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A CHARACTERISATION OF THE "SMITH IS HUQ" CONDITION IN THE POINTED MAL'TSEV SETTING

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dedicated to René Guitart on the occasion of his sixty-fifth birthday

Résumé. Nous donnons une caractérisation de la condition « Smith is Huq » pour une catégorie de Mal'tsev pointée $\mathbb{C}$ au moyen d'une propriété de la fibration des points $\P_{\mathbb{C}}\colon \mathsf{Pt}(\mathbb{C}) \to \mathbb{C}$, à savoir : tout foncteur changement de base $h^*\colon \mathsf{Pt}_Y(\mathbb{C}) \to \mathsf{Pt}_X(\mathbb{C})$ reflète la commutation des sous-objets normaux.

Abstract. We give a characterisation of the "Smith is Huq" condition for a pointed Mal'tsev category $\mathbb{C}$ by means of a property of the fibration of points $\P_{\mathbb{C}}\colon \mathsf{Pt}(\mathbb{C}) \to \mathbb{C}$, namely: any change of base functor $h^*\colon \mathsf{Pt}_Y(\mathbb{C}) \to \mathsf{Pt}_X(\mathbb{C})$ reflects commuting of normal subobjects.

Keywords. Fibration of points, Mal'tsev and protomodular category, commutation of subobjects, centralisation of equivalence relations, commutator theory, topological Mal'tsev model

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Introduction

It is well known that, given a group G and two subgroups H and K, they commute inside G (i.e., we have $h \cdot k = k \cdot h$, $\forall (h,k) \in H \times K$) if and only if the function $H \times K \to G\colon (h,k) \mapsto h \cdot k$ is a group homomorphism. When H and K are normal subgroups of G, and if R_H and R_K denote their associated

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equivalence relations on G, this is the case if and only if the equivalence relations R_H and R_K centralise each other (see [22, 21, 11]), namely if and only if the function $R_H \times_G R_K \to G\colon (xR_HyR_Kz) \mapsto x \cdot y^{-1} \cdot z$ is a group homomorphism, where $R_H \times_G R_K$ is defined by the following pullback:

$$\begin{tikzcd} R_H \times_G R_K \arrow[r, shift left] & R_K \arrow[l, shift left, "\delta_1^K"] \arrow[d, shift right, "d_0^K"'] \\ R_H \arrow[u, shift right] \arrow[r, shift left] & G \arrow[l, shift left, "d_1^H"] \arrow[u, shift right] \end{tikzcd}$$

with $\delta_0^H\colon R_H \times_G R_K \to R_H$ the left vertical arrow.

The commutation condition on subobjects is said to be *à la Huq* from [17], while the commutation condition on equivalence relations is said to be *à la Smith* from [22]. In the category Gp of groups, we just recalled that, in the case of normal subobjects, the two types of commutation are equivalent. This is the meaning of the "Smith is Huq" condition, which is far from being true in general.

It turns out that the right environment for the conceptual notion of centralisation of equivalence relations is the context of Mal'tsev categories [13, 14]. It was first shown in [11, Proposition 3.2] that, in a pointed Mal'tsev category, "Smith implies Huq", namely that if two equivalence relations R and S centralise each other (which we denote by $[R, S] = 0$), then necessarily their associated normal subobjects commute. But the converse is not true, as shown in [6, Proposition 6.1], from an example introduced by G. Janelidze in the pointed Mal'tsev category of *digroups*, namely sets endowed with two group structures only coinciding on the unit element.

The first conceptual setting where the "Smith is Huq" condition (SH) holds was pointed out in [11]: it is the context of of pointed *strongly protomodular* categories, of which the category Gp is an example. These are pointed categories $\mathbb{C}$ such that any change of base functor with respect to the fibration of points $\P_{\mathbb{C}} : \mathsf{Pt}(\mathbb{C}) \to \mathbb{C}$ is normal, i.e., conservative and reflecting normal subobjects. Further observations on the condition (SH) have been given in [19, 15, 16, 20].

So it is quite natural to ask for a characterisation of the (SH) condition, and more precisely to ask it in terms of a property of the change of base functors of the fibration $\P_{\mathbb{C}}$. Here we give an answer in the pointed Mal'tsev context: the property of *reflection of commutation of normal subobjects*. We show moreover that when a variety $\mathsf{Set}^{\mathbb{T}}$ of algebras over a Mal'tsev theory $\mathbb{T}$

satisfies this last condition, so does the category $\mathsf{Top}^{\mathbb{T}}$ of topological models of this theory, which implies that the category $\mathsf{Gp}(\mathsf{Top})$ of topological groups satisfies (SH).

We then extend some results already known for strongly protomodular categories [4] to the (SH) context. In particular, we show that, when they are defined, the Huq commutator and the Smith commutator coincide.

1 Unital categories and Mal'tsev categories

1.1 Unital categories and Huq commutation

In this section, $\mathbb{C}$ will be a pointed category, i.e., a category with a zero object 0. Let us recall from [3]:

Definition 1.1. Let $\mathbb{C}$ be a pointed category with finite products. Given two objects A and B in $\mathbb{C}$, consider the diagram

$$\begin{tikzcd} A \arrow[r, shift right, "\langle 1_A,0\rangle"'] & A \times B \arrow[l, shift right, "\pi_A"'] \arrow[r, shift left, "\pi_B"] & B. \arrow[l, shift left, "\langle 0,1_B\rangle"] \end{tikzcd}$$

The category $\mathbb{C}$ is said to be *unital* if, for every pair of objects $A, B \in \mathbb{C}$, the morphisms $\langle 1_A, 0\rangle$ and $\langle 0, 1_B\rangle$ are jointly strongly epimorphic.

