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ADJOINTS FOR SYMMETRIC CUBICAL CATEGORIES
(ON WEAK CUBICAL CATEGORIES, III)

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Dédié à René Guitart, en amitié

Résumé. En étandant un article précédent (avec R. Paré) sur les adjonctions pour les catégories doubles, on traite maintenant les catégories cubiques symétriques (de dimension infinie). Ici aussi, une "adjonction cubique" générale est formée d'un foncteur cubique *colax* qui est adjoint à gauche d'un foncteur cubique *lax*. Cela ne peut pas être envisagé comme une adjonction interne à une bicatégorie, car en composant des morphismes lax et colax on détruit leurs structure. Toutefois, comme dans le cas des adjonctions doubles, les adjonctions cubiques vivent dans une *catégorie double* intéressante; celle-ci est formée des catégories *cubiques* symétriques, avec les foncteurs cubiques *lax* et *colax* en tant que flèches horizontales et verticales, liées par des cellules doubles convenables.

Abstract. Extending a previous article (with R. Paré) on adjoints for double categories, we deal now with weak symmetric cubical categories (of infinite dimension). Also here, a general 'cubical adjunction' has a *colax* cubical functor left adjoint to a *lax* one. This cannot be viewed as an adjunction in some bicategory, because composing lax and colax morphisms destroys all comparisons. However, as in the case of double adjunctions, cubical adjunctions live in an interesting *double* category; this now consists of weak symmetric *cubical* categories, with lax and colax cubical functors as horizontal and vertical arrows, linked by suitable double cells.

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Introduction

This is the third paper in a series on weak symmetric cubical categories. The first [G4] explores the role of symmetries. The second [G5] deals with cubical limits and can be viewed as an infinite-dimensional extension of the study of double limits in [GP1]. We now investigate cubical adjunctions, extending the study of double adjoints in [GP2]. (See the acknowledgements at the end of this Introduction.)

Weak cubical categories were introduced in [G1-G3], as a basis for the study of cubical cospans in Algebraic Topology and higher cobordism. They have a cubical structure, with faces and degeneracies, weak compositions in countably many directions (indexed as 1, 2,..., n,...) and a strict composition in *one* direction, called the *transversal* one (and indexed as 0).

As a leading example, one can think of the weak cubical category $\omega\mathbb{S}p(\mathbf{X})$ of cubical spans in a category with pullbacks $\mathbf{X}$. An *n-dimensional object* is a functor $x\colon \mathsf{V}^n \to \mathbf{X}$, where V is the 'formal span' category

$$(1) \qquad -1 \leftarrow 0 \rightarrow 1 \qquad \mathsf{V}, \qquad \begin{array}{ccccc} (-1,-1) & \longleftarrow & (0,-1) & \longrightarrow & (1,-1) \\ \uparrow & & \uparrow & & \uparrow \\ (-1,0) & \longleftarrow & (0,0) & \longrightarrow & (1,0) \\ \downarrow & & \downarrow & & \downarrow \\ (-1,1) & \longleftarrow & (0,1) & \longrightarrow & (1,1) \end{array} \quad \mathsf{V}^2. \qquad \begin{array}{ll} \bullet \longrightarrow 1 \\ \downarrow 2 \end{array}$$

(In these diagrams, identities and composites are understood.) A *transversal n-map* is a natural transformation $f\colon x \to x'\colon \mathsf{V}^n \to \mathbf{X}$ of such functors; *notice that it is a diagram of dimension* $n+1$, as a functor defined on $\mathbf{2}\times\mathsf{V}^n$, and should be viewed as an *(n+1)-cell* of our structure.

The ordinary categories $\mathbb{S}p_n(\mathbf{X}) = \mathbf{Cat}(\mathsf{V}^n, \mathbf{X})$ form a cubical object in $\mathbf{Cat}$, with obvious faces and degeneracies. Moreover, n-dimensional spans (and their maps) have *cubical composition laws* (or *concatenations*)

$$(2) \quad x +_i y \qquad\qquad (f +_i g\colon x +_i y \to x' +_i y'),$$

in direction $i = 1,..., n$, that are computed with (a fixed choice of) pullbacks; these compositions are consistent with faces, *but only behave well up to invertible transversal maps*, the *comparisons* for associativity, unitarity and interchange.

As already stressed in [G1], $\omega\mathbb{S}p(\mathbf{X})$ is a weak *symmetric* cubical category, when equipped with the obvious action of the symmetric group S_n on the category $\mathbf{Cat}(\mathsf{V}^n, \mathbf{X})$, by permuting variables in V^n. These symmetries – *which only permute the* weak *directions* – reduce all faces, degeneracies and cubical compositions to the 1-indexed case (for instance), and allow us to simplify the coherence conditions.

Notice also that cubical 1-truncation keeps i-cubes and i-transversal maps (i.e. (i+1)-cells) for $i \leq 1$; it yields the weak *double* category $\mathbb{S}p(\mathbf{X}) = tr_1(\omega\mathbb{S}p(\mathbf{X}))$ of morphisms and ordinary spans, studied in [GP1, DPR2], with *one* weak direction *and* the strict transversal one. Here, all symmetries 'disappear', since the symmetric groups S_0 and S_1 acting on 0- and 1-objects (and their transversal maps) are trivial. In other words, *a weak double category is trivially symmetric*, in the present sense.

Outline. In Section 1 we review the construction of $\omega\mathbb{S}p(\mathbf{X})$, in order to clarify the general structure of symmetric cubical categories; a formal definition of this structure can be found in [G1] and [G4]. We also sketch a natural cubical colax/lax adjunction between cubical spans and cospans. (Other examples will be studied in a sequel.)

In Section 2 we introduce the strict *double* category $\mathbb{W}sc$ of weak sc-categories, lax and colax sc-functors and suitable double cells. Comma sc-categories are also considered. Both topics extend notions of weak double categories developed in [GP2]. It is interesting to note that the double category $\mathbb{W}sc$ *seems not to be the truncation of any cubical category of interest.*

Section 3 reviews the notions of companions and adjoints in a *double* category, from [GP2]. The next two sections introduce and study cubical colax/lax adjunctions, as adjoint arrows *in* the double category $\mathbb{W}sc$.

Finally, Section 6 deals with the preservation of (co)limits by adjoints, for weak sc-categories.

Literature. Besides the articles mentioned above, weak double categories - also called *pseudo* double categories - are studied in various works, like [BM, BMM, Da, DP1, DP2, DPR1, DPR2, Fi, FGK, P2]; the strict case was introduced by C. Ehresmann (see [E1, E2, BE]). Relations between weak cubical and globular (infinite dimensional) categories are dealt with in [GP5].

Acknowledgements. This paper is a natural extension of a joint work with R. Paré [GP2] on double adjunctions, for the cubical framework introduced in [G1, G4]. The paper was originally planned and discussed as a joint work with Bob. Then, his interests took other directions and the joint work will hopefully appear at a later date. He would like, however, to join me in wishing René all the best.

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1. Basic examples

The formal definition of a weak symmetric cubical category, or weak sc-category, is long and involved (see [G1, G4]). Rather than rewriting it, we use here - as a leading example - the weak symmetric cubical category $\omega\mathbb{S}\mathrm{p}(\mathbf{X})$ of higher cubical spans (on a category $\mathbf{X}$ with pullbacks), that extends the weak double category $\mathbb{S}\mathrm{p}(\mathbf{X})$ studied in [GP1]. This should be sufficient to give a clear idea of the general structure of a weak sc-category, and to show how it extends weak double categories to the infinite-dimensional case. The dual case of cospans is also mentioned.

Then we recall a colax/lax *double* adjunction $\mathbb{S}\mathrm{p}\mathbf{X} \rightleftarrows \mathbb{C}\mathrm{osp}\mathbf{X}$ studied in [GP2] and extend it to a colax/lax *cubical* adjunction $\omega\mathbb{S}\mathrm{p}\mathbf{X} \rightleftarrows \omega\mathbb{C}\mathrm{osp}\mathbf{X}$, preparing the way for the study of this notion.

Cubical structures have two faces ∂_i^α in each direction i, that are distinguished by a binary variable α; the values of the latter are written as $0, 1$ or $-, +$, and can be read as *lower* and *upper*.

1.1. Cubical spans. Let $\mathbf{X}$ be a category *with a (full) choice of distinguished pullbacks*: in other words, to every cospan (f, g) we assign *one distinguished* pullback (f', g'), in a symmetric way (assigning (g', f') to the cospan (g, f)).

The 'geometric model' of cubical spans of dimension n is the category V^n, a cartesian power of the *formal span* V

$$(1) \qquad -1 \leftarrow 0 \rightarrow 1 \quad \mathsf{V}, \qquad \begin{array}{ccccc} (-1,-1) & \longleftarrow & (0,-1) & \longrightarrow & (1,-1) \\ \uparrow & & \uparrow & & \uparrow \\ (-1,0) & \longleftarrow & (0,0) & \longrightarrow & (1,0) \\ \downarrow & & \downarrow & & \downarrow \\ (-1,1) & \longleftarrow & (0,1) & \longrightarrow & (1,1) \end{array} \quad \mathsf{V}^2. \qquad \begin{array}{cc} \bullet \longrightarrow & 1 \\ \downarrow 2 & \end{array}$$

(In these diagrams, identities and composites are understood.) An *n-cube* of $\omega\mathbb{S}\mathrm{p}(\mathbf{X})$ is a functor $x\colon \mathsf{V}^n \to \mathbf{X}$; in particular, a 0-cube 'is' an object of $\mathbf{X}$, and will also be called an *object* of $\omega\mathbb{S}\mathrm{p}(\mathbf{X})$.

A *transversal map* of n-cubes is a natural transformation $f\colon x \to y\colon \mathsf{V}^n \to \mathbf{X}$; it is also called an *n-map*, but *should be viewed as an (n+1)-dimensional cell*, as it is represented by the associated functor $f\colon \mathbf{2}\times\mathsf{V}^n \to \mathbf{X}$ (a diagram of dimension $n+1$).

These objects and maps form a category

$$(2) \quad \mathbb{S}\mathrm{p}_n(\mathbf{X}) \;=\; \mathbf{Cat}(\mathsf{V}^n, \mathbf{X}).$$

Its composition law, written g.f or gf, is called the *transversal composition* of $\omega\mathbb{S}p(\mathbf{X})$ in degree n and direction 0. The domain and codomain of a transversal n-map $f: x \to y$ are written as $\partial_0^-(f) = x$ and $\partial_0^+(f) = y$. The identity of x is written as id(x).

It is now easy to construct a symmetric cubical object in **Cat**, based on the structure of the category $\mathbf{V}$ as a *formal symmetric interval* (with respect to the cartesian product in **Cat**); this structure consists of two *faces* (∂^α), a *degeneracy* (e) and a *transposition* (s):

$$(3) \qquad \partial^\alpha: \mathbf{1} \rightrightarrows \mathbf{V}, \qquad e: \mathbf{V} \to \mathbf{1}, \qquad s: \mathbf{V}^2 \to \mathbf{V}^2 \qquad (\alpha = \pm),$$
$$\partial^\alpha(*) = \alpha 1, \qquad e(t) = *, \qquad s(t_1, t_2) = (t_2, t_1).$$

Faces, degeneracies and transpositions of n-cubes and n-maps are defined by (contravariant!) pre-composition with the corresponding maps between cartesian powers of $\mathbf{V}$ (for $\alpha = \pm$ and $i = 1,..., n$)

$$(4) \quad \partial_i^\alpha = \mathbf{V}^{i-1}\times\partial^\alpha\times\mathbf{V}^{n-i}: \mathbf{V}^{n-1} \to \mathbf{V}^n, \qquad \partial_i^\alpha(t_1,..., t_{n-1}) = (t_1,..., \alpha 1,..., t_{n-1}),$$
$$e_i = \mathbf{V}^{i-1}\times e\times\mathbf{V}^{n-i}: \mathbf{V}^n \to \mathbf{V}^{n-1}, \qquad e_i(t_1,..., t_n) = (t_1,..., \hat{t}_i,..., t_n),$$
$$s_i = \mathbf{V}^{i-1}\times s\times\mathbf{V}^{n-i}: \mathbf{V}^{n+1} \to \mathbf{V}^{n+1}, \qquad s_i(t_1,..., t_{n+1}) = (t_1,..., t_{i+1}, t_i,..., t_{n+1}),$$

so that the 2n faces of an n-cube $x: \mathbf{V}^n \to \mathbf{X}$ are $\partial_i^\alpha(x) = x.\partial_i^\alpha: \mathbf{V}^{n-1} \to \mathbf{X}$, and so on.

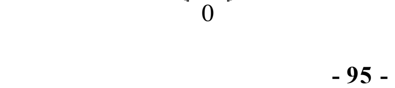
An n-cube has 2^n *vertices*, the objects $\partial_1^{\alpha_1}\partial_2^{\alpha_2}...\partial_n^{\alpha_n}(x)$. Similarly, a transversal n-map f has 2^n *vertices*, the 0-maps $\partial_1^{\alpha_1}\partial_2^{\alpha_2}...\partial_n^{\alpha_n}(f)$; f is said to be *special* if its vertices are identities. A 0-object x determines a sequence of *totally degenerate* cubes $e^n(x)$ of any dimension:

$$(5) \quad e^n(x) = e_n... e_1(x) = e_1... e_1(x).$$

The *i-concatenation* $x +_i y$, or *cubical composition in direction i*, of two n-cubes that are i-consecutive (i.e. $\partial_i^+(x) = \partial_i^-(y)$) is computed in the obvious way, by 3^{n-1} distinguished pullbacks whose 'vertices' are those of the common face (for i = 1,..., n). This operation is then extended to transversal n-maps, in the obvious way: $f +_i g$ is defined when $\partial_i^+(f) = \partial_i^-(g)$.

