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NORMALITY, COMMUTATION AND SUPREMA IN THE REGULAR MAL'TSEV AND PROTOMODULAR SETTINGS

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dedicated to René Guitart on the occasion of his sixty-fifth birthday

Résumé. Dans le contexte des catégories régulières de Mal'tsev et protomodulaires, nous développons les conséquences d'une caractérisation, acquise dans le simple cadre des catégories unitales sans condition de colimites, du fait que le sup de deux sous-objets qui commutent est leur commun codomaine. Nous retrouvons ainsi, mais avec des preuves conceptuelles, quelques résultats bien connus de la catégorie *Gp* des groupes.

Abstract. We develop, in the contexts of regular Mal'tsev and protomodular categories, the consequences of a characterization, obtained in a mere unital category without any cocompleteness assumption, of the fact that the supremum of two commuting subobjects is their common codomain. In this way we recover, with conceptual proofs, some well-known results in the category *Gp* of groups.

Keywords. Fibration of points, Mal'tsev and protomodular category, commutation of subobjects, centralisation of equivalence relations

Mathematics Subject Classification (2010). 08B05, 18C99, 18D30, 18D35, 20F12, 54H11

Introduction

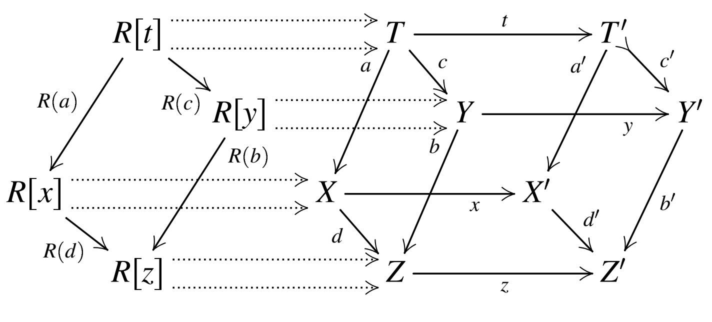
This work is devoted to unfold as far as possible the consequences of the mere observation according to which, in the unital setting, the construction of the supremum of two subobjects does not need on the ground category more cocompleteness than the regular [1] assumption provided that these two subobjects commute (Lemma 2.1). From that, in a pointed regular Mal'tsev setting, such a pair of subobjects is necessarily a pair of normal subobjects, and, in the homological setting, the supremum of two commuting normal subobjects is necessarily normal. We are then able to set in the homological setting a result already noticed in the much stricter context of semi-abelian categories by T. Everaert and M. Gran (and published in [14]), namely that, given two equivalence relations (R, S) on an object X such that the supremum of their normalizations is 1_X, they centralize each other if and only if their normalizations commute.

From that, we can derive two non-pointed applications:
1) in a regular Mal'tsev category, the centralizers of equivalence relations are stable under product provided that their base objects have global supports
2) in a regular protomodular category, the change of base functors f^* : $Pt_Y\mathbb{C} \to Pt_X\mathbb{C}$ with respect to the fibration of points, reflect the commutation of normal subobjects if and only if they reflect the mutual centralization of equivalence relations.
This last point extends to the non-pointed context some aspects of results obtained in different pointed situations in [9] and [19].

1 Direct image of a normal monomorphism

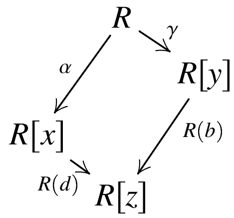
In this article any category will be assumed to be finitely complete. The aim of this section is to show that, in a regular Mal'tsev category, the direct image of a normal monomorphism along a regular epimorphism is still a normal monomorphism. Let us begin by the following observation:

Lemma 1.1. *Given any commutative right hand side cube in $\mathbb{E}$:*



where the face containing a and b is a pullback and the map c' is a monomorphism, the left hand side square given by the extensions to the kernel equivalence relations of the central quadrangles is a pullback.

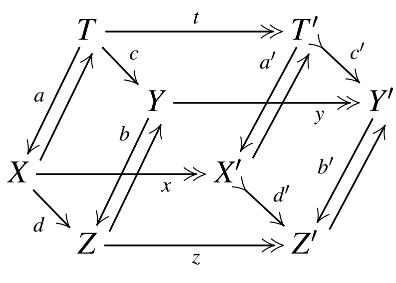
Proof. We denote by $R[f]$ the kernel equivalence relation of the morphism f. Now consider the following pullback of equivalence relations:



It produces an equivalence relation $R \rightrightarrows T$ on the object T and consequently a monomorphism of equivalence relations $j : R[t] \rightarrowtail R$. On the other hand $R \rightrightarrows T$ is coequalized by the map t since it is coequalized by $y.c = c'.t$ where c' is a monomorphism. Accordingly we have an inclusion $j' : R \rightarrowtail R[t]$ in the other direction. □

Recall that a Mal'tsev category is a category in which any reflexive relation is an equivalence relation, see [10], [11].

Proposition 1.1. *Suppose $\mathbb{C}$ is a regular Mal'tsev category. Consider any cube satisfying the previous conditions:*



and which, in addition, is such that x, y, z, t are regular epimorphisms and the maps a, b, a', b' are split epimorphisms such that all the squares are morphisms of split epimorphisms. Then the square with (d', c') is a pullback of split epimorphisms.

