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DUALITY FOR GENERIC ALGEBRAS

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Résumé. Les théorèmes de dualité affirment souvent que l'application canonique δ d'un objet dans son dual double (ou peut-être dual double convenablement "restreint") est un isomorphisme. Les deux dualisations utilisées dans la formation du dual double sont relatives à un objet basique R. Un exemple est la dualité de Gelfand. Dans la présente note, nous prouvons que l'algèbre générique R d'une théorie algébrique peut servir comme objet basique pour un tel théorème de dualité: le dual double (convenablement restreint) d'un objet représentable $y(C)$, dans le topos de préfaisceax $\mathcal{E}$ dans lequel R vit, est isomorphe, via δ, à $y(C)$ lui-même. La preuve utilise un "couplage complet", – une notion qui nous abstrayons de la preuve.
Parmi les corrolaires immédiats de ceci, on obtient que l'anneau générique R est un modèle pour la géométrie différentielle synthétique.

Abstract. Duality theorems often assert that the canonical map δ of an object into its double dual (possibly a "restricted" double dual) is an isomorphism. Both dualizations used in the formation of the double dual are with respect to some basic object R. An example is Gelfand duality. In this note, we prove that the generic algebra R for an algebraic theory serves as a basic object for such a duality theorem: the (suitably restricted) double dual of any representable object $y(C)$, in the presheaf topos $\mathcal{E}$ in which R lives, is isomorphic, via δ, to $y(C)$ itself. The proof goes via a "complete pairing" – a notion which we abstract from the proof.
Among the immediate corollaries of this is that the generic commutative ring R is a model for synthetic differential geometry.

Keywords. duality, generic algebra, classifying topos.
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The main result presented here (Theorem 5) was presented at the 17th PSSL in 1980 in Sussex, and announced in [7] (1981). I apologize for the long delay in publishing a complete account. I would like to thank Marta Bunge for several constructive suggestions.

1. Preliminaries

We consider a Cartesian closed category $\mathcal{E}$. For Q and R objects in $\mathcal{E}$, we denote the exponential object R^Q by $Q \pitchfork R$. Any map $k : P \times Q \to R$ gives rise to maps $i : Q \to P \pitchfork R$ and $j : P \to Q \pitchfork R$, the exponential adjoints (or exponential transposes) of k; the one comes about from the other by using the symmetry $P \times Q \cong Q \times P$. The maps $i : Q \to P \pitchfork R$ and $j : P \to Q \pitchfork R$ are called *twisted exponential adjoints* of each other. They are related to each other by a map δ, in a commutative diagram

$$\begin{tikzcd} P \arrow[r, "\delta"] \arrow[d, "j"'] & (P \pitchfork R) \pitchfork R \arrow[dl, "i \pitchfork R"] \\ Q \pitchfork R & \end{tikzcd}$$

where δ itself is the twisted exponential adjoint of the identity map of $P \pitchfork R$. We refer to δ as a "Dirac" map - a natural "embedding" of an object P into its double dual w.r.to a fixed R. (It need not be a monic, but often is.) There is a similar diagram with $Q \to (Q \pitchfork R) \pitchfork R$. All this belongs to the elementary theory of Cartesian closed categories, or to "pure lambda calculus".