This operation can be given a formal definition, based on the *model of binary composition* (for ordinary spans), the category $\mathbf{V}_2$ displayed in the commutative diagram below, with one non-trivial pullback

(6)



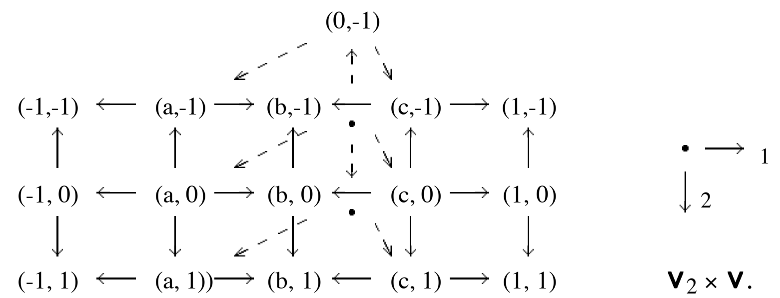
$\mathbf{V}_2$.

Indeed, two consecutive spans x, y in **X** define a functor $[x, y]: \mathbf{V}_2 \to \mathbf{X}$ (sending the previous pullback to a distinguished one, in **X**). Then the concatenation $x +_1 y: \mathbf{V} \to \mathbf{X}$ is obtained by pre-composing $[x, y]$ with the *concatenation map* $m: \mathbf{V} \to \mathbf{V}_2$, a full embedding (displayed in the diagram above by the names of the objects of $\mathbf{V}_2$). Similarly two transversal 1-maps $f: x \to x'$, $g: y \to y'$ define a natural transformation $[f, g]: [x, y] \to [x', y']: \mathbf{V}_2 \to \mathbf{X}$ and then:

$$f +_1 g = [f, g].m: x +_1 y \to x' +_1 y:' \mathbf{V} \to \mathbf{X}.$$

Then, i-concatenation of n-cubes (and n-maps) is based on the cartesian product $\mathbf{V}^{i-1} \times \mathbf{V}_2 \times \mathbf{V}^{n-i}$, as shown below for the concatenation of 2-cubes in direction $i = 1$

(7)



Comparisons for unitarity, associativity and interchange can be defined taking advantage of this formal construction (as proved in [G1], Section 3). These comparisons are invertible, special transversal maps:

(8) $\lambda_1 x: e_1\partial_1^- x +_1 x \to x$, $\quad \rho_1 x: x +_1 e_1\partial_1^+ x \to x$ (*unit 1-comparisons*),

$\kappa_1(x, y, x): x +_1 (y +_1 z) \to (x +_1 y) +_1 z$ (*associativity 1-comparison*),

$\chi_1(x, y, z, u): (x +_1 y) +_2 (z +_1 u) \to (x +_2 z) +_1 (y +_2 u)$

(*interchange 1-comparison*).

Of course, we are assuming that all concatenations above are legitimate.

The comparisons $\lambda_i, \rho_i, \kappa_i, \chi_i$ in the other directions are provided by transpositions, a fact that simplifies the structure and the coherence axioms. (The comparison χ_i deals with the interchange of $+_i$ and $+_{i+1}$.)

One can easily obtain a *unitary* (or normal) weak sc-category, where the unit comparisons are transversal identities, by assuming that our choice of distinguished pullbacks satisfies the *unitarity constraint*:

(*) *the distinguished pullback of the cospan* (f, 1) *is the span* (1, f) (and symmetrically).

1.2. Cubical cospans. Cubical *co*spans are obtained by the dual procedure, for a category $\mathbf{X}$ with distinguished pushouts:

(1) $\omega\mathbb{C}\mathrm{osp}(\mathbf{X}) = \omega\mathbb{S}\mathrm{p}(\mathbf{X}^{\mathrm{op}}), \qquad \mathbb{C}\mathrm{osp}_n(\mathbf{X}) = \mathbf{Cat}(\wedge^n, \mathbf{X}).$

Here the category $\wedge = \vee^{\mathrm{op}}$ is the *formal cospan*: $-1 \to 0 \leftarrow 1$. This case is of interest in Algebraic Topology and higher cubical cobordism, see [G1-G4].

Again, $\omega\mathbb{C}\mathrm{osp}(\mathbf{X})$ is a *unitary* weak sc-category if the choice of distinguished pushouts satisfies the *unitarity constraint*: the distinguished pushout of the span $(f, 1)$ is the cospan $(1, f)$ (and symmetrically).

From now on, *we adopt the unitarity constraint for pullbacks and pushouts*, for the sake of simplicity.

1.3. Truncation. Truncating a weak sc-category $\omega\mathbb{A}$ at the level of 1-cubes and 1-transversal maps (that are 2-dimensional!) we get a weak double category $\mathbb{A}$.

In particular, $\omega\mathbb{S}\mathrm{p}(\mathbf{X})$ gives the weak double category $\mathbb{S}\mathrm{p}(\mathbf{X})$ of sets, mappings and spans on $\mathbf{X}$, while $\omega\mathbb{C}\mathrm{osp}(\mathbf{X})$ gives the weak double category $\mathbb{C}\mathrm{osp}(\mathbf{X})$ of sets, mappings and cospans [GP1] (where the category $\mathbf{X}$ has distinguished pullbacks or pushouts, respectively).

More specifically, given a weak sc-category $\omega\mathbb{A}$:

- the 0-cubes and the transversal 0-maps $f\colon A \to A'$ of $\omega\mathbb{A}$ give the objects and the *horizontal* arrows of the weak double category $\mathbb{A} = \mathrm{tr}_1(\omega\mathbb{A})$;
- each 1-cube becomes a *vertical* arrow $u\colon A \to\!\!\!\!\bullet\; B$ (marked with a dot), where $A = \partial_1^-(u)$, $B = \partial_1^+(u)$;
- each transversal 1-map $a\colon u \to v$ becomes a *double cell* as below

(1)
$$\begin{array}{ccc} A & \xrightarrow{\;f\;} & A' \\ {\scriptstyle u}\downarrow\!\bullet & a & \bullet\!\downarrow{\scriptstyle v} \\ B & \xrightarrow[\;g\;]{} & B' \end{array} \qquad\qquad \begin{array}{l} \bullet \longrightarrow 0 \\ \downarrow 1 \end{array} \qquad\qquad \begin{array}{ll} u = \partial_0^-(a), & v = \partial_0^+(a), \\ & \\ f = \partial_1^-(a), & g = \partial_1^+(a), \end{array}$$

whose boundary is displayed as $a\colon (u\,{}^{f}_{g}\,v)$ or $a\colon u \to v$;

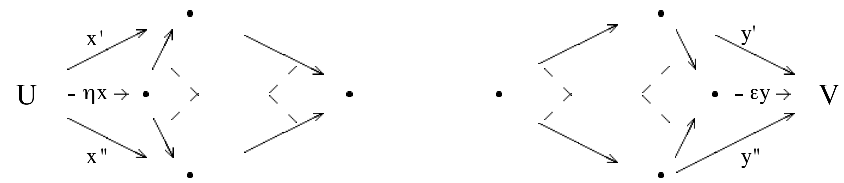
- the transversal composition (of 0-maps and 1-maps) becomes the *horizontal one*, while concatenation (of 1-cubes and 1-maps) becomes the *vertical composition*; the former is strictly categorical, while the latter is weakly categorical, up to the invertible special comparisons of $\omega\mathbb{A}$ (in degree 1).

1.4. A double adjunction. The weak double categories $\mathbb{S}p\mathbf{X}$ and $\mathbb{C}osp\mathbf{X}$ of spans and cospans on the category $\mathbf{X}$ (with distinguished pullbacks and pushouts) are linked by an obvious colax/lax adjunction

(1) $F\colon \mathbb{S}p\mathbf{X} \rightleftarrows \mathbb{C}osp\mathbf{X} : R, \qquad \eta\colon 1 \to RF, \quad \varepsilon\colon FR \to 1,$

that we describe here *in an informal way*. (Writing $\eta\colon 1 \to RF$ and $\varepsilon\colon FR \to 1$ is an abuse of notation, since we cannot compose the comparisons of F and R. The precise definition of a colax/lax adjunction of weak double categories can be found in [GP2]; but the reader will find here its cubical extension, in Section 4, and can easily recover the truncated notion.)

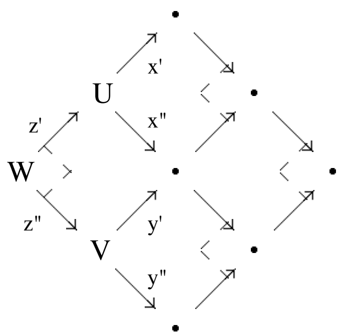
At the level 0 (of objects and arrows of $\mathbf{X}$), everything is an identity. At the level 1 (of 1-cubes and 1-maps), F operates by pushout and R by pullbacks; the special transversal 1-maps $\eta x\colon x \to RFx$ and $\varepsilon y\colon FRy \to y$ are obvious:

(2) 

The triangle identities are plainly satisfied:

$\varepsilon(Fx).F(\eta x) = id(Fx), \qquad R(\varepsilon y).(\eta Ry) = id(Ry).$

Finally it is easy to check that F is, in a natural way, a colax double functor (dually, R is lax). The comparison cell $\underline{F}(x, y)\colon F(x +_1 y) \to Fx +_1 Fy$ for concatenation is given by the natural mapping from the pushout of (x'z', y"z") to the cospan $Fx +_1 Fy$ (at the right-hand of the diagram below)

(3) 

Since we agreed to follow the unitarity constraint for the choice of pullbacks and pushouts in **X**, the adjunction is *unitary*, in the sense that so are the weak double categories $\mathbb{S}p\mathbf{X}$, $\mathbb{C}osp\mathbf{X}$ and the colax/lax double functors F, R. It is also *special*, in the sense that the restricted adjunction at the level of 0-objects and 0-maps

(4) $F_0\colon \mathbf{X} \rightleftarrows \mathbf{X} :R_0 \qquad \eta_0\colon 1 \to R_0F_0, \quad \varepsilon_0\colon F_0R_0 \to 1,$

is composed of identity functors and identity transformations.

Notice also that the natural transformations $F\eta$, εF, ηR, $R\varepsilon$ are invertible (which means that the adjunction is *idempotent*).

1.5. The cubical adjunction. The unitary colax double functor $F\colon \mathbb{S}p\mathbf{X} \to \mathbb{C}osp\mathbf{X}$ can be extended to a unitary colax sc-functor $F\colon \omega\mathbb{S}p\mathbf{X} \to \omega\mathbb{C}osp\mathbf{X}$.

Moving one degree up, for a 2-dimensional span x, $F_2(x)$ is constructed with the pushout of faces $F_1(\partial_i^\alpha x)$ and the colimit of the whole diagram x

(1) $\partial_i^\alpha(F_2(x)) = F_1(\partial_i^\alpha x) \qquad$ for $\alpha = 0, 1$ and $i = 1, 2$,

$F_2(x)(0, 0) = \mathrm{colim}(x).$

This colimit exists in **X**, since it can be easily constructed as a pushout of pushouts; but its choice must agree with transposition and preserve units. (See [P2] for a general characterisation of the dual topic: limits 'generated' by pullbacks).

The 2-dimensional span x of the left diagram below becomes thus the right-hand 2-dimensional cospan $F_2(x)$

$$\text{(2)}\quad \begin{array}{ccccc} x(-1,-1) & \leftarrow & x(0,-1) & \rightarrow & x(1,-1) \\ \uparrow & & \uparrow & & \uparrow \\ x(-1,0) & \leftarrow & x(0,0) & \rightarrow & x(1,0) \\ \downarrow & & \downarrow & & \downarrow \\ x(-1,1) & \leftarrow & x(0,1) & \rightarrow & x(1,1) \end{array} \qquad \begin{array}{ccccc} x(-1,-1) & \rightarrow & F(\partial_2^- x)(0) & \leftarrow & x(1,-1) \\ \downarrow & & \downarrow & & \downarrow \\ F(\partial_1^- x)(0) & \rightarrow & \mathrm{colim}(x) & \leftarrow & F(\partial_1^+ x)(0) \\ \uparrow & & \uparrow & & \uparrow \\ x(-1,1) & \rightarrow & F(\partial_2^+ x)(0) & \leftarrow & x(1,1) \end{array}$$

The definition of F_2 on transversal 2-maps is obvious, as well as the comparison cells for 1-directed concatenation $\underline{F}(x, y)\colon F(x +_1 y) \to Fx +_1 Fy$.

One proceeds in a similar way, defining F_n after F_{n-1}

(3) $\partial_i^\alpha(F_n(x)) = F_{n-1}(\partial_i^\alpha x) \qquad$ for $\alpha = 0, 1$ and $i = 1,..., n$,

$F_n(x)(\underline{0}) = \mathrm{colim}(x), \qquad$ for $\underline{0} = (0,..., 0) \in \mathbf{V}^n$.