In any finitely complete category this is equivalent to saying that the object $A \times B$ is the supremum of the two subobjects $\langle 1_A, 0\rangle$ and $\langle 0, 1_B\rangle$; namely, any monomorphism $j\colon J \rightarrowtail A \times B$ containing the two previous ones:

$$\begin{tikzcd} & J \arrow[d, tail, "j"'] & \\ A \arrow[ur, dotted] \arrow[r, shift right, "\langle 1_A,0\rangle"'] & A \times B \arrow[l, shift right, "\pi_A"'] \arrow[r, shift left, "\pi_B"] & B. \arrow[l, shift left, "\langle 0,1_B\rangle"] \arrow[ul, dotted] \end{tikzcd}$$

is an isomorphism. From this last remark, it is clear that the category Mon of monoids is unital. Unital categories give a setting where it is possible to express a categorical notion of commutation *à la Huq* [5]:

Definition 1.2 (Commutation à la Huq)**.** Let $\mathbb{C}$ be a unital category. Two morphisms with the same codomain, $f\colon X \to Z$ and $g\colon Y \to Z$, are said to cooperate (or to commute) if there exists a morphism $\varphi \colon X \times Y \to Z$ such that both triangles in the following diagram commute:

$$\begin{tikzcd} X \arrow[r, "\langle 1_X, 0\rangle"] \arrow[dr, "f"'] & X \times Y \arrow[d, dotted, "\varphi"] & Y \arrow[l, "\langle 0, 1_Y\rangle"'] \arrow[dl, "g"] \\ & Z. & \end{tikzcd}$$

The morphism φ is necessarily unique, because $\langle 1_X, 0\rangle$ and $\langle 0, 1_Y\rangle$ are jointly epimorphic, and it is called the *cooperator* of f and g.

The uniqueness of the cooperator makes commutation a property, rather than an additional structure on the category $\mathbb{C}$.

1.2 Mal'tsev categories and Smith commutation

A Mal'tsev category is a category in which every reflexive relation is an equivalence relation [13, 14]. The category Gp of groups is Mal'tsev. It is shown in [3] that a finitely complete category $\mathbb{C}$ is Mal'tsev if and only if any (necessarily pointed) fibre $\mathsf{Pt}_Y(\mathbb{C})$ of the fibration of points $\P_{\mathbb{C}} \colon \mathsf{Pt}(\mathbb{C}) \to \mathbb{C}$ is unital. Here $\mathsf{Pt}(\mathbb{C})$ is the category whose objects are the split epimorphisms in $\mathbb{C}$ and whose arrows are the commuting squares between such split epimorphisms, and $\P_{\mathbb{C}} \colon \mathsf{Pt}(\mathbb{C}) \to \mathbb{C}$ is the functor associating its codomain with any split epimorphism.

In this context, an equivalence relation R on an object X, coinciding with a reflexive relation on X, is just a subobject of the object $(p_0, s_0) \colon X \times X \rightleftarrows X$ in the fibre $\mathsf{Pt}_X(\mathbb{C})$:

$$\begin{tikzcd} R \arrow[r, tail, "\langle d_0^R, d_1^R\rangle"] \arrow[dr, shift right, "d_0^R"'] & X \times X \arrow[d, shift right, "p_0"'] \\ & X \arrow[u, shift right, "s_0"'] \arrow[ul, shift right, "s_0^R"'] \end{tikzcd}$$

Actually it is a normal subobject in this fibre since it is the *normalisation* (i.e., the class of the initial object in the pointed fibre $\mathsf{Pt}_X(\mathbb{C})$) of the following equivalence relation:

$$
\begin{tikzcd}
X \times R \arrow[r, shift left=2, "X \times d_1^R"] \arrow[dr, shift right, "p_X"'] & X \times X \arrow[l] \arrow[l, shift right=2, leftarrow, "X \times d_0^R"'] \arrow[d, shift right, "p_0"'] \\
& X \arrow[u, shift right, "s_0"'] \arrow[ul, shift right, "\langle 1_X, s_0^R \rangle"']
\end{tikzcd}
$$

We call this normal subobject the *local representation* of the equivalence relation R. Let us recall Proposition 3.4 of [6]:

Proposition 1.1 (Commutation à la Smith)**.** *Let $\mathbb{C}$ be a finitely complete Mal'tsev category, and (R,W) a pair of equivalence relations on an object X. The equivalence relations R and W centralise each other in $\mathbb{C}$ if and only if their (normal) local representations commute in the unital fibre $\mathsf{Pt}_X(\mathbb{C})$.*

Proof. In the unital fibre $\mathsf{Pt}_X(\mathbb{C})$, the subobjects

$$\langle d_1^R, d_0^R \rangle \colon R \rightarrowtail X \times X \quad \text{and} \quad \langle d_0^W, d_1^W \rangle \colon W \rightarrowtail X \times X$$

commute if there is a cooperator $R \times_X W \to X \times X$ in the fibre; it is necessarily of the form $\phi(xRyWz) = (x, p(xRyWz))$, satisfying the two equations $p(xRxWy) = y$ and $p(xRyWy) = x$. The morphism $p \colon R \times_X W \to X$ which, satisfying these equations, characterises the property that the equivalence relations R and W centralise each other in $\mathbb{C}$, is nothing but what is called the *connector* between R and W. (See [11] and also [21, 13, 14].) $\square$

As usual, we denote this situation by $[R,W] = 0$. It is worth noticing that, by construction of the pullback $R \times_X W$:

$$
\begin{tikzcd}
R \times_X W \arrow[r, shift right, "\delta_1^W"'] \arrow[d, shift right, "\delta_0^R"'] & W \arrow[l, shift right, "\sigma_0^W"'] \arrow[d, shift right, "d_0^W"'] \\
R \arrow[r, shift right, "d_1^R"'] \arrow[u, shift right, "\sigma_0^H"'] & X \arrow[l, shift right, "s_0^R"'] \arrow[u, shift right, "s_0^W"']
\end{tikzcd}
\qquad (*)
$$

the existence of the connector p does not depend on the possibly fibered context, namely on the fact that R and W are possibly in a fibre $\mathsf{Pt}_Y(\mathbb{C})$.

2 A characterisation of the "Smith is Huq" condition (SH)

2.1 Reflections of commutation

Let us introduce the following conditions:

(C) any change of base functor with respect to the fibration of points reflects the commutation of normal subobjects;

($\bar{\text{C}}$) any change of base functor with respect to the fibration of points reflects the centralisation of equivalence relations.

Recall that a protomodular category is such that any change of base functor with respect to the fibration of points reflects isomorphisms, and that any protomodular category is Mal'tsev. So, any protomodular category is such that any change of base functor with respect to the fibration of points reflects the inclusion of subobjects and, accordingly, the inclusion of equivalence relations.

