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# ABSTRACT ANNIHILATION GRAPHS

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**Résumé** Nous proposons un nouveau contexte, beaucoup plus général, pour l'étude des graphes diviseurs de zéro/ annulateurs d'idéaux, où les sommets des graphes ne sont pas des éléments/idéaux dún anneau commutatif, mais éléments dún ensemble ordonné abstrait (qui imite le treillis des idéaux), muni d'une loi binaire (qui imite le produit d'idéaux). On considère aussi le niveau intermédiaire des congruences des structures algébriques qui admettent une "bonne" théorie des commutateurs.

Abstract. We propose a new, widely generalized context for the study of the zero-divisor/annihilating-ideal graphs, where the vertices of graphs are not elements/ideals of a commutative ring, but elements of an abstract ordered set (imitating the lattice of ideals), equipped with a binary operation (imitating products of ideals). The intermediate level of congruences of any algebraic structure admitting a 'good' theory of commutators is also considered.

## introduction

Given a commutative ring $R$, one can form a graph whose vertices are (some) elements of $R$ and edges are pairs $(x, y)$ with $xy = 0$ in $R$. Or, one can replace elements with ideals of $R$ and do the same, that is, define edges as pairs $(A, B)$ with $AB = \{0\}$. The study of these so-called zero-divisor graphs and annihilating-ideal graphs were initiated by I. Beck [4] and M. Behboodi and Z. Rakeei [5] respectively, and then continued by various authors (see Section 2 for precise definitions and some references).

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*Key words and phrases.* annihilation graph, annihilating-ideal graph, zero-divisor graph, diameter of a graph, girth of a graph, poset, lattice, commutator, congruence.

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Since the product of two ideals of a commutative ring is nothing but their commutator in the sense of universal algebra (see Section 3 for details), it is natural to:

• replace ideals of a commutative ring with congruences of any algebraic structure that admits a good notion of commutator; "good" might have different meanings (see e.g. R. Freese and R. McKenzie [7] for different notions of a commutator), and the properties we actually need to hold are listed in Section 3;

• replace the property $AB = \{0\}$ above with the property $[\alpha, \beta] = 0$, where $\alpha$ and $\beta$ are congruences on a given algebraic structure, $[\alpha, \beta]$ their commutator, and 0 now denotes the equality relation (since it is the smallest congruence) on that given algebraic structure.

Although this replacement is itself a wide generalization, it immediately suggests a further wide generalization, where congruences on a given algebra are replaced with elements of an abstract lattice, or even just an ordered set, equipped with a binary operation. The condition that operation should be required to satisfy should then imitate the properties of commutators (as in our Section 3).

This two-step generalization in the study of *annihilation graphs* (we say "annihilation" instead of "annihilating-") is the author's PhD Thesis' theme, under supervision of Professor G. Janelidze, who suggested it. The results, to be considered as the first results of this project, are given in Section 5 of the present paper. Our proofs closely follow the ring case, suggesting that the purpose of the paper is first of all to provide a strong motivation for the whole project. The most surprising fact here is that the binary operation involved is not required to be associative, unlike the ring multiplication; this is important since the commutator operation is almost never associative, except the commutative ring case.

The paper is organized as follows: in Section 1 we recall standard definitions of graph theory that will be used below; Sections 2 and 3 are devoted to annihilating-ideal graphs and commutators respectively: we recall what we need and give some references; in Section 4 we introduce our new context of commutator posets, commutator lattices, and their annihilation graphs; as already mentioned, the main results are formulated and proved in Section 5 - in fact it is two results put together in Theorem 5.1.

Note also that, as suggested by the context considered in [1], we consider a 'relative version' of annihilation, where $ab = 0$ is replaced with $ab \leq c$ with a fixed $c$.

## 1. **Graphs**

By a graph we will mean a pair $G = (G_0, G_1)$, in which $G_0$ is a set and $G_1$ a binary irreflexive symmetric relation on $G_0$; the elements of $G_0$ and of $G_1$ will be called vertices and edges of $G$, respectively.

For a natural number $n$, a *path of length* $n$ in a graph $G$ is an $(n+1)$-tuple $(x_0, ..., x_n)$ of distinct vertices of G, such that $(x_{i-1}, x_i)$ is an edge of G, for each $i \in \{1, ..., n\}$.