We now assume that $\mathcal{E}$ is a topos with a natural number object. For $\mathbb{T}$ a finitary algebraic theory, the category $\mathbb{T}\text{-Alg}(\mathcal{E})$ of $\mathbb{T}$-algebra objects in $\mathcal{E}$ is monadic over $\mathcal{E}$, by an $\mathcal{E}$-enriched monad (cf. [11], [2]), and therefore the theory of [1], [4], [5], [6] etc. is available. Thus, if R is a $\mathbb{T}$-algebra in $\mathcal{E}$, and $P \in \mathcal{E}$ is any object, $P \pitchfork R$ inherits a $\mathbb{T}$-structure from that of R (here R also denotes the underlying object in $\mathcal{E}$ of the $\mathbb{T}$-algebra R). And $\mathbb{T}\text{-Alg}(\mathcal{E})$ is enriched over $\mathcal{E}$: if Q and R are $\mathbb{T}$-algebras in $\mathcal{E}$, the $\mathcal{E}$-valued hom functor thus gives an object $Q \pitchfork_{\mathbb{T}} R$, which in turn is a subobject of $Q \pitchfork R$. Also, for a map $k : P \times Q \to R$, (with Q and R $\mathbb{T}$-algebras) it makes sense to ask whether k is a $\mathbb{T}$-homomorphism in the second variable: it can be expressed that the exponential transpose $j : P \to Q \pitchfork R$ factors across the inclusion $Q \pitchfork_T R \subseteq Q \pitchfork R$, or equivalently, that the exponential transpose $i : Q \to P \pitchfork R$ is a $\mathbb{T}$-homomorphism.

For any finitary algebraic theory $\mathbb{T}$, one has a certain topos $\mathcal{E}$ with a $\mathbb{T}$-algebra object R in it, which classifies $\mathbb{T}$-algebra objects in arbitrary toposes;

this $\mathbb{T}$-algebra $R \in \mathcal{E}$ is called the *generic* $\mathbb{T}$-algebra. The description of this "classifying topos"$\mathcal{E}$, and the $\mathbb{T}$-algebra R in it, is simple and well known: $\mathcal{E}$ is the presheaf topos $[FP\mathbb{T}, \mathrm{Set}]$, where $FP\mathbb{T}$ is the category of finitely presented $\mathbb{T}$-algebras, and R is the "forgetful functor" $FP\mathbb{T} \to \mathrm{Set}$; see e.g. [13] Ch. VIII, or [3] Ch. D.3. For any $C \in FP\mathbb{T}$, we have two particular objects in $\mathcal{E}$, namely $y(C)$, where y is the Yoneda embedding, and $\gamma^*(C)$, where γ^* is left adjoint to the global sections functor $\gamma_* : \mathcal{E} \to \mathrm{Set}$. There is a canonical pairing

$$\gamma^*(C) \times y(C) \to R,$$

which we shall describe. The two exponential transposes of this map give rise to some duality isomorphisms.

2. Exponential objects in presheaf toposes

For the case where $\mathcal{E}$ is a presheaf topos $[\mathbb{A}, \mathrm{Set}]$, we shall recall one of the processes of exponential transposition in elementary terms (the other one then comes by the symmetry). The Set-valued hom-functor of $\mathbb{A}$ we denote by square brackets like $[X, Y]$. First, we describe the exponential object $Q \pitchfork R$ itself, namely for $B \in \mathbb{A}$,

$$(Q \pitchfork R)(B) = \int_{g \in B/\mathbb{A}} \mathrm{Hom}(Q(X), R(X)),$$

where X denotes the codomain of g, and where Hom denotes the hom functor for the category of sets. (We shall also use the notation $\int_{g:B\to X}$, for $\int_{g\in B/\mathbb{A}}$.) Thus, for an object $S \in [\mathbb{A}, \mathrm{Set}]$ to qualify for the name $Q \pitchfork R$, the object S should for each $B \in \mathbb{A}$ be equipped with maps $\pi_g : S(B) \to \mathrm{Hom}(Q(X), R(X))$ for each object $g \in B/\mathbb{A}$ (X denoting the codomain of g), subject to certain naturality conditions and a certain universal property. Then in terms of the maps π_g, the exponential transpose of a map $k : P \times Q \to R$ is given as the map $j = \hat{k} : P \to Q \pitchfork R$, with $(\hat{k})_B$ the unique map such that for each $g \in B/\mathbb{A}$, we have

$$\pi_g \circ (\hat{k})_B = \widehat{(k_X)} \circ P(g). \tag{1}$$

Here k_X is a map $P(X) \times Q(X) \to R(X)$ in the category of sets, so its exponential transpose $\widehat{(k_X)} : P(X) \to \mathrm{Hom}(Q(X), R(X))$ makes immediately sense.