Example 2.1. 1) According to Proposition 4.1 in [12], any *locally algebraically cartesian closed* (lacc: i.e., such that any change of base functor with respect to the fibration of points admits a right adjoint) protomodular category is such that any change of base functor with respect to the fibration of points *reflects the commutation of subobjects*, hence satisfies condition (C). The categories Gp of groups, ${}_R\mathsf{Lie}$ of Lie R-algebras, and $\mathsf{Gp}(\mathbb{E})$ of internal groups in a cartesian closed category $\mathbb{E}$ are examples of lacc protomodular categories.

2) According to Proposition 5.10 in [8], any *functorially action distinctive* protomodular category in the sense of [8] (again defined by a property of the change of base functors with respect to the fibration of points which we shall not detail here) is such that any change of base functor with respect to the fibration of points preserves the centralisers of equivalence relations, and, accordingly, satisfies condition ($\bar{\text{C}}$).

Proposition 2.1. *Let $\mathbb{C}$ be a finitely complete Mal'tsev category. Conditions* (C) *and* ($\bar{\text{C}}$) *are stable by slicing and coslicing, and consequently are still valid in any fibre* $\mathsf{Pt}_Y(\mathbb{C})$.

Proof. It is clear that given any morphism h in $\mathbb{C}/Y$ or in $Y/\mathbb{C}$ as below:

$$\begin{tikzcd} X \arrow[rr, "h"] \arrow[dr, "f"'] & & X' \arrow[dl, "f'"] \\ & Y & \end{tikzcd} \qquad \begin{tikzcd} X \arrow[rr, "h"] & & X' \\ & Y \arrow[ul, "s"] \arrow[ur, "s'"'] & \end{tikzcd}$$

we have:

$$\begin{tikzcd} \mathsf{Pt}_{f'}(\mathbb{C}/Y) \arrow[r, "h^*"] \arrow[d, equal] & \mathsf{Pt}_{f}(\mathbb{C}/Y) \arrow[d, equal] \\ \mathsf{Pt}_{X'}(\mathbb{C}) \arrow[r, "h^*"'] & \mathsf{Pt}_{X}(\mathbb{C}) \end{tikzcd} \qquad \begin{tikzcd} \mathsf{Pt}_{s'}(Y/\mathbb{C}) \arrow[r, "h^*"] \arrow[d, equal] & \mathsf{Pt}_{s}(Y/\mathbb{C}) \arrow[d, equal] \\ \mathsf{Pt}_{X'}(\mathbb{C}) \arrow[r, "h^*"'] & \mathsf{Pt}_{X}(\mathbb{C}) \end{tikzcd}$$

So the result is a consequence, on the one hand, of the fact that, as we recalled above, the condition $[R,W]= 0$ does not depend on the fibered context and, on the other hand, of the fact that the normality of a subobject in $\mathsf{Pt}_f(\mathbb{C}/Y)$ or $\mathsf{Pt}_s(Y/\mathbb{C})$ is given by a pullback condition in $\mathbb{C}$ which, accordingly, is still valid in $\mathsf{Pt}_X(\mathbb{C})$. The same observation holds for the commutation condition. □

Unlike in the stricter context of protomodular categories, a normal subobject in a pointed Mal'tsev category could be the normalisation of several equivalence relations; so the following, though it is not surprising, does deserve a proof:

Proposition 2.2. *Let $\mathbb{C}$ be a finitely complete Mal'tsev category. Condition* (C) *implies condition* ($\bar{\mathrm{C}}$).

Proof. Consider the following diagram in which R is an equivalence relation on the object (f,s) in $\mathsf{Pt}_Y(\mathbb{C})$, the kernel pair of f is denoted by $R[f]$ and any commutative square is a pullback:

$$\begin{tikzcd} R' \arrow[rr, "\tilde{x}"] \arrow[dd, shift right=2] \arrow[dd, shift left=2] \arrow[dr, "i_{R'}"] & & R \arrow[dd, shift right=2] \arrow[dd, shift left=2] \arrow[dr, "i_R"] & \\ & R[f'] \arrow[rr, "R(x)"'] \arrow[dl, shift right=2] \arrow[dl, shift left=2] & & R[f] \arrow[dl, shift right=2] \arrow[dl, shift left=2] \\ X' \arrow[uu] \arrow[ur] \arrow[rr, "x"] \arrow[d, shift right, "f'"'] & & X \arrow[uu] \arrow[ur] \arrow[d, shift right, "f"'] & \\ Y' \arrow[u, shift right, "s'"'] \arrow[rr, "y"'] & & Y \arrow[u, shift right, "s"'] & \end{tikzcd}$$

By Proposition 1.1, the inclusions $i_R \colon R \rightarrowtail R[f]$ and $i_{R'} \colon R' \rightarrowtail R[f']$ are normal subobjects in the fibres $\mathsf{Pt}_X(\mathbb{C})$ and $\mathsf{Pt}_{X'}(\mathbb{C})$. In addition, since any commutative square is a pullback, we have $R' = y^*(R)$, and also $i_{R'} = x^*(i_R)$ in the following diagram:

$$\begin{tikzcd}
R' \arrow[rr, "\tilde{x}"] \arrow[dr, "i_{R'}"] \arrow[dd, shift right, "d_0^{R'}"'] & & R \arrow[dr, "i_R"] \arrow[dd, shift right, "d_0^R"'] & \\
& R[f'] \arrow[rr] \arrow[dl, shift left] & & R[f] \arrow[dl, shift left] \\
X' \arrow[rr, "x"] \arrow[uu, shift right] \arrow[ur, shift left] & & X \arrow[uu, shift right] \arrow[ur, shift left] &
\end{tikzcd}$$

Now suppose we have another equivalence relation W on (f,s) in $\mathsf{Pt}_Y(\mathbb{C})$ with $W' = y^*(W)$ such that $[W',R'] = 0$ in $\mathsf{Pt}_{Y'}(\mathbb{C})$. This last property is equivalent to the commutation of the normal monomorphisms $i_{W'} = x^*(i_W)$ and $i_{R'} = x^*(i_R)$ in the fibre $\mathsf{Pt}_{X'}(\mathbb{C})$. Since the category $\mathbb{C}$ satisfies condition (C), the normal monomorphisms i_W and i_R commute in the fibre $\mathsf{Pt}_X(\mathbb{C})$ which means that we have $[W,R] = 0$ in $\mathsf{Pt}_Y(\mathbb{C})$. □

Even though the condition ($\bar{\mathrm{C}}$) may be weaker than (C), it is certainly not automatically satisfied, as shows the following result.

