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LOCALIFICATION PROCEDURE FOR AFFINE SYSTEMS

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Résumé. Motivé par le concept d'ensemble affine d'Y. Diers, cet article étudie la notion de système affine, qui généralise les systèmes topologiques de S. Vickers. La catégorie des ensembles affines est isomorphe à une sous-catégorie pleine coréflexive de la catégorie des systèmes affines. Nous donnons une condition nécessaire et suffisante pour que la catégorie duale de la variété d'algèbres, sous-jacent les ensembles affines, soit isomorphe à une sous-catégorie réflexive de la catégorie des systèmes affines. Par conséquent, nous arrivons à une reformulation de l'équivalence sobriété-spatialité pour les ensembles affines, selon le modèle de l'équivalence entre les catégories des espaces topologiques sobres et les "locales" spatiaux.

Abstract. Motivated by the concept of affine set of Y. Diers, this paper studies the notion of affine system, extending topological systems of S. Vickers. The category of affine sets is isomorphic to a full coreflective subcategory of the category of affine systems. We show the necessary and sufficient condition for the dual category of the variety of algebras, underlying affine sets, to be isomorphic to a full reflective subcategory of the category of affine systems. As a consequence, we arrive at a restatement of the sobriety-spatiality equivalence for affine sets, patterned after the equivalence between the categories of sober topological spaces and spatial locales.

Keywords. Adjoint situation, affine set, (co)reflective subcategory, sober topological space, spatial locale, state property system, T_0 topological space, topological system, variety.

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1. Introduction

In [20], S. Vickers introduced the notion of topological system as a common framework for both topological spaces and the underlying algebraic structures of their topologies – locales. In particular, the category of locales (resp. topological spaces) is isomorphic to a full (resp. co)reflective subcategory of the category of topological systems, which provides the so-called system localification (resp. spatialization) procedure.

In [9], Y. Diers has come out with the concept of algebraic or affine set, which included topological spaces as a particular example. Based in the already available results of [19], this paper presents the notion of affine system, which extends topological systems of S. Vickers, and also state property systems of D. Aerts [2], and shows that the category of affine sets is isomorphic to a full coreflective subcategory of the category of affine systems, thereby providing an affine analogue of the spatialization procedure for topological systems. The important difference of the setting of this manuscript from the setting of Y. Diers [9, 10, 11] is the buildup of both affine sets and systems over an arbitrary category instead of the category of sets.

The main contribution of this paper is the necessary and sufficient condition for the dual category of the variety of algebras, whose objects underly the structure of affine sets, to be isomorphic to a full reflective subcategory of the category of affine systems, thereby providing an affine analogue of the localification procedure for topological systems. As a consequence, one arrives at a restatement of the sobriety-spatiality equivalence for affine sets, which is patterned after the equivalence between the categories of sober topological spaces and spatial locales [14]. Moreover, the existence of the localification procedure for affine systems induces their category to be essentially algebraic. We also show a sufficient condition for the category of separated affine sets to make a reflective subcategory of the category of affine sets, extending the result that the category of T_0 topological spaces makes a reflective subcategory of the category of topological spaces.

All the category-theoretic notions of this paper (e.g., the concept of topological category) come from [1, 8].

2. Spatialization procedure for affine systems

This section introduces the notion of affine system and its respective spatialization procedure, motivated by the above-mentioned spatialization procedure for topological systems of S. Vickers.

2.1 Algebraic preliminaries

In this subsection, we briefly recall the algebraic notions, which will be used throughout the paper.

Definition 2.1. *Let $\Omega = (n_\lambda)_{\lambda \in \Lambda}$ be a family of cardinal numbers, which is indexed by a (possibly, proper or empty) class Λ. An* Ω-algebra *is a pair $(A, (\omega_\lambda^A)_{\lambda \in \Lambda})$, which comprises a set A and a family of maps $A^{n_\lambda} \xrightarrow{\omega_\lambda^A} A$ (n_λ*-ary primitive operations *on A). An* Ω-homomorphism $(A, (\omega_\lambda^A)_{\lambda \in \Lambda}) \xrightarrow{\varphi} (B, (\omega_\lambda^B)_{\lambda \in \Lambda})$ *is a map $A \xrightarrow{\varphi} B$, which makes the diagram*

$$\begin{array}{ccc} A^{n_\lambda} & \xrightarrow{\varphi^{n_\lambda}} & B^{n_\lambda} \\ {\scriptstyle \omega_\lambda^A}\downarrow & & \downarrow{\scriptstyle \omega_\lambda^B} \\ A & \xrightarrow[\varphi]{} & B \end{array}$$

commute for every $\lambda \in \Lambda$. **Alg**(Ω) *is the construct of Ω-algebras and Ω-homomorphisms.*