The unitary lax double functor $R\colon \mathbb{C}osp\mathbf{X} \to \mathbb{S}p\mathbf{X}$ is similarly extended, using distinguished limits instead of colimits, and gives a unitary lax sc-functor $R\colon \omega\mathbb{C}osp\mathbf{X} \to \omega\mathbb{S}p\mathbf{X}$.

One extends the unit $\eta: 1 \to RF$ by a similar inductive procedure:

(4) $\partial_i^\alpha(\eta(x)) \;=\; \eta(\partial_i^\alpha x)$ for $\alpha = 0, 1$ and $i = 1,..., n$,

$(\eta x)(\underline{0}): x(\underline{0}) \to RF(\underline{0}) = \lim(Fx)$, for $\underline{0} = (0,..., 0) \in \mathbf{V}^n$,

where the map $(\eta x)(\underline{0})$ is given by the universal property of the limit $\lim(Fx)$.

Analogously for the counit $\varepsilon: FR \to 1$. The triangular identities hold.

1.6. Transversal invariance. Extending a notion of double categories (introduced in [GP1], 2.4, under the name of *horizontal invariance*), we say that the weak sc-category $\mathbb{A}$ is *transversally invariant* if, for every n-cube x and every pair of transversal (n–1)-isomorphisms $f^\alpha: \partial_1^\alpha x \to y^\alpha$ (where $\alpha = \pm$), there exists some transversal n-isomorphism $f: x \to y$ with $\partial_1^\alpha f = f^\alpha$ (and therefore $\partial_1^\alpha y = y^\alpha$)

$$\text{(1)} \qquad \begin{array}{ccc} \bullet & \xrightarrow{f^-} & y^- \\ x\downarrow & f & \downarrow y \\ \bullet & \xrightarrow[f^+]{} & y^+ \end{array} \qquad\qquad \begin{array}{cc} \bullet \longrightarrow & 0 \\ \downarrow 1 & \end{array}$$

Of course, because of transpositions, the same property holds for every pair of i-directed faces ∂_i^α.

2. The double category of lax and colax symmetric cubical functors

In the 2-dimensional case, weak double categories, with lax and colax double functors and suitable double cells form a *strict* double category $\mathbb{D}$bl, a crucial, interesting structure introduced in [GP2] to define colax/lax double adjunctions.

We now extend this construction forming the strict *double* category $\mathbb{W}$sc of weak sc-categories, lax and colax sc-functors and suitable double cells, in order to define colax/lax adjunctions between weak sc-categories. Comma sc-categories are also considered, extending again the case of double categories dealt with in [GP2].

Notice that, as far as we can see, the double category $\mathbb{W}$sc *is not the truncation of any cubical category of interest*. Therefore we write it according to the notation for double categories used since [GP1]: the horizontal and vertical compositions of cells are written as $(\alpha \mid \beta)$ and $\binom{\alpha}{\gamma}$, or more simply as $\alpha|\beta$ and $\alpha\otimes\gamma$. Horizontal identities, of an object or a vertical arrow, are written as 1_A and 1_u: $(u \; {}^{A}_{B} \; u)$; vertical identities, of an object or a horizontal arrow, as $1^\bullet_A$ and $1^\bullet_f$: $(A \; {}^{f}_{f} \; A')$.

2.1. Lax sc-functors. As defined in [G5], 1.5, a *lax symmetric cubical functor* $R: \mathbb{X} \to \mathbb{A}$ between weak sc-categories, or *lax sc-functor,* strictly preserves faces, transpositions, transversal composition and transversal identities, but has special transversal maps, called *comparisons*, for the cubical operations, namely degeneracies (or units) and concatenation in direction 1 (those of the other cubical directions being generated by transpositions):

(1) $\underline{R}(x): e_1(Rx) \to R(e_1x)$ (for any cube x in $\mathbb{X}$),

$\underline{R}(x, y): Rx +_1 Ry \to R(z)$ (for any concatenation $z = x +_1 y$ in $\mathbb{X}$).

Recall that a transversal n-map is said to be *special* if its 2^n vertices are identities. Notice that in $\underline{R}(x)$, the cube x has an arbitrary degree $n \geq 0$, while in $\underline{R}(x, y)$ the cubes x, y must have degree ≥ 1 and be 1-consecutive: $\partial_1^+x = \partial_1^-y$.

These comparisons must satisfy the following axioms of coherence (writing $+_1$ as $+$ in most diagrams)

(i) (*naturality*) for a transversal n-map $f: x \to x'$ in $\mathbb{X}$ and a concatenation $f +_1 g$ (with $g: y \to y'$), we have the following commutative diagrams of transversal maps

(2)
$$\begin{array}{ccc} e_1(Rx) & \xrightarrow{e_1(Rf)} & e_1(Rx') \\ \downarrow{\scriptstyle \underline{R}(x)} & & \downarrow{\scriptstyle \underline{R}(x')} \\ R(e_1(x)) & \xrightarrow[R(e_1f)]{} & R(e_1(x')) \end{array} \qquad \begin{array}{ccc} Rx+Ry & \xrightarrow{Rf+Rg} & Rx'+Ry' \\ \downarrow{\scriptstyle \underline{R}(x,\, y)} & & \downarrow{\scriptstyle \underline{R}(x',\, y')} \\ R(x + y) & \xrightarrow[R(f+g)]{} & R(x' + y') \end{array}$$

(ii) (*coherence laws for degeneracies*) for an n-cube x in $\mathbb{X}$, with 1-indexed faces $\partial_1^-x = a$, $\partial_1^+x = b$, the following diagrams of transversal maps commute

(3)
$$\begin{array}{ccc} e_1(Ra) + Rx & \xrightarrow{\lambda_1(Rx)} & Rx \\ \downarrow{\scriptstyle \underline{R}(a)+id} & & \uparrow{\scriptstyle R(\lambda_1 x)} \\ R(e_1a) + Rx & \xrightarrow[\underline{R}(e_1a,\, x)]{} & R(e_1a + x) \end{array} \qquad \begin{array}{ccc} Rx + e_1(Rb) & \xrightarrow{\rho_1(Rx)} & Rx \\ \downarrow{\scriptstyle id+\underline{R}(b)} & & \uparrow{\scriptstyle R(\rho_1 x)} \\ Rx + R(e_1b) & \xrightarrow[R(x,\, e_1b)]{} & R(x + e_1b) \end{array}$$

(iii) (*coherence hexagon for associativity*) for 1-consecutive n-cubes x, y, z in $\mathbb{X}$, the following diagram of transversal maps is commutative (the index 1 is omitted in the labels of the arrows)

$$
(4) \qquad \begin{array}{ccc}
Rx + (Ry + Rz) & \xrightarrow{\kappa(Rx,Ry,Rz)} & (Rx + Ry) + Rz \\
{\scriptstyle id + \underline{R}(y,z)} \downarrow & & \downarrow {\scriptstyle \underline{R}(x,y) + id} \\
Rx + R(y + z) & & R(x + y) + Rz \\
{\scriptstyle \underline{R}(x,\, y+z)} \downarrow & & \downarrow {\scriptstyle \underline{R}(x+y,\, z)} \\
R((x + (y + z)) & \xrightarrow[R\kappa(x,y,z)]{} & R(x + y) + z
\end{array}
$$

(iv) (*coherence hexagon for cubical interchange*) for n-cubes x, y, z, u in $\mathbb{X}$ making the following concatenations legitimate, the following diagram of transversal maps is commutative (the indices 1, 2 are omitted in the labels of the arrows)

$$
(5) \qquad \begin{array}{ccc}
(Rx +_1 Ry) +_2 (Rz +_1 Ru) & \xrightarrow{\chi(Rx,Rz,Ru,Ru)} & (Rx +_2 Rz) +_1 (Ry +_2 Ru) \\
{\scriptstyle \underline{R}(x,y) + \underline{R}(z,u)} \downarrow & & \downarrow {\scriptstyle \underline{R}(x,z) + \underline{R}(y,u)} \\
R(x +_1 y) +_2 R(z +_1 u) & & R(x +_2 z) +_1 R(y +_2 u) \\
{\scriptstyle \underline{R}(x+y,z+u)} \downarrow & & \downarrow {\scriptstyle \underline{R}(x+z,y+u)} \\
R((x +_1 y) +_2 (z +_1 u)) & \xrightarrow[R(\chi(x,y,z,u))]{} & R((x +_2 z) +_1 (y +_2 u))
\end{array}
$$

A lax sc-functor R is said to be *unitary* if its unit comparisons $\underline{R}(x)$ are identities. If $\mathbb{X}$, $\mathbb{A}$ and R are unitary, the cells $\underline{R}(e_1\partial_1^- x,\, x)$ and $\underline{R}(x,\, e_1\partial_1^+ x)$ are also identities (by axiom (ii)).

A *colax sc-functor* F: $\mathbb{X} \to \mathbb{A}$ has comparisons in the opposite direction

(6) $\underline{F}(x)\colon F(e_1x) \to e_1(Fx), \qquad \underline{F}(x, y)\colon F(x +_1 y) \to Fx +_1 Fy.$

A *pseudo sc-functor* is a lax (or colax) sc-functor whose comparisons are invertible.

2.2. Transformations of lax sc-functors. A *transversal transformation* of lax sc-functors h: R → S: $\mathbb{X} \to \mathbb{A}$ assigns to every n-cube x of $\mathbb{X}$ an n-map hx: Rx → Sx in $\mathbb{A}$.

This family must be natural on transversal maps, commute with faces and transpositions and satisfy the coherence conditions (iii) for degeneracies and concatenations:

(i) for an n-map $f: x \to y$ in $\mathbb{A}$, $\qquad hy.Ff = Gf.hx,$

(ii) $\partial_i^\alpha(hx) = h(\partial_i^\alpha x),$ $\qquad h(s_i x) = s_i(hx),$

(iii) for an n-cube x and a 1-consecutive n-cube y in $\mathbb{X}$, the following squares of n-maps commute (again, we write $+$ for $+_1$):

$$(1) \qquad \begin{array}{ccc} e_1(Rx) & \xrightarrow{e_1(hx)} & e_1(Sx) \\ {\scriptstyle \underline{R}_1(x)}\downarrow & & \downarrow{\scriptstyle \underline{S}_1(x)} \\ R(e_1x) & \xrightarrow[h(e_1x)]{} & S(e_1x) \end{array} \qquad\qquad \begin{array}{ccc} Rx + Ry & \xrightarrow{hx+hy} & Sx + Sy \\ {\scriptstyle \underline{R}_1(x,\, y)}\downarrow & & \downarrow{\scriptstyle \underline{S}_1(x,\, y)} \\ R(x + y) & \xrightarrow[h(x+y)]{} & S(x + y) \end{array}$$

Weak sc-categories, lax sc-functors and their transversal transformations form a 2-category Lx**Wsc**.

Transversal transformations of colax sc-functors are defined in a similar way. Cx**Wsc** will denote the 2-category of weak sc-categories, colax sc-functors and their transversal transformations.

2.3. The double category $\mathbb{W}$sc. Lax and colax sc-functors do not compose well, since we cannot compose their comparisons. On the other hand, they can be organised in a *strict* double category $\mathbb{W}$sc, crucial for our study, where orthogonal adjunctions (recalled below, in Section 3) will provide our general notion of cubical adjunction (Section 4) while companion pairs amount to pseudo sc-functors (Section 5).

The objects of $\mathbb{W}$sc are the *weak* sc-categories $\mathbb{X}, \mathbb{A}, \mathbb{B}$,...; its horizontal arrows are the *lax* sc-functors R, S...; its vertical arrows are the *colax* sc-functors F, G... A cell α

$$(1) \qquad \begin{array}{ccc} \mathbb{X} & \xrightarrow{R} & \mathbb{A} \\ {\scriptstyle F}\,\downarrow^{\bullet} & \alpha & \downarrow^{\bullet}\,{\scriptstyle G} \\ \mathbb{B} & \xrightarrow[S]{} & \mathbb{C} \end{array}$$

is - very roughly speaking - a 'transformation' $\alpha: GR \to SF$.

But this is an abuse of notation, since the composites GR and SF are neither lax nor colax (just morphisms of symmetric 'face-cubical' sets, respecting the transversal structure): the coherence conditions of α require the *individual* knowledge of the four 'functors', including the comparison cells of each of them.

