr{C}$ denote the functor with $f^{\mathbb{Z}}(n) := X$ and $f^{\mathbb{Z}}(n \to n+1) := f$, i. e. the diagram

$$\cdots \to X \xrightarrow{\;f\;} X \xrightarrow{\;f\;} X \to \cdots \tag{4}$$

in $\mathscr{C}$. Then

$$S(f) = \underleftarrow{\mathrm{shlim}}\, f^{\mathbb{Z}}. \tag{5}$$

We denote the natural morphism $S(f) \to f^{\mathbb{Z}}(0)$ by i_f. So we have a commutative diagram

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & S(f) & \xrightarrow{\;f^{\times}\;} & S(f) & \xrightarrow{\;f^{\times}\;} & S(f) & \longrightarrow & \cdots \\ & & \Big\downarrow & & \Big\downarrow{\scriptstyle i_f} & & \Big\downarrow & & \\ \cdots & \longrightarrow & X & \xrightarrow{\;f\;} & X & \xrightarrow{\;f\;} & X & \longrightarrow & \cdots \end{array} \tag{6}$$

Definition 2. We define a commutative preadditive category $\mathscr{O}$ to be *pre-objective* if it is small and skeletal such that the following hold.

(O1) For every pair of objects U, V with $\mathrm{Hom}_{\mathscr{O}}(U,V) \neq 0$, there exists a short monomorphism $U \to V$.

(O2) The set of short monomorphisms is closed under composition.

Note that the factorization in (O1) is unique up to isomorphism, i. e. if $U \to U \xrightarrow{j} V$ is a second factorization of f, we have a commutative diagram

$$\begin{array}{ccccc} U & \longrightarrow & U & \xrightarrow{i} & V \\ \| & & \wr\downarrow e & & \| \\ U & \longrightarrow & U & \xrightarrow{j} & V \end{array}$$

with an automorphism e. In what follows, we write $\mathscr{O}(U)$ instead of $\mathrm{End}_{\mathscr{O}}(U)$ for an object U of $\mathscr{O}$.

Proposition 1. *Every short monomorphism $i\colon U \to V$ in a pre-objective category $\mathscr{O}$ defines a ring homomorphism $\rho_U^V\colon \mathscr{O}(V) \to \mathscr{O}(U)$ given by a commutative diagram*

$$\begin{array}{ccc} U & \xrightarrow{f|_U} & U \\ \downarrow i & & \downarrow i \\ V & \xrightarrow{f} & V \end{array} \tag{7}$$

where $f|_U := \rho_U^V(f)$ merely depends on $f \in \mathscr{O}(V)$ and $U \in \mathrm{Ob}\,\mathscr{O}$.

Proof. Let $f \in \mathscr{O}(V)$ be given. By (O1), the morphism fi has a factorization $fi\colon U \to U \xrightarrow{j} V$ with a short monomorphism j. Furthermore, $j = ie$ for some automorphism e of U. Hence fi factors through i, and so we get a commutative diagram (7). Since i is monic, $f|_U$ is unique. If i is replaced by $j = ie$, we have $fj = fie = i \cdot f|_U \cdot e = ie \cdot f|_U = j \cdot f|_U$. Thus $f|_U$ is determined by f and U. □

For short monomorphisms $U \to V \to W$ in $\mathscr{O}$, the ring homomorphism of Proposition 1 satisfies

$$\rho_U^V \rho_V^W = \rho_U^W \ ; \quad \rho_U^U = 1_{\mathscr{O}(U)}. \tag{8}$$

Proposition 2. *Let* $i\colon U \to V$ *and* $j\colon V \to W$ *be morphisms in a pre-objective category* $\mathscr{O}$ *such that* ji *is a short monomorphism. Then* i *is a short monomorphism. Every short monomorphism* $U \to U$ *is invertible.*

Proof. If $ji = 0$, then $U = 0$, and thus i is a short monomorphism. Otherwise, $i \neq 0$, and so there is a short monomorphism $i'\colon U \to V$ such that $i = i'e$ with an endomorphism e of U. Hence $ji' = ji \cdot f$ for some $f\colon U \to U$, and thus $ji \cdot fe = ji$, which gives $fe = 1$. Since $\mathscr{O}$ is commutative, e is invertible, whence i is a short monomorphism. The second assertion is trivial. □

Let $\mathscr{O}$ be a pre-objective category. For $U, V \in \operatorname{Ob}\mathscr{O}$, we write $U \leqslant V$ if there exists a monomorphism $U \to V$. By (O1), (O2), and Proposition 2, this makes $\operatorname{Ob}\mathscr{O}$ into a partially ordered set. In fact, if $U \leqslant V \leqslant U$, there are short monomorphisms $U \to V \to U$ by (O1). So $V \to U$ is a split epimorphism by Proposition 2, hence invertible.

If $U \in \operatorname{Ob}\mathscr{O}$ and $f \in \mathscr{O}(U)$, then a short limit $S(f)$ is equivalent to a greatest $V \leqslant U$ in $\operatorname{Ob}\mathscr{O}$ such that $f|_V$ is invertible. In other words, $S(f)$ exists if and only if the join $U_f := \bigvee\{V \leqslant U \mid \rho^U_V(f) \in \mathscr{O}(V)^\times\}$ exists and $f|_{U_f}$ is invertible.

For a pre-objective category $\mathscr{O}$, the subset $\operatorname{sh}(\mathscr{O})$ of short monomorphisms can be regarded as a fibered category [6] over the partially ordered set $\operatorname{Ob}\mathscr{O}$ such that the fiber over each $U \in \operatorname{Ob}\mathscr{O}$ consists of a single object, and every morphism in $\operatorname{sh}(\mathscr{O})$ is cartesian. In other words, $\operatorname{sh}(\mathscr{O}) \to \operatorname{Ob}\mathscr{O}$ is a linear extension in the sense of Baues and Wirsching [1].

By Eqs. (8), a pre-objective category $\mathscr{O}$ gives rise to a functor

$$\rho\colon \Omega^{\mathrm{op}} \to \mathbf{CRi} \tag{9}$$

from the dual of the partially ordered set $\Omega := \operatorname{Ob}\mathscr{O}$ to the category $\mathbf{CRi}$ of commutative rings. So we get a functor $\rho^\times\colon \Omega^{\mathrm{op}} \to \mathbf{Ab}$ into the category $\mathbf{Ab}$ of abelian groups which maps $U \in \Omega$ to the unit group $\rho(U)^\times$. Furthermore, we obtain a 2-cocycle. Namely, if we assign a short monomorphism $i_U^V\colon U \to V$ to each morphism $U \to V$ in Ω, any relation $U \leqslant V \leqslant W$ in Ω leads to an equation

$$i_V^W i_U^V = i_U^W \cdot c_{UVW} \tag{10}$$

with $c_{UVW} \in \mathscr{O}(U)^\times$. The associativity of composition yields

$$\rho_U^V(c_{VWY}) \cdot c_{UWY}^{-1} \cdot c_{UVY} \cdot c_{UVW}^{-1} = 1 \tag{11}$$

for $U \leqslant V \leqslant W \leqslant Y$, which means that c is a 2-cocycle with respect to the functor ρ. (In the terminology of [1], we have to regard $\rho^\times$ as a natural system which assigns $\rho(U)^\times$ to $U \leqslant V$.) If we set $i_U^U := 1_{\mathscr{O}(U)}$ for all $U \in \mathrm{Ob}\,\mathscr{O}$, then c will be *normalized*, i. e. it satisfies

$$c_{UUV} = c_{UVV} = 1. \tag{12}$$

If the i_U^V are replaced by $i_U^V \cdot d_{UV}$ with $d_{UV} \in \mathscr{O}(U)^\times$, then c changes by a 2-boundary. This leads to the following explicit description of pre-objective categories.

Proposition 3. *Up to isomorphism, there is a one-to-one correspondence between pre-objective categories and pairs (ρ, γ), where ρ is a functor (9) with a partially ordered set Ω such that $\rho(U) = 0$ for at most one U, and $\gamma \in H^2(\Omega, \rho^\times)$.*

Proof. Let (ρ, γ) be given. We define a pre-objective category $\mathscr{O}$ with $\mathrm{Ob}\,\mathscr{O} := \Omega$ as follows. For $U, V \in \Omega$, we set $\mathrm{Hom}_{\mathscr{O}}(U, V) = 0$ in case that $U \nleqslant V$. If $U \leqslant V$, we define $\mathrm{Hom}_{\mathscr{O}}(U, V) := \rho(U)$. Let $\gamma \in H^2(\Omega, \rho^\times)$ be represented by a 2-cocycle c. If we set $U = V = W$ or $V = W = Y$ in Eq. (11), we get $c_{UUU} = c_{UUV}$ and $\rho_U^V(c_{VVV}) = c_{UVV}$. Setting $d_{UV} := c_{UUU}$, we have a 2-boundary $(\delta d)_{UVW} = \rho_U^V(d_{VW}) \cdot d_{UW}^{-1} \cdot d_{UV} = \rho_U^V(c_{VVV})$. Therefore, we can normalize c by multiplying with $(\delta d)^{-1}$.