Definition 2.2. *Let $\mathcal{M}$ (resp. $\mathcal{E}$) be the class of Ω-homomorphisms with injective (resp. surjective) underlying maps. A* variety of Ω-algebras *is a full subcategory of* **Alg**(Ω)*, which is closed under the formation of products, $\mathcal{M}$-subobjects (subalgebras), and $\mathcal{E}$-quotients (homomorphic images). The objects (resp. morphisms) of a variety are called* algebras *(resp.* homomorphisms*).*

We provide some examples of varieties, which are relevant to this paper.

Example 2.3.

1. **CSLat**$(\bigvee)$ is the variety of $\bigvee$-*semilattices*, i.e., partially ordered sets, which have arbitrary suprema, and **CSLat**$(\bigwedge)$ is the variety of $\bigwedge$-*semilattices*, i.e., partially ordered sets, which have arbitrary infima.

2. **Frm** is the variety of *frames*, i.e., $\bigvee$-semilattices A, with singled out finite meets, and which additionally satisfy the distributivity condition $a \wedge (\bigvee S) = \bigvee_{s \in S}(a \wedge s)$ for every $a \in A$ and every $S \subseteq A$ [14].

3. **CBAlg** is the variety of *complete Boolean algebras*, i.e., complete lattices A such that $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$ for every a, b, $c \in A$, equipped with a unary operation $A \xrightarrow{(-)^*} A$ such that $a \vee a^* = \top_A$ and $a \wedge a^* = \bot_A$ for every $a \in A$, where $\top_A$ (resp. $\bot_A$) is the largest (resp. smallest) element of A.

4. **CSL** is the variety of *closure semilattices*, i.e., $\bigwedge$-semilattices, with the singled out bottom element.

2.2 Affine spaces

In this subsection, we provide an extension of the notion of affine set of Y. Diers [9, 10, 11].

Definition 2.4. *Given a functor* $\mathbf{X} \xrightarrow{T} \mathbf{B}^{op}$*, where* $\mathbf{B}$ *is a variety of algebras,* $\mathbf{Af\,Spc}(T)$ *is the concrete category over* $\mathbf{X}$*, whose objects (*T-affine spaces *or* T-spaces*) are pairs* (X, τ)*, where* X *is an* $\mathbf{X}$*-object and* τ *is a* $\mathbf{B}$*-subalgebra of* TX*; and whose morphisms (*T-affine morphisms *or* T-morphisms*)* $(X_1, \tau_1) \xrightarrow{f} (X_2, \tau_2)$ *are* $\mathbf{X}$*-morphisms* $X_1 \xrightarrow{f} X_2$ *with the property that* $(Tf)^{op}(\alpha) \in \tau_1$ *for every* $\alpha \in \tau_2$.

The following easy result will give rise to our main examples of T-spaces and T-morphisms.

Proposition 2.5. *Given a variety* $\mathbf{B}$*, every subcategory* $\mathbf{S}$ *of* $\mathbf{B}^{op}$ *induces a functor* $\mathbf{Set} \times \mathbf{S} \xrightarrow{\mathcal{P}_{\mathbf{S}}} \mathbf{B}^{op}$*,* $\mathcal{P}_{\mathbf{S}}((X_1, B_1) \xrightarrow{(f,\varphi)} (X_2, B_2)) = B_1^{X_1} \xrightarrow{\mathcal{P}_{\mathbf{S}}(f,\varphi)} B_2^{X_2}$*, where* $(\mathcal{P}_{\mathbf{S}}(f, \varphi))^{op}(\alpha) = \varphi^{op} \circ \alpha \circ f$.