Precisely, the cell α consists of the following data:

(a) two lax sc-functors R, S, with comparisons as follows:

$$R\colon \mathbb{X} \to \mathbb{A}, \qquad \underline{R}(x)\colon e_1(Rx) \to R(e_1x), \qquad \underline{R}(x, y)\colon Rx +_1 Ry \to R(x +_1 y),$$
$$S\colon \mathbb{B} \to \mathbb{C}, \qquad \underline{S}(x)\colon e_1(Sx) \to S(e_1x), \qquad \underline{S}(x, y)\colon Sx +_1 Sy \to S(x +_1 y),$$

(b) two colax sc-functors F, with comparisons as follows:

$$F\colon \mathbb{X} \to \mathbb{B}, \qquad \underline{F}(x)\colon F(e_1x) \to e_1(Fx), \qquad \underline{F}(x, y)\colon F(x +_1 y) \to Fx +_1 Fy,$$
$$G\colon \mathbb{A} \to \mathbb{C}, \qquad \underline{G}(x)\colon G(e_1x) \to e_1(Gx), \qquad \underline{G}(x, y)\colon G(x +_1 y) \to Gx +_1 Gy,$$

(c) a family of transversal n-maps $\alpha x\colon GR(x) \to SF(x)$ of $\mathbb{C}$ (for every n-cube x in $\mathbb{X}$), consistent with faces and transpositions

(2) $\alpha(\partial_i^\alpha x) = \partial_i^\alpha(\alpha x), \qquad\qquad \alpha(s_ix) = s_i(\alpha x).$

These data have to satisfy the naturality condition (c1) and the coherence conditions (c2), (c3) (with respect to 1-degeneracies and 1-concatenation, respectively)

(c1) $SFf.\alpha x = \alpha y.GRf\colon GR(x) \to SF(y)$ (for $f\colon x \to y$ in $\mathbb{X}$),

(c2) $S\underline{F}(x).\alpha e_1(x).G\underline{R}(x) = \underline{S}F(x).e_1(\alpha x).\underline{G}R(x)$ (for x in $\mathbb{X}$),

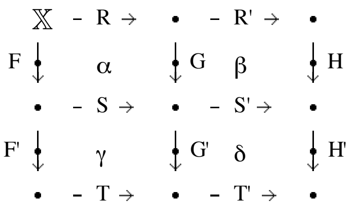
$$\begin{array}{ccccc}
Ge_1(Rx) & \xrightarrow{G\underline{R}(x)} & GR(e_1x) & \xrightarrow{\alpha e_1(x)} & SF(e_1x) \\
{\scriptstyle \underline{G}R(x)}\downarrow & & & & \downarrow{\scriptstyle S\underline{F}(x)} \\
e_1GR(x) & \xrightarrow[e_1\alpha(x)]{} & e_1SF(x) & \xrightarrow[\underline{S}F(x)]{} & Se_1F(x)
\end{array}$$

(c3) $S\underline{F}(x, y).\alpha z.G\underline{R}(x, y) = \underline{S}(Fx, Fy).(\alpha x +_1 \alpha y).\underline{G}(Rx, Ry)$

(for $z = x +_1 y$ in $\mathbb{X}$),

$$\begin{array}{ccccc}
G(Rx +_1 Ry) & \xrightarrow{G\underline{R}(x,y)} & GR(z) & \xrightarrow{\alpha z} & SF(z) \\
{\scriptstyle \underline{G}(Rx, Ry)}\downarrow & & & & \downarrow{\scriptstyle S\underline{F}(x, y)} \\
GRx +_1 GRy & \xrightarrow[\alpha x +_1 \alpha y]{} & SFx +_1 SFy & \xrightarrow[\underline{S}F(x, Fy)]{} & S(Fx +_1 Fy)
\end{array}$$

The horizontal composition $(\alpha \mid \beta)$ and the vertical composition $\alpha \otimes \gamma = \left(\frac{\alpha}{\gamma}\right)$ of double cells are both defined *via the composition of transversal maps* (in a weak sc-category)

(3) 

(4) $(\alpha \mid \beta)(x) \; = \; S'\alpha x.\beta Rx\colon HR'R(x) \to S'GR(x) \to S'SF(x)$,

$\left(\frac{\alpha}{\gamma}\right)(x) \; = \; \gamma Fx.G'\alpha x\colon G'GR(x) \to G'SF(x) \to TF'F(x)$ (for x in $\mathbb{X}$).

We verify below, in Theorem 2.4, that these compositions are well-defined, and satisfy the axioms of a double category.

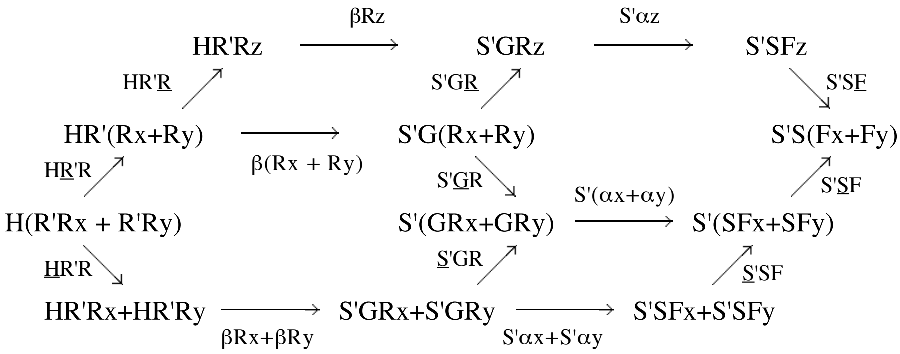
Within $\mathbb{W}$sc, we have the strict 2-category Lx**Wsc** of *weak sc-categories, lax sc-functors and transversal transformations*: namely, Lx**Wsc** is the restriction of $\mathbb{W}$sc to trivial vertical arrows.

Similarly, the strict 2-category Cx**Wsc** (resp. Ps**Wsc**) whose arrows are the colax (resp. pseudo) sc-functors, also lies in $\mathbb{W}$sc.

2.4. Theorem. $\mathbb{W}$sc, *as defined above, is indeed a strict double category.*

Proof. The argument is much the same as for $\mathbb{D}$bl, in [GP2].

First, to show that the double cells defined in 2.3.4 are indeed coherent, we verify the condition (c3) for $(\alpha \mid \beta)$, with respect to a concatenation $z = x +_1 y$ (written as $x + y$) in $\mathbb{X}$. Our property amounts to the commutativity of the outer diagram below, formed of transversal maps

(1) 

Indeed, the two hexagons commute applying (c3) to α and β, respectively; the upper parallelogram commutes by naturality of β; the lower one by consistency of S' with the cells αx, αy (by 2.1(i)).

Now, both compositions of double cells have been defined, in 2.3.4, *via the composition of transversal maps* (in a weak sc-category), and therefore are strictly unitary and associative.

Finally, to verify the middle-four interchange law on the four double cells of diagram 2.3.3, we compute the compositions $(\alpha \mid \beta)\otimes(\gamma \mid \delta)$ and $(\alpha\otimes\gamma) \mid (\beta\otimes\delta)$ on an n-cube x, and we obtain the two transversal maps $H'HR'Rx \to T'TF'Fx$ of the upper or lower path of the following diagram

$$(2) \qquad \begin{array}{ccccccc} H'HR'Rx & \xrightarrow{H'\beta Rx} & H'S'GRx & \xrightarrow{H'S'\alpha x} & H'S'SFx & & \\ & & \downarrow{\scriptstyle \delta GRx} & & \downarrow{\scriptstyle \delta SFx} & & \\ & & T'G'GRx & \xrightarrow[T'G'\alpha x]{} & T'G'SFx & \xrightarrow[T'\gamma Fx]{} & T'TF'Fx \end{array}$$

But these two paths coincide because the square commutes: it is a consequence of axiom (c1) for the double cell δ, namely the naturality of δ on the transversal map $\alpha x\colon GR(x) \to SF(x)$. □

2.5. Comma structures. Comma double categories (introduced in [GP2]) also have a natural extension to the cubical case. Given a *colax* sc-functor $F\colon \mathbb{A} \to \mathbb{C}$ and a *lax* sc-functor $R\colon \mathbb{X} \to \mathbb{C}$ with the same codomain, we can construct the *comma* weak sc-category $F{\downarrow}R$, where the projections P and Q are strict sc-functors, and π is a cell of $\mathbb{W}$sc

$$(1) \qquad \begin{array}{ccc} F{\downarrow}R & \xrightarrow{P} & \mathbb{A} \\ {\scriptstyle Q}\downarrow & \pi & \downarrow{\scriptstyle F} \\ \mathbb{X} & \xrightarrow[R]{} & \mathbb{C} \end{array}$$

An n-cube of $F{\downarrow}R$ is a triple $(a, x; c\colon Fa \to Rx)$ where a is an n-cube of $\mathbb{A}$, x is an n-cube of $\mathbb{X}$ and c is an n-map of $\mathbb{C}$. A transversal map $(h, f)\colon (a, x; c) \to (a', x'; c')$ comes from a pair of transversal maps $h\colon a \to a'$ (in $\mathbb{A}$,) and $f\colon x \to x'$ (in $\mathbb{X}$) that form in $\mathbb{C}$ a commutative square of transversal maps

$$(2) \qquad \begin{array}{ccc} Fa & \xrightarrow{c} & Rx \\ {\scriptstyle Fh}\downarrow & & \downarrow{\scriptstyle Rf} \\ Fa' & \xrightarrow[c']{} & Rx' \end{array} \qquad\qquad Rf.c \;=\; c'.Fh.$$

Faces, transposition and transversal composition are obvious. In the 1-concatenation

(3) $(a, x; c: Fa \to Rx) +_1 (b, y; d: Fb \to Ry)$
$= (a +_1 b, x +_1 y; u: F(a +_1 b) \to R(x +_1 y)),$

the transversal map u is the following composite, *defined using the fact that* F *is colax and* R *is lax*:

(4) $u = \underline{R}(x, y).(c +_1 d).\underline{F}(a, b):$
$F(a +_1 b) \to Fa +_1 Fb \to Rx +_1 Ry \to R(x +_1 y).$

The invertible associativity transversal map for 1-directed concatenation of three 1-consecutive cubes

$(a, x; c), \quad (a', x'; c'), \quad (a'', x''; c'')$

is given by the pair $(\alpha(\mathbf{a}), \xi(\mathbf{x}))$ of associativity isocells for our two triples of 1-consecutive cubes, namely $\mathbf{a} = (a, a', a'')$ in $\mathbb{A}$ and $\mathbf{x} = (x, x', x'')$ in $\mathbb{X}$ (we write $+_1$ as $+$)

(5) $(\alpha(\mathbf{a}), \xi(\mathbf{x})): ((a,x; c)+(a',x'; c'))+(a'',x''; c'') \to (a,x; c)+((a',x'; c')+(a'',x''; c'')).$

In fact, let $a_1 = (a+a') + a''$, $a_2 = a + (a'+a'')$, and similarly x_1, x_2. Let us consider the transversal maps $\Phi: Fa_1 \to Rx_1$ and $\Psi: Fa_2 \to Rx_2$ defined by the following transversal compositions:

$\Phi = \underline{R}(x+x', x'').(\underline{R}(x, x') + e_1Rx'')((c+c')+c'').(\underline{F}(a, a') + e_1Fa'').\underline{F}(a+a', a''):$

$Fa_1 \to F(a+a')+Fa'' \to (Fa+Fa')+Fa'' \to (Rx+Rx')+Rx'' \to R(x+x')+Rx'' \to Rx_1,$

$\Psi = \underline{R}(x, x'+x'').(e_1Rx + \underline{R}(x', x''))(c+(c'+c'')).(e_1Fa + \underline{F}(a', a'')).\underline{F}(a, a'+a''):$

$Fa_2 \to Fa+F(a'+a'') \to Fa+(Fa'+Fa'') \to Rx+(Rx'+Rx'') \to Rx+R(x'+Rx'') \to Rx_2.$

Then the coherence of the transversal map (5) is expressed by the equality

$\Psi.F\alpha(\mathbf{a}) = R\xi(\mathbf{x}).\Phi: \ Fa_1 \to Rx_2,$

that follows from the coherence axioms on F, R and $\mathbb{C}$.

Finally, the strict sc-functors P and Q are the obvious projections, while the component of the transversal transformation π on the n-cube $(a, x; c)$ of $F{\downarrow}R$ is the transversal map:

(6) $\pi(a, x; c) = c: Fa \to Rx.$

2.6. Theorem (Universal properties of commas). *(a) For a pair of* lax *sc-functors* S, T *and a cell* α *as below (in* $\mathbb{W}$sc*) there is a unique lax sc-functor* $L\colon \mathbb{Z} \to F{\downarrow}R$ *such that* $S = PL$, $T = QL$ *and* $\alpha = (\beta \mid \pi)$ *where the cell* β *is defined by the identity* $1\colon QL \to T$ *(a horizontal transformation of lax sc-functors)*

$$(1)\qquad \begin{array}{ccccc} \mathbb{Z} & & \xrightarrow{\quad S \quad} & & \mathbb{A} \\ {\scriptstyle 1}\downarrow & & \alpha & & \downarrow{\scriptstyle F} \\ \mathbb{Z} & \xrightarrow[T]{} & \mathbb{X} & \xrightarrow[R]{} & \mathbb{C} \end{array} \qquad = \qquad \begin{array}{ccccc} & & \overset{S}{\longrightarrow} & & \\ \mathbb{Z} & \xrightarrow{L} & F{\downarrow}R & \xrightarrow{P} & \mathbb{A} \\ {\scriptstyle 1}\downarrow & \beta & {\scriptstyle Q}\downarrow & \pi & \downarrow{\scriptstyle F} \\ \mathbb{Z} & \xrightarrow[T]{} & \mathbb{X} & \xrightarrow[R]{} & \mathbb{C} \end{array}$$

Moreover, L *is pseudo if and only if both* S *and* T *are.*

(b) *A similar property holds for a pair of* colax *sc-functors* G, H *and a double cell* $\alpha'\colon (G \,{}^{1}_{R}\, FH)$.