Proof. Since c' is a monomorphism and (d', c') is a morphism of split epimorphisms, the map d' is a monomorphism as well. Now let Θ be the vertex of the pullback of b' along d' and $\tau : T \rightarrow \Theta$ the induced factorization. According to the previous lemma we have $R[t] \simeq R[\tau]$:

$$\begin{array}{ccccc} R[t] & \underset{p_1}{\overset{p_0}{\rightleftarrows}} & T & \overset{t}{\twoheadrightarrow} & T' \\ \simeq\downarrow & & \| & & \downarrow j \\ R[\tau] & \underset{p_1}{\overset{p_0}{\rightleftarrows}} & T & \underset{\tau}{\rightarrow} & \Theta \end{array}$$

and the induced factorization j is a monomorphism. Since $\mathbb{C}$ is a regular Mal'tsev category and both T and Θ are the vertices of pullbacks of split epimorphisms, the factorization τ is a regular epimorphism (see Lemma 2.5.7 in [2]), since so are x, y and z; consequently the map j is also a regular epimorphism, and thus an isomorphism. □

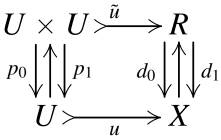
Given a finitely complete category $\mathbb{E}$, recall that $Pt(\mathbb{E})$ denotes the category whose objects are the split epimorphisms in $\mathbb{E}$ and whose arrows are the commuting squares between such split epimorphisms, and that $¶_{\mathbb{E}}$: $Pt(\mathbb{E}) \rightarrow \mathbb{E}$ denotes the functor associating its codomain with any split epimorphism: it is *the fibration of points*. The ¶-cartesian maps are nothing but the pullbacks of split epimorphisms. Recall that a category $\mathbb{C}$ is protomodular [3] when any change of base functor with respect to this fibration reflects isomorphisms.

Corollary 1.1. *Let $\mathbb{C}$ be a regular Mal'tsev category. Then, in the category $Pt\mathbb{C}$, the images along a regular epimorphism of a ¶-cartesian monomorphism is a ¶-cartesian monomorphism. In the category $Grd\mathbb{C}$, the images along a regular epimorphic functor of a monomorphic discrete fibration is a monomorphic discrete fibration.*

Proof. This comess from the fact that in $Pt\mathbb{C}$ the regular epimorphisms are levelwise, and in $Grd\mathbb{C}$ as well, see [13]. □

Recall from [3] the following:

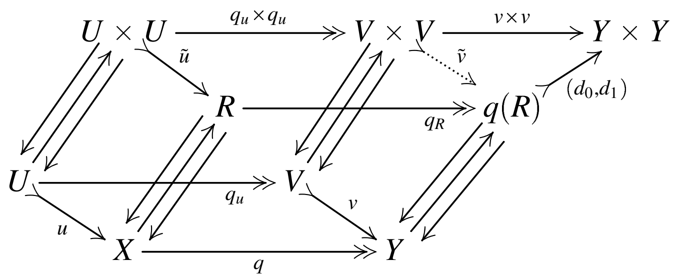
Definition 1.1. A monomorphism u in $\mathbb{E}$ is said to be normal to an equivalence relation R when: i) we have: $u^{-1}(R) = \nabla_U$ the indiscrete relation on U ii) the induced internal functor is a discrete fibration:



In the set theoretical context, when U is not empty, it is equivalent to saying that U is an equivalence class of R. Clearly, in this context, a monomorphism can be normal to several equivalence relations. It is also the case in a Mal'tsev setting. In a protomodular category a monomorphism is normal to at most one equivalence relation, see [2]; any kernel monomorphism is normal, but a normal monomorphism in this algebraic sense is not necessarily a kernel one.

Corollary 1.2. *Let $\mathbb{C}$ be a regular Mal'tsev category. Let $q : X \twoheadrightarrow Y$ be a regular epimorphism, R an equivalence relation on X and $u : U \rightarrowtail X$ a monomorphism which is normal to the equivalence relation R. Then the direct image $v : V \rightarrowtail Y$ of u along q is normal to the direct image $q(R)$ of R along q.*

Proof. Consider the following diagram:

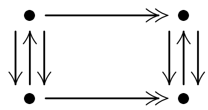


Since $q_u \times q_u$ is a regular epimorphism and (d_0, d_1) a monomorphism, there is a factorization $\tilde{v}$ which shows that $v^{-1}(q(R)) = \nabla_V$. The previous proposition shows that $(v, \tilde{v})$ is underlying a discrete fibration since so is $(u, \tilde{u})$. □

This corollary shows that a weaker version of the Hofmann axiom [15] following which the kernel monomorphisms are stable under direct image is already valid in the non-pointed context and under the mild assumption that the base category is a regular Mal'tsev one. Finally we get:

Proposition 1.2. *Let $\mathbb{C}$ be a regular Mal'tsev category. Then, in the category $Pt\mathbb{C}$, the direct image along a regular epimorphism of a ¶-cartesian equivalence relation is a ¶-cartesian equivalence relation.*