Two cases will be of particular interest, namely the case where Q is representable, and where Q is constant.

In the case where Q is representable, say $Q = y(C)$ for $C \in \mathbb{A}$, one has a well known explicit presentation of $Q \pitchfork R$, provided binary coproducts $\otimes$ exist in $\mathbb{A}$. Then $y(C) \pitchfork R$ may be taken to be $R \circ (- \otimes C) : \mathbb{A} \to \mathrm{Set}$; let us be explicit about the maps π_g which qualify $R \circ (- \otimes C)$ as $y(C) \pitchfork R$. So let $B \in \mathbb{A}$, and let $g : B \to X$. Then

$$\pi_g : R(B \otimes C) \to \mathrm{Hom}([C, X], R(X)) = \Pi_{f \in [C,X]} R(X)$$

is described by describing its f-coordinate, for $f \in [C, X]$:

$$p_f \circ \pi_g := R(\{g, f\}) \tag{2}$$

where $\{g, f\} : B \otimes C \to X$ denotes that map out of the coproduct whose components are g and f, respectively, and where p_f denotes the projection to the f- factor of the product (or, seeing the latter as $\mathrm{Hom}([C, X], R(X))$, as evaluation at the element $f \in [C, X]$).

In the case where Q is constant $Q = \gamma^*(C)$ for some set C, i.e. $Q(X) = C$ for all $X \in \mathbb{A}$, we have the following simple presentation of $(\gamma^*(C) \pitchfork R)(B)$; namely

$$(\gamma^*(C) \pitchfork R)(B) = \mathrm{Hom}(C, R(B)), \tag{3}$$

which qualifies for this name by virtue of $\pi_g = \mathrm{Hom}(C, R(g))$, for $g : B \to X$. This can also be seen from the $\int$ formula for $(\gamma^*(C) \pitchfork R)(B)$; for

$$\int_{g:B\to X} \mathrm{Hom}(C, R(X)) \cong \mathrm{Hom}(C, \int_{g:B\to X} R(X)) \cong \mathrm{Hom}(C, R(B)),$$

using that $\int_{g:B\to X} R(X) \cong R(B)$ via π_{1_B}, by Yoneda's Lemma.

3. $\mathbb{T}$-algebras in a topos

Let $\mathcal{E}$ be a topos. The category $\mathbb{T}\text{-}\mathrm{Alg}(\mathcal{E})$ of $\mathbb{T}$-algebras in $\mathcal{E}$ will be monadic over $\mathcal{E}$ ([11] and [2] Chapter V). This monad will in fact be $\mathcal{E}$-enriched; see (the proof of) Lemma 5.5 in [2]. We denote the $\mathcal{E}$ enriched hom functor of it by $\pitchfork_{\mathbb{T}}$. If X and Y are objects in $\mathcal{E}$ carrying $\mathbb{T}$-structures, we have $X \pitchfork_T$

$Y \subseteq X \pitchfork Y$. If $S \in \mathcal{E}$, and Y carries $\mathbb{T}$-structure, then $S \pitchfork Y$ inherits a $\mathbb{T}$-structure, in a canonical way. But for $\mathbb{T}$-algebras X and Y, the $\mathbb{T}$-structure, which $X \pitchfork Y$ inherits from Y, need not restrict to one on $X \pitchfork_{\mathbb{T}} Y$, unless $\mathbb{T}$ is a commutative theory (and e.g. the theory of commutative rings is not commutative).

If $R \in \mathbb{T}\text{-Alg}(\mathcal{E})$, we have the category $R/(\mathbb{T}\text{-Alg}(\mathcal{E}))$, which we call the category of R-algebras, denoted $R\text{-Alg}(\mathcal{E})$ or just $R\text{-Alg}$. It, too, is enriched in $\mathcal{E}$, with enriched hom functor denoted $\pitchfork_R$. (When $\mathbb{T}$ is the theory of commutative rings, and R is such a ring, the terminology "R-algebra" agrees with the standard use in commutative algebra; however, $\pitchfork_R$ may in this context also mean something different, namely the set (or object) of "R-linear maps", as relevant in the theory of Schwartz distributions, (typically with $R = \mathbb{R}$ or $= \mathbb{C}$); see also [8] and [10], which deal with such cases.)

