

## BIPULLBACKS AND CALCULUS OF FRACTIONS

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*Dedicated to Francis Borceux on the occasion of his 60th  
birthday*

RÉSUMÉ. Nous démontrons que la classe des équivalences faibles entre groupoïdes internes dans une catégorie régulière protomodulaire est une “congruence à biproduits fibrés” et, par conséquent, elle admet un calcul à droite des fractions. Comme application, nous montrons que les foncteurs monoïdaux entre groupoïdes internes dans les groupes et les homomorphismes entre 2-algèbres de Lie strictes sont les fractions des foncteurs internes par rapport aux équivalences faibles.

RÉSUMÉ. We prove that the class of weak equivalences between internal groupoids in a regular protomodular category is a bipullback congruence and, therefore, has a right calculus of fractions. As an application, we show that monoidal functors between internal groupoids in groups and homomorphisms of strict Lie 2-algebras are fractions of internal functors with respect to weak equivalences.

### 1. Introduction

It is well known that any monoidal category is monoidally equivalent to a strict one. This is not true for strong monoidal functors: not every strong monoidal functor is naturally isomorphic to a strict one (i.e., to a functor  $F$  such that the structural isomorphisms  $FA \otimes FB \rightarrow F(A \otimes B)$  and  $I \rightarrow FI$  are identities). An important example of this fact is given by Schreier theory of group extensions. In fact, let  $A$  and  $B$  be groups and write  $D(A)$  for  $A$  seen as a discrete internal groupoid in the category  $Grp$  of groups, and  $OUT(B)$  for the internal groupoid in  $Grp$  corresponding to the crossed module  $B \rightarrow Aut(B)$  of inner automorphisms.

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Then internal (= strict) functors from  $D(A)$  to  $OUT(B)$  correspond to split extensions of  $A$  through  $B$ , whereas monoidal functors from  $D(A)$  to  $OUT(B)$  correspond to arbitrary extensions of  $A$  by  $B$ .

The previous example leads to the following question: what is the precise relation between the 2-category of internal groupoids and internal functors in  $Grp$  and the 2-category of internal groupoids in  $Grp$  and monoidal functors? The same question can be asked working internally to the category  $Lie$  of Lie  $K$ -algebras (for  $K$  a fixed field), replacing monoidal functors by homomorphisms of strict Lie 2-algebras (precise definitions are in Section 7).

A possible answer to the previous questions is suggested by the fact that if  $F: \mathbb{C} \rightarrow \mathbb{D}$  is an internal functor in  $Grp$  which is a weak equivalence (i.e., full, faithful and essentially surjective on objects) then the quasi-inverse functor  $F^{-1}: \mathbb{D} \rightarrow \mathbb{C}$  is no longer an internal functor, but it is still a monoidal one. More precisely, we prove that:

1. The 2-category of internal groupoids in  $Grp$  and monoidal functors is the 2-category of fractions of the 2-category of internal groupoids and internal functors in  $Grp$  with respect to weak equivalences.
2. The 2-category of internal groupoids in  $Lie$  and homomorphisms is the 2-category of fractions of the 2-category of internal groupoids and internal functors in  $Lie$  with respect to weak equivalences.

The paper is organized as follows:

- In Section 2 we recall some basic facts on bicategories of fractions established by D. Pronk in [16]. We then revisit the right calculus of fractions for classes of 1-cells using bipullbacks.
- In Section 3 we show that, for a category  $\mathcal{C}$  with finite limits, the 2-category  $Grpd(\mathcal{C})$  of internal groupoids and internal functors has bipullbacks. More precisely, we show that the standard homotopy pullback in  $Grpd(\mathcal{C})$  also satisfies the universal property of a bipullback.
- Using bipullbacks, we show in Section 4 that if  $\mathcal{C}$  is regular, then the class of weak equivalences in  $Grpd(\mathcal{C})$  has a right calculus of fractions.

- In Section 5 we refine the previous result showing that if  $\mathcal{C}$  is regular and protomodular, then weak equivalences satisfy the “2  $\Rightarrow$  3” property and therefore they are a bipullback congruence, a notion inspired by Bénabou’s approach to categories of fractions (see [4]).
- In the last two sections we choose as base category  $\mathcal{C}$  the category of groups (Section 6) and the category of Lie  $K$ -algebras (Section 7) and we prove the results announced above.

Since *Grp* and *Lie* are Mal’cev categories, internal categories coincide with internal groupoids (see [11]). This is the reason why we restrict our attention to internal groupoids.

Let me finish with some comments. The result established in Section 6 is not at all a surprise. In fact, if we work with isomorphism classes of internal functors, then Proposition 6.4 becomes a result on categories of fractions (not on 2-categories of fractions) quite easy to prove directly and also easy to deduce using the Quillen model structures studied in [13] and in [15]. So, in my opinion, what is interesting is not the result *per se* but the fact that the 2-categorical nature of its proof requires the use of bipullbacks, whereas other kinds of 2-dimensional limits (like homotopy pullbacks) are not convenient in this context (see the Introduction in [4] for some comments on bilimits). Concerning the analogous result for Lie algebras stated in Section 7, I think it is interesting for a completely different reason. The notion of monoidal functor is a well-established one, whereas the notion of homomorphism of Lie 2-algebras is much more recent, so Proposition 7.4 could help to understand the 2-dimensional theory of Lie algebras.

*Notation:* the composite of  $f: A \rightarrow B$  and  $g: B \rightarrow C$  is written  $f \cdot g$  or  $fg$ .

*Terminology:* bicategory means bicategory with invertible 2-cells.

## 2. Bicategories of fractions

**2.1** Categories of fractions have been introduced by P. Gabriel and M. Zisman in [14] (see also Ch. 5 in [5]). If  $\mathcal{C}$  is a category and  $\Sigma$  a class of arrows in  $\mathcal{C}$ , the category of fractions of  $\mathcal{C}$  with respect to  $\Sigma$  is a functor

$$P_{\Sigma}: \mathcal{C} \rightarrow \mathcal{C}[\Sigma^{-1}]$$

universal among all functors  $\mathcal{F}: \mathcal{C} \rightarrow \mathcal{A}$  such that  $\mathcal{F}(s)$  is an isomorphism for all  $s \in \Sigma$ . This can be restated saying that for every category  $\mathcal{A}$

$$P_\Sigma \cdot - : \text{Funct}(\mathcal{C}[\Sigma^{-1}], \mathcal{A}) \rightarrow \text{Funct}_\Sigma(\mathcal{C}, \mathcal{A})$$

is an equivalence of categories, where  $\text{Funct}_\Sigma(\mathcal{C}, \mathcal{A})$  is the category of functors making the elements of  $\Sigma$  invertible. If the class  $\Sigma$  has a right calculus of fractions, then  $\mathcal{C}[\Sigma^{-1}]$  has a quite simple description:

**Proposition 2.2** (*Gabriel-Zisman*) *Assume that  $\Sigma$  satisfies the following conditions:*

*CF1.  $\Sigma$  contains all identities;*

*CF2.  $\Sigma$  is closed under composition;*

*CF3. For every pair  $f: A \rightarrow B \leftarrow C: g$  with  $g \in \Sigma$  there exist  $g': P \rightarrow A$  and  $f': P \rightarrow C$  such that  $g' \cdot f = f' \cdot g$  and  $g' \in \Sigma$ ;*

*CF4. If a pair of parallel arrows is coequalized by an element of  $\Sigma$ , then it is also equalized by an element of  $\Sigma$ .*

*Then the objects of  $\mathcal{C}[\Sigma^{-1}]$  are those of  $\mathcal{C}$  and an arrow from  $A$  to  $B$  is  $\mathcal{C}[\Sigma^{-1}]$  is a class of spans*

$$A \xleftarrow{s} I \xrightarrow{f} B$$

*with  $s \in \Sigma$ . Two spans  $(s, I, f)$  and  $(s', I', f')$  are equivalent if there exist arrows  $x, x'$  in  $\mathcal{C}$  such that  $x \cdot s = x' \cdot s' \in \Sigma$  and  $x \cdot f = x' \cdot f'$ .*

The analogous problem for bicategories has been solved by D. Pronk in [16]. For an introduction to bicategories see [3] or Ch. 7 in [5] where 2-categories are also discussed.

**Definition 2.3** (*Pronk*) Let  $\mathcal{B}$  be a bicategory and  $\Sigma$  a class of 1-cells in  $\mathcal{B}$ . The bicategory of fractions of  $\mathcal{B}$  with respect to  $\Sigma$  is a homomorphism of bicategories

$$P_\Sigma: \mathcal{B} \rightarrow \mathcal{B}[\Sigma^{-1}]$$

universal among all homomorphisms  $\mathcal{F}: \mathcal{B} \rightarrow \mathcal{A}$  such that  $\mathcal{F}(S)$  is an equivalence for all  $S \in \Sigma$ . This can be restated saying that for every bicategory  $\mathcal{A}$

$$P_\Sigma \cdot - : \text{Hom}(\mathcal{B}[\Sigma^{-1}], \mathcal{A}) \rightarrow \text{Hom}_\Sigma(\mathcal{B}, \mathcal{A})$$

is a biequivalence of bicategories, where  $\text{Hom}_\Sigma(\mathcal{B}, \mathcal{A})$  is the bicategory of those homomorphisms  $\mathcal{F}$  such that  $\mathcal{F}(S)$  is an equivalence for all  $S \in \Sigma$ .