For $U \leqslant V \leqslant W$ in Ω and morphisms $f \in \mathrm{Hom}_{\mathscr{O}}(U, V)$ and $g \in \mathrm{Hom}_{\mathscr{O}}(V, W)$ in $\mathscr{O}$, we define

$$gf := \rho_U^V(g) \cdot f \cdot c_{UVW}. \tag{13}$$

With this composition, $\mathscr{O}$ becomes a small commutative preadditive category. Since $\rho(U) = 0$ occurs at most once, the category $\mathscr{O}$ is skeletal.

Every morphism $U \to V$ in Ω can be associated to the short morphism $U \to V$ in $\mathscr{O}$ given by $1 \in \rho(U)$. This implies (O1). Assume that $U \xrightarrow{f} U \xrightarrow{i} V$ is a non-zero short monomorphism in $\mathscr{O}$. Then $i = if \cdot g$ for some $g\colon U \to U$. Thus $fg = 1$, and $if \cdot gf = if \cdot 1$ implies that $gf = 1$. Since $\rho(i)$ maps automorphisms to automorphisms, we get (O2). Now it is straighforward to verify that the correspondence is one-to-one. □

Corollary. *Let $\mathscr{O}$ be a pre-objective category with a greatest object X. Then the corresponding $\gamma \in H^2(\mathrm{Ob}\,\mathscr{O}, \rho^\times)$ is trivial, i. e. $\mathscr{O}$ is just given by the functor* (9).

Proof. For $U \in \mathrm{Ob}\,\mathscr{O}$, we choose a short monomorphism $i_U\colon U \to X$. Therefore, if $U \leqslant V$ in $\mathrm{Ob}\,\mathscr{O}$, there is a unique short monomorphism $i_U^V\colon U \to V$ with $i_U = i_V \cdot i_U^V$. With this normalization of short monomorphisms, we get

$$i_V^W i_U^V = i_U^W\,; \quad i_U^U = 1_U \tag{14}$$

for $U \leqslant V \leqslant W$. Hence γ is trivial. □

Example 2. In particular, the preceding corollary shows that every (commutatively) ringed space can be regarded as a pre-objective category with trivial 2-cocycle.

2 Affine objects

In what follows, we write $\mathbf{Mod}(\mathscr{O})$ for the category of *$\mathscr{O}$-modules*, i. e. additive functors $M\colon \mathscr{O}^{\mathrm{op}} \to \mathbf{Ab}$. We identify $\mathscr{O}$ with the full subcategory of representable functors via the Yoneda embedding

$$\mathscr{O} \hookrightarrow \mathbf{Mod}(\mathscr{O}) \tag{15}$$

which maps $U \in \mathrm{Ob}\,\mathscr{O}$ to $\mathrm{Hom}_{\mathscr{O}}(-, U)$.

Definition 3. Let $f\colon U \to U$ be an endomorphism in a pre-objective category $\mathscr{O}$. Assume that the short limit $S(f)$ exists. We call an $\mathscr{O}$-module M *regular* with respect to f or simply *f-regular* if the natural morphism

$$\varinjlim(M \circ f^{\mathbb{Z}}) \longrightarrow M(S(f)) \tag{16}$$

is invertible. We call M is *quasi-coherent* if M is f-regular for every morphism $f\colon U \to U$ in $\mathscr{O}$, provided that $S(f)$ exists for all such f.

Explicitly, the condition for $M \in \mathbf{Mod}(\mathscr{O})$ to be regular with respect to $f \in \mathscr{O}(U)$ states that the following are satisfied.

(L1) If a morphism $g\colon U \to M$ satisfies $gi_f = 0$, then $gf^n = 0$ for some $n \in \mathbb{N}$.

(L2) For any $g\colon S(f) \to M$ in $\mathbf{Mod}(\mathscr{O})$, there is an $n \in \mathbb{N}$ such that $g(f^\times)^n$ factors through i_f.

Definition 4. We say that an object U of a pre-objective category $\mathscr{O}$ is *covered* by a set $\mathscr{V} \subset \operatorname{Ob}\mathscr{O}$ if every $W \in \operatorname{Ob}\mathscr{O}$ with $V \leqslant W$ for all $V \in \mathscr{V}$ satisfies $U \leqslant W$. We call U *affine* if the short limit (5) exists for every $f \in \mathscr{O}(U)$ and the following are satisfied.

(A1) As an $\mathscr{O}$-module, U is f-regular for all $f \in \mathscr{O}(U)$.

(A2) If $V \leqslant U$, then $\bigvee\{S(f) \leqslant V \mid f \in \mathscr{O}(U)\} = V$.

(A3) If $f \in \mathscr{O}(U)$, then $S(f)$ is covered by $\mathscr{V} \subset \operatorname{Ob}\mathscr{O}$ if and only if the ideal of $\mathscr{O}(U)$ generated by the $g \in \mathscr{O}(U)$ with $S(g) \leqslant S(f)$ and $S(g) \leqslant V$ for some $V \in \mathscr{V}$ contains a power of f.

The full subcategory of affine objects in $\mathscr{O}$ will be denoted by $\mathscr{O}_{\text{aff}}$.

Proposition 4. *Let $\mathscr{O}$ be a pre-objective category, and let f, g be endomorphisms of $U \in \operatorname{Ob}\mathscr{O}$. Assume that the short limit $S(e)$ exists for all $e \in \mathscr{O}(U)$. Then*

$$S(fg) = S(f) \wedge S(g). \tag{17}$$

If U is affine, then $S(f+g)$ is covered by $\{S(f), S(g)\}$.

Proof. First, there are short monomorphisms $i_f\colon S(f) \to U$ and $i_g\colon S(g) \to U$. Since $f|_{S(fg)}$ is invertible, the short monomorphism $i_{fg}\colon S(fg) \to U$ factors through i_f and i_g. Hence $S(fg) \leqslant S(f), S(g)$ by Proposition 2. Now assume that $V \leqslant S(f), S(g)$. Then $f|_V$ and $g|_V$ are invertible. Hence $(fg)|_V$ is invertible, and thus $V \leqslant S(fg)$.

Let U be affine. Then we have $S((f+g)\cdot f) = S(f+g) \wedge S(f)$ and $S((f+g)\cdot g) = S(f+g) \wedge S(g)$. Since $(f+g)\cdot f + (f+g)\cdot g = (f+g)^2$, (A3) implies that $S(f+g)$ is covered by $\{S(f), S(g)\}$. □

In particular, Proposition 4 shows that $S(fg) = S(f)$ holds if g is invertible. The next proposition shows that for any $f \in \mathscr{O}(U)$, the ring homomorphism $\rho^U_{S(f)}$ of Proposition 1 can be regarded as a localization with respect to f.

Proposition 5. *Let $\mathscr{O}$ be a pre-objective category. For any $U \in \mathrm{Ob}\,\mathscr{O}_{\mathrm{aff}}$ and $f \in \mathscr{O}(U)$, there is a natural isomorphism*

$$\mathscr{O}(S(f)) \cong \mathscr{O}(U)_f. \tag{18}$$

Proof. For a given $h \in \mathscr{O}(S(f))$, there is an $n \in \mathbb{N}$ such that $h(f^\times)^n = g|_{S(f)}$ for some $g \in \mathscr{O}(U)$. We define a map $\varepsilon\colon \mathscr{O}(S(f)) \to \mathscr{O}(U)_f$ by $\varepsilon(h) := \frac{g}{f^n}$. By (L1), this map is well-defined, and it is easily checked that ε is a ring isomorphism. □

Proposition 6. *Let $\mathscr{O}$ be a pre-objective category, and let $f\colon U \to U$ be an endomorphism in $\mathscr{O}_{\mathrm{aff}}$. Then $S(f)$ is affine.*

Proof. Let $g \in \mathscr{O}(S(f))$ be given. To verify (A1) for $S(f)$, let $h\colon S(f) \to S(f)$ be an endomorphism with $hi_g = 0$. Since f satisfies (L2), there is an $n \in \mathbb{N}$ with $i_f \cdot g(f^\times)^n = u \cdot i_f$ and $i_f \cdot h(f^\times)^n = v \cdot i_f$. Hence $i_f \cdot g(f^\times)^{n+1} = f \cdot i_f \cdot g(f^\times)^n = fu \cdot i_f$. Since $S(fu) \leqslant S(f)$, this gives $S(fu) = S(g(f^\times)^{n+1}) = S(g)$. Therefore, $v \cdot i_{fu} \cong v \cdot i_f i_g = i_f \cdot h(f^\times)^n i_g = 0$ implies that $v(fu)^m = 0$ for some $m \in \mathbb{N}$. Hence $i_f hg^m (f^\times)^{mn+m+n} = v \cdot i_f g^m (f^\times)^{mn+m} = vu^m i_f (f^\times)^m = vu^m f^m i_f = 0$, and thus $hg^m = 0$. This proves (L1). Next let $h\colon S(g) \to S(f)$ be given. Then $S(fu) = S(g)$ and $fu|_{S(g)} = g^\times \cdot (f^\times)^{n+1}|_{S(g)}$ implies that $i_f h(g^\times)^k \cdot (f^\times)^{(n+1)k}|_{S(g)}$ factors through $i_f i_g$ for a suitable $k \in \mathbb{N}$. Hence $h(g^\times)^k \cdot (f^\times)^{(n+1)k}|_{S(g)}$ factors through i_g, and thus $h(g^\times)^k$ factors through i_g. This shows that $S(f)$ satisfies (A1).