Proposition 2.3. *Let $\mathbb{C}$ be a finitely complete pointed regular Mal'tsev category. Condition ($\bar{\mathrm{C}}$) implies that in $\mathbb{C}$, all extensions with abelian kernel are abelian extensions.*

Proof. We first consider the case of split epimorphisms. Let $(f,s) \colon X \rightleftarrows Y$ be an object in $\mathsf{Pt}_Y(\mathbb{C})$ such that the kernel K of f is abelian, meaning that the discrete equivalence relation Δ_K on K centralises itself. Then by ($\bar{\mathrm{C}}$) the kernel relation $R[f]$ of f—the relation associated to the kernel pair—centralises itself, which means that the extension f is abelian.

If now $g \colon Y \to Z$ is an extension with abelian kernel, i.e., a regular epimorphism in $\mathbb{C}$ of which the kernel K is abelian, then the kernel pair projection $g_0 \colon R[g] \to Y$ is an abelian extension by the above. Hence also g is abelian by [10, Proposition 4.1]. □

As a consequence, the counterexample from [6] in the category of digroups shows that a category may be semi-abelian without satisfying ($\bar{\mathrm{C}}$).

2.2 The characterisation

We are now ready for the characterisation:

Theorem 2.1. *Let* $\mathbb{C}$ *be a finitely complete pointed Mal'tsev category. The condition* (C) *is equivalent to the "Smith is Huq" condition* (SH).

Proof. The normalisation in $\mathbb{C}$ of an equivalence relation R on X is the image by the change of base along the initial morphism $\alpha_X \colon 1 \rightarrowtail X$ of the normal local representation in $\mathsf{Pt}_X(\mathbb{C})$:

$$\begin{tikzcd} R \arrow[r, tail, "\langle d_0^R, d_1^R \rangle"] \arrow[dr, shift right, "d_0^R"'] & X \times X \arrow[d, shift right, "p_0"'] \\ & X \arrow[ul, shift right, "s_0^R"'] \arrow[u, shift right, "s_0"'] \end{tikzcd}$$

So when $\mathbb{C}$ satisfies condition (C), we have $[R, W] = 0$, i.e., the local representations of R and W commute in $\mathsf{Pt}_X(\mathbb{C})$ as soon as their normalisations commute in $\mathbb{C}$.

Conversely, suppose that the condition (SH) holds. Let $(f, s) \colon X \rightleftarrows Y$ be an object in $\mathsf{Pt}_Y(\mathbb{C})$ and (R, W) a pair of equivalence relations on it. Denote by j_R and j_W their normalisations in $\mathsf{Pt}_Y(\mathbb{C})$:

$$\begin{tikzcd} I_R \arrow[r, tail, "j_R"] \arrow[dr, shift right, "\pi_R"'] & X \arrow[d, shift right, "f"'] & I_W \arrow[l, tail, "j_W"'] \arrow[dl, shift left, "\pi_W"] \\ & Y \arrow[ul, shift right, "\sigma^R"'] \arrow[u, shift right, "s"'] \arrow[ur, shift left, "\sigma^W"] & \end{tikzcd}$$

Supposing that their images by some change of base functor y^* commute implies that their images $\tilde{j}_R$ and $\tilde{j}_W$ by α_Y^*—that is to say, the respective kernels in the diagram below—commute in $\mathbb{C}$:

$$\begin{tikzcd} K[\pi_R] \arrow[r, tail, "\tilde{j}_R"] & K[f] \arrow[d, dashed, "k_f"'] & K[\pi_W] \arrow[l, tail, "\tilde{j}_W"'] \\ & X & \end{tikzcd}$$

Accordingly the two monomorphisms $k_f \circ \tilde{j}_R$ and $k_f \circ \tilde{j}_W$ do commute in $\mathbb{C}$. But $k_f \circ \tilde{j}_R$ and $k_f \circ \tilde{j}_W$ are the normalisations of R and W in $\mathbb{C}$. Now, since $\mathbb{C}$

satisfies (SH), then we get $[R, W] = 0$ in $\mathbb{C}$ and thus in $\mathsf{Pt}_Y(\mathbb{C})$, which implies that their normalisations j_R and j_W in $\mathsf{Pt}_Y(\mathbb{C})$ commute. □

Corollary 2.1. *If $\mathbb{C}$ is a finitely complete Mal'tsev category which satisfies* (C)*, then any fibre $\mathsf{Pt}_Y(\mathbb{C})$ satisfies* (SH)*. When, in addition, $\mathbb{C}$ is pointed, if it satisfies* (SH)*, then so does any fibre $\mathsf{Pt}_Y(\mathbb{C})$.*

Proof. This is a straightforward consequence of the previous theorem and of Proposition 2.1. □

2.3 Topological Mal'tsev models

Let $\mathbb{T}$ be a (finitary) Mal'tsev theory, $\mathsf{Set}^{\mathbb{T}}$ the corresponding variety of $\mathbb{T}$-algebras and $\mathsf{Top}^{\mathbb{T}}$ the category of topological $\mathbb{T}$-algebras. Recall that $\mathsf{Top}^{\mathbb{T}}$ is then a regular Mal'tsev category, see [18], whose regular epimorphisms are the open surjective morphisms. It is clearly finitely complete and cocomplete. In this section we shall show that when the variety $\mathsf{Set}^{\mathbb{T}}$ satisfies condition (C), so does $\mathsf{Top}^{\mathbb{T}}$. In particular, this will imply the well-known fact that the category $\mathsf{Gp}(\mathsf{Top})$ of topological groups (= $\mathsf{Top}^{\mathbb{T}}$ for $\mathbb{T}$ the theory of groups) satisfies (SH).

To see this, let us first recall that the functor $U\colon \mathsf{Top}^{\mathbb{T}} \to \mathsf{Set}^{\mathbb{T}}$ forgetting the topological data is topological [23] and, consequently, left exact. Hence it is cotopological [2, Proposition 7.3.6] and, consequently, right exact. This implies that the functor U is faithful.