A path $(x_0, ..., x_n)$ is also called a *path from* $x_0$ *to* $x_n$.

The *distance* $d(x, y)$ between vertices $x$ and $y$ of a graph $G$ is defined as follows:

- $d(x, y) = 0$, if $x = y$;
- $d(x, y) = n$, if $n$ is the smallest non-zero natural number for which there exists a path of length $n$ from $x$ to $y$.
- $d(x, y) = \infty$, if $x \neq y$ and there is no path from $x$ to $y$.

Accordingly, the distance between $x$ and $y$ is said to be *finite* if either $x = y$ or there exists a path from $x$ to $y$, and *infinite* otherwise.

A graph $G$ is said to be *connected*, if, for every two distinct vertices $x$ and $y$ of $G$, there exists a path from $x$ to $y$, or, equivalently, the distance $d(x, y)$ is finite. Note that, according to this definition, the empty graph is connected in contradiction with the categorical (and topological) notion of connectedness.

The *diameter* $\mathrm{diam}(G)$ of a graph $G$ is defined as the largest distance between its vertices.

For a natural $n \geq 3$, a *cycle of length* $n$ in a graph $G$ is an $(n+1)$-tuple $(x_0, ..., x_n)$ of vertices of $G$ that are distinct except $x_0 = x_n$, and such that $(x_{i-1}, x_i)$ is an edge of $G$, for each $i \in \{1, ..., n\}$.

The *girth* $\mathrm{gr}(G)$ of a graph $G$ is defined as follows:

- if $G$ has a cycle, then $\mathrm{gr}(G)$ is the smallest number $n$ such that $G$ has a cycle of length $n$.
- if $G$ has no cycle, then $\mathrm{gr}(G) = \infty$.

It is well known and easy to show that if $G$ has a cycle, then

$$\mathrm{gr}(G) \leq 2\ \mathrm{diam}(G) + 1; \tag{1.1}$$

however, in the contexts we shall consider, better inequalities are obtained (see Theorems 2.1 and 5.1; Theorem 2.1 is a generalization, from [1], of a result in [5]).

## 2. **The annihilating-ideal graph of a commutative ring with respect to an ideal**

The annihilating-ideal graph $AG(R)$ of a commutative ring $R$ (with 1), introduced by M. Behboodi and Z. Rakeei [5], is defined as follows:

• the vertices of $AG(R)$ are all non-zero ideals of $R$ with non-zero annihilators;
• a pair $(A, B)$ of distinct vertices of $AG(R)$ is an edge of $AG(R)$ if and only if $AB = \{0\}$.

This definition is a natural *ideal versus elements* version of the construction earlier introduced by D. F. Anderson and P. S. Livingston [2], which itself is a modified version of the construction first studied by I. Beck [4]. On the other hand, one can fix an ideal $I$ in $R$ and consider the graph $AG_I(R)$, called *the annihilating-ideal graph of $R$ with respect to the ideal $I$* in [1], in which:

• the vertices of $AG_I(R)$ are all ideals $A$ of $R$ not contained in $I$ and having an ideal $A'$ not containing in $I$ with $AA'$ containing in $I$;
• a pair $(A, B)$ of distinct vertices of $AG_I(R)$ is an edge of $AG_I(R)$ if and only if $AB \subseteq I$.

According to this definition, $AG(R) = AG_{\{0\}}(R)$.

Let us recall:

**Theorem 2.1.** (Theorem 3.3 in [1])

(a) *The graph $AG_I(R)$ is connected and diam$(AG_I(R)) \leq 3$.*
(b) *If $AG_I(R)$ contains a cycle, then gr$(AG_I(R)) \leq 4$.*

Similar inequalities were known before for the graph considered in [2]: see Theorem 2.3 in [2], Theorem 2.3 in [6], assertion (1.4) in [14], and Theorem

2.2 in [3] for various versions. They were also known for $I = \{0\}$, that is, for the graph $AG(R)$ (see Theorem 2.1 in [5]).