**Definition 2.4** (*Pronk*) Let  $\mathcal{B}$  be a bicategory and  $\Sigma$  a class of 1-cells in  $\mathcal{B}$ . The class  $\Sigma$  has a right calculus of fractions if the following conditions hold:

- BF1.  $\Sigma$  contains all equivalences;
- BF2.  $\Sigma$  is closed under composition;
- BF3. For every pair  $F: \mathbb{A} \rightarrow \mathbb{B} \leftarrow \mathbb{C}: G$  with  $G \in \Sigma$  there exist  $G': \mathbb{P} \rightarrow \mathbb{A}$ ,  $F': \mathbb{P} \rightarrow \mathbb{C}$  and  $\varphi: G' \cdot F \Rightarrow F' \cdot G$  with  $G' \in \Sigma$ ;
- BF4. For every  $\alpha: F \cdot W \Rightarrow G \cdot W$  with  $W \in \Sigma$  there exist  $V \in \Sigma$  and  $\beta: V \cdot F \Rightarrow V \cdot G$  such that  $V \cdot \alpha = \beta \cdot W$ , and for any other  $V' \in \Sigma$  and  $\beta': V' \cdot F \Rightarrow V' \cdot G$  such that  $V' \cdot \alpha = \beta' \cdot W$  there exist  $U, U'$  and  $\varepsilon: U \cdot V \Rightarrow U' \cdot V'$  such that  $U \cdot V \in \Sigma$  and

$$\begin{array}{ccc} U \cdot V \cdot F & \xrightarrow{U \cdot \beta} & U \cdot V \cdot G \\ \varepsilon \cdot F \downarrow & & \downarrow \varepsilon \cdot G \\ U' \cdot V' \cdot F & \xrightarrow{U' \cdot \beta'} & U' \cdot V' \cdot G \end{array}$$

commutes;

- BF5. If  $\alpha: F \Rightarrow G$  is a 2-cell, then  $F \in \Sigma$  if and only if  $G \in \Sigma$ .

If the class  $\Sigma$  has a right calculus of fractions, the bicategory  $\mathcal{B}[\Sigma^{-1}]$  can be described in a way similar to that recalled in Proposition 2.2. Here we do not give full details because what we will use in Sections 6 and 7 is the following useful result:

**Proposition 2.5** (*Pronk*) *Let  $\mathcal{B}$  be a bicategory and  $\Sigma$  a class of 1-cells in  $\mathcal{B}$  which has a right calculus of fractions. Consider a bifunctor  $\mathcal{F}: \mathcal{B} \rightarrow \mathcal{A}$  such that  $\mathcal{F}(S)$  is an equivalence for all  $S \in \Sigma$  and let  $\widehat{\mathcal{F}}: \mathcal{B}[\Sigma^{-1}] \rightarrow \mathcal{A}$  be its extension. Then  $\widehat{\mathcal{F}}$  is a biequivalence provided that  $\mathcal{F}$  satisfies the following conditions:*

*EF1.  $\mathcal{F}$  is surjective up to equivalence on objects;*

*EF2.  $\mathcal{F}$  is full and faithful on 2-cells;*

*EF3. For every 1-cell  $F$  in  $\mathcal{A}$  there exist 1-cells  $G$  and  $W$  in  $\mathcal{B}$  with  $W$  in  $\Sigma$  and a 2-cell  $\mathcal{F}(G) \Rightarrow \mathcal{F}(W) \cdot F$ .*

(In [16] it is stated that conditions EF1-EF3 are also necessary for  $\widehat{\mathcal{F}}$  being a biequivalence. This is not true, as proved by M. Dupont in [12].)

**2.6** Recall that a diagram

$$\begin{array}{ccc} \mathbb{P} & \xrightarrow{F'} & \mathbb{C} \\ G' \downarrow & \nearrow \varphi & \downarrow G \\ \mathbb{A} & \xrightarrow{F} & \mathbb{B} \end{array}$$

in a bicategory  $\mathcal{B}$  is a bipullback of  $F$  and  $G$  if for any other diagram

$$\begin{array}{ccc} \mathbb{X} & \xrightarrow{K} & \mathbb{C} \\ H \downarrow & \nearrow \psi & \downarrow G \\ \mathbb{A} & \xrightarrow{F} & \mathbb{B} \end{array}$$

there exists a fill-in, that is a triple  $(L: \mathbb{X} \rightarrow \mathbb{P}, \alpha: L \cdot G' \Rightarrow H, \beta: L \cdot F' \Rightarrow K)$  such that

$$\begin{array}{ccc} L \cdot G' \cdot F & \xrightarrow{L \cdot \varphi} & L \cdot F' \cdot G \\ \alpha \cdot F \downarrow & & \downarrow \beta \cdot G \\ H \cdot F & \xrightarrow{\psi} & K \cdot G \end{array}$$

commutes, and for any other fill-in  $(L', \alpha', \beta')$  there exists a unique  $\lambda: L' \Rightarrow L$  such that

$$\begin{array}{ccc}
 L' \cdot G' & \xrightarrow{\lambda \cdot G'} & L \cdot G' \\
 \searrow \alpha' & & \swarrow \alpha \\
 & H & 
 \end{array}
 \qquad
 \begin{array}{ccc}
 L' \cdot F' & \xrightarrow{\lambda \cdot F'} & L \cdot F' \\
 \searrow \beta' & & \swarrow \beta \\
 & K & 
 \end{array}$$

commute.

**Remark 2.7** 1. Bipullbacks are determined uniquely up to equivalence.

2. A 1-cell  $W: \mathbb{B} \rightarrow \mathbb{A}$  is called full and faithful if for every  $\mathbb{X}$  the hom-functor

$$\mathcal{B}(\mathbb{X}, W): \mathcal{B}(\mathbb{X}, \mathbb{B}) \rightarrow \mathcal{B}(\mathbb{X}, \mathbb{A})$$

is full and faithful in the usual sense. Consider now the following diagrams, the first one being a bipullback,

$$\begin{array}{ccc}
 \mathbb{K} & \xrightarrow{W_2} & \mathbb{B} \\
 w_1 \downarrow & \nearrow w & \downarrow W \\
 \mathbb{B} & \xrightarrow{W} & \mathbb{A}
 \end{array}
 \qquad
 \begin{array}{ccc}
 \mathbb{B} & \xrightarrow{id} & \mathbb{B} \\
 id \downarrow & \nearrow W & \downarrow W \\
 \mathbb{B} & \xrightarrow{W} & \mathbb{A}
 \end{array}$$

Let  $(D_W: \mathbb{B} \rightarrow \mathbb{K}, \delta_1: D_W \cdot W_1 \Rightarrow id, \delta_2: D_W \cdot W_2 \Rightarrow id)$  be the fill-in of the second diagram through the first one. Then  $W$  is full and faithful iff the second diagram is a bipullback iff the diagonal  $D_W$  is an equivalence.

**Proposition 2.8** Let  $\mathcal{B}$  be a bicategory with bipullbacks and  $\Sigma$  a class of 1-cells in  $\mathcal{B}$ . Assume that  $\Sigma$  satisfies the following conditions:

BP1.  $\Sigma$  contains all equivalences;

BP2.  $\Sigma$  is closed under composition;

BP3.  $\Sigma$  is stable under bipullbacks;

BP4. If  $W$  is in  $\Sigma$ , then the diagonal  $D_W$  is in  $\Sigma$ ;

BP5. If  $\alpha: F \Rightarrow G$  is a 2-cell, then  $F \in \Sigma$  if and only if  $G \in \Sigma$ .

Then  $\Sigma$  has a right calculus of fractions.

Proof. Clearly BP3 implies BF3. We have to show that BF4 holds. Consider the following diagrams, the first one being a bipullback,

$$\begin{array}{ccc}
 \begin{array}{ccc} \mathbb{K} & \xrightarrow{W_2} & \mathbb{B} \\ W_1 \downarrow & \nearrow w & \downarrow W \\ \mathbb{B} & \xrightarrow{W} & \mathbb{A} \end{array} & 
 \begin{array}{ccc} \mathbb{B} & \xrightarrow{id} & \mathbb{B} \\ id \downarrow & \nearrow W & \downarrow W \\ \mathbb{B} & \xrightarrow{W} & \mathbb{A} \end{array} & 
 \begin{array}{ccc} \mathbb{C} & \xrightarrow{G} & \mathbb{B} \\ F \downarrow & \nearrow \alpha & \downarrow W \\ \mathbb{B} & \xrightarrow{W} & \mathbb{A} \end{array}
 \end{array}$$

Let  $(D_W: \mathbb{B} \rightarrow \mathbb{K}, \delta_1, \delta_2)$  be the fill-in of the second diagram through the first one, and  $(H: \mathbb{C} \rightarrow \mathbb{K}, \alpha_1, \alpha_2)$  the fill-in of the third diagram through the first one. Consider also the bipullback

$$\begin{array}{ccc} \mathbb{D} & \xrightarrow{V} & \mathbb{C} \\ L \downarrow & \nearrow \varphi & \downarrow H \\ \mathbb{B} & \xrightarrow{D_W} & \mathbb{K} \end{array}$$

and define  $\beta: V \cdot F \Rightarrow V \cdot G$  as follows

$$VF \xrightarrow{V\alpha_1^{-1}} VHW_1 \xrightarrow{\varphi^{-1}W_1} LD_WW_1 \xrightarrow{L\delta_1} L \xrightarrow{L\delta_2^{-1}} LD_WW_2 \xrightarrow{\varphi W_2} VHW_2 \xrightarrow{V\alpha_2} VG$$

Observe that since  $W \in \Sigma$ , then  $D_W \in \Sigma$  by BP4, and then  $V \in \Sigma$  by BP3. Moreover, the condition  $V \cdot \alpha = \beta \cdot W$  follows from the fill-in condition on  $(D_W, \delta_1, \delta_2)$  and  $(H, \alpha_1, \alpha_2)$ .