When X and Y are R-algebras, we have $X \pitchfork_R Y \subseteq X \pitchfork_{\mathbb{T}} Y \subseteq X \pitchfork Y$.

We shall use $\otimes$ to denote finite coproducts in the category of $\mathbb{T}$-algebras, because one primary example is that of some category of commutative rings; and also, because the notation $+$ may incorrectly suggest that products $\times$ distribute over the coproduct $+$. There is an $\mathcal{E}$-enriched (monadic) adjointness between $\mathbb{T}$-algebras in $\mathcal{E}$, and R-algebras (in $\mathcal{E}$); the left adjoint is $R \otimes -$.

4. The pairing and its transposes

We now specialize to the case where $\mathbb{A}$ is some small category of $\mathbb{T}$-algebras, closed under finite coproducts (in the examples below, $\mathbb{A}$ is the category $FP\mathbb{T}$ of finitely presented $\mathbb{T}$-algebras, and the forgetful functor $R : FP\mathbb{T} \to \text{Set}$ lives in the topos $\mathcal{E} = [\mathbb{A}, \text{Set}]$ and is the generic $\mathbb{T}$-algebra). We consider $\mathcal{E} = [\mathbb{A}, \text{Set}]$ and we denote the Yoneda embedding $\mathbb{A}^{op} \to [\mathbb{A}, \text{Set}]$ by y.

Let γ_* be the global-sections functor of $\mathcal{E}$; it has a left adjoint $\gamma^* : \text{Set} \to [\mathbb{A}, \text{Set}] = \mathcal{E}$. It associates to a set S the functor $\mathbb{A} \to \text{Set}$ whose value is the functor with constant value S. Since γ^* preserves finite limits, it preserves algebraic structure, thus if $C \in \mathbb{A}$, C carries $\mathbb{T}$-structure, and therefore $\gamma^*(C)$ will carry structure of $\mathbb{T}$-algebra in $\mathcal{E}$. (We will also, as is costumary, use C as notation for underlying set of the $\mathbb{T}$- algebras; similarly $\gamma^*(C)$ denotes both a $\mathbb{T}$-algebra object in $\mathcal{E}$, and its underlying "unstructured" object.)

By R, we denote the "forgetful" functor $\mathbb{A} \to \text{Set}$. As an object in $\mathcal{E}$, it carries $\mathbb{T}$-algebra structure, and hence so does any object of the form $S \pitchfork R$.

Let $C \in \mathbb{A}$. We describe a map $k^C : \gamma^*(C) \times y(C) \to R$, and its two exponential transposes $i : y(C) \to \gamma^*(C) \pitchfork R$ and $j : \gamma^*(C) \to y(C) \pitchfork R$. These maps will be natural in C, (for k in the "extra"-sense of [12] IX 4). Since C will be fixed in the following, we shall omit the upper index C from notation. The map $k = k^C$ will be a $\mathbb{T}$-homomorphism in the first variable in the sense of [6]. Therefore, by loc.cit., i will factor through $\gamma^*(C) \pitchfork_{\mathbb{T}} R \subseteq \gamma^*(C) \pitchfork R$, and j will be a $\mathbb{T}$-homomorphism.