## 3. Commutators

The familiar group-theoretic notion of a commutator has been generalized to various contexts of universal algebra and category theory. The universal-algebraic references to commutators usually begin with J. D. H. Smith [16], and then mention various further generalizations of Smith's definition (see e.g. [7] and references therein, although there are many more recent ones). The categorical notions of commutators of subobjects and of internal equivalence relations first appear in S. A. Huq's papers (see [8]), and in M. C. Pedicchio's papers (see [15]), respectively.

As formulated in [9] (based on the approach of [12]), the commutator $[\alpha, \beta]$ of two congruences $\alpha$ and $\beta$ on an algebra $A$ in a Mal'tsev (=congruence permutable) variety $\mathbf{C}$ with a Mal'tsev term $p$ can be defined as the smallest congruence $\gamma$ on $A$ such that the map

$$\{(x, y, z) \in A^3 | (x, y) \in \alpha \,\&\, (y, z) \in \beta\} \to A/\gamma \tag{3.1}$$

sending $(x, y, z)$ to the $\gamma$-class of $p(x, y, z)$, is a homomorphism of algebras (that is, a morphism in $\mathbf{C}$ ).

As also mentioned in [9], this commutator has the following properties:

$$[\alpha, \beta] \leq \alpha \wedge \beta \tag{3.2}$$
$$[\alpha, \beta] = [\beta, \alpha], \tag{3.3}$$
$$[\alpha, \beta \vee \gamma] = [\alpha, \beta] \vee [\alpha, \gamma] \tag{3.4}$$

where $\wedge$ and $\vee$ are the meet and the join in the lattice of congruences on a given algebra $A$. It is well known that these properties also hold in various more general contexts, in particular for commutators in a congruence modular varieties (see e.g. [7]) and in an exact Mal'tsev category with coequalizers (see [15]).

When the ground variety **C** is semi-abelian in the sense of $G$. Janelidze, L. Marki, and W. Tholen [10] (and in some more general contexts, earlier studied by A. Ursini; see e.g. in [11] and references on Ursini's papers there), for each algebra $A$ in $C$, there is a lattice isomorphism

$$\mathrm{Con}(A) \approx \mathrm{NSub}(A) \tag{3.5}$$

between the lattice $\mathrm{Con}(A)$ of congruences on $A$ and the lattice $\mathrm{NSub}(A)$ of normal subalgebras of $A$, under which congruences correspond to their '0-classes'. This immediately allows us to define commutators in semi-abelian varieties using (3.1) as above, even though the so defined commutator will not necessarily coincide with the Huq commutator (see [13] for the clarification of their relationship). Accordingly, for normal subalgebras $H$ and $K$ of $A$ and the corresponding congruences $\alpha$ and $\beta$ on $A$, we shall write $[H,K]_{Smith}$ for the normal subalgebra of $A$ corresponding to $[\alpha,\beta]$.
Note also that, the so-defined $[H,K]_{Smith}$ is at the same time a special case of the commutator introduced by A. Ursini in [17], as shown in that paper.

Let us recall the simplest examples:

*Example* 3.1.

(a) if **C** is the variety of groups, then normal subalgebras of $A$ in **C** are the same as normal subgroups of $A$, and, for two normal subgroups $H$ and $K$ of $A$, $[H,K]_{Smith}$ is the ordinary commutator of $H$ and $K$. That is,

$[H,K]_{Smith}$ = the subgroup of $A$ generated by all $hkh^{-1}k^{-1}$ with $h \in H$ and $k \in K$.

(b) if **C** is the variety of commutative rings (here and below rings are not required to have an identity element), then normal subalgebras of $A$ in $\mathbf{C}$ are the same as ideals of $A$, and, for two ideals $H$ and $K$ of $A$,

$$[H, K]_{Smith} = HK,$$

the product of $H$ and $K$.

(c) if $\mathbf{C}$ is the variety of rings (not necessarily commutative), then normal subalgebras of $A$ in $\mathbf{C}$ are the same as ideals of $A$, and, for two ideals $H$ and $K$ of $A$,

$$[H, K]_{Smith} = HK + KH,$$

the product of $H$ and $K$.