Let  $\beta': V' \cdot F \Rightarrow V' \cdot G$  be such that  $V' \in \Sigma$  and  $V' \cdot \alpha = \beta' \cdot W$ . We obtain two fill-in of

$$\begin{array}{ccc} \mathbb{D}' & \xrightarrow{V' \cdot F} & \mathbb{B} \\ V' \cdot F \downarrow & \nearrow V'FW & \downarrow W \\ \mathbb{B} & \xrightarrow{W} & \mathbb{A} \end{array}$$

through the bipullback  $(\mathbb{K}, W_1, W_2, w)$ : the first one is

$$(\mathbb{D}' \xrightarrow{V'} \mathbb{C} \xrightarrow{F} \mathbb{B} \xrightarrow{D_W} \mathbb{K}, V' \cdot F \cdot \delta_1, V' \cdot F \cdot \delta_2)$$

and the second one is

$$(\mathbb{D}' \xrightarrow{V'} \mathbb{C} \xrightarrow{H} \mathbb{K}, V' \cdot \alpha_1, V'HW_2 \xrightarrow{V'\alpha_2} V'G \xrightarrow{(\beta')^{-1}} V'F)$$



By the universal property of  $(\mathbb{K}, W_1, W_1, w)$ , there exists a unique  $\beta^* : V' \cdot F \cdot D_W \Rightarrow V' \cdot H$  such that

$$\begin{array}{ccc} V' \cdot F \cdot D_W \cdot W_1 & \xrightarrow{\beta^* \cdot W_1} & V' \cdot H \cdot W_1 \\ & \searrow V' \cdot F \cdot \delta_1 & \swarrow V' \cdot \alpha_1 \\ & & V' \cdot F \end{array}$$

and

$$\begin{array}{ccc} V' \cdot F \cdot D_W \cdot W_2 & \xrightarrow{\beta^* \cdot W_2} & V' \cdot H \cdot W_2 \\ V' \cdot F \cdot \delta_2 \downarrow & & \downarrow V' \cdot \alpha_2 \\ V' \cdot F & \xrightarrow{\beta'} & V' \cdot G \end{array}$$

commute. Let  $(U : \mathbb{D}' \rightarrow \mathbb{D}, \eta : U \cdot L \Rightarrow V' \cdot F, \varepsilon : U \cdot V \Rightarrow V')$  be the fill-in of

$$\begin{array}{ccc} \mathbb{D}' & \xrightarrow{V'} & \mathbb{C} \\ V' \cdot F \downarrow & \nearrow \beta^* & \downarrow H \\ \mathbb{B} & \xrightarrow{D_W} & \mathbb{K} \end{array}$$

through the bipullback  $(\mathbb{D}, L, V, \varphi)$ . If we choose  $U' = id$ , we have  $\varepsilon : U \cdot V \Rightarrow U' \cdot V'$ . Since  $V' \in \Sigma$ , then also  $U' \cdot V'$  and  $U \cdot V$  are in  $\Sigma$  because of BP1, BP2 and BP5. It remains to check the compatibility of  $\varepsilon, \beta$  and  $\beta'$  as in BF4, but this is just a diagram chasing.  $\blacksquare$

### 3. Bipullbacks in $Grpd(\mathcal{C})$

The aim of this section is to prove the following result:

**Proposition 3.1** *Let  $\mathcal{C}$  be a category with finite limits, and let  $Grpd(\mathcal{C})$  be the 2-category of internal groupoids, internal functors and internal natural transformations in  $\mathcal{C}$ . The 2-category  $Grpd(\mathcal{C})$  has bipullbacks.*

**3.2** Let us fix notation (details can be found in Ch. 7 of [5] or in Appendix 3 of [7]):

- An internal groupoid  $\mathbb{C}$  is represented by

$$C_1 \times_{c,d} C_1 \xrightarrow{m} C_1 \begin{array}{c} \xrightarrow{d} \\ \xleftarrow{e} \\ \xrightarrow{c} \end{array} C_0 \quad C_1 \xrightarrow{i} C_1$$

where the following diagram is a pullback

$$\begin{array}{ccc} C_1 \times_{c,d} C_1 & \xrightarrow{\pi_2} & C_1 \\ \pi_1 \downarrow & & \downarrow d \\ C_1 & \xrightarrow{c} & C_0 \end{array}$$

- An internal functor  $F: \mathbb{C} \rightarrow \mathbb{D}$  is represented by

$$\begin{array}{ccc} C_1 & \xrightarrow{F_1} & D_1 \\ d \downarrow \downarrow c & & d \downarrow \downarrow c \\ C_0 & \xrightarrow{F_0} & D_0 \end{array}$$

- An internal natural transformation  $\alpha: F \Rightarrow G: \mathbb{C} \rightarrow \mathbb{D}$  is represented by

$$\begin{array}{ccc} C_1 & \xrightarrow{F_1} & D_1 \\ d \downarrow \downarrow c & \nearrow \alpha & d \downarrow \downarrow c \\ C_0 & \xrightarrow{F_0} & D_0 \\ & & \xrightarrow{G_0} \end{array}$$

**3.3** It is helpful to start recalling that in  $Grpd(Set)$  bipullbacks are comma-squares. With the notations of 2.6:

- an object in  $\mathbb{P}$  is a triple  $(a_0 \in A_0, b_1: F_0(a_0) \rightarrow G_0(c_0), c_0 \in C_0)$ ,
- an arrow from  $(a_0, b_1, c_0)$  to  $(a'_0, b'_1, c'_0)$  is a pair of arrows  $(a_1: a_0 \rightarrow a'_0, c_1: c_0 \rightarrow c'_0)$  such that  $F_1(a_1) \cdot b'_1 = b_1 \cdot G_1(c_1)$ ,
- $G': \mathbb{P} \rightarrow \mathbb{A}$  and  $F': \mathbb{P} \rightarrow \mathbb{C}$  are the obvious projections, and  $\varphi(a_0, b_1, c_0) = b_1$ ,

- $L_0(x_0) = (H_0(x_0), \psi(x_0), K_0(x_0)), L_1(x_1) = (H_1(x_1), K_1(x_1)), \alpha = id$  and  $\beta = id$ ,
- $\lambda(x_0) = (\alpha'(x_0), \beta'(x_0))$ .

**3.4** The description of bipullbacks in  $Grpd(Set)$  recalled in 3.3 indicates that the first step to obtain bipullbacks in  $Grpd(\mathcal{C})$  is to construct from an internal groupoid  $\mathbb{B}$  a new internal groupoid  $\vec{\mathbb{B}}$  whose objects are arrows in  $\mathbb{B}$  and whose arrows are commutative squares in  $\mathbb{B}$ . The construction of  $\vec{\mathbb{B}}$  is quite standard:

$$\vec{\mathbb{B}} = \left( \vec{B}_1 \times_{\vec{c}, \vec{d}} \vec{B}_1 \xrightarrow{\vec{m}} \vec{B}_1 \begin{array}{c} \xrightarrow{\vec{d}} \\ \xleftarrow{\vec{c}} \end{array} B_1 \quad \vec{B}_1 \xrightarrow{\vec{i}} \vec{B}_1 \right)$$

- $\vec{B}_1$  is defined by the following pullback

$$\begin{array}{ccc} \vec{B}_1 & \xrightarrow{m_2} & B_1 \times_{c,d} B_1 \\ m_1 \downarrow & & \downarrow m \\ B_1 \times_{c,d} B_1 & \xrightarrow{m} & B_1 \end{array}$$

- $\vec{d} = m_1 \cdot \pi_1$  and  $\vec{c} = m_2 \cdot \pi_2$ ,
- $\vec{e}$  is the unique factorization through  $\vec{B}_1$  of the following commutative diagram

$$\begin{array}{ccccc} B_1 & \xrightarrow{\langle d, 1 \rangle} & B_0 \times B_1 & \xrightarrow{e \times 1} & B_1 \times_{c,d} B_1 \\ \langle 1, c \rangle \downarrow & & & & \downarrow m \\ B_1 \times B_0 & \xrightarrow{1 \times e} & B_1 \times_{c,d} B_1 & \xrightarrow{m} & B_1 \end{array}$$

- we leave to the reader the task of describing  $\vec{m}$  and  $\vec{i}$ .