The case $\mathbf{S} = \{B \xrightarrow{1_B} B\}$ provides a functor $\mathbf{Set} \xrightarrow{\mathcal{P}_B} \mathbf{B}^{op}$, $\mathcal{P}_B(X_1 \xrightarrow{f} X_2) = B^{X_1} \xrightarrow{\mathcal{P}_B f} B^{X_2}$, where $(\mathcal{P}_B f)^{op}(\alpha) = \alpha \circ f$. In particular, if $\mathbf{B} = \mathbf{CBAlg}$, and $\mathbf{S} = \{2 \xrightarrow{1_2} 2\}$, then one obtains the well-known contravariant powerset functor $\mathbf{Set} \xrightarrow{\mathcal{P}} \mathbf{CBAlg}^{op}$, which is given on a map $X_1 \xrightarrow{f} X_2$ by $\mathcal{P}X_2 \xrightarrow{(\mathcal{P}f)^{op}} \mathcal{P}X_1$ with $(\mathcal{P}f)^{op}(S) = \{x \in X_1 \,|\, f(x) \in S\}$.

The following examples of the categories of the form $\mathbf{Af\,Spc}(T)$ will be relevant to this paper.

Example 2.6.

1. If $\mathbf{B} = \mathbf{Frm}$, then $\mathbf{Af\,Spc}(\mathcal{P}_2)$ is the category **Top** of topological spaces.
2. If $\mathbf{B} = \mathbf{CSL}$, then $\mathbf{Af\,Spc}(\mathcal{P}_2)$ is the category **Cls** of closure spaces [4].
3. $\mathbf{Af\,Spc}(\mathcal{P}_B)$ is the category $\mathbf{Af\,Set}(B)$ of affine sets of Y. Diers.
4. If $\mathbf{B} = \mathbf{Frm}$, then $\mathbf{Af\,Spc}(\mathcal{P}_\mathbf{S})$ is the category **S-Top** of variable-basis lattice-valued topological spaces of S. E. Rodabaugh [17].

Later on in this manuscript, we will use the following convenient property of the categories $\mathbf{Af\,Spc}(T)$.

Theorem 2.7. *Given a functor* $\mathbf{X} \xrightarrow{T} \mathbf{B}^{op}$*, the concrete category* $(\mathbf{Af\,Spc}(T), |-|)$ *is topological over* $\mathbf{X}$*.*

Proof. Given a $|-|$-structured source $\mathcal{S} = (X \xrightarrow{f_i} |(X_i, \tau_i)|)_{i \in I}$, the initial structure on X w.r.t. $\mathcal{S}$ is given by the subalgebra of TX, which is generated by the union $\bigcup_{i \in I} (Tf_i)^{op}(\tau_i)$. Given a $|-|$-structured sink $\mathcal{S} = (|(X_i, \tau_i)| \xrightarrow{f_i} X)_{i \in I}$, the final structure on X w.r.t. $\mathcal{S}$ is the intersection $\bigcap_{i \in I} ((Tf_i)^{op})^{-1}(\tau_i)$. □

As a consequence, one obtains the well-known result that all the categories of Example 2.6 are topological.

2.3 Affine systems

Following the ideas of [19], this subsection introduces the concept of affine system as an analogue of topological systems of S. Vickers [20].

Definition 2.8. *Given a functor* $\mathbf{X} \xrightarrow{T} \mathbf{B}^{op}$*,* $\mathbf{Af\,Sys}(T)$ *is the comma category* $(T \downarrow 1_{\mathbf{B}^{op}})$*, concrete over the product category* $\mathbf{X} \times \mathbf{B}^{op}$*, whose objects (*$T$-affine systems *or* T-systems*) are triples* (X, κ, B)*, which are made by* $\mathbf{B}^{op}$*-morphisms* $TX \xrightarrow{\kappa} B$*; and whose morphisms (*T-affine morphisms *or* T-morphisms*)* $(X_1, \kappa_1, B_1) \xrightarrow{(f,\varphi)} (X_2, \kappa_2, B_2)$ *are* $\mathbf{X} \times \mathbf{B}^{op}$*-morphisms*

$(X_1, B_1) \xrightarrow{(f,\varphi)} (X_2, B_2)$, *which make the diagram*

$$\begin{array}{ccc} TX_1 & \xrightarrow{Tf} & TX_2 \\ {\scriptstyle \kappa_1}\downarrow & & \downarrow{\scriptstyle \kappa_2} \\ B_1 & \xrightarrow[\varphi]{} & B_2 \end{array}$$

commute.

Example 2.9.