First, we describe k by describing $k_B : \gamma^*(C)(B) \times y(C)(B) \to R(B)$. Recall that $\gamma^*(C)(B) = C$ for all B, that $y(C)(B) = [C, B]$ (where square brackets denote the hom functor of $\mathbb{A}$), and recall that $R(B) = B$. Then the map $k_B : C \times [C, B] \to B$ is simply the evaluation map $(c, f) \mapsto f(c)$ for $c \in C$ and $f : C \to B$ in $\mathbb{A}$. It is a $\mathbb{T}$-homomorphism in the variable c because f is a $\mathbb{T}$-homomorphism. Thus, we have three maps

$$k : \gamma^*(C) \times y(C) \to R \text{ , a } \mathbb{T}\text{-homomorphism in the first variable} \tag{4}$$

$$i : y(C) \to \gamma^*(C) \pitchfork_{\mathbb{T}} R \subseteq \gamma^*(C) \pitchfork R \tag{5}$$

$$j : \gamma^*(C) \to y(C) \pitchfork R \text{ , a } \mathbb{T}\text{-homomorphism} \tag{6}$$

Using the explicit description of exponential transposition given above, and using (3), it is straightforward to see that $i_B : y(C)(B) \to (\gamma^*(C) \pitchfork R)(B)$ is given by the following recipe. We need to give a map $i_B : y(C)(B) \to \mathrm{Hom}(C, R(B))$; this is just the inclusion $[C, B] \subseteq \mathrm{Hom}(C, B)$, which is the B-component of the inclusion $\gamma^*(C) \pitchfork_{\mathbb{T}} R \subseteq \gamma^*(C) \pitchfork R$. This proves

Proposition 1. *The map $i : y(C) \to \gamma^*(C) \pitchfork_{\mathbb{T}} R$ is an isomorphism.*

The right hand side of this isomorphism deserves the name "Spec$_R C$", since it takes finite colimits to limits, and, for $C =$ the free $\mathbb{T}$-algebra in one generator, it gives R, cf. [9] I.12; the notion "Spec$_R C$" does not depend on the specifics of the topos $\mathcal{E}$.

Next, we study the $\mathbb{T}$-homomorphism $j : \gamma^*(C) \to y(C) \pitchfork R$. For $B \in \mathbb{A}$, we consider $j_B : \gamma^*(C)(B) \to (y(C) \pitchfork R)(B)$. Let us first note that the transpose of the set theoretic map $k_X : C \times [C, X] \to X$ is the "Dirac" map $\widehat{k_X} : C \to \mathrm{Hom}([C, X], X)$. Next, we utilize the "coproduct" description of $y(C) \pitchfork R = R(-\otimes C)$, being an exponential object $y(C) \pitchfork R$ by virtue of the maps π_g described in (2) above. In terms of this, we prove

Proposition 2. *The map $j : \gamma^*(C) \to y(C) \pitchfork R = R(- \otimes C)$ has for its B-component just the inclusion map $\mathrm{incl}_2 : C \to B \otimes C$ into the second component of the coproduct.*

Proof. Using the explicit characterization of exponential transposition given in (1), it suffices to see that incl_2 has the property that (for $g : B \to X$), $\pi_g \circ \mathrm{incl}_2 = \widehat{k_X}$ - note that the $P(g)$ occuring in (1) here is an identity map. We analyzed above that $\widehat{k_X}$ here is the relevant Dirac map δ, so the task is to prove that the upper triangle in the following diagram commutes:

$$\begin{tikzcd}
C \arrow[rr, "\mathrm{incl}_2"] \arrow[dr, "\delta"] \arrow[ddr, "f"'] & & B \otimes C \arrow[dl, "\pi_g"'] \arrow[ddl, "\{g, f\}"] \\
& ([C, X], X) \arrow[d, "p_f"] & \\
& X &
\end{tikzcd}$$

and this follows if for all $f : C \to X$, it commutes after postcomposition by p_f, displayed as the vertical arrow in the diagram. Here we write $([C, X], X)$ instead of $\mathrm{Hom}([C, X], X)$, for typographical reasons. The right hand triangle commutes, by construction of π_g, and the left hand triangle commutes, by lambda calculus. Finally, the outer triangle commutes, by definition of $\{g, f\}$. Therefore, the upper triangle commutes, and this shows that incl_2 is indeed the claimed exponential transpose of k. This proves the Proposition.