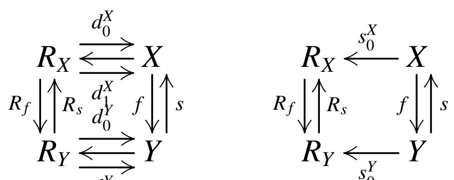
Proof. The regular epimorphisms in $Pt\mathbb{C}$ are levelwise. Consider the following diagram in $Pt\mathbb{C}$:



where the vertical parts are equivalence relations. Suppose the left hand side one is ¶-cartesian which means that any of its maps is ¶-cartesian. This is the case in particular for the vertical monomorphism. According to Corollary 1.1 the vertical monomorphism on the right hand side is ¶-cartesian as well. The fact that the whole relation is ¶-cartesian is a consequence of the following lemma. □

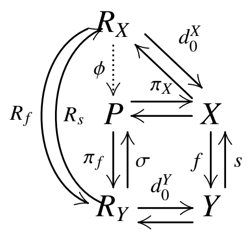
Lemma 1.2. *Suppose $\mathbb{C}$ is a Mal'tsev category. Then an equivalence relation in $Pt\mathbb{C}$ is ¶-cartesian if and only if its subdiagonal is ¶-cartesian. Any equivalence relation contained in ¶-cartesian one is itself ¶-cartesian.*

Proof. Let be given any equivalence relation in $Pt\mathbb{C}$ such that the right hand side square is a pullback:



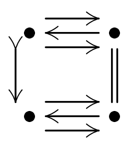
The pullback of R_f along s_0^Y is nothing but the equivalence relation $R_X \cap R[f]$. Accordingly, saying that the right hand side square is a pullback is equivalent to saying that the intersection $R_X \cap R[f]$ is the discrete equivalence relation

Δ_X. Now consider the following diagram where the lower square is a pullback and ϕ is the natural factorization:



Thanks to the Yoneda embedding, it is easy to check that, in any kind of category, $R_X \cap R[f] = \Delta_X$ implies that ϕ is a monomorphism. When, in addition, the category $\mathbb{C}$ is a Mal'tsev category, the factorization ϕ, being involved in a pullback of split epimorphisms, is necessarily a strong epimorphism. Accordingly ϕ is an isomorphism and the leg d_0 of the relation R in $Pt\mathbb{C}$ is ¶-cartesian. Clearly the same holds for the leg d_1.

Consider the following inclusion of equivalence relations in $Pt\mathbb{C}$:



Suppose the lower one is ¶-cartesian. Its subdiagonal is ¶-cartesian. Since the vertical left hand side map is a monomorphism, the leftward square is a pullback, so that the upper subdiagonal is ¶-cartesian, and so is the whole upper equivalence relation. □

2 Commutation and supremum

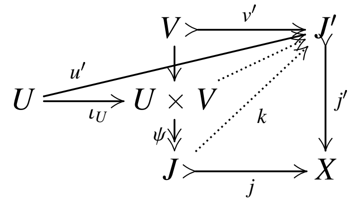
In this section, we shall show that, in a homological category, given two equivalence relations R and S on an object X, if the supremum of their normalizations is 1_X, then R and S centralize each other as soon as u and v commute in the sense of [16] and [4]. Let us begin by the following:

Lemma 2.1. *Let $\mathbb{C}$ be a unital category, and $(u : U \rightarrowtail X, v : U \rightarrowtail X)$ a pair of commuting subobjects with cooperator $\phi : U \times V \to X$. Then 1_X is the supremum of the pair (u, v) if and only if ϕ is an extremal epimorphism.*

Suppose, in addition, that $\mathbb{C}$ is regular; the supremum of a commuting pair (u, v) of monomorphisms is the image of their cooperator ϕ.

Proof. Suppose we have $1_X = u \vee v$ and a factorization $\phi = j.\psi$ with $j : J \rightarrowtail X$ a monomorphism. Then we have factorizations $\psi.\iota_U : U \rightarrowtail J$ and $\psi.\iota_V : V \rightarrowtail J$; and j is an isomorphism. Accordingly ϕ is an extremal epimorphism. Conversely suppose ϕ is an extremal epimorphism. and $j : J \rightarrowtail X$ a monomorphism with factorizations $u' : U \to J$, $v' : V \to J$. Since u and v commute and j is a monomorphism, so do u' and v'. Whence a map $\psi : U \times V \to J$ such that $\phi = j.\psi$. Since ϕ is an extremal epimorphism, the monomorphism j is an isomorphism.

Suppose in addition that $\mathbb{C}$ is regular. Let (u, v) be a pair of commuting subobjects with cooperator $\phi : U \times V \to X$, and $\phi = j.\psi$ its canonical regular decomposition. Now consider the following diagram:



where J' is a subobject containing U and V. Since j' is a monomorphism, and u and v commute, so do u' and v'; whence a cooperator $\psi' : U \times V \to J'$ such that $j'.\psi' = \phi = \psi.j$. Now, since ψ is a regular epimorphism and j' a monomorphism, we get the desired factorization k. □

From that, we can extend, rather unexpectedly, to any pointed regular Mal'tsev setting, a well known result of the category Gp of groups:

Proposition 2.1. *Let $\mathbb{C}$ be a pointed regular Mal'tsev category, and $(u : U \rightarrowtail X, v : V \rightarrowtail X)$ a pair of commuting subobjects such that their supremum is 1_X. Then u and v are normal to two equivalence relations on X which centralize each other.*