1. If $\mathbf{B} = \mathbf{Frm}$, then $\mathbf{Af\,Sys}(\mathcal{P}_2)$ is the category **TopSys** of topological systems of S. Vickers.

2. If $\mathbf{B} = \mathbf{Set}$, then $\mathbf{Af\,Sys}(\mathcal{P}_B)$ is the category $\mathbf{Chu}_B$ of Chu spaces over a set B of P.-H. Chu [6].

To provide another example of the categories of the form $\mathbf{Af\,Sys}(T)$, we need one additional notion.

Definition 2.10. *A T-system (X, κ, B) is called* separated *provided that $TX \xrightarrow{\kappa} B$ is an epimorphism in $\mathbf{B}^{op}$. $\mathbf{Af\,Sys}_s(T)$ is the full subcategory of $\mathbf{Af\,Sys}(T)$ of separated T-systems.*

We recall from, e.g., [5, pp. 393 – 394] that monomorphisms in every variety **B** are necessarily injective (given a monomorphism $B \xrightarrow{\varphi} B'$, the set $K = \{(b_1, b_2) \in B \times B \mid \varphi(b_1) = \varphi(b_2)\}$ is a subalgebra of $B \times B$ such that the respective projections $B \times B \underset{\pi_2}{\overset{\pi_1}{\rightrightarrows}} B$ satisfy $\varphi \circ \pi_1 = \varphi \circ \pi_2$, and therefore, $\pi_1 = \pi_2$).

Example 2.11. If $\mathbf{B} = \mathbf{CSL}$, then $\mathbf{Af\,Sys}_s(\mathcal{P}_2)$ is the category **SP** of state property systems of D. Aerts [4].

The nature of the category $\mathbf{Af\,Sys}(T)$ is quite different from the nature of the category $\mathbf{Af\,Spc}(T)$.

Theorem 2.12 ([1]). *A concrete category $(\mathbf{C}, |-|)$ over $\mathbf{Z}$ is essentially algebraic iff the following conditions are satisfied:*

1. $|-|$ *creates isomorphisms;*

2. $|-|$ *has a left adjoint;*

3. **C** *is (Epi, Mono-Source)-factorizable.*

The next result is a modification (in the formulation and, especially, in the proof) of [18, Theorem 44].

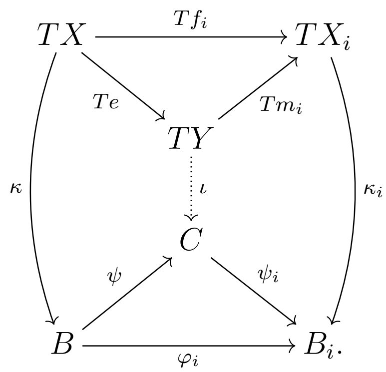
Theorem 2.13. *Suppose* **X** *is (Epi, Mono-Source)-factorizable, and* $\mathbf{X} \xrightarrow{T} \mathbf{B}^{op}$ *preserves epimorphisms. Then the concrete category* $(\mathbf{Af\,Sys}(T), |-|)$ *is essentially algebraic over the ground category* $\mathbf{X} \times \mathbf{B}^{op}$.

Proof.

Ad (1). Given an $\mathbf{X}\times\mathbf{B}^{op}$-isomorphism $(X_1, B_1) \xrightarrow{(f,\varphi)} |(X_2, \kappa_2, B_2)|$, the unique structure on (X_1, B_1), making (f, φ) an isomorphism in $\mathbf{Af\,Sys}(T)$, can be defined by $\kappa_1 = \varphi^{-1} \circ \kappa_2 \circ Tf$.

Ad (2). Given some $\mathbf{X} \times \mathbf{B}^{op}$-object (X, B), the $\mathbf{X} \times \mathbf{B}^{op}$-morphism $(X, B) \xrightarrow{\eta=(1_X,\mu_B)} |(X, \mu_{TX}, TX + B)|$, where $TX \xrightarrow{\mu_{TX}} TX + B \xleftarrow{\mu_B} B$ is the coproduct of TX and B in $\mathbf{B}^{op}$, is the required $|-|$-universal arrow.