**3.5** The internal groupoid  $\vec{\mathbb{B}}$  is equipped with two internal functors  $\delta, \gamma: \vec{\mathbb{B}} \rightarrow \mathbb{B}$  specified by

$$\begin{array}{ccc} \vec{B}_1 & \xrightarrow{\delta_1 = m_2 \cdot \pi_1} & B_1 \\ \vec{d} \downarrow \vec{c} & & \downarrow c \\ B_1 & \xrightarrow{\delta_0 = d} & B_0 \end{array} \quad \begin{array}{ccc} \vec{B}_1 & \xrightarrow{\gamma_1 = m_1 \cdot \pi_2} & B_1 \\ \vec{d} \downarrow \vec{c} & & \downarrow c \\ B_1 & \xrightarrow{\gamma_0 = c} & B_0 \end{array}$$

and it turns out that to give an internal natural transformation  $\alpha: F \Rightarrow G: \mathbb{A} \rightarrow \mathbb{B}$  is the same as giving an internal functor  $\alpha: \mathbb{A} \rightarrow \vec{\mathbb{B}}$  such that  $\alpha \cdot \delta = F$  and  $\alpha \cdot \gamma = G$ . Indeed, the internal functor  $\alpha$  is specified by

$$\begin{array}{ccc} A_1 & \xrightarrow{\alpha_1} & \vec{B}_1 \\ d \downarrow \downarrow c & & \vec{d} \downarrow \downarrow \vec{c} \\ A_0 & \xrightarrow{\alpha} & B_1 \end{array}$$

where  $\alpha_1$  is the unique factorization through  $\vec{B}_1$  of the following commutative diagram

$$\begin{array}{ccccc} A_1 & \xrightarrow{\langle 1, c \rangle} & A_1 \times A_0 & \xrightarrow{F_1 \times \alpha} & B_1 \times_{c,d} B_1 \\ \langle d, 1 \rangle \downarrow & & & & \downarrow m \\ A_0 \times A_1 & \xrightarrow{\alpha \times G_1} & B_1 \times_{c,d} B_1 & \xrightarrow{m} & B_1 \end{array}$$

**3.6** We are ready to prove Proposition 3.1. We use the notations of 2.6.

Proof. Given  $F: \mathbb{A} \rightarrow \mathbb{B}$  and  $G: \mathbb{C} \rightarrow \mathbb{B}$  in  $Grpd(\mathcal{C})$ , a bipullback

$$\begin{array}{ccc} \mathbb{P} & \xrightarrow{F'} & \mathbb{C} \\ G' \downarrow & \nearrow \varphi & \downarrow G \\ \mathbb{A} & \xrightarrow{F} & \mathbb{B} \end{array}$$

is given by the following limit in  $Grpd(\mathcal{C})$  (recall that  $Grpd(\mathcal{C})$  has limits computed componentwise in  $\mathcal{C}$ )

$$\begin{array}{ccccc} & & \mathbb{P} & & \\ & G' \swarrow & \downarrow \varphi & \searrow F' & \\ \mathbb{A} & & \vec{\mathbb{B}} & & \mathbb{C} \\ & F \searrow & \delta \swarrow & \gamma \swarrow & G \searrow \\ & & \mathbb{B} & & \mathbb{B} \end{array}$$

Indeed, any diagram

$$\begin{array}{ccc}
 \mathbb{X} & \xrightarrow{K} & \mathbb{C} \\
 H \downarrow & \nearrow \psi & \downarrow G \\
 \mathbb{A} & \xrightarrow{F} & \mathbb{B}
 \end{array}$$

produces a commutative diagram

$$\begin{array}{ccccc}
 & & \mathbb{X} & & \\
 & \swarrow H & \downarrow \psi & \searrow K & \\
 \mathbb{A} & & \mathbb{B} & & \mathbb{C} \\
 & \searrow F & \swarrow \delta & \searrow \gamma & \swarrow G \\
 & & \mathbb{B} & & \mathbb{B}
 \end{array}$$

so that following the universal property of  $\mathbb{P}$  as a limit there exists a unique  $L: \mathbb{X} \rightarrow \mathbb{P}$  such that  $L \cdot G' = H$ ,  $L \cdot F' = K$  and  $L \cdot \varphi = \psi$ . (In other words,  $(\mathbb{P}, G', F', \varphi)$  is the standard homotopy pullback of  $F$  and  $G$ .)

Clearly,  $(L, \alpha = id, \beta = id)$  is a fill-in of  $(\mathbb{X}, H, K, \psi)$  through  $(\mathbb{P}, G', F', \varphi)$ . Let  $(L', \alpha', \beta')$  be another fill-in of  $(\mathbb{X}, H, K, \psi)$  through  $(\mathbb{P}, G', F', \varphi)$ . We have to show that there exists a unique  $\lambda: L' \Rightarrow L$  such that  $\lambda \cdot G' = \alpha'$  and  $\lambda \cdot F' = \beta'$ . Define:

- $\tau_1$  to be the unique factorization through  $B_1 \times_{c,d} B_1$  of the following diagram

$$\begin{array}{ccccc}
 X_0 & \xrightarrow{\beta'} & C_1 & \xrightarrow{G_1} & B_1 \\
 L'_0 \downarrow & & & & \downarrow d \\
 P_0 & \xrightarrow{\varphi} & B_1 & \xrightarrow{c} & B_0
 \end{array}$$

- $\tau_2$  to be the unique factorization through  $B_1 \times_{c,d} B_1$  of the following diagram

$$\begin{array}{ccccc}
 X_0 & \xrightarrow{\psi} & B_1 & & \\
 \alpha' \downarrow & & & & \downarrow d \\
 A_1 & \xrightarrow{F_1} & B_1 & \xrightarrow{c} & B_0
 \end{array}$$

- $\tau$  to be the unique factorization through  $\vec{B}_1$  of the following diagram

$$\begin{array}{ccc} X_0 & \xrightarrow{\tau_2} & B_1 \times_{c,d} B_1 \\ \tau_1 \downarrow & & \downarrow m \\ B_1 \times_{c,d} B_1 & \xrightarrow{m} & B_1 \end{array}$$

Finally,  $\lambda$  is the unique factorization through  $P_1$  of the following diagram

$$\begin{array}{ccccc} & & X_0 & & \\ & \swarrow \alpha' & \downarrow \tau & \searrow \beta' & \\ A_1 & & \vec{B}_1 & & C_1 \\ & \searrow F_1 & \swarrow m_2 \cdot \pi_1 & \searrow m_1 \cdot \pi_2 & \swarrow G_1 \\ & & B_1 & & B_1 \end{array}$$

Clearly,  $\lambda \cdot G' = \alpha'$  and  $\lambda \cdot F' = \beta'$ . To check that  $\lambda \cdot d = L'_0$  and  $\lambda \cdot c = L_0$ , the naturality of  $\lambda$ , and its uniqueness is a diagram chasing using that  $\{G'_1, \varphi_1, F'_1\}$ ,  $\{m_1, m_2\}$  and  $\{\pi_1, \pi_2\}$  are jointly monomorphic. ■

#### 4. Weak equivalences in $Grpd(\mathcal{C})$

**Definition 4.1** (*Bunge-Paré*) Let  $F: \mathcal{C} \rightarrow \mathbb{B}$  be in  $Grpd(\mathcal{C})$ .

1.  $F$  is essentially surjective on objects if

$$C_0 \times_{F_0,d} D_1 \xrightarrow{t_2} D_1 \xrightarrow{c} D_0$$

is a regular epimorphism, where  $t_2$  is given by the following pull-back

$$\begin{array}{ccc} C_0 \times_{F_0,d} D_1 & \xrightarrow{t_2} & D_1 \\ t_1 \downarrow & & \downarrow d \\ C_0 & \xrightarrow{F_0} & D_0 \end{array}$$

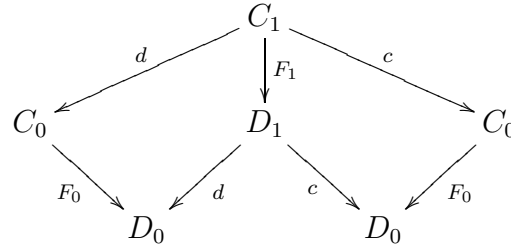
2.  $F$  is a weak equivalence if it is full and faithful (see 2.7) and essentially surjective on objects.

The previous definition is due to M. Bunge and R. Paré (see [10]). In [13] a more general notion of weak equivalence involving a Grothendieck topology on  $\mathcal{C}$  has been considered. Since in Sections 6 and 7 the base category  $\mathcal{C}$  is regular, I adopt for the moment the definition of Bunge and Paré. More on this point is contained in 5.10.

Next lemma is well-known and we only sketch the proof.