Proof. A pointed Mal'tsev category is unital. According to the previous lemma the cooperator $\phi : U \times Y \to X$ is a regular epimorphism. The inclusion $\iota_U : U \rightarrowtail U \times V$ is normal to $\nabla_U \times V$, while $\iota_V : V \rightarrowtail U \times V$ is normal to $U \times \nabla_V$. According to Corollary 1.2, the monomorphism u is normal to the direct image $\phi(\nabla_U \times V)$ while the monomorphism v is normal to the direct image $\phi(U \times \nabla_V)$. Now since we have $(\nabla_U \times V) \cap (U \times \nabla_V) = \Delta_{U \times V}$, the equivalence relations $\nabla_U \times V$ and $U \times \nabla_V$ centralize each other, and since ϕ is a regular epimorphism, it is the case for their direct images along ϕ, see [6]. □

In a pointed category, any equivalence relation R on X produces a normal subobject called its *normalization*, just take: $Kerd_0^R \overset{k}{\rightarrowtail} R \overset{d_1^R}{\rightarrow} X$.

Corollary 2.1. *Let $\mathbb{C}$ be a homological (i.e. pointed, regular and protomodular) category. Let R and S be two equivalence relations on X. Suppose that their normalizations $u : U \rightarrowtail X$ and $v : V \rightarrowtail X$ are such that 1_X is their supremum. Then R and S centralize each other if and only if u and v commute.*

Proof. We know that the normalizations of two equivalence relations which centralize each other do commute. Conversely suppose that the normalizations u and v of R and S do commute [7]. If, moreover, 1_X is their supremum, then, according to the previous proposition and the fact that any protomodular category is a Mal'tsev one, the unique R and S of which they are the normalizations centralize each other. □

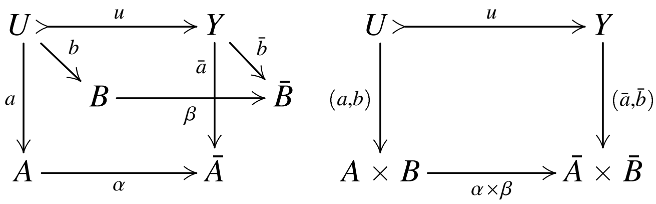
This last point was already observed in the stricter context of semi-abelian categories, see Proposition 4.6 in [14].

3 Supremum of two normal monomorphisms

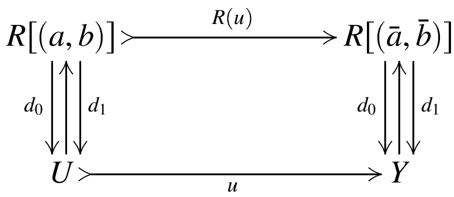
In this section, we shall show that, in a homological category, the supremum of two normal subobjects which commute is necessarily normal. Let us start with the following:

Lemma 3.1. *Let us consider, in a category $\mathbb{E}$, any left hand side diagram where any commutative square is a pullback and the map u is a monomorph-*

ism; then the right hand side square is a pullback:



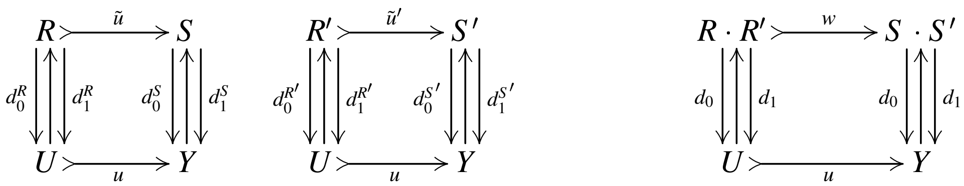
Accordingly the following diagram is a discrete fibration between equivalence relations:



Proof. The first assertion can be easily checked in Set. The second one is a consequence of the fact that the right hand side square is a pullback. □

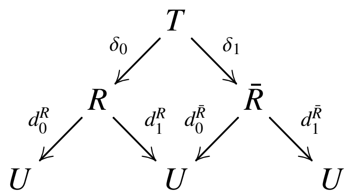
Recall that in a regular Mal'tsev category $\mathbb{C}$, the supremum of a pair (R, S) of equivalence relations on an object X is nothing but their composition $R \cdot S$.

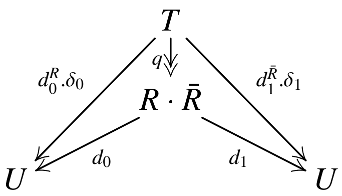
Proposition 3.1. *Let $\mathbb{C}$ be a regular Malt'sev category. Given any pair of monomorphic discrete fibrations between equivalence relations above a monomorphism u as on the left hand side below:*



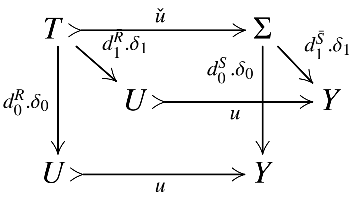
the induced monomorphism between the associated suprema, on the right hand side, is still a discrete fibration.

