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CORRECTIONS TO THE ARTICLE OPERADIC DEFINITION OF THE NON-STRICT CELLS published in Volume LII-4 (2011), pp. 269-316

by Camell KACHOUR

Résumé

Dans ces notes nous proposons une nouvelle approche de la contractibilité pour les ω-opérades colorées telle que définies dans l'article publié dans les Cahiers de Topologie et de Géométrie Différentielle Catégorique (2011), volume 4. Nous proposons aussi une autre façon de construire la monade des ω-opérades contractiles colorées libres.

Abstract

In this short notes we propose a new notion of contractibility for coloured ω-operad defined in the paper published in Cahiers de Topologie et de Géométrie Différentielle Catégorique (2011), volume 4. Also we propose an alternative direction to build the monad for free contractible coloured ω-operads,

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Introduction

Steve Lack has suggested to me to use the more common name *weak higher transformations* instead of *Non-strict cells* which were defined in [2]. More precisely, in this article we defined a coglobular complex of ω-operads

$$B^0 \underset{\kappa_0^1}{\overset{\delta_0^1}{\rightrightarrows}} B^1 \underset{\kappa_1^2}{\overset{\delta_1^2}{\rightrightarrows}} B^2 \cdots\rightrightarrows B^{n-1} \underset{\kappa_{n-1}^n}{\overset{\delta_{n-1}^n}{\rightrightarrows}} B^n \cdots$$

such that algebras for B^0 are the weak ω-categories, algebras for B^1 are the weak ω-functors, algebras for B^2 are the weak ω-natural transformations, etc. However André Joyal has pointed out to us that there are too many coherence cells for each B^n when $n \geqslant 2$, and gave us a simple example of a natural transformation which cannot be an algebra for the 2-coloured ω-operad B^2. In this section we propose a notion of contractibilty, slightly different from those used in [1, 2]. This new approach excludes the counterexample of André Joyal.

Furthermore the main theorem of the section 6 in [2] is false. I am indebted to Michael Batanin and to Mark Weber, to have shown us a counterexample which invalid this result. However this false theorem has no impact to main ideas of the article [2]. I am indebted to Michael Batanin who told us that the technics of the coproduct of monads was adapted to substitute technically the role of this false theorem, and to Steve Lack who gave us the precise result and references that we needed for this correction.

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Corrections

Here $\mathbb{T}$ design the monad of strict ω-categories on ω-graphs. Notions of $\mathbb{T}$-graphs, $\mathbb{T}$-categories, constant ω-graphs, can be found in [2, 5]. The category $T\text{-}\mathbb{G}\mathrm{r}_{p,c}$ of pointed T-graphs over constant ω-graphs, and the category $T\text{-}\mathbb{C}\mathrm{at}_c$ of $\mathbb{T}$-categories over constant ω-graphs are both defined in [2].

Definition 1 For any $\mathbb{T}$-graph (C,d,c) over a constant ω-graph G, a pair of cells (x,y) of $C(n)$ has the *the loop property* if: $s^n_0(x) = s^n_0(y) = t^n_0(x) = t^n_0(y)$ □

Remark 1 If G is a constant ω-graph (see section 1.4 of the article [2]) A p-cell of G is denoted by $g(p)$ and this notation has the following meaning: The symbol g indicates the "colour", and the symbol p point out that we must see $g(p)$ as a p-cell of G, because G has to be seen as an ω-graph even though it is just a set. $\square$

Definition 2 For any $\mathbb{T}$-graph (C,d,c) over a constant ω-graph G, we call *the root cells* of (C,d,c), those cells whose arities are the reflexivity of a 0-cell $g(0)$ of G, where here "g" indicates the colour (see section 1), or in other words, those cells $x \in C(n)$ $(n \geqslant 1)$ such that $d(x) = 1_n^0(g(0))$. $\square$

Here 1_n^0 design the reflexivity operators of free strict ω-category $\mathbb{T}(G)$ (see also [2]). These notions of *root cells* and *loop condition* are the keys for our new approach to contractibility. These observations motivate us to put the following definition of what should be a contractible $\mathbb{T}$-graphs (C,d,c). For each integers $k \geqslant 1$ let us note $\tilde{C}(k) = \{(x,y) \in C(k) \times C(k) : x \| y$ and $d(x) = d(y)$, and if also (x,y) is a pair of root cells then they also need to verify the *loop property*: $s_0^k(x) = t_0^k(y)\}$. Also we denote $\tilde{C}(0) = \{(x,x) \in C(0) \times C(0)\}$.

Definition 3 A contraction on the $\mathbb{T}$-graph (C,d,c), is the datum, for all $k \in \mathbb{N}$, of a map $\tilde{C}(k) \xrightarrow{[,]_k} C(k+1)$ such that

- $s([\alpha,\beta]_k) = \alpha, t([\alpha,\beta]_k) = \beta$,
- $d([\alpha,\beta]_k) = 1_{d(\alpha)=d(\beta)}$. $\square$

A $\mathbb{T}$-graph which is equipped with a contraction will be called contractible and we use the notation $(C,d,c;([,]_k)_{k\in\mathbb{N}})$ for a contractible $\mathbb{T}$-graph. Nothing prevents a contractible $\mathbb{T}$-graph from being equipped with several contractions. So here $CT\text{-}\mathbb{G}\mathrm{r}_c$ is the category of the contractible $\mathbb{T}$-graphs equipped with a specific contraction, and morphisms of this category preserves the contractions. One can also refer to the category $CT\text{-}\mathbb{G}\mathrm{r}_{c,G}$, where here contractible $\mathbb{T}$-graphs are only taken over a specific constant ∞-graph G. A pointed contractible $\mathbb{T}$-graphs (see section 1.2 of the article [2]) is denoted

$(C,d,c;p,([,]_k)_{k\in\mathbb{N}})$, and morphisms between two pointed contractible $\mathbb{T}$-graphs preserve contractibilities and pointings. The category of pointed contractible $\mathbb{T}$-graphs is denoted CT-$\mathbb{G}\mathrm{r}_{p,c}$. The categories T-$\mathbb{G}\mathrm{r}_{p,c}$ and CT-$\mathbb{G}\mathrm{r}_{p,c}$ are both locally finitely presentable and the forgetful functor V

$$H \dashv V : CT\text{-}\mathbb{G}\mathrm{r}_{p,c} \longrightarrow T\text{-}\mathbb{G}\mathrm{r}_{p,c}$$

is monadic, with induced monad $\mathbb{T}_C$ is finitary.