If $V \leqslant S(f)$, then there is a subset $F \subset \mathscr{O}(U)$ with $\bigvee\{S(g) \mid g \in F\} = V$. Hence $\bigvee\{S(g|_{S(f)}) \mid g \in F\} = V$, which proves (A2) for $S(f)$.

Finally, let $g \in \mathscr{O}(S(f))$ and $\mathscr{V} \subset \mathrm{Ob}\,\mathscr{O}$ be given. By multiplying g with a suitable power of $f^\times$, we can assume that $g = g'|_{S(f)}$ for some $g' \in \mathscr{O}(U)$, and $S(g) = S(g')$. Let I be the ideal of $\mathscr{O}(S(f))$ generated by the $h \in \mathscr{O}(S(f))$ with $S(g) \geqslant S(h) \leqslant V$ for some $V \in \mathscr{V}$, and let I' be the ideal of $\mathscr{O}(U)$ generated by the $h \in \mathscr{O}(U)$ with $S(g') \geqslant S(h) \leqslant V$ for some $V \in \mathscr{V}$. Then the ideal I is generated by $\rho^U_{S(f)}(I')$. Hence I contains a power of g if and only if I' contains a power of $g'f$. This proves (A3) for $S(f)$. □

Definition 5. We call an additive functor $F\colon \mathscr{O} \to \mathscr{O}'$ between pre-objective categories *objective* if the following are satisfied.

(F1) F maps short monomorphisms to short monomorphisms.
(F2) If $U \in \mathrm{Ob}\,\mathscr{O}$ and $f \in \mathscr{O}(U)$ such that $S(f)$ exists, then $FS(f) = S(Ff)$.
(F3) F respects finite meets whenever they exist.
(F4) For any $V \in \mathrm{Ob}\,\mathscr{O}$, the FU with affine $U \leqslant V$ cover every affine $W \leqslant FV$.

Let $\mathscr{C}$ be a preadditive category. Recall that a full subcategory $\mathscr{D}$ is said to be *dense* if every object C of $\mathscr{C}$ satisfies

$$C = \mathrm{Colim}(\mathscr{D}/C \to \mathscr{C}). \tag{19}$$

Here the *slice category* $\mathscr{D}/C$ has objects $f\colon D \to C$ with $D \in \mathrm{Ob}\,\mathscr{D}$. If $f'\colon D' \to C$ is a second object, a morphism $f \to f'$ in $\mathscr{D}/C$ is a morphism $g\colon D \to D'$ in $\mathscr{D}$ with $f'g = f$. The natural functor $\mathscr{D}/C \to \mathscr{C}$ maps $f\colon D \to C$ to D. Note that $\mathscr{D} \subset \mathscr{C}$ is dense if and only if the functor

$$\mathscr{C} \longrightarrow \mathbf{Mod}(\mathscr{D}) \tag{20}$$

which maps C to $\mathrm{Hom}_{\mathscr{D}}(-, C)$ is fully faithful (see [9], X.6, dual of Proposition 2).

Definition 6. We define an *objective* category to be a pre-objective category $\mathscr{O}$ which satisfies

(O3) Arbitrary joins exist in $\mathrm{Ob}\,\mathscr{O}$.

(O4) The full subcategory $\mathscr{O}_{\mathrm{aff}}$ is dense in $\mathscr{O}$.

(O3) implies that $\mathrm{Ob}\,\mathscr{O}$ is a complete lattice. In particular, there is a greatest object X. So the corollary of Proposition 3 implies that the morphisms $U \to V$ in the partially ordered set $\Omega := \mathrm{Ob}\,\mathscr{O}$ can be regarded as short monomorphisms $i_U^V \in \mathscr{O}$ via (14), i. e. Ω becomes a full subcategory of $\mathscr{O}$. In the sequel, we choose a fixed embedding $i\colon \Omega \hookrightarrow \mathscr{O}$ with (14) for any objective category $\mathscr{O}$. In particular, if $U \in \mathrm{Ob}\,\mathscr{O}$ and $f \in \mathscr{O}(U)$, we set $i_f := i_{S(f)}^U$.

Axiom (O4) is related to a recollement of sheaves.

Proposition 7. *Let $\mathscr{O}$ be an objective category. For every $Y \in \mathrm{Ob}\,\mathscr{O}$,*

$$Y = \bigvee \{U \in \mathrm{Ob}\,\mathscr{O}_{\mathrm{aff}} \mid U \leqslant Y\} \tag{21}$$

$$\mathscr{O}(Y) = \varprojlim \{\mathscr{O}(U) \mid Y \geqslant U \in \mathrm{Ob}\,\mathscr{O}_{\mathrm{aff}}\}. \tag{22}$$

Proof. By definition, $Y = \mathrm{Colim}(\mathscr{O}_{\mathrm{aff}}/Y \to \mathscr{O})$. Let $Z \in \mathrm{Ob}\,\mathscr{O}$ be an object with $U \leqslant Z$ for all affine $U \leqslant Y$. Then the map $i_U^Y f \mapsto i_U^Z f$ defines a cocone over $\mathscr{O}_{\mathrm{aff}}/Y \to \mathscr{O}$. Hence there is a unique morphism $h\colon Y \to Z$ with $h \cdot i_U^Y = i_U^Z$ for all affine $U \leqslant Y$. If $Y = 0$, then $Y \leqslant Z$. Otherwise, there exists a non-zero affine $U \leqslant Y$. Therefore, $i_U^Z \neq 0$, which gives $h \neq 0$. Thus $Y \leqslant Z$. This proves (21).

To verify (22), we have to show that the restrictions $\rho_U^Y\colon \mathscr{O}(Y) \to \mathscr{O}(U)$ with $Y \geqslant U \in \mathrm{Ob}\,\mathscr{O}_{\mathrm{aff}}$ form a limit cone. Thus let $f_U \in \mathscr{O}(U)$ be given for each affine $U \leqslant Y$ such that $f_U|_V = f_V$ for affine $V \leqslant U$. Then $i_U^Y f \mapsto i_U^Y f \cdot f_U$ defines a cocone $(\mathscr{O}_{\mathrm{aff}}/Y \to \mathscr{O}) \to Y$. So there is a unique $g\colon Y \to Y$ with $g i_U^Y = i_U^Y f_U$ for all affine $U \leqslant Y$. Whence $g|_U = f_U$ for all U. □

Note that Definition 5(F4) can be regarded as a relative version of (O4).

3 The associated scheme

In this section, we associate a scheme $\operatorname{Spec}\mathscr{O}$ to any pre-objective category $\mathscr{O}$. In Section 4, we will prove that $\operatorname{Spec}\mathscr{O}$ determines $\mathscr{O}$ if $\mathscr{O}$ is objective.

Definition 7. We define a *point* of a pre-objective category $\mathscr{O}$ to be a non-empty subset x of $\operatorname{Ob}\mathscr{O}_{\mathrm{aff}}$ such that the following are satisfied.

(P0) $0 \notin x$.

(P1) $U \geqslant V \in x \;\Rightarrow\; U \in x$.

(P2) $U, V \in x \;\Rightarrow\; \exists W \in x\colon W \leqslant U, V$.

(P3) $U \in x,\ f \in \mathscr{O}(U) \;\Rightarrow\; \big(S(f) \in x \;\text{ or }\; S(1-f) \in x\big)$.

The set of points of $\mathscr{O}$ will be denoted by $\operatorname{Spec}\mathscr{O}$.