**Lemma 4.2** *Let  $F: \mathbb{C} \rightarrow \mathbb{D}$  be in  $\text{Grpd}(\mathcal{C})$ .*

1.  *$F$  is full and faithful if and only if the following is a limit diagram*



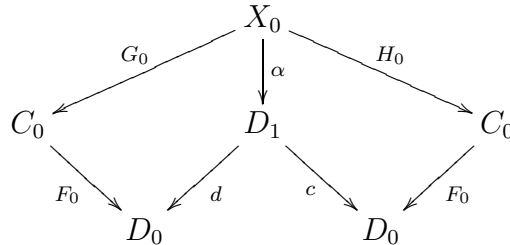
2.  *$F$  is an equivalence if and only if it is full and faithful and*

$$C_0 \times_{F_0, d} D_1 \xrightarrow{t_2} D_1 \xrightarrow{c} D_0$$

*is a split epimorphism.*

Proof. 1. If the diagram is a limit diagram and  $\alpha: G \cdot F \Rightarrow H \cdot F: \mathbb{X} \rightarrow \mathbb{D}$  is an internal natural transformation, then  $\alpha \cdot d = G_0 \cdot F_0$  and  $\alpha \cdot c = H_0 \cdot F_0$ . By the universal property of  $C_1$  we get a unique  $\beta: X_0 \rightarrow C_1$  such that  $\beta \cdot d = G_0$ ,  $\beta \cdot c = H_0$  and  $\beta \cdot F_1 = \alpha$ . So we have  $\beta: G \Rightarrow H$  such that  $\beta \cdot F = \alpha$ . (The naturality of  $\beta$  follows from that of  $\alpha$ .)

Conversely, any commutative diagram



gives rise to internal functors  $G, H: \mathbb{X} \rightarrow \mathbb{C}$  with discrete domain

$$\begin{array}{ccc}
 X_0 & \begin{array}{c} \xrightarrow{G_0 \cdot e} \\ \xrightarrow{H_0 \cdot e} \end{array} & C_1 \\
 \begin{array}{c} \downarrow 1 \\ \downarrow 1 \end{array} & & \begin{array}{c} \downarrow d \\ \downarrow c \end{array} \\
 X_0 & \begin{array}{c} \xrightarrow{G_0} \\ \xrightarrow{H_0} \end{array} & C_0
 \end{array}$$

and to an internal natural transformation  $\alpha: G \cdot F \Rightarrow H \cdot F$ . To give an internal natural transformation  $\beta: G \Rightarrow H$  such that  $\beta \cdot F = \alpha$  means precisely to give a factorization  $\beta: X_0 \rightarrow C_1$  of  $(G_0, \alpha, H_0)$  through  $(d, F_1, c)$ .

2. Let  $F$  be an equivalence and consider an internal natural transformation  $\beta: G \cdot F \Rightarrow Id_{\mathcal{D}}$ . Since  $\beta \cdot d = G_0 \cdot F_0$ , there exists a unique  $j: D_0 \rightarrow C_0 \times_{F_0, d} D_1$  such that  $j \cdot t_1 = G_0$  and  $j \cdot t_2 = \beta$ . Therefore  $j \cdot t_2 \cdot c = \beta \cdot c = id$ .

Conversely, if  $j: D_0 \rightarrow C_0 \times_{F_0, d} D_1$  such that  $j \cdot t_2 \cdot c = id$ , we can construct a quasi-inverse internal functor  $G: \mathbb{D} \rightarrow \mathbb{C}$  as follows: first define  $G_0$  by

$$G_0 = j \cdot t_1: D_0 \rightarrow C_0 \times_{F_0, d} D_1 \rightarrow C_0$$

Then, define  $j_1: D_1 \rightarrow D_1$  by

$$j_1 = \langle d \cdot j \cdot t_2, 1, c \cdot j \cdot t_2 \cdot i \rangle \cdot (m \times 1) \cdot m: D_1 \rightarrow D_1 \times_{c, d} D_1 \times_{c, d} D_1 \rightarrow D_1$$

Finally, since  $F$  is full and faithful, by the first part of the lemma we get a unique arrow  $G_1: D_1 \rightarrow C_1$  such that  $G_1 \cdot d = d \cdot G_0$ ,  $G_1 \cdot F_1 = j_1$  and  $G_1 \cdot c = c \cdot G_0$ .  $\blacksquare$

**Corollary 4.3** *Every equivalence in  $Grpd(\mathcal{C})$  is a weak equivalence. The converse is true provided that in  $\mathcal{C}$  the axiom of choice holds (i.e., regular epimorphisms split).*

**4.4** Regular categories have been introduced by M. Barr in [2] (see also Ch. 2 in [6]). In a regular category regular epimorphisms behave well: they are closed under composition and finite products, stable under pullbacks, and if a composite arrow  $f \cdot g$  is a regular epimorphism, then  $g$  is a regular epimorphism. It follows that if  $F: \mathbb{C} \rightarrow \mathbb{D}$  is in  $Grpd(\mathcal{C})$  with  $\mathcal{C}$  regular and if  $F_0$  is a regular epimorphism, then  $F$  is essentially surjective on objects.



**Proposition 4.5** *Let  $\mathcal{C}$  be a regular category and let  $\Sigma$  be the class of weak equivalences in  $Grpd(\mathcal{C})$ . Then  $\Sigma$  has a right calculus of fractions.*

Proof. Since by Proposition 3.1  $Grpd(\mathcal{C})$  has bipullbacks, to prove that  $\Sigma$  has a right calculus of fractions we check conditions BP1–BP5 in Proposition 2.8.

BP1 is given by Corollary 4.3, BP4 follows from 2.7 and BP5 is an exercise for the reader.

BP2: full and faithful internal functors are closed under composition because so they are in  $Grpd(Set)$ . Assume now that  $F: \mathbb{A} \rightarrow \mathbb{B}$  and  $G: \mathbb{B} \rightarrow \mathbb{C}$  are essentially surjective. Consider the following pullbacks

$$\begin{array}{ccc}
 A_0 \times_{F_0, d} B_1 \xrightarrow{t_2} B_1 & B_0 \times_{G_0, d} C_1 \xrightarrow{t_2} C_1 & A_0 \times_{F_0 G_0, d} C_1 \xrightarrow{\tau_2} C_1 \\
 t_1 \downarrow & t_1 \downarrow & \tau_1 \downarrow \\
 A_0 \xrightarrow{F_0} B_0 & B_0 \xrightarrow{G_0} C_0 & A_0 \xrightarrow{F_0 \cdot G_0} C_0
 \end{array}$$

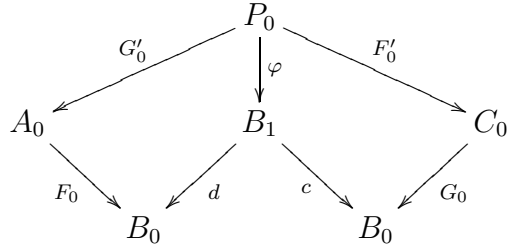
The essential surjectivity of  $F \cdot G$  comes from the commutativity of the following diagram

$$\begin{array}{ccccc}
 A_0 \times_{F_0, d} B_1 \times_{G_1, d} C_1 & \xrightarrow{t_2 \times 1} & B_1 \times_{G_1, d} C_1 & \xrightarrow{c \times 1} & B_0 \times_{G_0, d} C_1 \\
 1 \times G_1 \times 1 \downarrow & & & & \downarrow t_2 \\
 A_0 \times_{F_0 G_0, d} C_1 \times_{c, d} C_1 & & & & C_1 \\
 1 \times m \downarrow & & & & \downarrow c \\
 A_0 \times_{F_0 G_0, d} C_1 & \xrightarrow{\tau_2} & C_1 & \xrightarrow{c} & C_0
 \end{array}$$

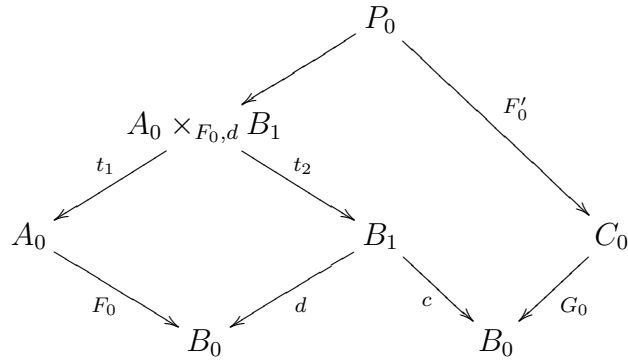
BP3: full and faithful internal functors are stable under bipullbacks because so they are in  $Grpd(Set)$  (use 3.3) and  $Grpd(\mathcal{C})(\mathbb{X}, -): Grpd(\mathcal{C}) \rightarrow Grpd(Set)$  preserves bipullbacks. Consider now a bipullback

$$\begin{array}{ccc}
 \mathbb{P} & \xrightarrow{F'} & \mathbb{C} \\
 G' \downarrow & \nearrow \varphi & \downarrow G \\
 \mathbb{A} & \xrightarrow{F} & \mathbb{B}
 \end{array}$$

and assume that  $F$  is essentially surjective. Following the description of  $\mathbb{P}$  given at the beginning of 3.6, we have a limit diagram in  $\mathcal{C}$



But such a limit can be obtained performing two pullbacks as follows



Since by assumption  $t_2 \cdot c: A_0 \times_{F_0, d} B_1 \rightarrow B_1 \rightarrow B_0$  is a regular epimorphism,  $F'_0$  also is a regular epimorphism and then  $F'$  is essentially surjective (see 4.4). ■

## 5. Bipullback congruences

Next definition is the direct bicategorical generalization of the notion of pullback congruence introduced by J. Bénabou in [4].