We already know from more abstract reasons that j is a $\mathbb{T}$-homomorphism; this also appears explicitly from the above Proposition, since the coproduct inclusion $C \to B \otimes C$ is a $\mathbb{T}$-homomorphism. Now $B \otimes C$ is not only a $\mathbb{T}$-algebra, but it is a B-algebra by virtue of the coproduct inclusion $\mathrm{incl}_1 : B \to B \otimes C$. Any $\mathbb{T}$-algebra X extends uniquely to a B-algebra $B \otimes X$, the "free B-algebra in X". The canonical extension of the $\mathbb{T}$-algebra morphism $i_2 : C \to B \otimes C$ to a B-algebra morphism $B \otimes C \to B \otimes C$ is clearly the identity map. The R-algebra structure $R \to y(C) \pitchfork R$ of

$y(C) \pitchfork R = (- \otimes C)$ has for its B-component just incl_1. Since coproducts $\otimes$ of $\mathbb{T}$-algebras in a presheaf topos are calculated coordinatewise, the free R-algebra $R \otimes \gamma^*(C)$ on $\gamma^*(C)$ has for its B component $B \otimes C$. This proves

Theorem 3. *The extension of the* $\mathbb{T}$*-homomorphism* $j : \gamma^*(C) \to y(C) \pitchfork R$ *to an* R*-algebra morphism* $\overline{j} : R \otimes \gamma^*(C) \to y(C) \pitchfork R$ *is an isomorphism.*

Example 1. If $\mathbb{T}$ is the theory of commutative rings, and C is the ring of dual numbers $\mathbb{Z}[\epsilon]$, then in the commutative ring classifier topos $[FP\mathbb{T}, \mathrm{Set}]$, the isomorphism $\overline{j}$ in this Theorem gives in particular the isomorphism of the simplest KL axiom, saying that the map $R \times R \to R^D$ is an isomorphism of R-algebras (with algebra structure on $R \times R$ being "the ring of dual numbers $R[\epsilon]$). (Here $D = y(\mathbb{Z}[\epsilon])$). – Similarly $R[X]$ (= the free R-algebra in one generator) is isomorphic, via $\overline{j}$ for $\mathbb{Z}[X]$, to $R \pitchfork R$.

Example 2. If $\mathbb{T}$ is the initial algebraic theory (so $\mathbb{T}$-algebras are just sets), the category $FP\mathbb{T}$ is the category S_0 of finite sets, and the generic algebra R is called the generic *object*. The classifying topos $[S_0, \mathrm{Set}]$ is called the *object classifier*, cf. [2] Ch. IV (they write U rather than R). Coproducts $\otimes$ of "algebras" are here better denoted $+$; and the isomorphism $\overline{j}$ in this case is a map $R + 1 \to R \pitchfork R$. The "added" point in the domain of this $\overline{j}$ is mapped by $\overline{j}$ to the identity map of R.

If X is a $\mathbb{T}$-algebra in $\mathcal{E}$, and Z is an R-algebra, we have an isomorphism in $\mathcal{E}$ between $X \pitchfork_{\mathbb{T}} Z$ and $(R \otimes X) \pitchfork_R Z$, expressing the enrichment of the adjointness between $\mathbb{T}$-algebras in $\mathcal{E}$ and R-algebras (in $\mathcal{E}$). Using the notion of R-algebra and this isomorphism, we may reformulate Proposition 1. There is no harm in denoting the isomorphism $y(C) \to \gamma^* C \pitchfork_{\mathbb{T}} R$ of Proposition 1 and the isomorphism $y(C) \to (R \otimes \gamma^* C) \pitchfork_T R$, by the same symbol i; so Proposition 1 is reformulated:

Theorem 4. *The map* $i : y(C) \to (R \otimes \gamma^*(C)) \pitchfork_R R$ *is an isomorphism.*