Proof. (a) L is defined as follows on an n-cube z and an n-map $f\colon z \to z'$ of $\mathbb{Z}$

(2) $L(z) = (Sz, Tz; \alpha z\colon FSz \to RTz),$ $\qquad\qquad$ $L(f) = (Sf, Tf).$

The comparison transversal maps $\underline{L}$ for z and $z = x +_1 y$ in $\mathbb{Z}$, are constructed with the laxity transversal maps $\underline{S}$ and $\underline{T}$ (and are invertible if and only if the latter are)

(3) $\underline{L}z = (\underline{S}z, \underline{T}z)\colon\ e_1(Lz) \to Le_1(z),$

$\underline{L}(x, y) = (\underline{S}(x, y), \underline{T}(x, y))\colon\ Lx +_1 Ly \to L(z).$

Here, $Lx +_1 Ly$ is the n-cube defined as below (following 2.5.3-4)

(4) $Lx +_1 Ly = (Sx, Tx; \alpha x\colon FSx \to RTx) + (Sy, Ty; \alpha y\colon FSy \to RTy)$

$= (Sx +_1 Sy, Tx +_1 Ty; u),$

$u = \underline{R}(Tx, Ty).(\alpha x +_1 \alpha y).\underline{F}(Sx, Sy)\colon$

$F(Sx +_1 Sy) \to FSx +_1 FSy \to RTx +_1 RTy \to R(Tx +_1 Ty).$

The coherence condition 2.5.2 on the transversal map $\underline{L}(x, y) = (\underline{S}(x, y), \underline{T}(x, y))$ of $F{\downarrow}R$

$$(5)\qquad \begin{array}{ccc} F(Sx +_1 Sy) & \xrightarrow{\ u\ } & R(Tx +_1 Ty) \\ {\scriptstyle F\underline{S}(x, y)}\downarrow & & \downarrow{\scriptstyle R\underline{T}(x, y)} \\ FS(z) & \xrightarrow[\alpha z]{} & RT(z) \end{array} \qquad R\underline{T}(x, y).u = \alpha z.F\underline{S}(x, y),$$

follows from the coherence condition (c3) of α as a double cell in $\mathbb{W}sc$

(6) $\underline{(RT)}(x, y).(\alpha x +_1 \alpha y).\underline{F}(Sx, Sy) \; = \; \alpha z.F\underline{S}(x, y)$,

$$\begin{array}{ccccc}
F(Sx +_1 Sy) & \xrightarrow{F\underline{S}(x,y)} & FS(z) & \xrightarrow{\alpha z} & RT(z) \\
{\scriptstyle \underline{F}(Sx,\, Sy)} \downarrow & & & & \downarrow {\scriptstyle 1} \\
FSx +_1 FSy & \xrightarrow[\alpha x \, +_1 \alpha y]{} & RTx +_1 RTy & \xrightarrow[\underline{RT}(x,\, y)]{} & RT(z)
\end{array}$$

where $\underline{(RT)}(x, y) = R\underline{T}(x, y).\underline{R}(Tx, Ty)$.

The uniqueness of L is obvious. □

3. Companions and adjoints in double categories

This section, taken from [GP2], Section 1, studies the connections between horizontal and vertical morphisms in a double category: horizontal morphisms can have vertical *companions* and vertical *adjoints*. Such phenomena are interesting in themselves and typical of double categories.

$\mathbb{A}$ is always a weak double category, that we assume to be *unitary* (in the sense that the identities are strict units), for the sake of simplicity.

3.1. Orthogonal companions. In the weak double category $\mathbb{A}$, the horizontal morphism $f: A \to B$ and the vertical morphism $u: A \nrightarrow B$ are made (orthogonal) *companions* by assigning a pair (η, ε) of cells as below, called the *unit* and *counit*, that satisfy the identities $\eta | \varepsilon = 1^{\bullet}_f$ and $\eta \otimes \varepsilon = 1_u$

$$(1) \qquad \begin{array}{ccc}
A & = & A \\
{\scriptstyle 1} \downarrow & \eta & \downarrow {\scriptstyle u} \\
A & \xrightarrow[f]{} & B
\end{array}
\qquad\qquad
\begin{array}{ccc}
A & \xrightarrow{f} & B \\
{\scriptstyle u} \downarrow & \varepsilon & \downarrow {\scriptstyle 1} \\
B & = & B
\end{array}$$

Given f, this is equivalent (*by unitarity*) to saying that the pair (u, ε) satisfies the following universal property:

(a) for every cell $\varepsilon': (u' \genfrac{}{}{0pt}{}{f}{g} B)$ there is a unique cell $\lambda: (u' \genfrac{}{}{0pt}{}{A}{g} u)$ such that $\varepsilon' = \lambda | \varepsilon$

$$(2) \qquad \begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle u'}\downarrow & \varepsilon' & \downarrow{\scriptstyle 1} \\ A' & \xrightarrow[g]{} & B \end{array} \qquad = \qquad \begin{array}{ccccc} A & = & A & \xrightarrow{f} & B \\ {\scriptstyle u'}\downarrow & \lambda & \downarrow{\scriptstyle u} & \varepsilon & \downarrow{\scriptstyle 1} \\ A' & \xrightarrow[g]{} & B & = & B \end{array}$$

In fact, given (η, ε), we can (and must) take $\lambda = \eta \otimes \varepsilon'$; on the other hand, given (a), we define $\eta\colon (A \,{}^{A}_{f}\, u)$ by the equation $\eta | \varepsilon = 1^{\bullet}_{f}$ and deduce that $\eta \otimes \varepsilon = 1_u$ because $(\eta \otimes \varepsilon) \mid \varepsilon = (\eta|\varepsilon) \otimes \varepsilon = \varepsilon = (1_u \mid \varepsilon)$.

Similarly, also the pair (u, η) is characterised by a universal property

(b) for every cell $\eta'\colon (A \,{}^{g}_{f}\, u')$ there is a unique cell $\mu\colon (u \,{}^{g}_{B}\, u')$ such that $\eta' = \eta|\mu$.

Therefore, if f has a vertical companion, this is determined up to a unique special isocell, *and will often be written as* f_*. Companions compose in the obvious (covariant) way: if $g\colon B \to C$ also has a companion, then $g_* f_*\colon A \,\dashrightarrow\, C$ is companion to $gf\colon A \to C$, with unit $\left(\dfrac{\eta \mid 1}{1^{\bullet} \mid \eta'}\right)\colon (A \,{}^{A}_{gf}\, g_* f_*)$.

Companionship is preserved by *unitary* lax or colax double functors.

We say that $\mathbb{A}$ *has vertical companions* if every horizontal arrow has a vertical companion. The weak double categories recalled in Section 1 have vertical companions, given by the obvious embedding of horizontal arrows into the vertical ones.

Companionship is simpler for horizontal *iso*morphisms. If f is one and has a companion u, then its unit and counit are also horizontally invertible and determine each other:

$$(3) \quad (\varepsilon \mid 1^{\bullet}_{g} \mid \eta) \;=\; \eta \otimes \varepsilon \;=\; 1_u \qquad\qquad (g \;=\; f^{-1}),$$

as it appears rewriting $(\varepsilon \mid 1^{\bullet}_{g} \mid \eta)$ as follows, and then applying middle-four interchange

$$\begin{array}{ccccccc} A & \xrightarrow{f} & B & \xrightarrow{g} & A & = & A \\ {\scriptstyle 1}\downarrow & 1^{\bullet}_{f} & \downarrow{\scriptstyle 1} & 1^{\bullet}_{g} & \downarrow{\scriptstyle 1} & \eta & \downarrow{\scriptstyle u} \\ A & \xrightarrow{f} & B & \xrightarrow{g} & A & \xrightarrow{f} & B \\ {\scriptstyle u}\downarrow & \varepsilon & \downarrow{\scriptstyle 1} & 1^{\bullet}_{g} & \downarrow{\scriptstyle 1} & 1^{\bullet}_{f} & \downarrow{\scriptstyle 1} \\ B & = & B & \xrightarrow{g} & A & \xrightarrow{f} & B \end{array}$$

Conversely, the existence of a horizontally invertible cell $\eta\colon (A\ {}^{A}_{f}\ u)$ implies that f is horizontally invertible, with companion u and counit as above.

3.2. Orthogonal adjoints. Transforming companionship by vertical (or horizontal) duality, the arrows $f\colon A \to B$ and $v\colon B \rightarrowtriangle A$ are made *orthogonal adjoints* by a pair (α, β) of cells as below

$$(1) \qquad \begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle 1}\downarrow & \alpha & \downarrow{\scriptstyle v} \\ A & = & A \end{array} \qquad\qquad \begin{array}{ccc} B & = & B \\ {\scriptstyle v}\downarrow & \beta & \downarrow{\scriptstyle 1} \\ A & \xrightarrow[f]{} & B \end{array}$$

with $\alpha|\beta = 1^{\bullet}_{f}$ and $\beta\otimes\alpha = 1_{v}$. Then, f is the *horizontal adjoint* and v the *vertical* one. (In the general case, there is no reason of distinguishing 'left' and 'right', unit and counit; see the examples below).

Again, given f, these relations can be described by universal properties for (v, β) or (v, α)

(a) for every cell $\beta'\colon (v'\ {}^{g}_{f}\ B)$ there is a unique cell $\lambda\colon (v'\ {}^{g}_{A}\ v)$ such that $\beta' = \lambda|\beta$,

(b) for every cell $\alpha'\colon (A\ {}^{f}_{g}\ v')$ there is a unique cell $\mu\colon (v\ {}^{B}_{g}\ v')$ such that $\alpha' = \alpha|\mu$.

The vertical adjoint of f is determined up to a special isocell and will often be written as f^{*}; vertical adjoints compose, contravariantly: $(gf)^{*}$ can be constructed as $f^{*}g^{*}$.

We say that $\mathbb{A}$ *has vertical adjoints* if every horizontal arrow has a vertical adjoint. Plainly, this is the case for the weak double categories recalled in Section 1.

3.3. Proposition. *Let* $f\colon A \to B$ *have a vertical companion* $u\colon A \rightarrowtriangle B$. *Then* $v\colon B \rightarrowtriangle A$ *is vertical adjoint to* f *if and only if* $u \dashv v$ *in the bicategory* $\mathbf{V}\mathbb{A}$ *(of vertical arrows and special cells).*

Proof. Given four cells $\eta, \varepsilon, \alpha, \beta$ as above (in 3.1, 3.2), we have two special cells

$$\eta\otimes\alpha\colon 1^{\bullet} \to u\otimes v, \qquad\qquad \beta\otimes\varepsilon\colon u\otimes v \to 1^{\bullet},$$

that are easily seen to satisfy the triangle identities in $\mathbf{V}\mathbb{A}$. The converse is similarly obvious. □

4. Cubical adjunctions

A colax/lax cubical adjunction is now defined as an orthogonal adjunction in the strict double category $\mathbb{W}$sc.

4.1. Colax/lax adjunctions. An orthogonal adjunction (F, R) *in* $\mathbb{W}$sc (3.2) gives a notion of *cubical adjunction* (η, ε): F $\dashv$ R between weak sc-categories, which occurs naturally in various situations, as already seen in Section 1.

The left adjoint F: $\mathbb{A} \to \mathbb{X}$ is a *colax* sc-functor, the right adjoint R: $\mathbb{X} \to \mathbb{A}$ is *lax*, and we have two $\mathbb{W}$sc-cells η, ε

$$(1) \qquad \begin{array}{ccc} \mathbb{A} & = & \mathbb{A} \\ {\scriptstyle F}\downarrow^{\bullet} & \eta & \| \\ \mathbb{X} & \xrightarrow[R]{} & \mathbb{A} \end{array} \qquad\qquad \begin{array}{ccc} \mathbb{X} & \xrightarrow{R} & \mathbb{A} \\ \| & \varepsilon & \downarrow^{\bullet}{\scriptstyle F} \\ \mathbb{X} & = & \mathbb{X} \end{array}$$

satisfying the triangle equalities $\eta \otimes \varepsilon = 1_F$ and $\varepsilon | \eta = 1^{\bullet}_R$. (As in 2.3, the arrow of a colax sc-functor is marked with a dot *when displayed vertically*, in a diagram of $\mathbb{W}$sc.)

This general adjunction will be said to be of *colax/lax type*. We speak of a *pseudo/lax* (resp. a *colax/pseudo*) adjunction when the left (resp. right) adjoint is pseudo, and of a *pseudo* adjunction when both adjoints are pseudo (replacing *pseudo* with *strict* when appropriate).

From general properties (see 3.2), we already know that the left adjoint of a lax sc-functor R is determined up to isomorphism (a special invertible cell between vertical arrows in $\mathbb{W}$sc) and that left adjoints compose, contravariantly. Similarly for right adjoints.

As in 2.3, we may write the unit of the adjunction as η: 1 $\to$ RF, by abuse of notation; *but one should recall that the coherence conditions of such a transformation work through the interplay of the comparison cells of* F *and* R. Similarly for the counit ε: FR $\to$ 1. Therefore (as with double categories, in [GP2]), a *general* colax/lax *adjunction cannot be seen as an adjunction in some bicategory*; but we shall prove in the next section that this is possible in particular cases, a *pseudo/lax* or a *colax/pseudo* adjunction.

4.2. Description. To make the previous definition explicit, a *colax/lax adjunction* (η, ε): F $\dashv$ R between the weak sc-categories $\mathbb{A}$, $\mathbb{X}$ consists of the following items.