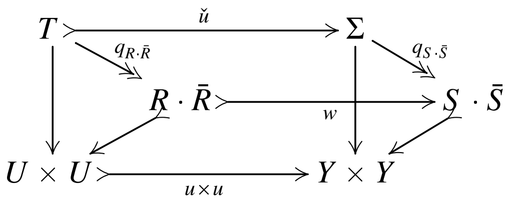
Proof. The category $\mathbb{C}$ being a regular Mal'tsev one, the supremum of the pair $(R, \bar{R})$ is $R \cdot \bar{R}$ which is given by the following construction where the upper quadrangle on the left hand side is a pullback and the pair (d_0, d_1) on the right hand side is jointly monic:



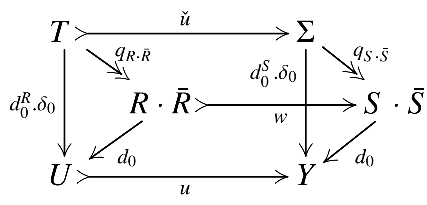


Accordingly, when we have discrete fibrations between equivalence relations, the following left hand side diagram is such that any of the commutative squares is a pullback, where Σ is defined in the same way as T with respect to $S \cdot S'$:





So, according to the previous lemma, the induced vertical rectangle on the right hand side is a pullback. Now, the category being regular, the right hand side upper quadrangle is a pullback, and the factorization w is necessarily a monomorphism. Finally, since the following rectangle is also a pullback, the lower quadrangle is a pullback as well:



which means that the morphism of equivalence relations $R \cdot \bar{R} \rightarrowtail S \cdot \bar{S}$ above u is a discrete fibration. □

Theorem 3.1. *Let $\mathbb{C}$ be a homological category. Suppose that two normal subobjects u and v commute. Then their surpremum (which exists by Lemma 2.1) is a normal monomorphism.*

Proof. Let $\phi : U \times V \to X$ be the cooperator of the commuting pair; their supremum is given by the image $j : J \rightarrowtail X$ of ϕ. Let R and S be the equivalence relations to which u and v are normal. The inverse image of the normal monomorphism u along j is $\psi.\iota_U$; it is normal to $R' = j^{-1}(R)$ which is equal to the direct image $\psi(\nabla_U \times V)$ by Lemma 2.1, since $\mathbb{C}$ is protomodular. Similarly $S' = j^{-1}(S)$ is $\psi(U \times \nabla_V)$. On the other hand the supremum of $\nabla_U \times V$ and $U \times \nabla_V$ is the indiscrete equivalence relation $\nabla_{U\times V}$, and the image of the equivalence relations along the regular epimorphism ψ preserves the supremum. Then the supremum of R' and S' is the indiscrete equivalence relation ∇_J. Now consider the diagram:

$$\begin{array}{ccccc} U\times U & \rightarrowtail & R' & \overset{\tilde{j}}{\rightarrowtail} & R \\ p_0 \downarrow\uparrow\downarrow p_1 & & d_0^{R'} \downarrow\uparrow\downarrow d_1^{R'} & & d_0^{R} \downarrow\uparrow\downarrow d_1^{R} \\ U & \underset{\psi.\iota_U}{\rightarrowtail} & J & \underset{j}{\rightarrowtail} & X \end{array}$$

The category $\mathbb{C}$ being protomodular, the right hand side part of the diagram is a discrete fibration, since so are the left hand part and the whole diagram. The same holds for S and $S' = j^{-1}(S)$. Now, according to Proposition 3.1 the following diagram is a discrete fibration:

$$\begin{array}{ccc} J\times J = R'\cdot S' & \rightarrowtail & R\cdot S \\ p_0 \downarrow\uparrow\downarrow p_1 & & d_0 \downarrow\uparrow\downarrow d_1 \\ J & \underset{j}{\rightarrowtail} & X \end{array}$$

which means that the supremum j of u and v is normal to $R \cdot S$. □

4 Applications

4.1 Action distinctive categories

Here we shall investigate the product of centralizers of equivalence relations in the regular Mal'tsev setting. Recall, from [5], the following:

Definition 4.1. A Mal'tsev category $\mathbb{C}$ is said to be action distinctive when, in the category $Pt\mathbb{C}$, any object (f,s) admits a largest ¶-cartesian equivalence relation on it, called its *action distinctive* equivalence relation.

Given any split epimorphism $(f,s) : X \rightleftarrows Y$ in $\mathbb{C}$, we shall denote its associated action distinctive equivalence relation $\mathbb{D}[f,s]$ in $Pt\mathbb{C}$ in the following way in $\mathbb{C}$:

$$\begin{array}{ccc} D_X[f,s] & \overset{\delta_0^X}{\underset{\delta_1^X}{\rightleftarrows}} & X \\ {\scriptstyle D_f}\downarrow\uparrow{\scriptstyle D_s} & & {\scriptstyle f}\downarrow\uparrow{\scriptstyle s} \\ D_Y[f,s] & \overset{\delta_0^Y}{\underset{\delta_1^Y}{\rightleftarrows}} & Y \end{array}$$

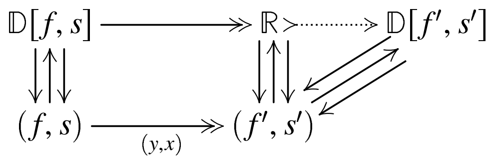
A Mal'tsev category $\mathbb{C}$ is action distinctive if and only if any equivalence relation $(d_0,d_1) : R \rightrightarrows X$ admits a centralizer [5] which is nothing but the ground equivalence relation of the action distinctive equivalence relation of the split epimorphism $(d_0,s_0) : R \rightleftarrows X$:

$$\begin{array}{ccc} D_R[d_0,s_0] & \overset{\delta_0^R}{\underset{\delta_1^R}{\rightleftarrows}} & R \\ {\scriptstyle D_{d_0}}\downarrow\uparrow{\scriptstyle D_{s_0}} & & {\scriptstyle d_0}\downarrow\uparrow{\scriptstyle s_0} \quad \overset{d_1}{\dashrightarrow} \\ D_X[d_0,s_0] & \overset{\delta_0^X}{\underset{\delta_1^X}{\rightleftarrows}} & X \end{array}$$