**Definition 5.1** Let  $\mathcal{B}$  be a bicategory with bipullbacks and  $\Sigma$  a class of 1-cells in  $\mathcal{B}$ . The class  $\Sigma$  is a bipullback congruence if the following conditions hold:

BC1.  $\Sigma$  contains all equivalences;

BC2.  $\Sigma$  satisfies the “2  $\Rightarrow$  3” property: let  $F: \mathbb{C} \rightarrow \mathbb{D}$  and  $G: \mathbb{D} \rightarrow \mathbb{E}$  be 1-cells in  $\mathcal{B}$ ; if two of  $F$ ,  $G$  and  $F \cdot G$  are in  $\Sigma$ , then the third one is in  $\Sigma$ ;

BC3.  $\Sigma$  is stable under bipullbacks;

BC4. If  $\alpha: F \Rightarrow G$  is a 2-cell, then  $F \in \Sigma$  if and only if  $G \in \Sigma$ .

**Proposition 5.2** *Let  $\mathcal{B}$  be a bicategory with bipullbacks. Any bipullback congruence has a right calculus of fractions.*

Proof. It is enough to prove that a bipullback congruence  $\Sigma$  satisfies condition BP3 in Proposition 2.8. Let  $W: \mathbb{B} \rightarrow \mathbb{A}$  be in  $\Sigma$  and let  $(D_W: \mathbb{B} \rightarrow \mathbb{K}, \delta_1: D_W \cdot W_1 \Rightarrow id, \delta_2: D_W \cdot W_2 \Rightarrow id)$  be the diagonal fill-in as in 2.7. By BC1,  $id \in \Sigma$ , and then by BC4  $D_W \cdot W_1 \in \Sigma$ . Since by BC3  $W_1 \in \Sigma$ , we conclude by BC2 that  $D_W \in \Sigma$ . ■

**5.3** Protomodular categories have been introduced by D. Bourn in [8] (see also [7]). Since we are concerned only with regular categories, we can consider the next lemma, proved in [9], as a definition of protomodular category. This lemma makes also evident the analogy between bipullback congruences and regular protomodular categories: in a regular protomodular category pullbacks satisfies the “2  $\Rightarrow$  3” property. This analogy will be made precise in Proposition 5.5.

**Lemma 5.4** *(Bourn-Gran) Let  $\mathcal{C}$  be a regular category. The following conditions are equivalent:*

1.  $\mathcal{C}$  is protomodular;
2. In any commutative diagram

$$\begin{array}{ccc} \longrightarrow & & \longrightarrow \\ \downarrow & & \downarrow \\ \longrightarrow & & \longrightarrow \\ \downarrow & & \downarrow \\ & b & \end{array}$$

where  $b$  is a regular epimorphism, if the left hand square and the outer rectangle are pullbacks, then the right hand square is a pullback.

**Proposition 5.5** *Let  $\mathcal{C}$  be a regular protomodular category. The class of weak equivalences in  $\text{Grpd}(\mathcal{C})$  is a bipullback congruence.*

Let  $F: \mathbb{A} \rightarrow \mathbb{B}$  and  $G: \mathbb{B} \rightarrow \mathbb{C}$  be in  $\text{Grpd}(\mathcal{C})$ . In order to prove Proposition 5.5 we need two lemmas on the shape of certain limits. The proof is routine.

**Lemma 5.6** *Consider the pullbacks*

$$\begin{array}{ccc} A_0 \times_{F_0, d} B_1 & \xrightarrow{t_2} & B_1 \\ t_1 \downarrow & & \downarrow d \\ A_0 & \xrightarrow{F_0} & B_0 \end{array} \quad \begin{array}{ccc} B_1 \times_{c, F_0} A_0 & \xrightarrow{s_2} & A_0 \\ s_1 \downarrow & & \downarrow F_0 \\ B_1 & \xrightarrow{c} & B_0 \end{array}$$

and the commutative diagrams

$$\begin{array}{ccc} A_1 & \xrightarrow{c} & A_0 \\ \langle d, F_1 \rangle \downarrow & (1) & \downarrow F_0 \\ A_0 \times_{F_0, d} B_1 & \xrightarrow{t_2} & B_1 \xrightarrow{c} B_0 \end{array} \quad \begin{array}{ccc} A_1 & \xrightarrow{d} & A_0 \\ \langle F_1, c \rangle \downarrow & (2) & \downarrow F_0 \\ B_1 \times_{c, F_0} A_0 & \xrightarrow{s_1} & B_1 \xrightarrow{d} B_0 \end{array}$$

Then  $F: \mathbb{A} \rightarrow \mathbb{B}$  is full and faithful iff (1) is a pullback iff (2) is a pullback.

**Lemma 5.7** *Consider the pullbacks*

$$\begin{array}{ccc} A_0 \times_{F_0, d} B_1 & \xrightarrow{t_2} & B_1 \\ t_1 \downarrow & & \downarrow d \\ A_0 & \xrightarrow{F_0} & B_0 \end{array} \quad \begin{array}{ccc} C_1 \times_{c, G_0} B_0 & \xrightarrow{s_2} & B_0 \\ s_1 \downarrow & & \downarrow G_0 \\ C_1 & \xrightarrow{c} & C_0 \end{array} \quad \begin{array}{ccc} A_0 \times_{F_0 G_0, d} C_1 & \xrightarrow{\tau_2} & C_1 \\ \tau_1 \downarrow & & \downarrow d \\ A_0 & \xrightarrow{F_0 \cdot G_0} & C_0 \end{array}$$

and the commutative diagrams

$$\begin{array}{ccc} A_0 \times_{F_0, d} B_1 & \xrightarrow{t_2} & B_1 \xrightarrow{\langle G_1, c \rangle} C_1 \times_{c, G_0} B_0 \\ t_1 \downarrow & (3) & \downarrow s_1 \\ A_0 & \xrightarrow{F_0} & B_0 \xrightarrow{G_0} C_0 \end{array} \quad \begin{array}{ccc} A_0 \times_{F_0, d} B_1 & \xrightarrow{t_2} & B_1 \xrightarrow{c} B_0 \\ 1 \times G_1 \downarrow & (4) & \downarrow G_0 \\ A_0 \times_{F_0 G_0, d} C_1 & \xrightarrow{\tau_2} & C_1 \xrightarrow{c} C_0 \end{array}$$

Then (3) is a pullback iff (4) is a pullback.

**5.8** We are ready to prove Proposition 5.5.

Proof. Let  $\Sigma$  be the class of weak equivalences in  $Grpd(\mathcal{C})$ . We have to show that condition BC2 holds, since the other conditions have been checked in the proof of Proposition 4.5. More precisely, given  $F: \mathbb{A} \rightarrow \mathbb{B}$  and  $G: \mathbb{B} \rightarrow \mathbb{C}$  in  $Grpd(\mathcal{C})$  such that  $F \cdot G \in \Sigma$ , we have to prove that  $F \in \Sigma$  iff  $G \in \Sigma$ . There are two not obvious steps. (The protomodularity of  $\mathcal{C}$  is needed only for the first step.)

1. If  $F \cdot G$  is full and faithful and  $F$  is a weak equivalence, then  $G$  is full and faithful. Consider the following commutative diagram

$$\begin{array}{ccccc}
 A_1 & \xrightarrow{c} & A_0 & & \\
 \langle d, F_1 \rangle \downarrow & & \downarrow F_0 & & \\
 A_0 \times_{F_0, d} B_1 & \xrightarrow{t_2} & B_1 & \xrightarrow{c} & B_0 \\
 1 \times G_1 \downarrow & & & & \downarrow G_0 \\
 A_0 \times_{F_0 G_0, d} C_1 & \xrightarrow{\tau_2} & C_1 & \xrightarrow{c} & C_0
 \end{array}$$

Since  $F$  is full and faithful, by Lemma 5.6 the top square is a pullback. Since  $F \cdot G$  is full and faithful, by Lemma 5.6 the outer rectangle is a pullback. Since  $F$  is essentially surjective, the second row is a regular epimorphism. Following Lemma 5.4 the bottom square is a pullback. Therefore, by Lemma 5.7, the outer rectangle of the following commutative diagram is a pullback

$$\begin{array}{ccccc}
 A_0 \times_{F_0, d} B_1 & \xrightarrow{t_2} & B_1 & \xrightarrow{\langle G_1, c \rangle} & C_1 \times_{c, G_0} B_0 \\
 \downarrow t_1 & & \downarrow d & & \downarrow s_1 \\
 A_0 & \xrightarrow{F_0} & B_0 & \xrightarrow{G_0} & C_0 \\
 & & & & \downarrow d
 \end{array}$$

Since the left hand square is a pullback by definition and the second column is a split epimorphism, by Lemma 5.4 the right hand square is a pullback. By Lemma 5.6 again we conclude that  $G$  is full and faithful.