(a) A *colax* sc-functor $F: \mathbb{A} \to \mathbb{X}$, with comparison cells

$$\underline{F}(a): F(e_1 a), \to e_1(Fa), \qquad \underline{F}(a, b): F(a +_1 b) \to Fa +_1 Fb.$$

(b) A *lax* sc-functor $R: \mathbb{X} \to \mathbb{A}$, with comparison cells

$$\underline{R}(x): e_1(Rx) \to R(e_1 x), \qquad \underline{R}(x, y): Rx +_1 Ry \to R(x +_1 y).$$

(c) An ordinary adjunction in every degree $n \geq 0$

$$\eta_n: 1 \to R_n F_n: \mathbf{A}_n \to \mathbf{A}_n, \qquad \varepsilon_n: F_n R_n \to 1: \mathbf{B}_n \to \mathbf{B}_n$$

$$\varepsilon_n F_n . F_n \eta_n = 1_{F_n}, \qquad R_n \varepsilon_n . \eta_n R_n = 1_{R_n}.$$

Explicitly this means that we are assigning:

- transversal maps $\eta a: a \to RFa$ in $\mathbb{A}$ (for a in $\mathbb{A}$),
- transversal maps $\varepsilon x: FRx \to x$ in $\mathbb{X}$ (for x in $\mathbb{X}$),

satisfying the naturality conditions (c1) and the triangle identities (c2), for $h: a \to b$ in $\mathbb{A}$ and $f: x \to y$ in $\mathbb{X}$

(c1) $\eta b.h = RFh.\eta a,$ $\qquad \varepsilon y.FRf = f.\varepsilon x,$

(c2) $\varepsilon Fa.F\eta a = 1_{Fa},$ $\qquad R\varepsilon x.\eta Rx = 1_{Rx}.$

(d) These families $\eta = (\eta_n)$ and $\varepsilon = (\varepsilon_n)$ must respect faces and transpositions, and be coherent with the cubical operations (in terms of the comparison cells of F and R):

(1) $\eta(\partial_i^\alpha x) = \partial_i^\alpha(\eta x),$ $\qquad \eta(s_i x) = s_i(\eta x),$

(2) $\varepsilon(\partial_i^\alpha x) = \partial_i^\alpha(\varepsilon x),$ $\qquad \varepsilon(s_i x) = s_i(\varepsilon x).$

(d1') (*coherence of* η *with identities*) for a in $\mathbb{A}$:

(3) $R\underline{F}a.\eta(e_1 a) = \underline{R}Fa.e_1(\eta a)$ $\qquad$ ($\eta(e_1 a) = e_1(\eta a)$, if F and R are unitary),

(d1") (*coherence of* ε *with identities*) for x in $\mathbb{X}$:

(4) $\varepsilon(e_1 x).F\underline{R}x = e_1(\varepsilon x).\underline{F}Rx$ $\qquad$ ($\varepsilon(e_1 x) = e_1(\varepsilon x)$, if F and R are unitary);

(d2') (*coherence of* η *with concatenation*) for $c = a +_1 b$ in $\mathbb{A}$:

(5)
$$\begin{array}{ccc} c & \xrightarrow{\eta c} & RFc \\ {\scriptstyle \eta a + \eta b}\downarrow & & \downarrow{\scriptstyle R\underline{F}(a,b)} \\ RFa + RFb & \xrightarrow[\underline{R}F(a,b)]{} & R(Fa + Fb) \end{array} \qquad \begin{array}{l} R\underline{F}(a, b).\eta c \\ \quad = \underline{R}(Fa, Fb).(\eta a +_1 \eta b), \end{array}$$

(d2") (*coherence of* ε *with concatenation*) for $z = x +_1 y$ in $\mathbb{X}$:

$$(6)\qquad \begin{array}{ccc} F(Rx + Ry) & \xrightarrow{F\underline{R}(x,\,y)} & FRz \\ {\scriptstyle \underline{F}(Rx,\,Ry)}\downarrow & & \downarrow{\scriptstyle \varepsilon z} \\ FRx + FRy & \xrightarrow[\varepsilon x\,+\,\varepsilon y]{} & z \end{array} \qquad \begin{array}{l} \varepsilon z.F\underline{R}(x, y) \\ \quad = (\varepsilon x +_1 \varepsilon y).\underline{F}(Rx, Ry). \end{array}$$

4.3. A remark. In this colax/lax adjunction, the comparison maps of R, together with the unit η, determine the comparison maps of F. In fact, the equation of (d1') says that the adjoint map of $\underline{F}a$, that is $(\underline{F}a)' = R\underline{F}a.\eta e_1 a$, must be equal to $\underline{R}Fa.e_1(\eta a)$. Similarly for $\underline{F}(a, b)$, from (d2').

Dually, the comparison maps of F and the counit ε determine the comparison maps of R.

4.4. Theorem (Characterisation by transversal hom-sets). *An adjunction* (η, ε): $F \dashv R$ *can equivalently be given by a colax sc-functor* $F: \mathbb{A} \to \mathbb{X}$, *a lax sc-functor* $R: \mathbb{X} \to \mathbb{A}$ *and a sequence of functorial isomorphisms* H_n

(1) $H_n: \mathbf{X}_n(F_n-, \cdot) \to \mathbf{A}_n(-, R_n\cdot): \mathbf{A}_n^{op} \times \mathbf{X}_n \to \mathbf{Set}$ $(n \geq 0)$,

$H_n(a, x): \mathbf{X}_n(Fa, x) \to \mathbf{A}_n(a, Rx)$,

whose components $H_n(a, x)$ *(that we often write as* H*) commute with faces and transpositions, and are coherent with the cubical operations (through the comparison cells of* F *and* R*), i.e. satisfy the following conditions*:

(ad.1) $H_n(\partial_i^\alpha a, \partial_i^\alpha x) = \partial_i^\alpha(H_n(a, x))$, $\qquad H_n(s_i a, s_i x) = s_i(H_n(a, x))$,

(ad.2) $H(e_1(f).\underline{F}(a)) = \underline{R}(x).e_1(Hf)$ (for $f: Fa \to x$ in $\mathbb{X}$),

$$Fe_1(a) \xrightarrow{\underline{F}(a)} e_1(Fa) \xrightarrow{e_1(f)} e_1(x) \qquad e_1(a) \xrightarrow{e_1(Hf)} e_1(Rx) \xrightarrow{\underline{R}(x)} R(x)$$

(ad.3) $H((f + g).\underline{F}(a, b)) = \underline{R}(x, y).(Hf + Hg)$ (for $f: Fa \to x$, $g: Fb \to y$),

$$F(a+b) \xrightarrow{\underline{F}(a,\,b)} Fa+Fb \xrightarrow{f+g} x+y \qquad a+b \xrightarrow{Hf+Hg} Rx+Ry \xrightarrow{\underline{R}(x,\,y)} R(x+y).$$

In this equivalence, $H_n(a, x)$ *is defined by the unit* η *as*

(2) $H_n(a, x)(f) = Rf.\eta a: a \to RFa \to Rx$ (for $f: Fa \to x$ in $\mathbb{X}$),

while the component $\eta_n: 1 \to R_nF_n: \mathbf{A}_n \to \mathbf{A}_n$ *of the unit is defined by* H *as*

(3) $\eta_n(a) = H_n(a, Fa)(\mathrm{id}Fa): a \to RF(a)$ (for a in $\mathbf{A}_n$).

Proof. We have only to verify the equivalence of 4.2.1-6 with the conditions above.

To show, for instance, that 4.2.5 implies (ad.2), let $f\colon Fa \to x$ and $g\colon Fb \to y$ be 1-consecutive transversal maps in $\mathbb{X}$, and apply H as defined above, in (2):

$$
\begin{aligned}
(4)\quad H((f + g).\underline{F}(a, b)) &= R(f + g).R\underline{F}(a, b).\eta(a + b) \\
&= R(f + g).\underline{R}(Fa, Fb).(\eta a + \eta b) && \text{(by 4.2.5)} \\
&= \underline{R}(x, y).(Rf + Rg).(\eta a + \eta b) && \text{(by 2.1(i))} \\
&= \underline{R}(x, y).(Hf + Hg) && \square
\end{aligned}
$$

4.5. Corollary (Characterisation by commas). *An adjunction amounts to an isomorphism of weak sc-categories* $H\colon F{\downarrow}\mathbb{X} \to \mathbb{A}{\downarrow}R$ *over the product* $\mathbb{A} \times \mathbb{X}$

$$
(1)\qquad \begin{array}{ccc}
F{\downarrow}\mathbb{X} & \xrightarrow{\ H\ } & \mathbb{A}{\downarrow}R \\
& \searrow \quad \swarrow & \\
& \mathbb{A} \times \mathbb{X} &
\end{array}
$$

Proof. It is a straightforward consequence of the previous theorem. $\square$

4.6. Theorem (Right adjoint by universal properties). *Given a colax sc-functor* $F\colon \mathbb{A} \to \mathbb{X}$, *the existence and choice of a right adjoint lax sc-functor* R *amounts to a sequence of conditions and choices* (rad.n):

(rad.n) *for every* n*-cube* x *in* $\mathbb{X}$ *there is a universal arrow* $(Rx, \varepsilon x\colon F(Rx) \to x)$ *from the functor* F_n *to the object* x (and we choose one),

It is also assumed that these choices commute with faces and transpositions *(which can be realised starting in degree zero and going up, one degree at a time).*

Explicitly, the universal property means that, for each n*-cube* a *in* $\mathbb{A}$ *and transversal map* $f\colon Fa \to x$ *in* $\mathbb{X}$ *there is a unique* $h\colon a \to Rx$ *such that* $f = \varepsilon x.Fh\colon Fa \to F(Rx) \to x$.

The comparison special transversal maps of R

(1) $\underline{R}(x)\colon e_1(Rx) \to R(e_1x)$, $\qquad \underline{R}(x, y)\colon Rx +_1 Ry \to R(x +_1 y)$,

are then provided by the universal property of ε, *as the unique solution of the equations* 4.2.4, 4.2.6, *respectively; and* R *is pseudo if and only if all such cells are special isocells.*

Proof. The conditions (rad.n) are plainly necessary.

Conversely, (rad.n) provides an ordinary adjunction $(\eta_n, \varepsilon_n)\colon F_n \dashv R_n$ for the categories $\mathbf{A}_n$, $\mathbf{X}_n$, so that R, η and ε are correctly defined – as far as cubes, transversal maps, faces, transpositions, transversal composition and transversal identities are concerned.

Now, we define the R-comparison maps $\underline{R}$ as specified in the statement, so that the coherence properties of ε are satisfied (4.2.4, 4.2.6). One verifies easily, for such transversal maps, the axioms of naturality and coherence (2.1).

Finally, we have to prove that $\eta\colon 1 \to RF$ satisfies the coherence property 4.2.5

(2) $R\underline{F}(a, b).\eta c = \underline{R}(Fa, Fb).(\eta a +_1 \eta b)$,

with respect to a concatenation $c = a +_1 b$ of n-cubes in $\mathbb{A}$ (similarly one proves 4.2.3). By the universal property of ε, it will suffice to show that the composite $\varepsilon(Fa + Fa').F(-)$ takes the same value on both terms of (2). In fact, on the left-hand term we get $\underline{F}(a, b)$

(3) $\varepsilon(Fa + Fa').FR\underline{F}(a, b).F\eta c = \underline{F}(a, b).\varepsilon Fc.F\eta c = \underline{F}(a, b)$;

but we get the same on the right-hand term of (2), using 4.2.6, the naturality of $\underline{F}$, the middle four interchange in $\mathbb{X}$ and a triangle identity

(4) $\varepsilon(Fa + Fa').F\underline{R}(Fa, Fb).F(\eta a +_1 \eta b) = (\varepsilon Fa + \varepsilon Fa').\underline{F}(RFa, RFb).F(\eta a +_1 \eta b)$
$= (\varepsilon Fa + \varepsilon Fa').(F\eta a +_1 F\eta b).\underline{F}(a, b) = \underline{F}(a, b)$. □

4.7. Theorem (Factorisation of adjunctions). *Let* $F \dashv R$ *be a colax/lax adjunction between* $\mathbb{A}$ *and* $\mathbb{X}$. *Then, using the isomorphism of weak sc-categories* $H\colon F{\downarrow}\mathbb{X} \to \mathbb{A}{\downarrow}R$ (in Corollary 4.5), *we can factor the adjunction as*

$$(1)\quad \mathbb{A} \underset{P}{\overset{F'}{\rightleftarrows}} F{\downarrow}\mathbb{X} \underset{H^{-1}}{\overset{H}{\rightleftarrows}} \mathbb{A}{\downarrow}R \underset{R'}{\overset{Q}{\rightleftarrows}} \mathbb{X} \qquad F = QHF', \quad R = PH^{-1}R'.$$

- *a* coreflective colax/strict adjunction $F' \dashv P$ *(with unit* $PF' = 1$*),*
- *an* isomorphism $H \dashv H^{-1}$,
- *a* reflective strict/lax adjunction $Q \dashv R'$ *(with counit* $QR' = 1$*),*

where the comma projections P *and* Q *are strict sc-functors.*

Proof. We define the lax sc-functor $R'\colon \mathbb{X} \to \mathbb{A}{\downarrow}R$ by the strong universal property of commas (2.6(a)), applied to $R\colon \mathbb{X} \to \mathbb{A}$, $1\colon \mathbb{X} \to \mathbb{X}$ and $\alpha = 1^{\bullet}_R$ as in the diagram below

$$(2)\qquad \begin{array}{ccccc} \mathbb{X} & & \xrightarrow{\quad R \quad} & & \mathbb{A} \\ {\scriptstyle 1}\downarrow & & \alpha & & \downarrow{\scriptstyle 1} \\ \mathbb{X} & \xrightarrow[T]{} & \mathbb{X} & \xrightarrow[R]{} & \mathbb{A} \end{array} \quad = \quad \begin{array}{ccccc} & & \overset{R}{\longrightarrow} & & \\ \mathbb{X} & \xrightarrow{R'} & 1{\downarrow}R & \xrightarrow{P} & \mathbb{A} \\ {\scriptstyle 1}\downarrow & \beta & {\scriptstyle Q}\downarrow & \pi & \downarrow{\scriptstyle 1} \\ \mathbb{X} & \xrightarrow[T]{} & \mathbb{X} & \xrightarrow[R]{} & \mathbb{A} \end{array}$$