Let $\mathscr{O}$ be a pre-objective category. We introduce a topology on $\operatorname{Spec}\mathscr{O}$ with basic open sets

$$\widetilde{U} := \{x \in \operatorname{Spec}\mathscr{O} \mid U \in x\} \tag{23}$$

for each $U \in \operatorname{Ob}\mathscr{O}_{\mathrm{aff}}$. If a point $x \in \operatorname{Spec}\mathscr{O}$ satisfies $x \in \widetilde{U} \cap \widetilde{V}$ for two objects U, V of $\mathscr{O}_{\mathrm{aff}}$, then $U, V \in x$, and so there exists some $W \in x$ with $W \leqslant U, V$. Thus $x \in \widetilde{W} \subset \widetilde{U} \cap \widetilde{V}$. So the $\widetilde{U}$ form a basis of open sets. For arbitrary $V \in \operatorname{Ob}\mathscr{O}$, we define

$$\widetilde{V} := \{x \in \operatorname{Spec}\mathscr{O} \mid \exists\, U \in x\colon U \leqslant V\}. \tag{24}$$

If V is affine, this definition coincides with (23).

Proposition 8. *Let $\mathscr{O}$ be a pre-objective category, and let $U \in \operatorname{Ob}\mathscr{O}_{\mathrm{aff}}$ be covered by $\mathscr{V} \subset \operatorname{Ob}\mathscr{O}$. For every $x \in \widetilde{U}$, there exists some $V \in \mathscr{V}$ with $x \in \widetilde{V}$.*

Proof. Since $U = S(1_U)$ is affine, (A3) implies that the $g \in \mathscr{O}(U)$ with $S(g) \leqslant V$ for some $V \in \mathscr{V}$ generate $\mathscr{O}(U)$. Hence $1_U = \sum_{i=1}^n a_i g_i$ with $a_i, g_i \in \mathscr{O}(U)$ and $S(g_i) \leqslant V_i \in \mathscr{V}$. Suppose that $S(a_i g_i) \notin x$ for all $i \in \{1, \ldots, n\}$. We set $f_k := \sum_{i=1}^k a_i g_i$. Since $S(1_U) \in x$, there is a minimal $m > 1$ with $S(f_m) \in x$. With $a := f_{m-1}|_{S(f_m)} \cdot (f_m^\times)^{-1}$ and $b := a_m g_m|_{S(f_m)} \cdot (f_m^\times)^{-1}$, we have $a + b = 1$. Therefore, (P3) implies that $S(a) \in x$ or $S(b) \in x$. From $S(a) = S(f_{m-1}|_{S(f_m)}) \leqslant S(f_{m-1})$ and $S(b) \leqslant S(a_m g_m)$, we get $S(f_{m-1}) \in x$ or $S(a_m g_m) \in x$, a contradiction. Hence $S(a_i g_i) \in x$ for some $i \in \{1, \ldots, n\}$. Since $S(a_i g_i) \leqslant S(g_i) \leqslant V_i$, it follows that $x \in \widetilde{V}_i$. □

For any $x \in \widetilde{U}$ with $U \in \operatorname{Ob} \mathscr{O}_{\mathrm{aff}}$, we define

$$\mathfrak{p}_x := \{f \in \mathscr{O}(U) \mid S(f) \notin x\}. \tag{25}$$

Proposition 9. *Let $\mathscr{O}$ be a pre-objective category. Then $x \mapsto \mathfrak{p}_x$ gives a homeomorphism*

$$\mathfrak{p} \colon \widetilde{U} \longrightarrow \operatorname{Spec} \mathscr{O}(U) \tag{26}$$

for every affine object U of $\mathscr{O}$.

Proof. We show first that $\mathfrak{p}_x$ is a prime ideal of $\mathscr{O}(U)$ for any $x \in \widetilde{U}$. Thus let $f, g \in \mathscr{O}(U)$ be given. Assume that $f \in \mathfrak{p}_x$. Then $S(f) \notin x$, and $S(fg) \leqslant S(f)$ by Proposition 4. Hence (P1) of Definition 7 gives $S(fg) \notin x$ and thus $fg \in \mathfrak{p}_x$. In particular, $-f \in \mathfrak{p}_x$. Furthermore, (P0) implies that $0 \in \mathfrak{p}_x$. To show that $\mathfrak{p}_x$ is an ideal of $\mathscr{O}(U)$, suppose that $f, g \in \mathfrak{p}_x$ and $f + g \notin \mathfrak{p}_x$. Then $S(f), S(g) \notin x$ but $S(f+g) \in x$. Proposition 4 implies that $S(f+g)$ is covered by $\{S(f), S(g)\}$. So we get a contradiction to Proposition 8.

To show that $\mathfrak{p}_x$ is a prime ideal, assume that $f, g \in \mathscr{O}(U)$ satisfy $fg \in \mathfrak{p}_x$. Then $S(fg) \notin x$. Suppose that $f, g \notin \mathfrak{p}_x$, i. e. $S(f), S(g) \in x$. By (P2), we find an object $W \in x$ with $W \leqslant S(f), S(g) \leqslant U$. Proposition 4 implies that $W \leqslant S(f) \wedge S(g) = S(fg)$. Thus $S(fg) \in x$, a contradiction. Finally, $x \in \widetilde{U}$ implies that $S(1_U) = U \in x$, whence $1 \notin \mathfrak{p}_x$.

Conversely, let $\mathfrak{p}$ be a prime ideal of $\mathscr{O}(U)$. We define

$$x := \{V \in \operatorname{Ob} \mathscr{O}_{\mathrm{aff}} \mid \exists f \in \mathscr{O}(U) \smallsetminus \mathfrak{p} \colon S(f) \leqslant V\} \tag{27}$$

and show that x is a point of $\mathscr{O}$. If $f \in \mathscr{O}(U) \smallsetminus \mathfrak{p}$, then f is not nilpotent. Hence, the (L1) part of (A1) implies that $f \cdot i_f \neq 0$, which yields $S(f) \neq 0$. This proves (P0). As (P1) is trivial, let us prove (P2). If $U, V \in x$, there are $f, g \in \mathscr{O}(U) \smallsetminus \mathfrak{p}$ with $S(f) \leqslant U$ and $S(g) \leqslant V$. Hence $fg \notin \mathfrak{p}$ and $U, V \geqslant S(f) \wedge S(g) = S(fg) \in x$.

To verify (P3), let $V \in x$ and $f \in \mathscr{O}(V)$ be given. So there exists some $g \in \mathscr{O}(U) \smallsetminus \mathfrak{p}$ with $S(g) \leqslant V$. By the (L2) part of (A1), there is some $n \in \mathbb{N}$ with $f|_{S(g)} \cdot (g^{\times})^n = h|_{S(g)}$ for some $h \in \mathscr{O}(U)$. Since $g^{n+1} \notin \mathfrak{p}$, it follows that $hg \notin \mathfrak{p}$ or $g^{n+1} - hg \notin \mathfrak{p}$. Hence $S(hg) \in x$ or $S(g^{n+1} - hg) \in x$. Furthermore, $S(hg) \leqslant S(g)$ and $S(g^{n+1} - hg) \leqslant S(g)$ implies that $S(hg) = S(f|_{S(g)} \cdot (g^{\times})^{n+1}) \leqslant S(f|_{S(g)}) \leqslant S(f)$ and $S(g^{n+1} - hg) = S((g^{\times})^{n+1} - f|_{S(g)} \cdot (g^{\times})^{n+1}) \leqslant S(1 - f|_{S(g)}) \leqslant S(1-f)$. Therefore, we get $S(f) \in x$ or $S(1-f) \in x$, which completes the proof of (P3). Thus x is a point of $\mathscr{O}$. Since $1_U \in \mathscr{O}(U) \smallsetminus \mathfrak{p}$ and $S(1_U) = U$, we have $U \in x$, i. e. $x \in \widetilde{U}$. Next we show that $\mathfrak{p}_x = \mathfrak{p}$.

If $f \in \mathfrak{p}_x$, then $S(f) \notin x$, which gives $f \in \mathfrak{p}$. Conversely, assume that $f \notin \mathfrak{p}_x$. Then $S(f) \in x$. So there exists some $g \in \mathscr{O}(U) \smallsetminus \mathfrak{p}$ with $S(g) \leqslant S(f)$. Thus $f|_{S(g)}$ is invertible. By (A1), there exists some $n \in \mathbb{N}$ such that $i_g \cdot f|_{S(g)}^{-1} \cdot (g^{\times})^n = hi_g$ for some $h \in \mathscr{O}(U)$. Hence $fhi_g = i_g \cdot (g^{\times})^n = g^n i_g$. By (A1), this gives $fhg^m = g^{m+n}$ for a suitable $m \in \mathbb{N}$. Therefore, we get $f \notin \mathfrak{p}$, which proves that $\mathfrak{p}_x = \mathfrak{p}$.