2. If  $F \cdot G$  is essentially surjective and  $G$  is full and faithful, then  $F$  is

essentially surjective. Consider the following pullback (notations as in Lemma 5.7)

$$\begin{array}{ccc}
 Q & \xrightarrow{\lambda_2} & B_0 \\
 \lambda_1 \downarrow & & \downarrow G_0 \\
 A_0 \times_{F_0 G_0, d} C_1 & \xrightarrow{\tau_2} C_1 \xrightarrow{c} & C_0
 \end{array}$$

By assumption  $\tau_2 \cdot c$  is a regular epimorphism, so that  $\lambda_2$  also is a regular epimorphism. Since  $G$  is full and faithful, there exists  $\lambda: Q \rightarrow B_1$  such that  $\lambda \cdot d = \lambda_1 \cdot \tau_1 \cdot F_0$ ,  $\lambda \cdot G_1 = \lambda_1 \cdot \tau_2$  and  $\lambda \cdot c = \lambda_2$ . From the first equation on  $\lambda$ , we deduce the existence of  $\mu: Q \rightarrow A_0 \times_{F_0, d} B_1$  such that  $\mu \cdot t_1 = \lambda_1 \cdot \tau_1$  and  $\mu \cdot t_2 = \lambda$ . Finally,  $\mu \cdot t_2 \cdot c = \lambda \cdot c = \lambda_2$ , so that  $t_2 \cdot c$  is a regular epimorphism. (Note that we need only the existence of  $\lambda$ , not its uniqueness. In other words we only use the “fullness” of  $G$ , and not its “faithfulness”.) ■

**5.9** Observe that, contrarily to Lemma 5.4, Proposition 5.5 is not a characterization of regular protomodular categories. Indeed, if  $\mathcal{C}$  is *Set* (more generally, if in  $\mathcal{C}$  the axiom of choice holds) then weak equivalences in  $Grpd(\mathcal{C})$  are the same that equivalences (see Corollary 4.3), and the class of equivalences obviously is a bipullback congruence.

**5.10** G. Janelidze pointed out to me that condition 2 in Lemma 5.4 holds in any protomodular (not necessarily regular) category  $\mathcal{C}$  provided that the arrow  $b$  is a pullback stable strong epimorphism. This fact has an interesting consequence. Indeed, Proposition 4.5 holds when  $\mathcal{C}$  is any finitely complete category and  $\Sigma$  is the class of “weak  $\mathcal{E}$ -equivalences”, where:

- $\mathcal{E}$  is any class of arrows that behaves well (in the sense explained in 4.4) and contains the split epimorphisms,
- an internal functor  $F$  is a weak  $\mathcal{E}$ -equivalence if it is full and faithful and essentially  $\mathcal{E}$ -surjective (that is, the arrow  $t_2 \cdot c: C_0 \times_{F_0, d} D_1 \rightarrow D_1 \rightarrow D_0$  of Definition 4.1 is in  $\mathcal{E}$ ).

Therefore, Proposition 5.5 holds for weak  $\mathcal{E}$ -equivalences in any protomodular category  $\mathcal{C}$  provided that  $\mathcal{E}$  behaves well, contains the split

epimorphisms and is contained in the class of pullback stable strong epimorphisms. Examples are:

- i. the class of pullback stable regular epimorphisms,
- ii. the class of pullback stable regular epimorphisms that are effective descent morphisms.

## 6. Monoidal functors

All along this section we fix  $\mathcal{C} = Grp$ , the category of groups, which is a regular and protomodular category. I use additive notation for groups.

**6.1** The aim of this section is to prove that the 2-category  $MON$  described hereunder is the bicategory of fractions of  $Grpd(\mathcal{C})$  with respect to weak equivalences.

1. Objects of  $MON$  are internal groupoids in  $Grp$ . Note that since the forgetful functor  $Grp \rightarrow Set$  preserves finite limits, any object of  $MON$  is also a groupoid in the usual sense.
2. 1-cells  $F: \mathbb{A} \rightarrow \mathbb{B}$  in  $MON$  are monoidal functors, that is, pairs  $(F, F_2)$  where  $F$  is a (not necessarily internal) functor and

$$F_2 = \{F_2^{a,b}: Fa + Fb \rightarrow F(a + b)\}_{a,b \in A_0}$$

is a natural family of arrows in  $\mathbb{B}$  satisfying the cocycle condition

$$\begin{array}{ccc} Fa + Fb + Fc & \xrightarrow{1+F_2^{b,c}} & Fa + F(b + c) \\ F_2^{a,b+1} \downarrow & & \downarrow F_2^{a,b+c} \\ F(a + b) + Fc & \xrightarrow{F_2^{a+b,c}} & F(a + b + c) \end{array}$$

(and suitable  $F_0: 0 \rightarrow F0$  is uniquely determined by  $F$  and  $F_2$ ).

3. 2-cells  $\lambda: F \Rightarrow G$  in  $MON$  are monoidal natural transformations, that is, natural transformations such that the following diagram

commutes

$$\begin{array}{ccc}
 Fa + Fb & \xrightarrow{F_2^{a,b}} & F(a + b) \\
 \lambda_a + \lambda_b \downarrow & & \downarrow \lambda_{a+b} \\
 Ga + Gb & \xrightarrow{G_2^{a,b}} & G(a + b)
 \end{array}$$

- Remark 6.2**
1. The 2-category  $Grpd(\mathcal{C})$  embeds into the 2-category  $MON$ : internal functors  $F: \mathbb{A} \rightarrow \mathbb{B}$  are precisely those monoidal functors for which all the  $F_2^{a,b}$  are identities. Indeed, in this case the naturality of  $F_2$  corresponds to the fact that  $F_1: A_1 \rightarrow B_1$  is a group homomorphism, and the cocycle condition is verified because  $e: B_0 \rightarrow B_1$  is a group homomorphism.
  2. The embedding  $\mathcal{F}: Grpd(\mathcal{C}) \rightarrow MON$  is full and faithful on 2-cells. Indeed, if  $F_2^{a,b} = id = G_2^{a,b}$ , then the fact that  $\lambda$  is monoidal corresponds to the fact that  $\lambda: A_0 \rightarrow B_1$  is a group homomorphism.
  3. The embedding  $\mathcal{F}: Grpd(\mathcal{C}) \rightarrow MON$  preserves weak equivalences. In fact, the forgetful functor  $Grp \rightarrow Set$  preserves and reflects finite limits and regular epimorphisms (this is because  $Grp$  is an algebraic category, see Ch. 3 in [6]), so that weak equivalences in  $Grpd(\mathcal{C})$  and in  $MON$  are 1-cells which are full, faithful and essentially surjective in the usual sense.
  4. In  $MON$  weak equivalences coincide with equivalences. Indeed, if  $F: \mathbb{A} \rightarrow \mathbb{B}$  is a weak equivalence, any quasi-inverse  $G: \mathbb{B} \rightarrow \mathbb{A}$  can be equipped with a monoidal structure as follows: choose, for each  $x \in B_0$ , an arrow  $\beta_x: F(Gx) \rightarrow x$  so to have a natural transformation  $\beta: G \cdot F \Rightarrow Id$ . Then define

$$G_2^{x,y}: Gx + Gy \rightarrow G(x + y)$$

to be the unique arrow making the following diagram commutative

$$\begin{array}{ccc}
 F(Gx + Gy) & \xrightarrow{F(G_2^{x,y})} & F(G(x + y)) \\
 F_2^{Gx, Gy} \uparrow & & \downarrow \beta_{x+y} \\
 F(Gx) + F(Gy) & \xrightarrow{\beta_x + \beta_y} & x + y
 \end{array}$$



It is straightforward to check naturality and cocycle condition for  $G_2$  and that  $\beta$  is monoidal. Moreover, we get a monoidal natural transformation  $\alpha: F \cdot G \Rightarrow Id$  via the equation  $F(\alpha_a) = \beta_{Fa}$ .

5. The above construction of  $G_2$  makes clear that even if  $F$  is a weak equivalence in  $Grpd(\mathcal{C})$  in general  $G$  is in  $MON$  but not in  $Grpd(\mathcal{C})$ .

**Lemma 6.3** *The 2-category  $MON$  has bipullbacks. Moreover, given 1-cells  $F: \mathbb{A} \rightarrow \mathbb{B}$  and  $G: \mathbb{C} \rightarrow \mathbb{B}$ , it is possible to choose a bipullback of  $F$  and  $G$*

$$\begin{array}{ccc}
 \mathbb{P} & \xrightarrow{F'} & \mathbb{C} \\
 G' \downarrow & \nearrow \varphi & \downarrow G \\
 \mathbb{A} & \xrightarrow{F} & \mathbb{B}
 \end{array}$$

in such a way that  $F'$  and  $G'$  are internal functors in  $Grp$ .