## 4. **Commutator lattices and their graphs**

As suggested by commutator theory, we introduce

**Definition 4.1.** (a) A commutator lattice is a (bounded) lattice $L$ equipped with a binary operation $[-,-]$, also written as $[x, y] = xy$, and satisfying the conditions similar to (3.2)-(3.4), that is, satisfying

$$xy \leq x, \tag{4.1}$$
$$xy = yx \tag{4.2}$$
$$x(y \vee z) = (xy) \vee (xz) \tag{4.3}$$

for all $x, y, z$ in $L$.

(b) More generally, a commutator poset is a poset (=ordered set) $L$ equipped with a binary operation $[-,-]$, also written as $[x, y] = xy$, and satisfying (4.1), (4.2), and

$$x \leq y \Rightarrow xz \leq yz, \tag{4.4}$$

for all $x, y, z$ in $L$.

Our obvious examples of interest of a commutator lattice are:

*Example* 4.2.

(a) For an algebra $A$ in a Mal'tsev variety $\mathbf{C}$, the lattice $\mathrm{Con}(A)$ of congruences on $A$, equipped with the commutator operation defined as in the previous section, is a commutator lattice. The same is obviously true in all those contexts where commutators satisfy properties (3.2)-(3.4), including the context of congruence modular varieties considered in [7].

(b) For an algebra $A$ in a semi-abelian variety $\mathbf{C}$, the lattice $\mathrm{NSub}(A)$ of normal subalgebras of $A$, equipped with the commutator operation $[-,-]_{Smith}$ defined as in the previous section, is a commutator lattice. In particular, this is the case for the varieties considered in Example 3.1 with the commutators described there.

Let us mention two other obvious examples:

*Example* 4.3. An arbitrary lattice $L$ becomes a commutator lattice if we put either

(a) $xy = x \wedge y$ for all $x, y \in L$, provided $L$ is distributive, or

(b) $xy = 0$ for all $x, y \in L$.

As suggested by commutator theory, we might call these two kinds of commutator lattices *arithmetical* and *abelian*, respectively.

The definition of annihilating-ideal graph $AG_I(R)$ of a ring $R$ with respect to an ideal $I$ immediately extends to the context of a commutator lattice as follows:

**Definition 4.4.** For an element $c$ in a commutator poset $L$ we define the annihilation graph of $L$ with respect to $c$ as the graph $AG_c(L)$, in which:

the vertices of $AG_c(L)$ are all elements $x$ of $L$ not less-or-equal than $c$ and having an element $y$ in $L$ not less-or-equal than $c$ with $xy \leq c$. A pair $(x, y)$ of distinct vertices of $AG_c(L)$ is an edge of $AG_c(L)$ if and only if $xy \leq c$.

We shall also write $AG(L) = AG_0(L)$, and call this graph the annihilation graph of $L$.

In particular we have

$$AG_I(R) = AG_I(L) \text{ and } AG(R) = AG(L), \tag{4.5}$$

where $AG_I(R)$ and $AG(R)$ are as in Section 2, while $L$ is the commutator lattice of ideals of $R$ with the commutator operation as in Example 3.1(b).

## 5. The ring-theoretic results extend to the context of commutator posets

The purpose of this section is to extend Theorem 2.1 to the context of commutator posets, that is, to prove the following:

**Theorem 5.1.** *Let $L$ be a commutator poset and $c$ an element in $L$. Then:*

(a) *The graph $AG_c(L)$ is connected and diam$(AG_c(L)) \leq 3$.*

(b) *If $AG_c(L)$ contains a cycle, then gr$(AG_c(L)) \leq 4$.*

*Proof.* (a): We have to show that, for every two distinct vertices a and b of the graph $AG_c(L)$ with $ab \nleq c$, either there exists a vertex $x$ of $AG_c(L)$ with ($a \neq x \neq b$ and)

$$ax, xb \leq c, \tag{5.1}$$

or there exist distinct vertices $x$ and $y$ of $AG_c(L)$ with ($a \neq x \neq b$, $a \neq y \neq b$, and)

$$ax, xy, yb \leq c. \quad (5.2)$$

Our arguments will depend on the satisfaction of the inequalities $a^2 \leq c$ and $b^2 \leq c$, and therefore we have to consider four cases:

Case 1: $a^2 \leq c$ and $b^2 \leq c$. In this case $x = ab$ satisfies (5.1). Indeed, $ax = a(ab) \leq a^2 \leq c$, and similarly $xb \leq c$;

Case 2: $a^2 \leq c$ and $b^2 \nleq c$. Here we first choose $t$ to be any vertex of $AG_c(L)$ with $tb \leq c$, which is possible by definition of $AG_c(L)$, and continue depending on the satisfaction of the inequality $at \leq c$. If $at \leq c$, then $x = t$ satisfies (5.1). Indeed, $ax = a(at) \leq a^2 \leq c$ while $xb = (at)b \leq tb \leq c$.

Case 3: $a^2 \nleq c$ and $b^2 \leq c$ - is trivially similar.

Case 4: $a^2 \nleq c$ and $b^2 \nleq c$. Now we first choose vertices $u$ and $v$ of $AG_c(L)$ with ($a \neq u \neq b$, $a \neq v \neq b$, and)

$$au, vb \leq c. \quad (5.3)$$

which is possible by definition of $AG_c(L)$. Next, if $u = v$, then $x = u = v$ satisfies (5.1). Therefore we can assume $u \neq v$. But $u \neq v$ together with $uv \leq c$ would imply that $x = u$ and $y = v$ satisfy (5.2). Therefore we can also assume $uv \nleq c$. However, under these assumptions $x = uv$ satisfies (5.1). Indeed, $ax = a(uv) \leq au \leq c$, and similarly $xb \leq c$.

(b): Suppose $AG_c(L)$ contains a cycle $(x_0, ..., x_n)$ of length $n$ (hence, in particular, $x_0 = x_n$). Since we only need to show that there exists a cycle of length $\leq 4$ in $AG_c(L)$, we can assume, without loss of generality, that $n \geq 5$, and that the cycle $(x_0, ..., x_n)$ has the minimal length. Using this minimality, we observe:

• $x_0x_3 \nleq c$. Indeed, if $x_0x_3 \leq c$, then the sequence obtained from $(x_0, ..., x_n)$ by removing $x_1$ and $x_2$ would a cycle in $AG_c(L)$;
• $x_0x_3 \neq x_1$. Indeed, $x_0x_3 = x_1$ would imply $x_1 \leq x_0$ and then $x_1x_{n-1} \leq x_0x_{n-1} = x_nx_{n-1} \leq c$, making the sequence obtained from $(x_0, ..., x_n)$ by removing $x_0$ a cycle in $AG_c(L)$;
• $x_0x_3 \neq x_2$, which can be proved similarly to $x_0x_3 \neq x_1$;

Now, under our assumptions, we can prove that $AG_c(L)$ has a cycle of length 3: specifically, so is $(x_1, x_0x_3, x_2, x_1)$. Indeed:

since $x_0x_3 \leq c$ and $x_0x_3 \leq x_0$, $x_0x_3$ is a vertex of $AG_c(L)$; we already know that $x_1$, $x_0x_3$, and $x_2$ are all distinct from each other;

• $x_1(x_0x_3) \leq x_1x_0 \leq c$;
• $(x_0x_3)x_2 \leq x_3x_2 \leq c$;
• $x_2x_1 \leq c$. □

*Remark* 5.2. There are simple counter-examples showing that $\mathrm{gr}(AG_c(L)) \leq 3$ is not always true, even if there are cycles of length $\geq 5$. For instance, if $R$ is an integral domain that is not a field, then, obviously, $\mathrm{gr}(AG(R \times R)) = 4$ (where $AG(...)$ is as in [5]; see Section 2). In this well-known case all cycles in $AG(R \times R)$ have even number of elements, and, for every $n$-tuple $(A_1, \cdots, A_n)$ of distinct ideals in $R$, the sequence

$$(A_0 \times \{0\}, \{0\} \times A_0, A_1 \times \{0\}, \{0\} \times A_1, \cdots, A_{n-1} \times \{0\}, \{0\} \times A_{n-1}, A_n \times \{0\}, \{0\} \times A_n),$$

where $n \geq 2$ and $A_0 = A_n$, is a cycle of length $2n$ in $AG(R \times R)$.

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