Ad (3). Let $\mathcal{S} = ((X, \kappa, B) \xrightarrow{(f_i,\varphi_i)} (X_i, \kappa_i, B_i))_{i\in I}$ be a source in $\mathbf{Af\,Sys}(T)$. By the assumption, there exists an (Epi, Mono-Source)-factorization $X \xrightarrow{f_i} X_i = X \xrightarrow{e} Y \xrightarrow{m_i} X_i$. Since **B** is a variety, it is strongly complete, and therefore, by [1, Corollary 15.17], **B** is an (ExtrEpi-Sink, Mono)-category. Then $\mathbf{B}^{op}$ is an (Epi, ExtrMono-Source)-category, and thus, there exists an (Epi, ExtrMono-Source)-factorization $B \xrightarrow{\varphi_i} B_i = B \xrightarrow{\psi} C \xrightarrow{\psi_i} B_i$, and a unique $\mathbf{B}^{op}$-morphism $TY \xrightarrow{\iota} C$ such that the next diagram commutes



Thus, $(X, \kappa, B) \xrightarrow{(f_i,\varphi_i)} (X_i, \kappa_i, B_i) = (X, \kappa, B) \xrightarrow{(e,\psi)} (Y, \iota, C) \xrightarrow{(m_i,\psi_i)} (X_i, \kappa_i, B_i)$ is an (Epi, Mono-Source)-factorization of $\mathcal{S}$. $\square$

One gets that the categories of Example 2.9 are essentially algebraic. We also recall from [1] that essentially algebraic categories have convenient properties, e.g., their respective underlying functors detect colimits, and preserve and create limits. Additionally, it is easy to see that items (1), (3) of the proof of Theorem 2.13 hold in case of the category $\mathbf{Af\,Sys}_s(T)$ as well, but not item (2), for which one can show the following.

Proposition 2.14. *The forgetful functor* $\mathbf{Af\,Sys}_s(T) \xrightarrow{|-|} \mathbf{X} \times \mathbf{B}^{op}$ *does not have in general a left adjoint.*

Proof. An easy counterexample provides the case of the functor $\mathbf{Set} \xrightarrow{\mathcal{P}_2} \mathbf{Set}^{op}$. More precisely, if the underlying functor $\mathbf{Af\,Sys}_s(\mathcal{P}_2) \xrightarrow{|-|} \mathbf{Set} \times \mathbf{Set}^{op}$ has a left adjoint, then there exists a $|-|$-universal arrow $(\varnothing, 2) \xrightarrow{(f,\varphi)} |(X, \kappa, B)|$. In particular, one has the following commutative triangle

$$\begin{array}{ccc} (\varnothing, 2) & \xrightarrow{(f,\varphi)} & |(X, \kappa, B)| \\ & {\scriptstyle (!,\psi)} \searrow & \Big\downarrow {\scriptstyle (\hat{!},\hat{\psi})} \\ & & |(\varnothing, !, 1)|, \end{array}$$

in which ! denotes the unique map, and ψ^{op} is any of the possible two maps. Since $X \xrightarrow{\hat{!}} \varnothing$ is a map, $X = \varnothing$, and thus, since κ^{op} is a monomorphism, $B \cong 1$, i.e., one can assume that $\hat{\psi}^{op}$ is the identity map. Commutativity of the above triangle gives then $\varphi = \psi$, i.e., $\mathbf{Set}(1, 2) = \{\varphi^{op}\}$, which is a contradiction. $\square$

We recall now the concept of algebraic category of [1, Definition 23.19].

Definition 2.15. *An essentially algebraic concrete category* $(\mathbf{C}, |-|)$ *over* $\mathbf{Z}$ *is called* algebraic *provided that* $|-|$ *preserves extremal epimorphisms.*

The next result is a modification (both in the formulation and in the proof) of [18, Corollary 48].

Theorem 2.16. *Suppose that* $\mathbf{X}$ *is (Epi, Mono-Source)-factorizable,* $\mathbf{X} \xrightarrow{T} \mathbf{B}^{op}$ *preserves epimorphisms, and, moreover, the following three equivalent conditions hold:*

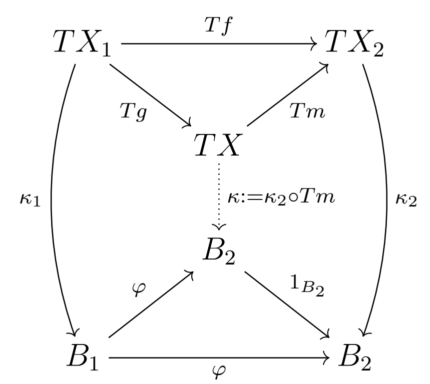
1. $\mathbf{B}$ *has the (Epi, Mono)-diagonalization property;*
2. $ExtrEpi(\mathbf{B}) = Epi(\mathbf{B})$*;*
3. *epimorphisms in* $\mathbf{B}$ *are surjective.*

Then the concrete category $(\mathbf{Af\,Sys}(T), |-|)$ *is algebraic over the ground category* $\mathbf{X} \times \mathbf{B}^{op}$.