Proof. The construction of the pullback  $\mathbb{P}$  is as in 3.3. The interesting point is that, even if  $F$  and  $G$  are monoidal (not necessarily internal) functors,  $\mathbb{P}$  is an internal groupoid in  $Grp$  and not just a monoidal category. Indeed, if

$$(a, f: Fa \rightarrow Gx, x) \quad \text{and} \quad (b, g: Fb \rightarrow Gy, y)$$

are objects in  $\mathbb{P}$ , their tensor product  $(a, f: Fa \rightarrow Gx, x) + (b, g: Fb \rightarrow Gy, y)$  is given by

$$(a + b, F(a + b) \xrightarrow{(F_2^{a,b})^{-1}} Fa + Fb \xrightarrow{f+g} Gx + Gy \xrightarrow{G_2^{x,y}} G(x + y), x + y)$$

If  $(c, h: Fc \rightarrow Gz, z)$  is a third object in  $\mathbb{P}$ , to check that the above tensor product is strictly associative easily reduces to the commutativity of the

following diagram

$$\begin{array}{ccccc}
 & & F(a+b+c) & & \\
 & \nearrow^{F_2^{a,b+c}} & & \nwarrow_{F_2^{a+b,c}} & \\
 Fa + F(b+c) & & & & F(a+b) + Fc \\
 & \nwarrow_{1+F_2^{b,c}} & & \nearrow^{F_2^{a,b}+1} & \\
 & & Fa + Fb + Fc & & \\
 & \swarrow_{f+(F_2^{b,c})^{-1} \cdot (g+h) \cdot G_2^{y,z}} & \downarrow_{f+g+h} & \searrow_{(F_2^{a,b})^{-1} \cdot (f+g) \cdot G_2^{x,y}+h} & \\
 & & Gx + Gy + Gz & & \\
 & \swarrow_{1+G_2^{y,z}} & & \searrow_{G_2^{x,y}+1} & \\
 Gx + G(y+z) & & & & G(x+y) + Gz \\
 & \swarrow_{G_2^{x,y+z}} & & \nwarrow_{G_2^{x+y,z}} & \\
 & & G(x+y+z) & & 
 \end{array}$$

that is, to the cocycle condition on  $F_2$  and  $G_2$ .

The fact that  $F'$  and  $G'$  are internal functors is obvious. ■

**Proposition 6.4** *The embedding  $\mathcal{F}: Grpd(\mathcal{C}) \rightarrow MON$  is the bicategory of fractions of  $Grpd(\mathcal{C})$  with respect to the class of weak equivalences.*

Proof. Let  $\Sigma$  be the class of weak equivalences in  $Grpd(\mathcal{C})$ . From Proposition 4.5 we know that  $\Sigma$  has a right calculus of fractions. Moreover, by 6.2.3 and 6.2.4,  $\mathcal{F}(W)$  is an equivalence for every  $W \in \Sigma$ . It remains to check conditions EF1–EF3 in Proposition 2.5: EF1 is obvious and EF2 is precisely 6.2.2. As far as EF3 is concerned, consider a 1-cell  $F: \mathbb{A} \rightarrow \mathbb{B}$  in  $MON$  and perform the bipullback of  $F$  along the identity 1-cell  $I$  as in Lemma 6.3

$$\begin{array}{ccc}
 \mathbb{P} & \xrightarrow{G} & \mathbb{B} \\
 W \downarrow & \nearrow \varphi & \downarrow I \\
 \mathbb{A} & \xrightarrow{F} & \mathbb{B}
 \end{array}$$

so that both  $W$  and  $G$  are internal functors. Since equivalences are stable under bipullbacks,  $W$  is an equivalence in  $MON$  and therefore it

is a weak equivalence in  $Grpd(\mathcal{C})$ . Finally,  $\varphi: \mathcal{F}(W) \cdot F \Rightarrow \mathcal{F}(G)$  is the 2-cell needed in EF3. Following Proposition 2.5,  $\mathcal{F}: Grpd(\mathcal{C}) \rightarrow MON$  is the bicategory of fractions with respect to  $\Sigma$ . ■

**Remark 6.5** Observe that we cannot expect to describe a class larger than the class of monoidal functors as fractions of internal functors with respect to weak equivalences. Indeed, the existence of a 2-cell  $\mathcal{F}(W) \cdot F \Rightarrow \mathcal{F}(G)$  as in condition EF3 implies that  $F$  is monoidal.

## 7. Homomorphisms of strict Lie 2-algebras

In this section the base category  $\mathcal{C}$  is the category  $Lie$  of Lie algebras over a fixed field  $K$ , which is a regular and protomodular category. The situation is completely analogous to the situation described in Section 6 for groups. The reason is that the forgetful functors  $Lie \rightarrow Vect$  (where  $Vect$  is the category of vector spaces over  $K$ ) and  $Vect \rightarrow Set$  preserve and reflect finite limits and regular epimorphisms (because  $Lie$  and  $Vect$  are algebraic categories) and moreover in  $Vect$  the axiom of choice holds (because every vector space is free and therefore regular projective).

**7.1** The aim of this section is to prove that the 2-category  $LIE$  described hereunder is the bicategory of fractions of  $Grpd(\mathcal{C})$  with respect to weak equivalences.

1. Objects of  $LIE$  are internal groupoids in  $Lie$ , also called strict Lie 2-algebras in [1].
2. 1-cells  $F: \mathbb{A} \rightarrow \mathbb{B}$  in  $LIE$  are internal functors in  $Vect$  equipped with a family of arrows in  $\mathbb{B}$

$$F_2 = \{F_2^{a,b}: [Fa, Fb] \rightarrow F[a, b]\}_{a,b \in A_0}$$

which is natural, bilinear, antisymmetric, and satisfies the follow-

ing Jacobi condition

$$\begin{array}{ccc}
 [Fa, [Fb, Fc]] & \xlongequal{\quad} & [[Fa, Fb], Fc] + [Fb, [Fa, Fc]] \\
 \downarrow [1, F_2^{b,c}] & & \downarrow [F_2^{a,b}, 1] + [1, F_2^{a,c}] \\
 [Fa, F[b, c]] & & [F[a, b], Fc] + [Fb, F[a, c]] \\
 \downarrow F_2^{a,[b,c]} & & \downarrow F_2^{[a,b],c} + F_2^{b,[a,c]} \\
 F[a, [b, c]] & \xlongequal{\quad} & F[[a, b], c] + F[b, [a, c]]
 \end{array}$$

These 1-cells are simply called homomorphisms in [1], where in fact they are defined for more general semi-strict Lie 2-algebras.

3. 2-cells  $\lambda: F \Rightarrow G$  in *LIE* are internal natural transformations in *Vect* such that the following diagram commutes

$$\begin{array}{ccc}
 [Fa, Fb] & \xrightarrow{F_2^{a,b}} & F[a, b] \\
 \downarrow [\lambda_a, \lambda_b] & & \downarrow \lambda_{[a,b]} \\
 [Ga, Gb] & \xrightarrow{G_2^{a,b}} & G[a, b]
 \end{array}$$

**Remark 7.2** 1. The 2-category  $Grpd(\mathcal{C})$  embeds into the 2-category *LIE*: internal functors  $F: \mathbb{A} \rightarrow \mathbb{B}$  are precisely those homomorphisms for which all the  $F_2^{a,b}$  are identities. The embedding  $\mathcal{F}: Grpd(\mathcal{C}) \rightarrow LIE$  is full and faithful on 2-cells, and preserves weak equivalences.

2. In *LIE* weak equivalences coincide with equivalences. Indeed, let  $F: \mathbb{A} \rightarrow \mathbb{B}$  be a weak equivalence in *LIE*. Then  $F$  is also a weak equivalence in the 2-category of internal groupoids and internal functors in *Vect*. Since in *Vect* the axiom of choice holds,  $F$  has a quasi-inverse  $G: \mathbb{B} \rightarrow \mathbb{A}$  which is an internal functor in *Vect* (see Corollary 4.3). Now  $G$  can be equipped with a structure of homomorphism as follows: consider the internal (in *Vect*) natural transformation  $\beta: G \cdot F \Rightarrow Id$  and define

$$G_2^{x,y}: [Gx, Gy] \rightarrow G[x, y]$$

to be the unique arrow making the following diagram commutative

$$\begin{array}{ccc}
 F[Gx, Gy] & \xrightarrow{F(G_2^{x,y})} & F(G[x, y]) \\
 F_2^{Gx, Gy} \uparrow & & \downarrow \beta_{[x,y]} \\
 [F(Gx), F(Gy)] & \xrightarrow{[\beta_x, \beta_y]} & [x, y]
 \end{array}$$

**Lemma 7.3** *The 2-category  $LIE$  has bipullbacks. Moreover, given 1-cells  $F: \mathbb{A} \rightarrow \mathbb{B}$  and  $G: \mathbb{C} \rightarrow \mathbb{B}$ , it is possible to choose a bipullback of  $F$  and  $G$*

$$\begin{array}{ccc}
 \mathbb{P} & \xrightarrow{F'} & \mathbb{C} \\
 G' \downarrow & \nearrow \varphi & \downarrow G \\
 \mathbb{A} & \xrightarrow{F} & \mathbb{B}
 \end{array}$$

*in such a way that  $F'$  and  $G'$  are internal functors in  $Lie$ .*

Proof. Once again the point is that, even if  $F$  and  $G$  are homomorphisms, the bipullback  $\mathbb{P}$  constructed as in 3.3 is an internal groupoid in  $Lie$  and not just a semi-strict Lie 2-algebra. Indeed, the Lie operation in  $\mathbb{P}$  is defined by

$$([a, b], F[a, b] \xrightarrow{(F_2^{a,b})^{-1}} [Fa, Fb] \xrightarrow{[f,g]} [Gx, Gy] \xrightarrow{G_2^{x,y}} G[x, y], [x, y])$$

and the Jacobi identity is strictly verified thanks to the Jacobi condition on  $F_2$  and  $G_2$ . ■

**Proposition 7.4** *The embedding  $\mathcal{F}: Grpd(\mathbb{C}) \rightarrow LIE$  is the bicategory of fractions of  $Grpd(\mathbb{C})$  with respect to the class of weak equivalences.*

Proof. The proof is analogous to that of Proposition 6.4 and we omit details. ■

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