For the bijectivity of (26), it remains to be shown that

$$x = \{V \in \operatorname{Ob} \mathscr{O}_{\mathrm{aff}} \mid \exists f \in \mathscr{O}(U) \smallsetminus \mathfrak{p}_x \colon S(f) \leqslant V\} \tag{28}$$

holds for any point $x \in \widetilde{U}$. The inclusion "$\supset$" follows by (25). Conversely, assume that $V \in x$. Then (P2) yields an object $W \in x$ such that $W \leqslant U, V$. By (A2), it follows that W is covered by the $S(f)$ with $f \in \mathscr{O}(U)$ and $S(f) \leqslant W$. Furthermore, Proposition 6 implies that these $S(f)$ are affine. Hence Proposition 8 yields some $f \in \mathscr{O}(U)$ with $S(f) \leqslant W$ and $S(f) \in x$. Thus (25) gives $f \in \mathscr{O}(U) \smallsetminus \mathfrak{p}_x$ and $S(f) \leqslant V$. This proves (28).

Finally, (A2) of Definition 4 implies that the $\widetilde{S(f)}$ with $f \in \mathscr{O}(U)$ form a basis of $\widetilde{U}$. Therefore, Proposition 5 shows that the map (26) is a homeomorphism. □

By Proposition 9, every affine object U of a pre-objective category $\mathscr{O}$ gives rise to an embedding

$$\operatorname{Spec} \mathscr{O}(U) \hookrightarrow \operatorname{Spec} \mathscr{O} \tag{29}$$

such that $\operatorname{Spec} \mathscr{O}(U)$ can be identified with $\widetilde{U} \subset \operatorname{Spec} \mathscr{O}$. To any basic open set $\widetilde{U}$ of $\operatorname{Spec} \mathscr{O}$, we associate the commutative ring

$$\mathscr{O}(\widetilde{U}) := \mathscr{O}(U). \tag{30}$$

By Eqs. (8), this makes $\mathscr{O}$ into a presheaf on $\operatorname{Spec} \mathscr{O}$. So we obtain

Theorem 1. *For a pre-objective category $\mathscr{O}$, the associated presheaf makes* $\operatorname{Spec} \mathscr{O}$ *into a scheme.*

Proof. For an open set $V \subset \operatorname{Spec} \mathscr{O}$, we define

$$\mathscr{O}(V) := \varprojlim_{\widetilde{U} \subset V} \mathscr{O}(\widetilde{U}), \tag{31}$$

where U runs through $\operatorname{Ob} \mathscr{O}_{\text{aff}}$. By Proposition 5, the $\widetilde{U}$ are affine schemes. Hence $\operatorname{Spec} \mathscr{O}$ is a scheme with structure sheaf (30) by [5], 0.3.2.2. □

4 Objective categories

In the sequel, we write **POb** for the category of pre-objective categories with a greatest object and with objective functors as morphisms. By the corollary of Proposition 3, the objects of **POb** can be regarded as functors $\rho\colon \Omega^{\text{op}} \to \mathbf{CRi}$ such that the partially ordered set Ω has a greatest element, and $\rho(a) = 0$ for at most one element $a \in \Omega$. Note that by (F3) of Definition 5, a morphism $\mathscr{O} \to \mathscr{O}'$ in **POb** respects the greatest object $\bigwedge \varnothing$ of $\mathscr{O}$. By **Obj** we denote the full subcategory of **POb** consisting of the objective categories. The category of schemes as locally ringed spaces (see [7], II.2) will be denoted by **Sch**.

Proposition 10. *Let $\mathscr{O}$ be a pre-objective category. For $U, V \in \operatorname{Ob}\mathscr{O}$ with U affine,*

$$U \leqslant V \iff \widetilde{U} \subset \widetilde{V}. \tag{32}$$

If $\mathscr{O}$ is objective, the equivalence holds for all $U, V \in \operatorname{Ob}\mathscr{O}$, and every open set of $\operatorname{Spec}\mathscr{O}$ *is of the form $\widetilde{V}$ for some $V \in \operatorname{Ob}\mathscr{O}$.*

Proof. The implication "$\Rightarrow$" follows by (24). Assume that $\widetilde{U} \subset \widetilde{V}$, and let $\mathfrak{p}$ be any prime ideal of $\mathscr{O}(U)$. By Proposition 9, this implies that $\mathfrak{p} = \mathfrak{p}_x$ for some $x \in \widetilde{U} \subset \widetilde{V}$. So there is an affine $W \in x$ with $W \leqslant V$. By Eq. (27), we find some $f \in \mathscr{O}(U) \smallsetminus \mathfrak{p}$ with $S(f) \leqslant W \leqslant V$. Therefore, the $f \in \mathscr{O}(U)$ with $S(f) \leqslant V$ generate $\mathscr{O}(U)$. Thus (A3) implies that $U = S(1_U) \leqslant V$. If $\mathscr{O}$ is objective, the restriction on U can be dropped by virtue of Proposition 7.

Now let $\mathscr{O}$ be objective, and let V' be an open set of $\operatorname{Spec}\mathscr{O}$. Then $V' = \bigcup\{\widetilde{U} \mid U \in \mathscr{V}\}$ for some $\mathscr{V} \subset \operatorname{Ob}\mathscr{O}_{\text{aff}}$. We show that $V := \bigvee \mathscr{V}$ satisfies $\widetilde{V} = V'$. For all $U \in \mathscr{V}$, we have $U \leqslant V$, hence $\widetilde{U} \subset \widetilde{V}$, and thus $V' \subset \widetilde{V}$. Conversely, assume that $x \in \widetilde{V}$. By (24), there is some $U \in x$ with $U \leqslant V$. Hence $U = S(1_U)$ is covered by $\mathscr{V}$. So Proposition 8 implies that $x \in \widetilde{W}$ for some $W \in \mathscr{V}$. Hence $x \in \widetilde{W} \subset V'$. □

As an immediate consequence, Proposition 10 yields

Corollary. *For an objective category $\mathscr{O}$, the map $V \mapsto \widetilde{V}$ is a lattice isomorphism between the complete lattice* $\operatorname{Ob}\mathscr{O}$ *and the set of open sets of* $\operatorname{Spec}\mathscr{O}$.

Now we are ready to prove

Theorem 2. *The category* **Obj** *of objective categories is dual to the category* **Sch** *of schemes.*

Proof. We show first that the map which associates a scheme to an objective category (Theorem 1) extends to a functor

$$\mathrm{Spec}\colon \mathbf{Obj} \longrightarrow \mathbf{Sch}^{\mathrm{op}}. \tag{33}$$

Thus let $F\colon \mathscr{O} \to \mathscr{O}'$ be a morphism in **Obj**. We define

$$\mathrm{Spec}\, F\colon \mathrm{Spec}\, \mathscr{O}' \to \mathrm{Spec}\, \mathscr{O} \tag{34}$$

as follows. For $x' \in \mathrm{Spec}\, \mathscr{O}'$, we set

$$(\mathrm{Spec}\, F)(x') := \{U \in \mathscr{O}_{\mathrm{aff}} \mid x' \in \widetilde{FU}\}. \tag{35}$$

We show that $x := (\mathrm{Spec}\, F)(x') \in \mathrm{Spec}\, \mathscr{O}$. Since F is additive, (P0) holds for x. As F is monotonous by (F3), we get (P1). If $U, V \in x$, then $x' \in \widetilde{FU} \cap \widetilde{FV}$. So there exists some $W' \in x'$ with $W' \leqslant FU \wedge FV = F(U \wedge V)$. By (F4), the FW with affine $W \leqslant U \wedge V$ cover W'. Therefore, Proposition 8 implies that $x' \in \widetilde{FW}$ for some affine $W \leqslant U \wedge V$. Hence $W \in x$, which proves (P2) for x. To verify (P3), let $U \in x$ and $f \in \mathscr{O}(U)$ be given. Then $FU \geqslant U'$ for some $U' \in x'$. Hence $g := (Ff)|_{U'}$ satisfies $S(g) \in x'$ or $S(1-g) \in x'$. Now $S(g) \leqslant S(Ff) = FS(f)$, and similarly, $S(1-g) \leqslant FS(1-f)$. Hence $S(f) \in x$ or $S(1-f) \in x$. So the map (34) is well-defined.

To show that $\mathrm{Spec}\, F$ is continuous, let $V \in \mathrm{Ob}\, \mathscr{O}$ be given. Then (24) yields

$$\begin{aligned} x' \in (\mathrm{Spec}\, F)^{-1}(\widetilde{V}) &\Leftrightarrow (\mathrm{Spec}\, F)(x') \in \widetilde{V} \\ &\Leftrightarrow \exists\, U \in \mathrm{Ob}\, \mathscr{O}_{\mathrm{aff}}\colon V \geqslant U \in (\mathrm{Spec}\, F)(x') \\ &\Leftrightarrow \exists\, U \in \mathrm{Ob}\, \mathscr{O}_{\mathrm{aff}}\colon U \leqslant V, x' \in \widetilde{FU}. \end{aligned}$$

By Eq. (21), we have $V = \bigvee\{U \in \mathrm{Ob}\, \mathscr{O}_{\mathrm{aff}} \mid U \leqslant V\}$. Therefore, (F4) gives $FV = \bigvee\{FU \mid V \geqslant U \in \mathrm{Ob}\, \mathscr{O}_{\mathrm{aff}}\}$, and the corollary of Proposition 10 yields $\widetilde{FV} = \bigcup\{\widetilde{FU} \mid V \geqslant U \in \mathrm{Ob}\, \mathscr{O}_{\mathrm{aff}}\}$. So we get

$$(\mathrm{Spec}\, F)^{-1}(\widetilde{V}) = \widetilde{FV} \tag{36}$$

which shows that the map (34) is continuous.