Lemma 2.1. *Let $\mathbb{T}$ be a Mal'tsev theory and the following diagram a pullback of split epimorphisms in $\mathsf{Top}^{\mathbb{T}}$:*

$$\begin{tikzcd}
P \arrow[r, shift right, "\phi"'] \arrow[d, shift right, "\phi'"'] & X' \arrow[l, shift right, "\sigma"'] \arrow[d, shift right, "f'"'] \\
X \arrow[r, shift right, "f"'] \arrow[u, shift right, "\sigma'"'] & Y \arrow[l, shift right, "s"'] \arrow[u, shift right, "s'"']
\end{tikzcd}$$

then P is endowed with the final topology with respect to the pair

$$\begin{tikzcd}
U(P) & U(X'), \arrow[l, "U(\sigma)"'] \\
U(X) \arrow[u, "U(\sigma')"] &
\end{tikzcd}$$

namely, P is the $\mathbb{T}$-algebra $U(P)$ endowed with the finest topology making $U(P)$ a topological algebra and the pair $(U(\sigma), U(\sigma'))$ a pair of continuous homomorphisms.

Proof. Since the functor U is cotopological, we can endow the $\mathbb{T}$-algebra $U(P)$ with the final topology with respect to the pair in question. This defines the object $\bar{P}$ and the following lower diagram in $\mathsf{Top}^{\mathbb{T}}$ above the given pair:

$$\begin{tikzcd} P & & \\ & \bar{P} \arrow[ul, dotted] & X' \arrow[l, "\bar{\sigma}"] \arrow[ull, "\sigma"'] \\ & X \arrow[u, "\bar{\sigma}'"'] \arrow[uul, "\sigma'"] & \end{tikzcd}$$

By the universal property of the final topology, there exists a factorisation $\iota\colon \bar{P} \to P$ making the diagram above commute. In other words, the topology $\bar{P}$ on $U(P)$ is finer than the topology P and $\iota = 1_{U(P)}\colon \bar{P} \to P$ is continuous. This morphism is clearly a monomorphism in $\mathsf{Top}^{\mathbb{T}}$. Now $\mathsf{Top}^{\mathbb{T}}$ is a Mal'tsev category, the fibre $\mathsf{Pt}_Y(\mathsf{Top}^{\mathbb{T}})$ is unital, and the pair (σ, σ') is jointly strongly epic, which implies that the monomorphism ι is an isomorphism, and means that the topologies P and $\bar{P}$ on $U(P)$ coincide. □

Proposition 2.4. *Let $\mathbb{T}$ be a Mal'tsev theory such that the variety $\mathsf{Set}^{\mathbb{T}}$ satisfies condition* (C)*. Then so does the category $\mathsf{Top}^{\mathbb{T}}$.*

Proof. Let us consider the following pair of normal monomorphisms in the fibre $\mathsf{Pt}_Y(\mathsf{Top}^{\mathbb{T}})$:

$$\begin{tikzcd} I_R \arrow[r, tail, "j_R"] \arrow[dr, shift right, "\pi_R"'] & X \arrow[d, shift right, "f"'] & I_S \arrow[l, tail, "j_S"'] \arrow[dl, shift left, "\pi_S"] \\ & Y \arrow[u, shift right, "s"'] \arrow[ul, shift right, "\sigma^R"'] \arrow[ur, shift left, "\sigma^S"] & \end{tikzcd}$$

and the following pullback in $\mathsf{Top}^{\mathbb{T}}$:

$$\begin{tikzcd} X' \arrow[r, "x"] \arrow[d, shift right, "f'"'] & X \arrow[d, shift right, "f"'] \\ Y' \arrow[r, "y"'] \arrow[u, shift right, "s'"'] & Y \arrow[u, shift right, "s"'] \end{tikzcd}$$

Suppose that $y^*(j_R)$ and $y^*(j_S)$ commute in the fibre $\mathsf{Pt}_{Y'}(\mathsf{Top}^{\mathbb{T}})$; then this is the case for their images by the functor U. Since $\mathsf{Set}^{\mathbb{T}}$ satisfies condition (C), the images $U(j_R)$ and $U(j_S)$ commute in $\mathsf{Set}^{\mathbb{T}}$. This means that there is a $\mathbb{T}$-homomorphism ϕ such that $\phi \circ U(s_R) = U(j_R)$ and $\phi \circ U(s_S) = U(j_S)$ in the following diagram, where the whole quadrangle is the image by U of a pullback of split epimorphisms in $\mathsf{Top}^{\mathbb{T}}$:

$$\begin{tikzcd}
& U(I_R \times_Y I_S) \arrow[dl, shift right, "U(p_R)"'] \arrow[dr, shift left, "U(p_S)"] \arrow[d, "\phi"'] & \\
U(I_R) \arrow[ur, shift right, "U(s_R)"'] \arrow[r, tail, "U(j_R)"] \arrow[dr, shift right, "U(\pi_R)"'] & U(X) \arrow[d, shift right, "U(f)"'] & U(I_S) \arrow[ul, shift left, "U(s_S)"] \arrow[l, tail, "U(j_S)"'] \arrow[dl, shift left, "U(\pi_S)"] \\
& U(Y) \arrow[u, shift right, "U(s)"'] \arrow[ul, shift right, "U(\sigma^R)"'] \arrow[ur, shift left, "U(\sigma^S)"] &
\end{tikzcd}$$

This means that the "restrictions" $\phi \circ U(s_R)$ and $\phi \circ U(s_S)$ of the $\mathbb{T}$-homomorphism ϕ to the "subobjects" $U(I_R)$ and $U(I_S)$ are the continuous $\mathbb{T}$-homomorphisms j_R and j_S. By the previous lemma, $I_R \times_Y I_S$ is endowed with the final topology with respect to the pair $U(s_R)$ and $U(s_S)$, which implies that the $\mathbb{T}$-homomorphism ϕ is itself continuous: $I_R \times_Y I_S \to X$ and actually lies in $\mathsf{Top}^{\mathbb{T}}$. This means precisely that j_R and j_S commute in the fibre $\mathsf{Pt}_Y(\mathsf{Top}^{\mathbb{T}})$. □

3 Applications of the condition (SH)

In this section we shall extend some results known for strongly protomodular categories to a context merely satisfying the condition (SH).