Examples. From the previous observation, it is clear that the categories Gp of groups is action distinctive; starting with any split epimorphism (f,s) : $X \rightleftarrows Y$, the equivalence relation $D_Y[f,s]$ is nothing but the kernel equivalence relation of its associated canonical action $\phi : Y \to AutK$, where K is the kernel of the homomorphism f. The categories R-Lie of Lie R-algebras, Rg of non-commutative rings, Rg^* of non-commutative rings with unit and the category $TopGp$ of topological groups are action distinctive as well. It is clear also that this notion is stable under coslicing and easy to show that it is stable under slicing (given an equivalence relation R on the object h of $\mathbb{C}/T$, its centralizer in $\mathbb{C}/T$ is $Z[R] \cap R[h]$). Accordingly the notion of action distinctiveness is stable under the passage to the fibres $Pt_Y\mathbb{C}$.

Proposition 4.1. *Let $\mathbb{C}$ be a regular Mal'tsev category which is action distinctive. Any regular epimorphism in $Pt\mathbb{C}$ has an extension up to the level of* *the action distinctive equivalence relations.*

Proof. Consider the following diagram in $Pt\mathbb{C}$ where (y, x) is a regular epimorphism and $\mathbb{R}$ the image of $\mathbb{D}[f, s]$ along it:



By Proposition 1.2 the equivalence relation $\mathbb{R}$ is ¶-cartesian; accordingly there is the dotted factorization. □

Theorem 4.1. *Let $\mathbb{C}$ be a regular Mal'tsev category which is action distinctive. The action distinctive equivalence relations are stable under finite products provided that the base objects of the split epimorphisms have global supports. If, in addition, the category is pointed, the action distinctive equivalence relations are stable under finite products without any restriction.*

Proof. Given any pair (f, s) and (f', s') of split epimorphisms, then the product $\mathbb{D}[f, s] \times \mathbb{D}[f', s']$ is a ¶-cartesian equivalence relation and produces an inclusion $\mathbb{D}[f, s] \times \mathbb{D}[f', s'] \subset \mathbb{D}[f \times f', s \times s']$. Suppose the base object Y of the split epimorphism (f, s) has a global support; its domain X has a global support as well. If Y' and X' denote respectively the codomain and the domain of (f', s'), then the projections $p_{Y'} : Y \times Y' \to Y'$ and $p_{X'} : X \times X' \to X'$ are regular epimorphisms and, according to the previous proposition, we get an extension: $\mathbb{D}[f \times f', s \times s'] \to \mathbb{D}[f', s']$. Similarly when Y' has a global support we get an extension: $\mathbb{D}[f \times f', s \times s'] \to \mathbb{D}[f, s]$. These two extensions led to an inverse factorization $\mathbb{D}[f \times f', s \times s'] \subset \mathbb{D}[f, s] \times \mathbb{D}[f', s']$. □

Corollary 4.1. *Let $\mathbb{C}$ be a regular Mal'tsev category which is action distinctive. The centralizers of equivalence relations are stable under finite products provided that the base objects of these relations have global supports. If, in addition, the category is pointed, the centralizers of equivalence relations are stable under finite products without any restriction.*

An action distinctive category $\mathbb{C}$ is said to be *functorially action distinctive* when, in addition, there is a (unique) functorial extension of any ¶-cartesian map up to the level of the ¶-distinctive equivalence relations. Any action accessible category [8] is functorially action distinctive. According to Proposition 5.10 in [5], any functorially action distinctive Mal'tsev category is such that any fibre $Pt_Y\mathbb{C}$ is functorially action distinctive and any change of base functor with respect to the fibration of points preserves the centralizers of equivalence relations. Let us end this section by a result we shall need as an illustration of the next section:

Corollary 4.2. *Let $\mathbb{C}$ be any functorially action distinctive protomodular category. Then any change of base functor with respect to the fibration of points reflects the centralization of equivalence relations.*

Proof. In a protomodular category the change of base functors with respect to the fibration of points reflects the inclusion of equivalence relations since it is the case for any left exact functor which reflects the isomorphisms. If moreover $\mathbb{C}$ is functorially action distinctive, the change of base functors preserve the centralizers; accordingly they reflect the centralization of equivalence relations. □

4.2 Reflection of commutation and centralization

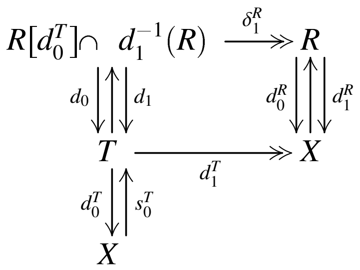
In the pointed Mal'tsev setting, when two equivalence relations centralize each other [23] [22] [6], then their normalizations commute, see [7]. A pointed Mal'tsev category is said to satisfy the condition (SH) when the converse is true. The condition (SH) is satisfied in any pointed strongly protomodular category (see Theorem 6.1 in [7]), in any action accessible category [8], [12] and in any category of interest [20], [21]. See also [18] for other remarks. Let us recall the following conditions introduced in [9]:

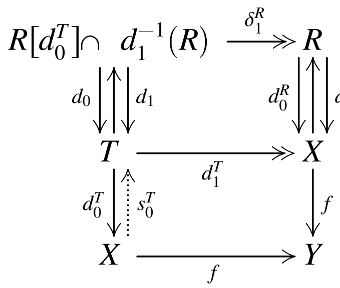
- (C) any change of base functor with respect to the fibration of points reflects the commutation of normal subobjects;
- ($\bar{\mathrm{C}}$) any change of base functor with respect to the fibration of points reflects the centralization of equivalence relations.

We noticed at the end of the previous section that any functorially action distinctive protomodular category satisfies condition ($\bar{\mathrm{C}}$). It is showed in [9] that, in a Mal'tsev category $\mathbb{C}$, (C) implies ($\bar{\mathrm{C}}$) and that, when in addition the category $\mathbb{C}$ is pointed, the first condition is equivalent to the condition (SH).

It is also proved that the two conditions are stable under slicing and coslicing. This section will be devoted to prove that, in the regular protomodular setting, we have the converse, namely: ($\bar{\mathrm{C}}$) implies (C). Let us begin with the two following lemmas:

Lemma 4.1. *Let $R \subset T$ be two equivalence relations on X in $\mathbb{E}$. Then the following left hand side upper diagram determines a regular epic discrete fibration in $\mathbb{E}$:*

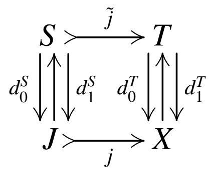




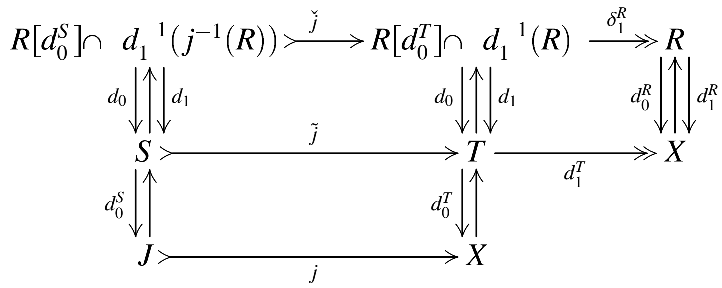
such that $R[d_0^T] \cap d_1^{-1}(R)$ is an equivalence relation on the point (d_0^T, s_0^T) in $Pt_X\mathbb{E}$. The equivalence relation $R[d_0^T] \cap d_1^{-1}(R)$ is the unique one to satisfy these two properties. Accordingly, when the equivalence relation T is effective, namely equal to some $R[f]$, the equivalence $R[d_0^T] \cap d_1^{-1}(R)$ is obtained by the right hand side iterated pullbacks above.

Proof. The fact that $R[d_0^T] \cap d_1^{-1}(R)$ is an equivalence relation on the point (d_0^T, s_0^T) in $Pt_X\mathbb{E}$ means that d_0^T coequalizes the pair (d_0, d_1). By the Yoneda Lemma, it is sufficient to check the assertion of the lemma in Set which is straightforward. □

Lemma 4.2. *Given a monomorphic discrete fibration between equivalence relations:*

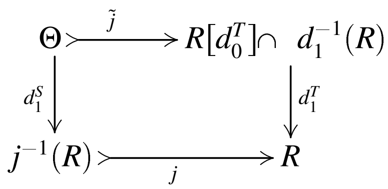


and an equivalence relation $R \subset T$, there is a unique map $\check{j}$:



making the left hand side upper part a monomorphic discrete fibration between equivalence relations; namely the left hand side vertical diagram in the fibre $Pt_J\mathbb{E}$ is the image by the change of base funtor $j^* : Pt_X\mathbb{E} \to Pt_J\mathbb{E}$ of the middle vertical diagram in the fibre $Pt_X\mathbb{E}$.

Proof. Consider the following pullback of equivalence relations in $\mathbb{E}$ where the morphisms of equivalence relations are only labelled by their underlying maps in $\mathbb{E}$:



The equivalence relation Θ is a relation on the object S which is coequalized by d_0^S since it is coequalized by $d_0^T.\tilde{j} = j.d_0^S$ and j is a monomorphism. On the other hand, the discrete fibrations being stable under pullback, the morphism labelled by d_1^S is a discrete fibration since so is the one labelled by d_1^T.Then, according to the previous lemma, we have $\Theta = R[d_0^S] \cap\ d_1^{-1}(j^{-1}(R))$. Moreover, since $j : S \rightarrowtail T$ is a discrete fibration, and $R \subset T$, the morphism $j^{-1}(R) \rightarrowtail R$ is a discrete fibration, and consequently so is:

$$\tilde{j} : \Theta = R[d_0^S] \cap\ d_1^{-1}(j^{-1}(R)) \rightarrowtail R[d_0^T] \cap\ d_1^{-1}(R)$$

□

Then we get:

Proposition 4.2. *Suppose $\mathbb{C}$ is a homological category. Then the condition* $(\bar{\mathrm{C}})$ *implies the condition (SH) (which is equivalent to the condition* (C)*). Accordingly, in the homological setting, the conditions* (C) *and* $(\bar{\mathrm{C}})$ *are equivalent.*

Proof. Suppose that (R, S) is a pair of equivalence relations on X such that their normalizations u and v commute. Let us denote by $j : J \rightarrowtail X$ their supremum. Then, according to the construction of Lemma 2.1, the factorizations $u' : U \rightarrowtail J$ of u and $v' : V \rightarrowtail J$ of v commute and their supremum is 1_J; according to Corollary 2.1 the equivalence relations $R' = j^{-1}(R)$ and $S' = j^{-1}(S)$ on J centralize each other. Moreover the supremum j is normal by Theorem 1.2 via the following discrete fibration :

$$\begin{array}{ccc} J \times J & \overset{\tilde{j}}{\rightarrowtail} & R \cdot S \\ p_0 \downarrow\uparrow\downarrow p_1 & & d_0 \downarrow\uparrow\downarrow d_1 \\ J & \underset{j}{\rightarrowtail} & X \end{array}$$

Now, R and S being contained in their supremum $R \cdot S$, we can apply them Lemma 4.2 with respect to this discrete fibration. So, considering the change of base functor $j^* : Pt_X\mathbb{C} \to Pt_J\mathbb{C}$, we have:

$$j^*(R[d_0^{R \cdot S}] \cap d_1^{-1}(R)) = R[p_0^J] \cap d_1^{-1}(R')$$

On the other hand, by Lemma 4.1, $R[p_0^J] \cap d_1^{-1}(R')$ is nothing but the pullback of R' along the terminal map $\tau_J : J \to 1$ since the indiscrete equivalence relation ∇_J is effective. Accordingly since R' and S' commute in $\mathbb{C}$, so do $R[p_0^J] \cap d_1^{-1}(R') = j^*(R[d_0^{R \cdot S}] \cap d_1^{-1}(R))$ and $R[p_0^J] \cap d_1^{-1}(S') = j^*(R[d_0^{R \cdot S}] \cap d_1^{-1}(S))$ in the fibre $Pt_J\mathbb{C}$. Now since the category $\mathbb{C}$ satisfies the condition (C), the equivalence relations $R[d_0^{R \cdot S}] \cap d_1^{-1}(R)$ and $R[d_0^{R \cdot S}] \cap d_1^{-1}(S)$ commute in the fibre $Pt_X\mathbb{C}$ and thus in $\mathbb{C}$. Now the direct images of these equivalence relations (in $\mathbb{C}$) along the regular epimorphism $d_1^{R \cdot S}$ are nothing but R and S (see Lemma 4.1), which consequently commute in $\mathbb{C}$. □

This equivalence is set in the stricter context of semi-abelian categories in [19].

Theorem 4.2. *Let $\mathbb{C}$ be any regular protomodular category. Then the condition* (C) *is equivalent to the condition* $(\bar{\mathrm{C}})$.

Proof. In any Mal'tsev context, (C) implies ($\bar{\text{C}}$). Suppose now $\mathbb{C}$ is regular, protomodular and satisfies ($\bar{\text{C}}$). Then any fibre $Pt_X\mathbb{C}$ is homological. Since the condition ($\bar{\text{C}}$) is stable under slicing and coslicing, any fibre $Pt_X\mathbb{C}$ satisfies ($\bar{\text{C}}$). According to the previous proposition any fibre $Pt_X\mathbb{C}$ satisfies (SH). The fact that, for any change of base functor $f^* : Pt_Y\mathbb{C} \to Pt_X\mathbb{C}$, the two conditions are equivalent is now a consequence of the following lemma. □

Lemma 4.3. *Let $U : \mathbb{C} \to \mathbb{D}$ be a left exact functor between pointed Mal'tsev categories. When $\mathbb{C}$ satisfies (SH), then U reflects the centralizing equivalence relations as soon as it reflects the commutation of normal monomorphisms. When $\mathbb{D}$ satisfies (SH), then U reflects the commutation of normal monomorphisms as soon as it reflects the centralized equivalence relations. When both $\mathbb{C}$ and $\mathbb{D}$ satisfy (SH), the two conditions on U are equivalent.*

Proof. Suppose $\mathbb{C}$ satisfies (SH) and U reflects the commutation of normal monomorphisms. Start with a pair (R, S) of equivalence relations whose images $(U(R), U(S))$ centralize each other. Denote by u and v their normalizations. Then the normalizations $U(u)$ and $U(v)$ of $U(R)$ and $U(S)$ commute in $\mathbb{D}$. Since U reflects the commutation of normal monomorphisms, u and v commute in $\mathbb{C}$; since $\mathbb{C}$ satisfies (SH), (R, S) centralize each other.

Suppose $\mathbb{D}$ satisfies (SH) and U reflects the centralized equivalence relations. Denote by u and v the normalizations of two equivalence relations R and S in $\mathbb{C}$ and suppose that $U(u)$ and $U(v)$ commute. Then $U(R)$ and $U(S)$ centralize each other in $\mathbb{D}$. Since U reflects the centralized equivalence relations, R and S centralize each other in $\mathbb{C}$. So their normalizations u and v do commute. □

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