Now F induces a ring homomorphism $\mathscr{O}(U) \to \mathscr{O}'(FU)$ for any $U \in \operatorname{Ob}\mathscr{O}$. For a short monomorphism $i\colon V \to U$, we have a commutative diagram

$$\begin{array}{ccc} \mathscr{O}(U) & \longrightarrow & \mathscr{O}'(FU) \\ \Big\downarrow \rho_V^U & & \Big\downarrow \rho_{FV}^{FU} \\ \mathscr{O}(V) & \longrightarrow & \mathscr{O}'(FV) \end{array}$$

by (F1). Therefore, Eq. (36) implies that F induces a morphism $\mathscr{O} \to (\operatorname{Spec} F)_* \mathscr{O}'$ of sheaves, which yields a morphism of schemes. This establishes the functor (33).

Conversely, we construct a functor

$$\mathbf{Sch}^{\mathrm{op}} \longrightarrow \mathbf{Obj}. \tag{37}$$

Let X be a scheme with structure sheaf $\mathscr{O}_X$. According to Proposition 3 and its corollary, the presheaf X defines a pre-objective category $\mathscr{O}$ with trivial 2-cocycle. Therefore, we can regard every inclusion $U \subset V$ with $U, V \in \operatorname{Ob}\mathscr{O}$ as a short morphism $i_U^V\colon U \to V$ in $\mathscr{O}$. Furthermore, $\mathscr{O}$ has a unique zero object $\varnothing$.

Let $f\colon U \to U$ be an endomorphism in $\mathscr{O}$, i. e. $f \in \mathscr{O}_X(U)$. We define $S(f)$ to be the set of points $x \in U$ such that the germ f_x of f at x is invertible in $\mathscr{O}_{X,x}$. Thus $S(f)$ is the maximal open subset V of U such that $f|_V$ is invertible. Hence $S(f) = \underleftarrow{\operatorname{shlim}}\, f^{\mathbb{Z}}$ (see [7], chap. II, Exercise 2.16). Furthermore, $\mathscr{O}$ satisfies (O3).

Now let $U \subset X$ be an affine open set. For any $f \in \mathscr{O}(U)$, the set $S(f) \subset U$ consists of the prime ideals $\mathfrak{p}$ of $\mathscr{O}(U)$ with $f \notin \mathfrak{p}$. Hence $\mathscr{O}(S(f)) = \mathscr{O}(U)_f$, and thus U satisfies (A1) of Definition 4. Moreover, (A2) and (A3) are easily verified. Hence $U \in \operatorname{Ob}\mathscr{O}_{\mathrm{aff}}$. Conversely, the open sets in $\operatorname{Ob}\mathscr{O}_{\mathrm{aff}}$ are affine by Proposition 5. To verify (O4), let $V, W \subset X$ be open such that for any affine open $U \subset V$ and a morphism $i_U^V \cdot f\colon U \to V$, there is a corresponding morphism $i_U^W \cdot f'\colon U \to W$. Assume that these maps form a cocone over $\mathscr{O}_{\mathrm{aff}}/V \to \mathscr{O}$. This means that for each affine open $U \subset V$, there is a section $f_U := (1_U)' \in \mathscr{O}_X(U)$ such that for every affine open $U' \subset U$, $f_{U'} = f_U|_{U'}$. Since $\mathscr{O}_X$ is a sheaf, there exists a unique section $f \in \mathscr{O}_X(V)$ with $f|U = f_U$ for all affine open $U \subset V$. So we get a morphism $i_V^W f \colon V \to W$ which completes the proof of (O4).

Next let $(\varphi, \vartheta) \colon X' \to X$ be a morphism of schemes, i. e. $\varphi \colon X' \to X$ is continuous, and $\vartheta \colon \mathscr{O}_X \to \varphi_* \mathscr{O}_{X'}$ is a morphism of sheaves. Let $\mathscr{O}', \mathscr{O}$ be the corresponding objective categories. We define a functor $F \colon \mathscr{O} \to \mathscr{O}'$ as follows. For $U \in \operatorname{Ob} \mathscr{O}$, we set $FU := \varphi^{-1}(U)$, and for a morphism $i_U^V \cdot f \colon U \to V$ in $\mathscr{O}$, we define $F(i_U^V \cdot f) := i_{FU}^{FV} \cdot \vartheta_U(f)$. Thus F respects addition of morphisms, and for a morphism $i_V^W \cdot g \colon V \to W$, we have $F(i_V^W g \cdot i_U^V f) = F(i_U^W \cdot g|_U \cdot f) = i_{FU}^{FW} \vartheta_U(g|_U \cdot f) = i_{FU}^{FW} \vartheta_U(g|_U) \vartheta_U(f) = i_{FU}^{FW} \vartheta_V(g)|_{FU} \cdot \vartheta_U(f) = i_{FV}^{FW} \vartheta_V(g) \cdot i_{FU}^{FV} \vartheta_U(f) = F(i_V^W \cdot g) F(i_U^V \cdot f)$. Since $F(i_U^V) = i_{FU}^{FV}$, it follows that F is an additive functor which satisfies (F1) of Definition 5. For $f \in \mathscr{O}_X(U)$, we have $FS(f) = \varphi^{-1}(S(f)) = S(\vartheta_U(f)) = S(Ff)$ since (φ, ϑ) is a morphism of locally ringed spaces. This proves (F2). As (F3) and (F4) are trivial, the functor F is objective. It is straightforward to check that the functor (37) is inverse to (33). $\square$

Remark. The preceding proof shows that for a scheme X, an open subset U of X is affine if and only if the object U of the corresponding objective category is affine in the sense of Definition 4.

By virtue of Theorem 2, the following result locates the category of schemes within the category of pre-objective categories.

Theorem 3. *The category* **Obj** *of objective categories is a reflective full subcategory of the category* **POb** *of preobjective categories with a greatest object.*

Proof. For a pre-objective category $\mathscr{O}$ with a greatest object X, Theorem 2 implies that the associated scheme $\operatorname{Spec} \mathscr{O}$ corresponds to an objective category $\widetilde{\mathscr{O}}$. The objects of $\widetilde{\mathscr{O}}$ are the open sets of $\operatorname{Spec} \mathscr{O}$. By the corollary of Proposition 3, $\mathscr{O}$ admits a subcategory of short monomorphisms $i_U^V \colon U \to V$ for $U \leqslant V$ such that the relations (14) are satisfied. For any $V \in \operatorname{Ob} \mathscr{O}$, Eq. (24) can be rewritten as

$$\widetilde{V} = \bigcup \{ \widetilde{U} \mid V \geqslant U \in \mathscr{O}_{\text{aff}} \}. \tag{38}$$

Every $f \in \mathscr{O}(V)$ gives rise to a system of $f|_U \in \mathscr{O}(U) = \mathscr{O}(\widetilde{U})$ for all affine $U \leqslant V$. By recollement, this yields an endomorphism $\widetilde{f} \in \widetilde{V}$ with $\widetilde{f}|_{\widetilde{U}} = f|_U$ for all U. So we get an additive functor $H\colon \mathscr{O} \to \widetilde{\mathscr{O}}$ which maps $i_V^W \cdot f\colon V \to W$ with $f \in \mathscr{O}(V)$ to $\widetilde{f} \in \mathscr{O}(\widetilde{V}) = \mathrm{Hom}_{\widetilde{\mathscr{O}}}(\widetilde{V}, \widetilde{W})$ according to Proposition 3. By construction, H satisfies (F1) of Definition 5. Assume that $S(f)$ exists for some $f \in \mathscr{O}(V)$ with $V \in \mathrm{Ob}\,\mathscr{O}$. To verify (F2), we have to show that $S(Hf) \subset HS(f)$. This means that $\widetilde{U} \subset HS(f)$ for every $U \in \mathscr{O}_{\mathrm{aff}}$ with $\widetilde{U} \subset S(Hf) = S(\widetilde{f})$. For such U, the restriction $\widetilde{f}|_{\widetilde{U}} = f|_U$ is invertible. Hence $U \subset S(f)$, and thus $\widetilde{U} \subset HS(f)$. Properties (F3) and (F4) are immediate consequences of (38). Thus $H\colon \mathscr{O} \to \widetilde{\mathscr{O}}$ is a morphism in **POb**.