$R'(x) \ = \ (Rx, x; 1\colon Rx \to Rx),$

$\underline{R}'(f, g) \ = \ (\underline{R}(f, g), 1)\colon (Rf + Rg, f + g; \underline{R}(f, g)) \to (R(f + g), f + g; 1).$

Similarly, we define the colax sc-functor $F'\colon \mathbb{A} \to F{\downarrow}\mathbb{X}$ by the dual result (2.6(b))

(3) $F'(a) \ = \ (a, Fa; 1\colon Fa \to Fa),$

$\underline{F}'(a, b) \ = \ (1, \underline{F}(a, b))\colon (a + b, F(a + b); 1) \to (a + b, Fa + Fb; \underline{F}(a, b)).$

The coreflective adjunction $F' \dashv P$ is obvious

(4) $\eta'a \ = \ 1_a\colon a \to PF'a,$

$\varepsilon'(a, x; f\colon Fa \to x) \ = \ (1_a, f)\colon \ (a, Fa; 1\colon Fa \to Fa) \to (a, x; f\colon Fa \to x),$

as well as the reflective adjunction $Q \dashv R'$ and the factorisation above. □

5. Cubical adjunctions and pseudo sc-functors

We consider now sc-adjunctions where the left or right adjoint is a pseudo sc-functor. Adjoint equivalences of weak sc-categories are introduced.

5.1. Comments. Let us recall, from 4.1, that a *pseudo/lax* sc-adjunction $F \dashv R$ is a colax/lax adjunction between weak sc-categories where the left adjoint F is pseudo.

Then, the comparison cells of F are horizontally invertible and the composites RF and FR are lax sc-functors; it follows (from definition 2.3) that the unit and counit are horizontal transformations of such functors. Therefore, *a pseudo/lax sc-adjunction gives an adjunction in the 2-category* Lx**Wsc** of weak sc-categories, lax sc-functors and transversal transformations (2.3); and we shall prove that these two facts are actually equivalent (Theorem 5.3).

Dually, a *colax/pseudo* sc-adjunction, where the right adjoint R is pseudo, will amount to an adjunction in the 2-category Cx**Wsc** of weak sc-categories, colax sc-functors and transversal transformations. Finally, a *pseudo* sc-adjunction, where both F and R are pseudo, will be the same as an adjunction in the 2-category Ps**Wsc**, whose arrows are the pseudo sc-functors.

5.2. Theorem (Companions in $\mathbb{W}$sc). *A lax sc-functor* R *has an orthogonal companion* F in $\mathbb{W}$sc *if and only if it is pseudo; then one can define* $F = R_*$ *as the colax sc-functor which coincides with* R *except for comparison maps transversally inverse to those of* R.

Proof. If R is pseudo, it is obvious that R_*, as defined above, is an orthogonal companion.

Conversely, suppose that $R: \mathbb{X} \to \mathbb{A}$ (lax) has an orthogonal companion F (colax). There are thus two cells η, ε in $\mathbb{W}$sc)

(1)
$$\begin{array}{ccc} \mathbb{X} & = & \mathbb{X} \\ {\scriptstyle 1}\downarrow & \eta & \downarrow{\scriptstyle F} \\ \mathbb{X} & \xrightarrow[R]{} & \mathbb{A} \end{array} \qquad\qquad \begin{array}{ccc} \mathbb{X} & \xrightarrow{R} & \mathbb{A} \\ {\scriptstyle F}\downarrow & \varepsilon & \downarrow{\scriptstyle 1} \\ \mathbb{A} & = & \mathbb{A} \end{array}$$

which satisfy the identities $\eta|\varepsilon = 1^{\bullet}_R$, $\eta\otimes\varepsilon = 1_F$.

This means two 'transformations' $\eta: F \to R$, $\varepsilon: R \to F$, as defined in 2.3; for every n-cube x in $\mathbb{X}$, we have two transversal maps ηx and εx in $\mathbb{A}$

(2) $\eta x: Fx \to Rx, \qquad\qquad \varepsilon x: Rx \to Fx,$

consistently with faces and transpositions. These maps are transversally inverse, because of the previous identities (cf. 2.3.4)

(3) $\eta x.\varepsilon x = (\eta \mid \varepsilon)(x) = 1_{Rx}, \qquad\qquad \varepsilon x.\eta x = (\eta\otimes\varepsilon)(x) = 1_{Fx}.$

Applying now the coherence condition (c3) (in 2.3), for the transformations η, ε and the concatenation $z = x +_1 y$ in $\mathbb{X}$, we find

(4) $\eta z = \underline{R}(x, y).(\eta x +_1 \eta y).\underline{F}(x, y): Fz \to Rz,$

$\varepsilon x +_1 \varepsilon y = \underline{F}(x, y).\varepsilon z.\underline{R}(x, y): Rx +_1 Ry \to Fx +_1 Fy.$

Since all the components of η and ε are transversally invertible, this proves that $\underline{R}(x, y)$ has a left inverse and a right inverse transversal map. Similarly for degeneracies.

Therefore R is pseudo (and F is transversally isomorphic to R_*). □

5.3. Theorem. (a) (Pseudo/lax adjunctions) *Any adjunction* $F \dashv R$ *in the 2-category* Lx**Wsc** *has* F *pseudo and is a pseudo/lax sc-adjunction in the sense of* 4.1 (*or* 5.1).

(b) (Colax/pseudo adjunctions) *Any adjunction* $F \dashv R$ *in the 2-category* Cx**Wsc** *has* R *pseudo and is a colax/pseudo sc-adjunction in the sense of* 4.1 (*or* 5.1).

More formally, (a) *can be rewritten saying that, in the double category* $\mathbb{W}$sc, *if the horizontal arrow* R *has a 'horizontal left adjoint'* F *(within the horizontal 2-category* **H**$\mathbb{W}$sc = Lx**Wsc**), *then it also has an orthogonal adjoint* G *(colax). (Then, applying* 3.3, *it would follow that* F *and* G *are companions, whence F is pseudo, by* 5.2, *and isomorphic to* G.*)*

Proof. It suffices to prove (a); again, we only deal with the comparisons of a concatenation.

Let the lax structures of $F: \mathbb{A} \to \mathbb{X}$ and $R: \mathbb{X} \to \mathbb{A}$ be given by the following comparison maps, where $c = a +_1 b$ and $z = x +_1 y$

(1) $\lambda(a, b): Fa +_1 Fb \to F(a +_1 b), \qquad \underline{R}(x, y): Rx +_1 Ry \to R(x +_1 y),$

so that we have:

(2) $\eta c \; = \; R\lambda(a, b).\underline{R}(Fa, Fb).(\eta a +_1 \eta b):$

$$c \to RFa +_1 RFb \to R(Fa +_1 Fb) \to RFc,$$

$\varepsilon x +_1 \varepsilon y \; = \; \varepsilon z.F\underline{R}(x, y).\lambda(Rx, Ry):$

$$FRx +_1 FRy \to F(Rx +_1 Ry) \to FR(x +_1 y) \to z.$$

We construct now a colax structure $\underline{F}$ for F

(3) $\underline{F}(a, b) \; = \; \varepsilon(Fa +_1 Fb).F\underline{R}(Fa, Fb).F(\eta a +_1 \eta b):$

$$Fc \to F(RFa +_1 RFb) \to FR(Fa +_1 Fb) \to Fa +_1 Fb,$$

and prove that $\underline{F}(a, b)$ and $\lambda(a, b)$ are transversally inverse:

(4) $\lambda(a, b).\underline{F}(a, b) \; = \; \lambda(a, b).\varepsilon(Fa +_1 Fb).F\underline{R}(Fa, Fb).F(\eta a +_1 \eta b)$

$= \; \varepsilon Fc.FR\lambda(a, b).F\underline{R}(Fa, Fb).F(\eta a +_1 \eta b)$ (by naturality of ε, cf. 4.2),

$= \; \varepsilon F(c).F(\eta c) \; = \; 1_{Fc}$ (by (2) and a triangle identity);

(5) $\underline{F}(a, b).\lambda(a, b) \; = \; \varepsilon(Fa +_1 Fb).F\underline{R}(Fa, Fb).F(\eta a +_1 \eta b).\lambda(a, b)$

$= \; \varepsilon(Fa +_1 Fb).F\underline{R}(Fa, Fb).\lambda(RFa, RFb).(F\eta a +_1 F\eta b)$ (by naturality of λ),

$= \; (\varepsilon Fa +_1 \varepsilon Fb).(F\eta a +_1 F\eta b)$ (by (2)),

$= \; \varepsilon Fa.F\eta a +_1 \varepsilon Fb.F\eta b$

(by interchange of transversal composition and concatenation),

$= \; 1_{Fa} +_1 1_{Fb} \; = \; 1_{Fa + Fb}$ (by a triangle identity and unitarity of $\mathbb{X}$). □

5.4. Equivalences of weak sc-categories. An *adjoint equivalence* between two weak sc-categories $\mathbb{A}$ and $\mathbb{X}$ will be a pseudo sc-adjunction (η, ε): F ⊣ R where the transversal transformations $\eta: 1_{\mathbb{A}} \to RF$ and $\varepsilon: FR \to 1_{\mathbb{X}}$ are invertible.

The following properties of an sc-functor $F: \mathbb{A} \to \mathbb{X}$ will allow us to characterise this fact in the usual way, assuming the axiom of choice and under the mild restriction of *transversal invariance* (cf. 1.6):

(a) We say that F is *faithful* if all the ordinary functors $F_n: \mathbf{A}_n \to \mathbf{X}_n$ (between the categories of n-cubes and their transversal maps) are: given two transversal maps $h, k: a \to b$ of $\mathbb{A}$ between the same n-cubes, $F(h) = F(k)$ implies $h = k$.

(b) Similarly, we say that F is *full* if all the ordinary functors $F_n: \mathbf{A}_n \to \mathbf{X}_n$ are: for every transversal map $f: F(a) \to F(b)$ in $\mathbb{X}$, there is a transversal map $h: a \to b$ in $\mathbb{A}$ such that $F(h) = f$.

(c) Finally, we say that F is *essentially surjective on cubes* if every F_n is essentially surjective on objects: for every n-cube x in $\mathbb{X}$, there is some n-cube a in $\mathbb{A}$ and some invertible transversal map $f: F(a) \cong x$ in $\mathbb{X}$.

5.5. Theorem (Characterisations of equivalences). *Let* $F: \mathbb{A} \to \mathbb{X}$ *be a pseudo sc-functor between two* transversally invariant *weak sc-categories (see* 1.6*). The following conditions are equivalent (under the axiom of choice):*

(i) $F: \mathbb{A} \to \mathbb{X}$ is *(i.e. belongs to) an adjoint equivalence of weak sc-categories;*

(ii) F *is faithful, full and essentially surjective on cubes (cf.* 5.4*);*

(iii) *every ordinary functor* $F_n: \mathbf{A}_n \to \mathbf{X}_n$ *(between the transversal categories of* n*-cubes) is an equivalence of categories.*

Proof. By our previous definitions, in 5.4, conditions (ii) and (iii) are about the sequence of ordinary functors (F_n) and are well-known to be equivalent (assuming (AC)). Moreover, if F belongs to an adjoint equivalence (η, ε): F ⊣ R, every F_n is obviously an equivalence of categories.

Conversely, let us assume that every F_n is an equivalence of ordinary categories and let us extend the pseudo sc-functor F to an adjoint equivalence, proceeding by induction on the degree $n \geq 0$.

First, F_0 is an equivalence of categories and we begin by constructing an adjoint quasi-inverse $R_0: \mathbf{X}_0 \to \mathbf{A}_0$ in the usual way.

In other words, we choose for every 0-cube x some R(x) and some isomorphism $\varepsilon x: FR(x) \to x$; then a transversal map $g: x \to y$ in $\mathbb{X}$ is sent to the unique $\mathbb{A}$-map $R(g): R(x) \to R(y)$ coherent with the previous choices (since F_0 is full and faithful). Finally the isomorphism $\eta a: a \to RF(a)$ is determined by the triangle equations (for every 0-cube a of $\mathbb{A}$).

Assume now that the components of R, ε and η have been defined up to degree $n - 1 \geq 0$, and let us define them in degree n, taking care that the new choices be consistent with the previous ones.

First, for every n-cube $x: x^- \to_1 x^+$ in $\mathbb{X}$ we want to choose some n-cube $R(x): R(x^-) \to_1 R(x^+)$ in $\mathbb{A}$ and some n-isomorphism $\varepsilon x: FR(x) \cong x$ in $\mathbb{X}$. In fact, there exists (and we choose) some n-cube $a: a^- \to_1 a^+$ and some transversal isomorphism $i: F(a) \cong x$; but then $F(a^\alpha) \cong x^\alpha \cong FR(x^\alpha)$ and there are two transversal (n–1)-isomorphisms $h^\alpha: R(x^\alpha) \cong a^\alpha$.