5. Duality

The theme of double dualization occurs in many guises in many areas of mathematics. In a Cartesian closed category, the simplest is the full double dualization functor $(- \pitchfork R) \pitchfork R$ into an object R; there is a natural transformation, whose instantiation at X is a map $\delta_X : X \to (X \pitchfork R) \pitchfork R$, described in Section 1 (the notation "δ" is for "Dirac"). There are restricted variants of δ, in case R carries some algebraic structure, say, of $\mathbb{T}$-algebra. Then one has $X \to (X \pitchfork R) \pitchfork_{\mathbb{T}} R$ (as studied above); and in case that also X carries $\mathbb{T}$-structure, we have a $\mathbb{T}$-homomorphism $X \to (X \pitchfork_{\mathbb{T}} R) \pitchfork R$, obtained by postcomposing $\delta_X : X \to (X \pitchfork R) \pitchfork R$ with $s \pitchfork R$, where s denotes the inclusion of $X \pitchfork_{\mathbb{T}} R$ into $X \pitchfork R$. This composite will also be denoted δ_X. Similarly, if X is an R-algebra, we have an R-algebra homomorphism $\delta_X : X \to (X \pitchfork_R R) \pitchfork R$.

In the context of classifying toposes and generic algebras, as studied above, the dualization functors (are contravariant and) go from "geometric objects" (objects in $\mathcal{E}$) to "algebraic objects" ($\mathbb{T}$-algebras), and vice versa; the object R is, as a geometric object, the *line*, but it is canonically endowed with a $\mathbb{T}$-algebra structure, so it lives in both worlds. Similarly, $C \in \mathbb{A}$ is a $\mathbb{T}$-algebra, but it represents a geometric object $y(C)$. This is the reason for the title of the announcement [7].

Duality Theorems often have as conclusion that one or the other of the Dirac maps mentioned above is an isomorphism. Such duality results occur in our context as Corollaries of the results above. Combining the isomorphisms $y(C) \cong \gamma^*C \pitchfork_{\mathbb{T}} R$ ($\cong (R \otimes \gamma^*C) \pitchfork_R R$) (Theorem 4) with $R \otimes \gamma^*C \cong y(C) \pitchfork R$ (Theorem 3), we conclude $y(C) \cong (y(C) \pitchfork R) \pitchfork_R R$. However, the following main Theorem says something more precise, namely that the canonical "Dirac" map is an isomorphism.

Theorem 5. *For any $C \in \mathbb{A}$, we have that*

$$\delta_{y(C)} : y(C) \to (y(C) \pitchfork R) \pitchfork_R R$$

is an isomorphism in $\mathcal{E}$.

This one may see as a "Gelfand duality" result; it will follow from a duality result concerning the R-algebra $R \otimes \gamma^*(C)$:

Theorem 6. *For any $C \in \mathbb{A}$, we have that*

$$\delta_{R\otimes\gamma^*(C)} : R \otimes \gamma^*(C) \to ((R \otimes \gamma^*(C)) \pitchfork_R R) \pitchfork R$$

is an isomorphism of R-algebras in $\mathcal{E}$.

We begin by proving Theorem 6. We replace the pairing $k : \gamma^*(C) \times y(C) \to R$ (which is a $\mathbb{T}$-homomorphism in the first variable) by its extension to a pairing

$$\overline{k} : (R \otimes \gamma^*(C)) \times y(C) \to R, \tag{7}$$

(which is an R-algebra morphism in the first variable), and its two exponential transposes $\overline{i}$ and $\overline{j}$; here, $\overline{i}$ factors as

$$y(C) \xrightarrow{\ i\ } (R \otimes \gamma^*C) \pitchfork_R R \xrightarrow{s} (R \otimes \gamma^*C) \pitchfork R$$

where s denotes the inclusion of the $\pitchfork_R$ into $\pitchfork$; and $\overline{j}$ is the extension of the $\mathbb{T}$-homomorphism j to an R-homomorphism. By "pure lambda calculus", as stated in Section 1, we have commutativity of the upper left triangle in

$$\begin{tikzcd}
R \otimes \gamma^*C \arrow[r, "\delta"] \arrow[d, "\overline{j}"', "\cong"] & ((R \otimes \gamma^*C) \pitchfork R) \pitchfork R \arrow[dl, "\overline{i} \pitchfork R"'] \arrow[d, "s \pitchfork R"] \\
yC \pitchfork R & ((R \otimes \gamma^*C) \pitchfork_R R) \pitchfork R. \arrow[l, "\cong"', "i \pitchfork R"]
\end{tikzcd}$$

The composite $(s \pitchfork R) \circ \delta$ in this diagram is the Dirac map δ considered in the statement of the Theorem. From the commutativity of the diagram, and the fact that $\overline{j}$ and $i \pitchfork R$ are isomorphisms (Theorems 3 and 4), we deduce that the δ of the Theorem is an isomorphism, as claimed.