Now let $F\colon \mathscr{O} \to \mathscr{O}'$ be a morphism in **POb** with $\mathscr{O}'$ objective. For $Y \in \mathrm{Ob}\,\widetilde{\mathscr{O}}$, we define

$$F'Y := \bigvee\{FU \mid U \in \mathscr{O}_{\mathrm{aff}},\ \widetilde{U} \subset Y\}. \tag{39}$$

For any $V \in \mathrm{Ob}\,\mathscr{O}$, Proposition 10 gives $F'\widetilde{V} = \bigvee\{FU \mid U \in \mathscr{O}_{\mathrm{aff}},\ \widetilde{U} \subset \widetilde{V}\} = \bigvee\{FU \mid V \geqslant U \in \mathscr{O}_{\mathrm{aff}}\} = FV$. Choose a system of short monomorphisms $i_V\colon V \to FX$ for any $V \in \mathrm{Ob}\,\mathscr{O}'$ such that $i_{FU} = Fi_U^X$ holds for $U \in \mathrm{Ob}\,\mathscr{O}$. So there is a unique set of short monomorphisms $i_V^W\colon V \to W$ in $\mathscr{O}'$ which satisfy (14) and $i_{FU}^{FV} = F(i_U^V)$ for $U \leqslant V$ in $\mathscr{O}$. Every morphism $f \in \widetilde{\mathscr{O}}(Y)$ restricts to a system of endomorphisms $f|_{\widetilde{U}} \in \mathscr{O}(\widetilde{U}) = \mathscr{O}(U)$ with $U \in \mathrm{Ob}\,\mathscr{O}_{\mathrm{aff}}$ and $\widetilde{U} \subset Y$. By Theorem 2, $\mathscr{O}'$ can be regarded as a scheme such that the short monomorphisms $i_U^V \in \mathscr{O}'$ are to be viewed as inclusions $U \hookrightarrow V$. Therefore, the $F(f|_{\widetilde{U}}) \in \mathscr{O}(FU)$ admit a recollement $f' \in \mathscr{O}'(F'Y)$. For any inclusion $i\colon Y \subset Z$ in $\mathrm{Ob}\,\widetilde{\mathscr{O}}$, we set $F'(if) := i_{F'Y}^{F'Z} \cdot f'$. This gives an additive functor $F'\colon \widetilde{\mathscr{O}} \to \mathscr{O}'$ with $F'H = F$ which satisfies (F1).

Assume that $f \in \widetilde{\mathscr{O}}(Y)$. With $\mathscr{V} := \{U \in \mathrm{Ob}\,\mathscr{O}_{\mathrm{aff}} \mid \widetilde{U} \subset Y\}$, we have $F'S(f) = \bigvee\{FU \mid U \in \mathscr{V},\ f|_{\widetilde{U}} \in \mathscr{O}(U)^\times\} = \bigvee\{FS(f|_{\widetilde{U}}) \mid U \in \mathscr{V}\}$. Since $FS(f|_{\widetilde{U}}) = S(F(f|_{\widetilde{U}})) = S(F'f|_{FU})$, we get

$$F'S(f) = \bigvee\{S(F'f|_{FU}) \mid U \in \mathscr{V}\} = \bigvee_{U \in \mathscr{V}} \big(S(F'f) \wedge FU\big) = S(F'f).$$

Thus F' satisfies (F2). Furthermore, Eq. (39) implies that (F3) and (F4) for F carry over to F'. Hence F' is objective and unique. $\square$

5 Quasi-coherent sheaves

Let X be a scheme with structure sheaf $\mathscr{O}_X$, and let $\mathscr{O}$ be the corresponding objective category. By the remark of section 4, the scheme X is affine if and only if the largest object X of $\mathscr{O}$ is affine. For an object Y of $\mathscr{O}$, we denote the full subcategory of objects $Z \leqslant Y$ by $\mathscr{O}|_Y$. Thus $\mathscr{O}|_Y$ is affine if and only if $Y \in \operatorname{Ob} \mathscr{O}_{\mathrm{aff}}$.

A presheaf of $\mathscr{O}_X$-modules is just an object of $\mathbf{Mod}(\mathscr{O})$, i. e. an additive functor $M\colon \mathscr{O}^{\mathrm{op}} \to \mathbf{Ab}$. In fact, if $U \in \operatorname{Ob} \mathscr{O}$, then the ring homomorphism $\mathscr{O}(U) \to \operatorname{End}(M(U))$ makes $M(U)$ into an $\mathscr{O}(U)$-module, and for $V \leqslant U$ in $\operatorname{Ob} \mathscr{O}$, the restriction $M(i_V^U)\colon M(U) \to M(V)$ is $\mathscr{O}(U)$-linear if $M(V)$ is regarded as an $\mathscr{O}(U)$-module via $\rho_V^U\colon \mathscr{O}(U) \to \mathscr{O}(V)$. Furthermore, any pair of $\mathscr{O}$-modules M, N has a tensor product $M \otimes_{\mathscr{O}} N$ given by $(M \otimes_{\mathscr{O}} N)(U) := M(U) \otimes_{\mathscr{O}(U)} N(U)$ for all $U \in \operatorname{Ob} \mathscr{O}$ and the obvious restrictions. In the sequel, we write $\operatorname{Hom}_{\mathscr{O}}(M, N)$ instead of $\operatorname{Hom}_{\mathbf{Mod}(\mathscr{O})}(M, N)$.

Proposition 11. *Let X be a scheme with structure sheaf $\mathscr{O}_X$, and let $\mathscr{O}$ be the corresponding objective category. Up to isomorphism, there is a natural bijection between quasi-coherent sheaves on X and quasi-coherent $\mathscr{O}_{\mathrm{aff}}$-modules (see Definition 3).*

Proof. Every quasi-coherent sheaf on X restricts to an $\mathscr{O}_{\mathrm{aff}}$-module M. For an endomorphism $f\colon U \to U$ in $\mathscr{O}_{\mathrm{aff}}$, Proposition 5 implies that $\mathscr{O}(S(f)) \cong \mathscr{O}(U)_f$. As an $\mathscr{O}(U)$-module, $\mathscr{O}(U)_f \cong \mathscr{O}(U)[t]/(1 - ft)$ is the direct limit of the diagram

$$\cdots \to \mathscr{O}(U) \xrightarrow{f} \mathscr{O}(U) \xrightarrow{f} \mathscr{O}(U) \to \cdots$$

Hence $\varinjlim(M \circ f^{\mathbb{Z}}) \cong \mathscr{O}(U)_f \otimes_{\mathscr{O}(U)} M(U) \cong M(S(f))$.

Conversely, let $M \in \mathbf{Mod}(\mathscr{O}_{\text{aff}})$ be quasi-coherent. To show that M defines a sheaf of $\mathscr{O}_X$-modules via (31), we use [5], 0.3.2.2. By [5], I, Theorem 1.4.1, the conditions (L1) and (L2) after Definition 3 imply that for any $U \in \operatorname{Ob} \mathscr{O}_{\text{aff}}$, the restriction of M to $\mathbf{Mod}(\mathscr{O}_{\text{aff}}|_U)$ coincides with the associated sheaf of an $\mathscr{O}(U)$-module. Hence M defines a sheaf of $\mathscr{O}_X$-modules which is quasi-coherent by [5], I, Proposition 2.2.1. $\square$

Proposition 11 shows that the category of quasi-coherent sheaves on X can be identified with the full subcategory $\mathbf{Qcoh}(\mathscr{O}) \subset \mathbf{Mod}(\mathscr{O}_{\text{aff}})$ of quasi-coherent $\mathscr{O}_{\text{aff}}$-modules. Since direct limits in $\mathbf{Mod}(\mathscr{O}_{\text{aff}})$ are exact, the full subcategory $\mathbf{Qcoh}(\mathscr{O})$ is closed under kernels and colimits (cf. [5], I, Corollary 2.2.2). Furthermore, $\mathbf{Qcoh}(\mathscr{O})$ is closed with respect to the tensor product, and the greatest object X of $\mathscr{O}$ (which corresponds to the structure sheaf $\mathscr{O}_X$) belongs to $\mathbf{Qcoh}(\mathscr{O})$. Hence $\mathbf{Qcoh}(\mathscr{O})$ is a cocomplete abelian tensor category.

For $M \in \mathbf{Qcoh}(\mathscr{O})$ and $x \in \operatorname{Spec} \mathscr{O}$, the localization

$$M_x := \varinjlim_{U \in x} M(U) \tag{40}$$

can be regarded as an object of $\mathbf{Qcoh}(\mathscr{O})$, given by the skyscraper sheaf

$$M_x(U) := \begin{cases} M_x & \text{for } x \in \widetilde{U} \\ 0 & \text{for } x \notin \widetilde{U}. \end{cases} \tag{41}$$

Moreover, there is a natural morphism $M \to M_x$ in $\mathbf{Qcoh}(\mathscr{O})$.