3.1 Discrete fibrations of reflexive graphs

First observe that in a finitely complete category $\mathbb{E}$, any split epimorphism $(f, s)\colon X \rightleftarrows Y$ is actually the domain of the kernel of a split epimorphism in the pointed fibre $\mathsf{Pt}_Y(\mathbb{E})$:

$$\begin{tikzcd}
X \arrow[r, tail, "{\langle f,1_X\rangle}"] \arrow[dr, shift right, "f"'] & Y \times X \arrow[r, shift left, "Y\times f"] \arrow[d, "p_Y"'] & Y \times Y \arrow[l, tail, shift left, "Y\times s"] \arrow[dl, shift right, "p_Y"'] \\
& Y \arrow[ul, shift right, "s"'] \arrow[ur, shift right, "s_0"'] &
\end{tikzcd}$$

and thus it produces the normal monomorphism $\langle f, 1_X\rangle$ in $\mathsf{Pt}_Y(\mathbb{E})$.

On the other hand, it is known from [14] that, in a Mal'tsev category $\mathbb{C}$, a reflexive graph is endowed with at most one structure of internal category, and that any internal category is a groupoid. Let us recall (from [9]) another proof of this result which sheds a new light on the nature of the uniqueness of the groupoid structure. From any reflexive graph

$$\begin{tikzcd}
Y_1 \arrow[r, shift left=2, "d_0"] \arrow[r, shift right=2, "d_1"'] & Y_0 \arrow[l, "s_0" description]
\end{tikzcd}$$

in $\mathbb{C}$, we get two normal subobjects in $\mathsf{Pt}_{Y_0}(\mathbb{C})$:

$$\begin{tikzcd}
Y_1 \arrow[r, tail, "{\langle d_0,1_{Y_1}\rangle}"] \arrow[dr, shift right, "d_0"'] & Y_0 \times Y_1 \arrow[d, shift right, "p_{Y_0}"'] & Y_1 \arrow[l, tail, "{\langle d_1,1_{Y_1}\rangle}"'] \arrow[dl, shift left, "d_1"] \\
& Y_0 \arrow[ul, shift right, "s_0"'] \arrow[u, shift right, "{\langle 1_{Y_0},s_0\rangle}"'] \arrow[ur, shift left, "s_0"] &
\end{tikzcd}$$

We can now assert the following:

Proposition 3.1. *Let $\mathbb{C}$ be a finitely complete Mal'tsev category. The reflexive graph in question is a groupoid if and only if these two normal subobjects commute in $\mathsf{Pt}_{Y_0}(\mathbb{C})$.*

Proof. The two subobjects commute in $\mathsf{Pt}_{Y_0}(\mathbb{C})$ if and only if they have a cooperator $\phi\colon Y_1 \times_{Y_0} Y_1 \to Y_0 \times Y_1$, i.e. a morphism satisfying $\phi\circ s_0 = \langle d_1, 1_{Y_1}\rangle$

and $\phi\circ s_1 = \langle d_0, 1_{Y_1}\rangle$:

$$\begin{tikzcd}
 & Y_1 \times_{Y_0} Y_1 \arrow[dl, shift right, "d_2"'] \arrow[dr, shift left, "d_0"] \arrow[d, "\phi"] & \\
Y_1 \arrow[ur, shift right, "s_1"'] \arrow[r, tail, "\langle d_0, 1_{Y_1}\rangle"] \arrow[dr, shift right, "d_0"'] & Y_0 \times Y_1 \arrow[d, shift right, "p_{Y_0}"'] & Y_1 \arrow[ul, shift left, "s_0"] \arrow[l, tail, "\langle d_1, 1_{Y_1}\rangle"'] \arrow[dl, shift left, "d_1"] \\
 & Y_0 \arrow[ul, shift right, "s_0"'] \arrow[u, shift right, "\langle 1_{Y_0}, s_0\rangle"'] \arrow[ur, shift left, "s_0"] &
\end{tikzcd}$$

where the whole quadrangle is a pullback in $\mathbb{C}$. Hence the morphism ϕ is a pair $\langle d_0\circ d_2, d_1\rangle$, where $d_1\colon Y_1 \times_{Y_0} Y_1 \to Y_1$ is such that $d_1\circ s_0 = 1_{Y_1}$ and $d_1\circ s_1 = 1_{Y_1}$. Since the morphism d_1 satisfies these two identities, it makes the reflexive graph in question *multiplicative* in the sense of [14]. And, according to Theorem 2.2 in [14], in a Mal'tsev category, any multiplicative reflexive graph is a groupoid. Conversely, the composition morphism $d_1\colon Y_1 \times_{Y_0} Y_1 \to Y_1$ of an internal category satisfies the previous two identities and produces the cooperator $\phi = \langle d_0\circ d_2, d_1\rangle$. □

Now let us consider a morphism of reflexive graphs

$$\begin{tikzcd}
X_1 \arrow[r, "f_1"] \arrow[d, shift right=2, "d_0"'] \arrow[d, shift left=2, "d_1"] & Y_1 \arrow[d, shift right=2, "d_0"'] \arrow[d, shift left=2, "d_1"] \\
X_0 \arrow[u] \arrow[r, "f_0"'] & Y_0 \arrow[u]
\end{tikzcd}$$

and recall the following result from [3, Proposition 14]:

Proposition 3.2. *When $\mathbb{C}$ is a finitely complete Mal'tsev category, then, given any morphism of reflexive graphs as above, the square indexed by* 0 *is a pullback if and only if the square indexed by* 1 *is a pullback. In such a situation this morphism is said to be a* discrete fibration *between reflexive graphs.*

We can now extend a result already known in strongly protomodular categories, see [4, Consequence B, p. 216]:

Proposition 3.3. *Let $\mathbb{C}$ be a finitely complete Mal'tsev category satisfying condition* (C)*. Given any discrete fibration of reflexive graphs, the codomain reflexive graph Y_1 is a groupoid as soon as so is the domain reflexive graph X_1.*