To prove Theorem 5, we apply the dualization functor $- \pitchfork_R R$ to the isomorphism of Theorem 6, and conclude that we get an isomorphism $\delta_X \pitchfork_R R$ (in $\mathcal{E}$) from the right to the left in

$$\begin{tikzcd}
(R \otimes \gamma^*C) \pitchfork_R R \arrow[r, shift right, "\delta_Y"'] & (((R \otimes \gamma^*C) \pitchfork_R R) \pitchfork)R) \pitchfork_R R \arrow[l, shift right, "\delta_X \pitchfork_R R"']
\end{tikzcd}$$

where $X := R \otimes \gamma^*C$ and $Y := (R \otimes \gamma^*C) \pitchfork_R R$. However, the map δ_Y here is a splitting of $\delta_X \pitchfork_R R$, by the triangle identity for the adjointness

$$\begin{tikzcd}
\mathcal{E} \arrow[r, shift left, "- \pitchfork R"] & (R\text{-Alg})^{op} \arrow[l, shift left, "- \pitchfork_R R"]
\end{tikzcd}$$

(or by "pure lambda-calculus"). But a splitting of an isomorphism is an isomorphism, so we conclude that δ_Y is an isomorphism. Now by Theorem 4, the Y here is isomorphic to $y(C)$, whence also $\delta_{y(C)}$ is an isomorphism, proving Theorem 5.

Note that since $\mathbb{A}$ is assumed to have finite coproducts, it follows that the representable objects $y(C)$ are atoms, in the sense that $y(C) \pitchfork - : \mathcal{E} \to \mathcal{E}$ have right adjoints. One might conjecture that the (Gelfand-type) duality Theorem 5 applies not only to the representables $y(C)$, but to any atom in $\mathcal{E}$. In this case, the Theorem for $\mathcal{E}, R$ can be formulated without reference to the specific construction of $\mathcal{E}$.

6. Complete pairings

The Theorems 3 and 4 together provide an example of a *complete pairing* in a sense to be described now. I don't (yet) know many examples, but the notion itself seems to have an aesthetic value.

Let T_1 and T_2 be $\mathcal{E}$-enriched (= strong) monads on a Cartesian closed category $\mathcal{E}$, and let $R \in \mathcal{E}$ be an object equipped with algebra structures for both the monads; these two structures should commute with each other, in the sense described in [5], Section 4. Let P be a T_1-algebra and Q a T_2-algebra, and let $k : P \times Q \to R$ be a map which is a T_1-homomorphism in the first variable, and a T_2-homomorphism in the second varaible. There results, by general theory, a T_1- homomorphism $j : P \to Q \pitchfork_{T_2} R$, and a T_2-homomorphism $i : Q \to P \pitchfork_{T_1} R$. Then k deserves the name *complete pairing* if both i and j are isomorphisms. A complete pairing gives rise to two Dirac maps, both of which are isomorphisms.

If T_1 is the monad whose algebras are R-algebras, as considered above, and if T_2 is the identity monad, then the $\overline{k}$ considered in (7) satisfies the conditions, by the Theorems 3 and 4, and these theorems imply the Theorems 5 and 6. Another example is with T_1 the theory of boolean algebras, T_2 the inital theory, $\mathcal{E}$ the category of sets, and $R = 2$. Then for any finite set C, one has a complete pairing, namely the evaluation map $(C \pitchfork R) \times C \to R$. This example one may see as the origin of Stone duality.

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