Now we briefly discuss how to recover an objective category $\mathscr{O}$ from the abelian category $\mathbf{Qcoh}(\mathscr{O})$. Our method is more explicit than the reconstruction of Rosenberg [11] who considered various non-commutative generalizations [11, 12]. Recall that an object Q of an abelian category is said to be *quasi-injective* [10] if for morphisms $f, i\colon A \to Q$ with i monic there is an endomorphism e of Q with $f = ei$.

Definition 8. Let $\mathscr{A}$ be an abelian category. We call an object P of $\mathscr{A}$ a *point* of $\mathscr{A}$ if P is quasi-injective, $\operatorname{End}_{\mathscr{A}}(P)$ is a field, and every subobject $A \neq 0$ of P generates P. By $\operatorname{Spec} \mathscr{A}$ we denote a skeleton of the full subcategory of points.

For a non-commutative generalization in the affine case, see [2].

Proposition 12. *Let $\mathscr{O}$ be an objective category. There is a natural bijection $\kappa\colon \operatorname{Spec}\mathscr{O} \xrightarrow{\sim} \operatorname{Spec}\mathbf{Qcoh}(\mathscr{O})$.*

Proof. Let X denote the corresponding scheme with structure sheaf $\mathscr{O}_X$, and let P be a point of $\mathbf{Qcoh}(\mathscr{O})$. Choose an affine $U \in \operatorname{Ob}\mathscr{O}$ with $P(U) \neq 0$. By Proposition 9, there exists a point $x \in \widetilde{U}$ with $P_x \neq 0$. So we have an exact sequence

$$0 \to P' \to P \to P_x$$

in $\mathbf{Qcoh}(\mathscr{O})$. Since P' is invariant under $\operatorname{End}_{\mathscr{O}}(P)$ and $P' \neq P$, we have $P' = 0$. Hence $P \cong P_x$. Thus P can be regarded as a module over the local ring $\mathscr{O}_{X,x} = \varinjlim_{x\in\widetilde{U}} \mathscr{O}(U)$. For any $f \in \mathscr{O}_{X,x}$, the submodule $fP \subset P$ is fully invariant. Hence $fP = P$ or $fP = 0$. Therefore, the annihilator $\mathfrak{p} := \operatorname{Ann}(P) \subset \mathscr{O}_{X,x}$ is prime. If $fP = P$, then f is invertible on P since every non-zero submodule of P generates P. So we can assume that $\mathfrak{p} = \operatorname{Rad}\mathscr{O}_{X,x}$, and P is a vector space over the residue field $\kappa(x)$ of $\mathscr{O}_{X,x}$. Since $\operatorname{End}_{\mathscr{O}}(P)$ is a field, P must be one-dimensional over $\kappa(x)$. Conversely, the skyscraper sheaf with stalk $\kappa(x)$ at x is a point in $\mathbf{Qcoh}(\mathscr{O})$. So we get a bijection $\operatorname{Spec}\mathscr{O} \xrightarrow{\sim} \operatorname{Spec}\mathbf{Qcoh}(\mathscr{O})$. □

Let $\mathscr{A}$ be an abelian category. For a subset $U \subset \operatorname{Spec}\mathscr{A}$, let $\mathscr{T}_U$ denote the full subcategory of $\mathscr{A}$ consisting of the objects X such that $\operatorname{Hom}_{\mathscr{A}}(Y,P) = 0$ for all subobjects Y of X and $P \in U$. Thus $\mathscr{T}_U$ is a Serre subcategory of $\mathscr{A}$. In particular, we write $\mathscr{T}_P := \mathscr{T}_{\{P\}}$ for any $P \in \operatorname{Spec}\mathscr{A}$.

Definition 9. Let $\mathscr{A}$ be an abelian category. For an object X of $\mathscr{A}$ and $P \in \operatorname{Spec}\mathscr{A}$, we call a set F of morphisms $f\colon Y \to X$ *P-epic* if every epimorphism $g\colon X \to Z$ with $gf = 0$ for all $f \in F$ satisfies $Z \in \mathscr{T}_P$. We say that $X \in \operatorname{Ob}\mathscr{A}$ is of *finite type* if for every $P \in \operatorname{Spec}\mathscr{A}$, any P-epic set F of morphisms $Y \to X$ has a finite P-epic subset.

For $X \in \operatorname{Ob}\mathscr{A}$ of finite type, we define a subset $U_X \subset \operatorname{Spec}\mathscr{A}$ by

$$U_X := \{P \in \operatorname{Spec}\mathscr{A} \mid X \in \mathscr{T}_P\}. \tag{42}$$

If $X, Y \in \mathrm{Ob}\,\mathscr{A}$ are of finite type, then $X \oplus Y$ is of finite type, and

$$U_X \cap U_Y = U_{X \oplus Y}. \tag{43}$$

Therefore, with respect to inclusion, the U_X form a partially ordered set

$$\Omega_{\mathscr{A}} := \{U_X \mid X \in \mathrm{Ob}\,\mathscr{A} \text{ of finite type}\}. \tag{44}$$

which is a basis of open sets for a topology on $\mathrm{Spec}\,\mathscr{A}$. We endow $\mathrm{Spec}\,\mathscr{A}$ with this topology. In particular, $\Omega_{\mathscr{A}}$ has a greatest element $U_0 = \mathrm{Spec}\,\mathscr{A}$.

Recall that the *center* $Z(\mathscr{C})$ of a preadditive category $\mathscr{C}$ is the ring of natural endomorphisms of the identity functor $1\colon \mathscr{C} \to \mathscr{C}$. If $\mathscr{C}$ is small, then $Z(\mathscr{C}) \in \mathbf{CRi}$. We define

$$\mathscr{O}_{\mathscr{A}}(U) := Z(\mathscr{A}/\mathscr{T}_U) \tag{45}$$

for any $U \in \Omega_{\mathscr{A}}$. If $U \subset V$ holds in $\Omega_{\mathscr{A}}$, then $\mathscr{T}_V \subset \mathscr{T}_U$, which induces an additive functor $\mathscr{A}/\mathscr{T}_V \to \mathscr{A}/\mathscr{T}_U$, and thus a ring homomorphism $\rho^V_U\colon \mathscr{O}_{\mathscr{A}}(V) \to \mathscr{O}_{\mathscr{A}}(U)$. So we get a functor

$$\rho\colon \Omega^{\mathrm{op}}_{\mathscr{A}} \to \mathbf{CRi}. \tag{46}$$

If $\rho(U) = 0$, then $\mathscr{A}/\mathscr{T}_U = 0$, which implies that $U = \varnothing$. Hence by the corollary of Proposition 3, the functor (46) defines a pre-objective category $\mathscr{O}_{\mathscr{A}} \in \mathbf{POb}$.

The following theorem shows that an objective category and its corresponding scheme (cf. [11]) can be recovered from the category $\mathbf{Qcoh}(\mathscr{O})$.

Theorem 4. *Every objective category $\mathscr{O}$ is isomorphic to $\mathscr{O}_{\mathbf{Qcoh}(\mathscr{O})}$.*

Proof. We set $\mathscr{A} := \mathbf{Qcoh}(\mathscr{O})$ and $X := \mathrm{Spec}\,\mathscr{O}$. For a point $P = \kappa(x)$ of $\mathscr{A}$, the Serre subcategory $\mathscr{T}_P$ consists of the quasi-coherent sheaves M with $M_x = 0$. Therefore, $E \in \mathrm{Ob}\,\mathscr{A}$ is of finite type in the sense of Definition 9 if and only if E is of finite type as a quasi-coherent sheaf (see [5], 0.5.2). By [5], Chap. 0, Proposition 5.2.2, such an E has a closed support, i. e. $U_E \subset \operatorname{Spec} \mathscr{A}$ corresponds to an open set $\kappa^{-1}(U_E) \subset X$. This shows that the map κ of Proposition 12 is continuous.

Conversely, by Proposition 10, every open set in X is of the form $\widetilde{U}$ for some $U \in \operatorname{Ob} \mathscr{O}$. Let X_U be the corresponding quasi-coherent ideal of $\mathscr{O}_X$ which annihilates $X \smallsetminus \widetilde{U}$. Then there is a short exact sequence $0 \to X_U \to X \to E \to 0$ in $\mathbf{Qcoh}(\mathscr{O})$ with E of finite type and $U_E = \kappa(\widetilde{U})$. This shows that κ is a homeomorphism. Furthermore, $\mathscr{A}/\mathscr{T}_U \approx \mathbf{Qcoh}(\mathscr{O}|_U)$, whence $\mathscr{O}_{\mathscr{A}}(U) = \mathscr{O}(U)$. This proves that $\mathscr{O}_{\mathscr{A}} \cong \mathscr{O}$. □

Note: The preceding proof shows that $\Omega_{\mathscr{A}}$ is not only a basis, but the totality of all open sets of $\operatorname{Spec} \mathscr{A}$.

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