Proof. Since $\mathbb{C}$ satisfies condition (C), it is enough to show that the images under the change of base functor along f_0 of the two normal monomorphisms associated with the codomain reflexive graph Y_1 do commute in the fibre $\mathsf{Pt}_{X_0}(\mathbb{C})$. The two images in question are the following ones:

$$\begin{tikzcd}
X_1 \arrow[r, tail, "\langle d_0, f_1 \rangle"] \arrow[dr, shift right, "d_0"'] & X_0 \times Y_1 \arrow[d, shift right, "p_{X_0}"'] & X_1 \arrow[l, tail, "\langle d_1, f_1 \rangle"'] \arrow[dl, shift left, "d_1"] \\
& X_0 \arrow[ul, shift right, "s_0"'] \arrow[u, shift right, "\langle 1_{X_0}, f_1 \circ s_0 \rangle"'] \arrow[ur, shift left, "s_0"] &
\end{tikzcd}$$

since the morphism of reflexive graphs is a discrete fibration. They do commute in $\mathsf{Pt}_{X_0}(\mathbb{C})$, being given by the following composition in this fibre, where the horizontal part commutes since the reflexive graph X_1

$$\begin{tikzcd}
X_1 \arrow[r, tail, "\langle d_0, 1_{X_1} \rangle"] \arrow[dr, "\langle d_0, f_1 \rangle"'] & X_0 \times X_1 \arrow[d, "X_0 \times f_1"] & X_1 \arrow[l, tail, "\langle d_1, 1_{X_1} \rangle"'] \arrow[dl, "\langle d_1, f_1 \rangle"] \\
& X_0 \times Y_1 &
\end{tikzcd}$$

is a groupoid. □

3.2 The condition (SH) and commutators

In this section we shall prove that, as expected, in the Mal'tsev context, under (SH) the Smith and the Huq commutators in the sense of [6] do coincide.

3.2.1 The Huq commutator in a unital category

We shall suppose here that $\mathbb{C}$ is a unital category which is moreover finitely cocomplete. In this context, in [6] there was given a construction, for any pair $f\colon X \to Z$, $g\colon Y \to Z$ of morphisms with the same codomain, of a morphism which universally makes them cooperate. Indeed consider the following diagram, where $Q[\![f,g]\!]$ is the colimit of the diagram made of the plain arrows:

$$
\begin{tikzcd}
& X \arrow[dl, "l_X"'] \arrow[dr, "f"] \arrow[d, dotted, "\bar{\phi}_X"'] & \\
X \times Y \arrow[r, dotted, "\bar{\phi}"'] & Q[\![f,g]\!] & Z \arrow[l, dotted, "\bar{\psi}"] \\
& Y \arrow[ul, "r_Y"] \arrow[ur, "g"'] \arrow[u, dotted, "\bar{\phi}_Y"] &
\end{tikzcd}
$$

Clearly the morphisms $\bar{\phi}_X$ and $\bar{\phi}_Y$ are completely determined by the pair $(\bar{\phi}, \bar{\psi})$, and clearly the morphism $\bar{\phi}$ is the cooperator of the pair $(\bar{\psi}\circ f, \bar{\psi}\circ g)$. On the other hand, the strong epimorphism $\bar{\psi}$ measures how far the pair (f,g) is from cooperating, and we have [6]:

Proposition 3.4. *Suppose $\mathbb{C}$ finitely cocomplete and unital. Then $\bar{\psi}$ is the universal morphism which, by composition, makes the pair (f,g) cooperate. The morphism $\bar{\psi}$ is an isomorphism if and only if the pair (f,g) cooperates.*

Since the morphism $\bar{\psi}$ is a strong epimorphism, its distance from being an isomorphism is its distance from being a monomorphism, which is exactly measured by its kernel relation $R[\bar{\psi}]$, whence the following definition:

Definition 3.1 (Huq commutator)**.** Given any pair (f,g) of morphisms with the same codomain in a finitely cocomplete unital category $\mathbb{C}$, their *Huq commutator* $[\![f,g]\!]$ is the kernel relation $R[\bar{\psi}]$.

When the category $\mathbb{C}$ is moreover regular [1], i.e., such that the strong epimorphisms are stable by pullback and any effective equivalence relation (= kernel pair) admits a quotient, we can add some piece of information. First, any morphism $f\colon X \to Z$ has a canonical regular epi/mono factorisation $X \twoheadrightarrow f(X) \rightarrowtail Z$, and the morphism $f(X) \rightarrowtail Z$ is then called the *image* of the morphism f. Secondly, two morphisms f and g cooperate if and only if their images $f(X) \rightarrowtail Z$ and $g(Y) \rightarrowtail Z$ do.

3.2.2 The Smith commutator in a Mal'tsev category

We shall suppose here that $\mathbb{C}$ is finitely complete and cocomplete, regular Mal'tsev category. In a regular Mal'tsev category, given a regular epimorphism $f\colon X \twoheadrightarrow Y$, any equivalence relation R on X has a *direct image* $f(R)$

along f on Y. It is given by the regular epi/mono factorisation of the morphism

$$\langle f\circ d_0, f\circ d_1\rangle \colon R \twoheadrightarrow f(R) \rightarrowtail Y \times Y$$

Clearly in any regular category $\mathbb{C}$, the relation $f(R)$ is reflexive and symmetric; when moreover $\mathbb{C}$ is Mal'tsev, $f(R)$ is an equivalence relation.

Now let us recall the following results and definition from [6]: first consider the following diagram, in which $Q[R,S]$ is the colimit of the plain arrows:

$$\begin{tikzcd} & R \arrow[dl, "l_R"'] \arrow[d, dotted, "\phi_R"] \arrow[dr, "d_0^R"] & \\ R \times_X S \arrow[r, dotted, "\phi"'] & Q[R,S] & X \arrow[l, dotted, "\psi"] \\ & S \arrow[ul, "r_S"] \arrow[u, dotted, "\phi_S"'] \arrow[ur, "d_1^S"'] & \end{tikzcd}$$

Notice that, here, in consideration of the pullback defining $R \times_X S$ (diagram $(*)$), the roles of the projections d_0 and d_1 have been interchanged. As in the section above, the morphisms ϕ_R and ϕ_S are completely determined by the pair (ϕ, ψ) and the morphism ψ is a strong epimorphism (and thus a regular epimorphism in our regular context). This morphism ψ measures how far the equivalence relations R and S are from centralising each other:

Proposition 3.5. *Let $\mathbb{C}$ be a finitely complete and cocomplete, regular Mal'tsev category. The morphism ψ is the universal regular epimorphism which makes the direct images $\psi(R)$ and $\psi(S)$ centralise each other (i.e. $[R,S]=0$). The equivalence relations R and S centralise each other if and only if ψ is an isomorphism.*

Since the morphism ψ is a regular epi, its distance from being an isomorphism is its distance from being a monomorphism, which is exactly measured by its kernel relation $R[\psi]$. Accordingly, it is meaningful to introduce the following definition:

Definition 3.2 (Smith commutator)**.** Let $\mathbb{C}$ be a finitely complete and cocomplete, regular Mal'tsev category. Consider in $\mathbb{C}$ two equivalence relations $(d_0^R, d_1^R) \colon R \rightrightarrows X$ and $(d_0^S, d_1^S) \colon S \rightrightarrows X$ on the same object X. The kernel relation $R[\psi]$ of the morphism ψ is called the *Smith commutator* of R and S. We shall use the classical notation $[R,S]$ for this commutator.

Example 3.1. If we suppose moreover that the category $\mathbb{C}$ is Barr exact [1]—namely such that any equivalence relation is effective, i.e., the kernel relation of some morphism—then, thanks to Theorem 3.9 in [21], the previous definition is equivalent to the definition of [21], and accordingly to the definition of Smith [22] in the Mal'tsev-varietal context. On the other hand, one of the advantages of this definition is that it extends the meaning and existence of commutator from the exact Mal'tsev context to the regular Mal'tsev context, enlarging the range of examples to the Mal'tsev quasi-varieties and to the topological Mal'tsev models, as the category $\mathsf{Gp}(\mathsf{Top})$ of topological groups for instance.

The example $\mathsf{Gp}(\mathsf{Top})$ is also interesting for the following reason. In a Mal'tsev category $\mathbb{C}$, given any pair (R, S) of equivalence relations on an object X, we obtain $[R, S] = 0$ as soon as the intersection $R \cap S$ is the discrete equivalence relation Δ_X. When the Mal'tsev $\mathbb{C}$ is not only regular, but also exact (which means that any equivalence relation is *effective*, i.e., the kernel relation of some morphism), this implies that we necessarily have $[R, S] \leqslant R \cap S$. Indeed, since $\mathbb{C}$ is exact, we may take the quotient $q\colon X \twoheadrightarrow Q$ of the equivalence relation $R \cap S$; then we see that $q(R) \cap q(S) = \Delta_Q$ as in any regular category. Accordingly, $[q(R), q(S)] = 0$. When, in addition, $\mathbb{C}$ is finitely cocomplete Mal'tsev, we have a factorisation $\xi\colon Q[R, S] \twoheadrightarrow Q$ and the inclusion $[R, S] \leqslant R \cap S$. The regular (but not exact) Mal'tsev category $\mathsf{Gp}(\mathsf{Top})$ provides a setting in which this inclusion does not hold: see [7, Proposition 5.3].

3.3 Commutators in the pointed Mal'tsev setting

From now on $\mathbb{C}$ will be a regular pointed Mal'tsev category. Recall from [6] that, on the one hand, if $f\colon X \twoheadrightarrow Y$ is a regular epimorphism and R an equivalence relation on X, then the normal subobject $j(f(R))$ associated with $f(R)$ is the direct image $f(j(R))$ along f of the normal subobject $j(R)$ associated with R. On the other hand, we get:

Proposition 3.6. *Let $\mathbb{C}$ be a finitely complete and cocomplete, regular, pointed Mal'tsev category. Then, given any pair (R, S) of equivalence relations on an object X, there is a natural comparison $\zeta\colon Q[\![j(R), j(S)]\!] \twoheadrightarrow Q[R, S]$,* *and consequently we have* $[\![j(R), j(S)]\!] \leq [R, S]$, *namely an inclusion of the Huq commutator into the Smith commutator.*

Proof. Consider the morphism $\psi \colon X \twoheadrightarrow Q[R, S]$. We have $[\psi(R), \psi(S)] = 0$, so that

$$[\![\psi(j(R)), \psi(j(S))]\!] = [\![j(\psi(R)), j(\psi(S))]\!] = 0.$$

Hence the two morphisms $\psi \circ j(R)$ and $\psi \circ j(S)$ commute. Now thanks to the universal property of the morphism $\bar{\psi} \colon X \twoheadrightarrow Q[\![j(R), j(S)]\!]$, there is a unique factorisation $\zeta \colon Q[\![j(R), j(S)]\!] \to Q[R, S]$ such that $\zeta \circ \bar{\psi} = \psi$, and thus an inclusion $[\![j(R), j(S)]\!] \leqslant [R, S]$ of the Huq commutator into the Smith commutator. □

Exactly in the same way as for strongly protomodular categories [6], we can now assert:

Theorem 3.1. *Let* $\mathbb{C}$ *be a finitely complete and cocomplete, pointed and regular Mal'tsev category satisfying* (SH)*. Then, given any pair* (R, S) *of equivalence relations on an object X, the natural comparison* $\zeta \colon Q[\![j(R), j(S)]\!] \to Q[R, S]$ *is an isomorphism, and consequently we have* $[\![j(R), j(S)]\!] = [R, S]$, *namely the Smith and the Huq commutators coincide.*

Proof. Consider the morphism $\bar{\psi} \colon X \twoheadrightarrow Q[\![j(R), j(S)]\!]$. Then we get:

$$[\![j(\bar{\psi}(R)), j(\bar{\psi}(S))]\!] = [\![\bar{\psi}(j(R)), \bar{\psi}(j(S))]\!] = 0$$

Now thanks to condition (SH), we have that $[\bar{\psi}(R), \bar{\psi}(S)] = 0$. Then the universal property of the morphism $\psi \colon X \twoheadrightarrow Q[R, S]$ produces a unique factorisation $\theta \colon Q[R, S] \to Q[\![j(R), j(S)]\!]$ which is necessarily an inverse of ζ (see Proposition 3.6), and thus an isomorphism $[R, S] \cong [\![j(R), j(S)]\!]$. Hence the two notions of commutator coincide. □

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