By transversal invariance in $\mathbb{A}$, we can choose a transversal n-isomorphism h as in the left square below, and we define $R(x) = \partial_1^- h$

$$
(1)\qquad
\begin{array}{ccc}
Rx^- & \xrightarrow{h^-} & a^- \\
{\scriptstyle Rx}\downarrow & h & \downarrow{\scriptstyle a} \\
Rx^+ & \xrightarrow[h^+]{} & a^+
\end{array}
\qquad
\begin{array}{ccccc}
FRx^- & \xrightarrow{Fh^-} & Fa^- & \longrightarrow & x^- \\
{\scriptstyle FRx}\downarrow & Fh & \downarrow{\scriptstyle Fa} & i & \downarrow{\scriptstyle x} \\
FRx^+ & \xrightarrow[Fh^+]{} & Fa^+ & \longrightarrow & x
\end{array}
\qquad
\begin{array}{cc}
\bullet \longrightarrow & 0 \\
\downarrow & 1
\end{array}
$$

Then we define $\varepsilon x = i.Fh: FR(x) \cong x$, as in the right diagram above.

Now, since F_n is full and faithful, a transversal n-map $f: x \to y$ in $\mathbb{X}$ is sent to the unique $\mathbb{A}$-map $R(f): R(x) \to R(y)$ satisfying the condition $\varepsilon y.F(Rf) = f.\varepsilon x$ (naturality of ε).

Again, the n-isomorphism $\eta a: a \to RF(a)$ is determined by the triangle equations, for every n-cube a of $\mathbb{A}$.

The comparison n-maps $\underline{R}$ are uniquely determined by their coherence conditions (4.2), for an (n–1)-cube x and a 1-concatenation of n-cubes $z = x +_1 y$ in $\mathbb{X}$

(2) $\varepsilon e_1 x.F\underline{R}x \ = \ e_1(\varepsilon x).\underline{F}Rx, \qquad \varepsilon z.F\underline{R}(x, y) \ = \ (\varepsilon x +_1 \varepsilon y).\underline{F}(Rx, Ry).$

The construction of R, ε and η is now achieved.

One ends by proving that R is indeed a pseudo sc-functor, for a cube a and a 1-concatenation $c = a +_1 b$ in $\mathbb{A}$

(3) $R\underline{F}a.\eta e_1 a \ = \ \underline{R}Fa.e_1(\eta a), \qquad R\underline{F}(a, b).\eta c \ = \ \underline{R}(Fa, Fb).(\eta a +_1 \eta b),$

and that ε, η are coherent with the comparison cells of F and R. □

6. Limits and adjoints for weak sc-categories

We recall the definition of cones and limits from [G5], Section 3, and prove that right adjoints preserve the limits of lax sc-functors.

All weak sc-categories are assumed to be *pre-unitary*, in the sense that all the unitarity comparisons $e_1(x) +_1 e_1(x) \to e_1(x)$ are identities.

6.1. The shift. A crucial fact in the theory of weak *symmetric* cubical categories (inherited from *symmetric* cubical sets, see [G4]) is the presence of *one* strict double functor *of paths* (or *cocylinder*)

(1) $P\colon \mathbf{Wsc} \to \mathbf{Wsc}$,

$P\mathbb{A} = ((\mathbb{A}_{n+1}), (\partial^\alpha_{i+1}), (e_{i+1}), (s_{i+1}), (+_{i+1}), \lambda_2, \rho_2, \kappa_2, \chi_2)$,

that shifts down all components, discarding the structure of index 1. (On the other hand, cubical structures *without symmetries* have *a left* and *a right* path functor [G4], which makes things complicated.) Plainly, P preserves cubical adjunctions.

The faces and degeneracies of index 1 are then used to build three transversal transformations, the *faces* and *degeneracy* of P

(2) $\partial^\alpha = \partial^\alpha_1\colon P\mathbb{A} \to \mathbb{A}$, $\qquad e = e_1\colon \mathbb{A} \to P\mathbb{A}$.

Here, ∂^α and e are strict sc-functors: $\partial^\alpha_i\partial^\alpha_1 = \partial^\alpha_1\partial^\alpha_{i+1}$, etc.

6.2. Cones. Let $\mathbb{X}$ and $\mathbb{A}$ be weak sc-categories, and let $\mathbb{X}$ be small. Consider the *diagonal* functor (of ordinary categories)

(1) $D = D_\mathbb{A}\colon tv_0\mathbb{A} \to \mathrm{Lx}\mathbf{Wsc}(\mathbb{X}, \mathbb{A})$,

where $tv_0\mathbb{A}$ is the ordinary category of 0-cubes (objects) of $\mathbb{A}$ and their transversal maps.

D takes each 0-object a to the constant sc-functor $\mathbb{X} \to \mathbb{A}$, defined as follows on n-objects x and n-maps f of $\mathbb{X}$

(2) $Da\colon \mathbb{X} \to \mathbb{A}, \qquad Da(x) = e^n(a), \qquad Da(f) = id(e^n a) \qquad (x, f \text{ in } tv_n\mathbb{X})$.

It also takes every 0-map $t\colon a \to b$ in $\mathbb{A}$ to the diagonal transversal transformation

(3) $Dt\colon Da \to Db\colon \mathbb{X} \to \mathbb{A}, \qquad (Dt)(x) = e^n(t)\colon e^n(a) \to e^n(b) \qquad (x \text{ in } tv_n\mathbb{X})$.

Note that $Da\colon \mathbb{X} \to \mathbb{A}$ is strict because $\mathbb{A}$ is assumed to be pre-unitary.

Let $T\colon \mathbb{X} \to \mathbb{A}$ be a *lax* sc-functor (2.1), with comparison special cells

$\underline{T}(x)\colon e_1(Tx) \to T(e_1(x)), \qquad \underline{T}(x, y)\colon Tx +_1 Ty \to T(x +_1 y)$.

A *(transversal) sc-cone* for T is a pair $(a, h: Da \to T)$ consisting of an object a of $\mathbb{A}$. (the *vertex* of the cone) and a transversal transformation

$$h: Da \to T: \mathbb{X} \to \mathbb{A}.$$

In other words, (a, h) is an object of the ordinary comma category $(D \downarrow T)$, where T is viewed as an object of the category $Lx\mathbf{Wsc}(\mathbb{X}, \mathbb{A})$.

By definition (2.2), this amounts to assigning the following data:

- a transversal n-map $hx: e^n(a) \to Tx$ of $\mathbb{A}$, for every n-object x in $\mathbb{X}$,

subject to the following axioms:

(scc.1) $Tf.hx = hy$ (for $f: x \to y$ in $\mathbb{X}$);

(scc.2) h commutes with faces and transpositions and $h(e_1(x)) = \underline{T}(x).(e_1(h(x))$;

(scc.3) $h(x +_1 y) = \underline{T}(x, y).(hx +_1 hy): e^n(a) \to T(x +_1 y)$ $(\partial_1^+ x = \partial_1^- y)$,

$$(4) \qquad \begin{array}{ccc} e^{n+1}(a) & \xrightarrow{e_1(hx)} & e_1(Tx) \\ \downarrow 1 & & \downarrow \underline{T}(x) \\ e^{n+1}(a) & \xrightarrow[h(e_1 x)]{} & T(e_1(x)) \end{array} \qquad \begin{array}{ccc} e^n(a) +_1 e^n(a) & \xrightarrow{hx +_1 hy} & Tx +_1 Ty \\ \downarrow 1 & & \downarrow \underline{T}(x, y) \\ e^n(a) & \xrightarrow[h(x +_1 y)]{} & T(x +_1 y) \end{array}$$

6.3. Definition (Limits and cubical limits). A *(transversal) limit* $\lim(T) = (a, h)$ of the lax sc-functor $T \in Lx\mathbf{Wsc}(\mathbb{X}, \mathbb{A})$ is a universal cone $(a, h: Da \to T)$. In other words:

(tl.0) a is an object of $\mathbb{A}$ and $h: Da \to T: \mathbb{X} \to \mathbb{A}$, is a transversal transformation of lax sc-functors;

(tl.1) for every cone $(a', h': Da' \to T)$ there is precisely one 0-map $t: a' \to a$ in $\mathbb{A}$ such that $h.Dt = h'$.

We say that $\mathbb{A}$ has *limits of degree zero* on $\mathbb{X}$ if all these limits exist. We say that $\mathbb{A}$ has *limits of all degrees* on $\mathbb{X}$ if all sc-categories $P^n\mathbb{A}$ satisfy this condition, for $n \geq 0$.

We say that $\mathbb{A}$ has *symmetric cubical limits on* $\mathbb{X}$, or *lax functorial sc-limits on* $\mathbb{X}$, if:

(i) $\mathbb{A}$ has limits of all degrees on $\mathbb{X}$;

(ii) the limit-functors $\lim_n: Lx\mathbf{Wsc}(\mathbb{X}, P^n\mathbb{A}) \to tv_n\mathbb{A}$ commute with faces, degeneracies and transpositions.

Then the universal property gives a unitary *lax* sc-functor

(1) $\lim = (\lim_n)_{n\geq 0}\colon\ \mathrm{Lx}\mathbf{Wsc}(\mathbb{X}, \mathrm{P}^{\bullet}\mathbb{A}) \to \mathbb{A}.$

We say that $\mathbb{A}$ has *pseudo functorial sc-limits on* $\mathbb{X}$ if this lax sc-functor happens to be a pseudo sc-functor.

Without symmetries, things would become complicated. While the condition of having limits of degree zero can be directly extended to cubical categories, *the conditions (i), (ii) should (perhaps) be rewritten replacing each* P^n *with the family of all path functors of degree* n, namely $\mathrm{P}^n_i = \mathrm{P}^{n-i}.\mathrm{SP}^i\mathrm{S}$ for $i = 0,..., n$ (cf. [G4], 1.8), where S is the transposer endofunctor of cubical structures (that reverses the order of faces). We will not deal with such a situation

6.4. Theorem (Preservation of limits). *Let* $(\eta, \varepsilon)\colon F \dashv R$ *be a colax/lax cubical adjunction, where* both functors are unitary.

Then $R\colon \mathbb{B} \to \mathbb{A}$ *preserves all (the existing) transversal limits of lax sc-functors* $T\colon \mathbb{X} \to \mathbb{B}$.

Proof. Let $(b, k\colon D_{\mathbb{B}}(b) \to T)$ be a limit of T in $\mathbb{B}$. We want to prove that the pair

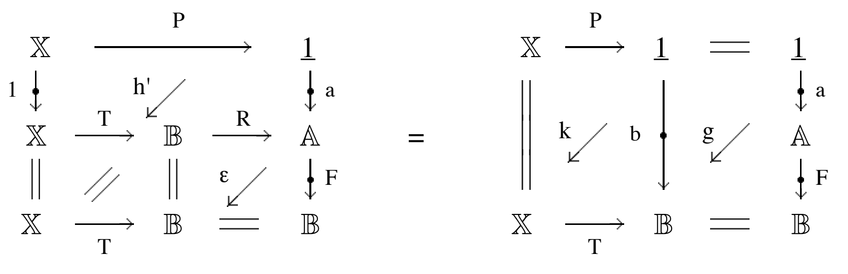
(1) $(Rb, Rk\colon R.D_{\mathbb{B}}(b) \to RT),$

is a limit of RT in $\mathbb{A}$. First, since R is unitary, $RD_{\mathbb{B}}(b) = D_{\mathbb{A}}(Rb)$, so that the pair (1) is indeed a cone of the lax sc-functors $RT\colon \mathbb{X} \to \mathbb{A}$.

Moreover, given an n-cone $(a, h'\colon D_{\mathbb{A}}(a) \to RT)$ of RT, with transversal components $h'x\colon e^n(a) \to RTx$, for every n-object x in $\mathbb{X}$, the adjunction gives a family $k'x\colon Fe^n(a) \to Tx$, that is a cone $(Fa, k'\colon D_{\mathbb{B}}(Fa) \to T)$ in $\mathbb{B}$. Therefore there is precisely one transversal map $g\colon Fa \to b$ in $\mathbb{B}$ such that $k.Dg = k'$. This means precisely one transversal map $f\colon a \to Rb$ in $\mathbb{A}$ such that $Rk.Df = h'$.

The proof can be rewritten using double cells of the double category $\mathbb{W}$sc. Given h', the pasting of the left diagram can be uniquely factorised as at the right

(2)



Now, the adjoint transversal n-map $f = Rg.\eta a: a \to RFa \to Re^n(B)$ is the unique transversal map of $\mathbb{A}$ such that $h = (Rk \mid f): aQ \to RT$ (as one sees pasting the $\mathbb{W}$sc-cell $\eta: (F \overset{1}{\underset{R}{}} 1)$ at the right of both diagrams above). □

6.5. Remark. Since the n-shift double functor P^n: **Wsc** → **Wsc** preserves cubical adjunctions, it follows that, if the weak sc-category $\mathbb{B}$ has *symmetric cubical limits on* $\mathbb{X}$, these are preserved by the right adjoint $R: \mathbb{B} \to \mathbb{A}$ (letting P^nR act on the cones of $P^n\mathbb{B}$, of course).

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