Also the category T-$\mathbb{C}\mathrm{at}_c$ is locally finitely presentable and the forgetful functor U

$$M \dashv U : T\text{-}\mathbb{C}\mathrm{at}_c \longrightarrow T\text{-}\mathbb{G}\mathrm{r}_{p,c}$$

is monadic, with induced monad $\mathbb{T}_M$ is finitary.

A $\mathbb{T}$-category is contractible if its underlying pointed $\mathbb{T}$-graph lies in CT-$\mathbb{G}\mathrm{r}_{p,c}$. Morphisms between two contractible $\mathbb{T}$-categories are morphisms of $\mathbb{T}$-categories which preserve contractibilities. Let us write CT-$\mathbb{C}\mathrm{at}_c$ for the category of contractible $\mathbb{T}$-categories. Also consider the pullback in $\mathbb{C}AT$

$$\begin{array}{ccc} CT\text{-}\mathbb{G}\mathrm{r}_{p,c} \underset{T\text{-}\mathbb{G}\mathrm{r}_{p,c}}{\times} T\text{-}\mathbb{C}\mathrm{at}_c & \xrightarrow{p_1} & T\text{-}\mathbb{C}\mathrm{at}_c \\ \downarrow{\scriptstyle p_2} & & \downarrow{\scriptstyle U} \\ CT\text{-}\mathbb{G}\mathrm{r}_{p,c} & \xrightarrow[V]{} & T\text{-}\mathbb{G}\mathrm{r}_{p,c} \end{array}$$

We have an equivalence of categories

$$CT\text{-}\mathbb{G}\mathrm{r}_{p,c} \underset{T\text{-}\mathbb{G}\mathrm{r}_{p,c}}{\times} T\text{-}\mathbb{C}\mathrm{at}_c \simeq CT\text{-}\mathbb{C}\mathrm{at}_c$$

Furthermore we have the general fact (which can be found in the articles [3, 4])

Proposition 1 (Max Kelly) *Let K be a locally finitely presentable category, and $Mnd_f(K)$ the category of finitary monads on K and strict morphisms of* *monads. Then $Mnd_f(K)$ is itself locally finitely presentable. If T and S are object of $Mnd_f(K)$, then the coproduct $T \amalg S$ is algebraic, which means that $K^T \underset{K}{\times} K^S$ is equal to $K^{T\amalg S}$ and the diagonal of the pullback square*

$$\begin{array}{ccc} K^T \underset{K}{\times} K^S & \xrightarrow{p_1} & K^S \\ {\scriptstyle p_2}\downarrow & & \downarrow{\scriptstyle U} \\ K^T & \xrightarrow[V]{} & K \end{array}$$

is the forgetful functor $K^{T\amalg S} \longrightarrow K$. Furthermore the projections $K^T \underset{K}{\times} K^S \longrightarrow K^T$ and $K^T \underset{K}{\times} K^S \longrightarrow K^S$ are monadic. □

Remark 2 According to Steve Lack this result can be easily generalise for monads having ranks in the context of locally presentable category. □

We apply this proposition to the diagram above which shows that $CT\text{-}\mathbb{C}\mathrm{at}_c$ is a locally presentable category, and also that the forgetful functor

$$CT\text{-}\mathbb{C}\mathrm{at}_c \xrightarrow{O} T\text{-}\mathbb{G}\mathrm{r}_{p,c}$$

is monadic. Denote by F the left adjoint of O. If we apply the functor F to the coglobular complex of $T\text{-}\mathbb{G}\mathrm{r}_{p,c}$ build in the article [2]

$$C^0 \underset{\kappa_0^1}{\overset{\delta_0^1}{\rightrightarrows}} C^1 \underset{\kappa_1^2}{\overset{\delta_1^2}{\rightrightarrows}} C^2 \cdots\rightrightarrows C^{n-1} \underset{\kappa_{n-1}^n}{\overset{\delta_{n-1}^n}{\rightrightarrows}} C^n \cdots$$

we obtain the coglobular complex of the coloured ω-operads of the weak higher transformations with our corrected notion of contractibility

$$B_C^0 \underset{\kappa_0^1}{\overset{\delta_0^1}{\rightrightarrows}} B_C^1 \underset{\kappa_1^2}{\overset{\delta_1^2}{\rightrightarrows}} B_C^2 \cdots\rightrightarrows B_C^{n-1} \underset{\kappa_{n-1}^n}{\overset{\delta_{n-1}^n}{\rightrightarrows}} B_C^n \cdots$$

Remark 3 It is evident that the ω-operad B_C^0 of Michael Batanin is still initial in the category of contractible ω-operads equipped with a composition system, where our new approach of contractibility is considered. □

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Camell KACHOUR
Department of Mathematics, Macquarie University
North Ryde, NSW 2109 Australia.
Phone: 00 612 9850 8942
Email:camell.kachour@gmail.com