Proof. Equivalence of items (1), (2), (3) of the theorem is a consequence of the facts that, first, $\mathbf{B}$ is an (ExtrEpi-Sink, Mono)-category (cf. the proof of Theorem 2.13), and, second, monomorphisms in $\mathbf{B}$ are necessarily injective. Given now an extremal epimorphism $(X_1, \kappa_1, B_1) \xrightarrow{(f,\varphi)} (X_2, \kappa_2, B_2)$ in $\mathbf{Af\,Sys}(T)$, we show that f and φ are extremal epimorphisms in their respective categories.

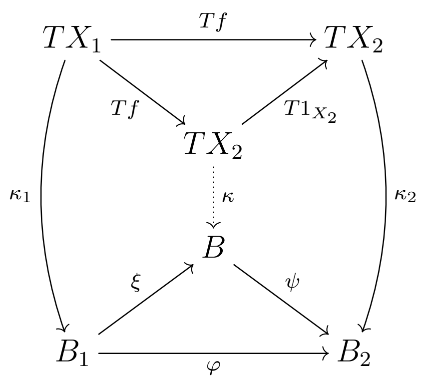
The assumptions of the theorem provide the (Epi, Mono)-factorization $(X_1, \kappa_1, B_1) \xrightarrow{(f,\varphi)} (X_2, \kappa_2, B_2) = (X_1, \kappa_1, B_1) \xrightarrow{(e,\psi)} (X, \kappa, B) \xrightarrow{(m,\xi)} (X_2, \kappa_2, B_2)$ constructed in the proof of Theorem 2.13. Then (m, ξ) is an isomorphism, and therefore, (f, φ) is "essentially" the morphism (e, ψ), which is an epimorphism in $\mathbf{X} \times \mathbf{B}^{op}$.

To show that f is extremal in $\mathbf{X}$, let $X_1 \xrightarrow{f} X_2 = X_1 \xrightarrow{g} X \xrightarrow{m} X_2$ be a (– , Mono)-factorization. The commutative diagram



gives the (–, Mono)-factorization $(X_1, \kappa_1, B_1) \xrightarrow{(f,\varphi)} (X_2, \kappa_2, B_2) = (X_1, \kappa_1, B_1) \xrightarrow{(g,\varphi)} (X, \kappa, B_2) \xrightarrow{(m,1_{B_2})} (X_2, \kappa_2, B_2)$ in **Af Sys**(T). It follows that $(m, 1_{A_2})$ is an isomorphism, and therefore, m must be as well.

To show that φ is extremal in $\mathbf{B}^{op}$, let $B_1 \xrightarrow{\varphi} B_2 = B_1 \xrightarrow{\xi} B \xrightarrow{\psi} B_2$ be a (–, Mono)-factorization. By assumption (1), there exists a $\mathbf{B}^{op}$-morphism $TX_2 \xrightarrow{\kappa} B$, which makes the following diagram commute



and therefore providing the (–, Mono)-factorization $(X_1, \kappa_1, B_1) \xrightarrow{(f,\varphi)} (X_2, \kappa_2, B_2) = (X_1, \kappa_1, B_1) \xrightarrow{(f,\xi)} (X_2, A, \kappa) \xrightarrow{(1_{X_2},\psi)} (X_2, \kappa_2, B_2)$ in **Af Sys**(T). Then $(1_{X_2}, \psi)$ is an isomorphism, and thus, ψ is as well. $\square$

We notice that while **Frm** does have non-surjective epimorphisms [15, Proposition 2.4.1], **CSL** does not [18, Lemma 49].

2.4 Affine spatialization procedure

Following the results of [19], this subsection shows an affine analogue of the topological system spatialization procedure of S. Vickers.

Theorem 2.17. *$\mathbf{Af\,Spc}(T) \overset{E}{\hookrightarrow} \mathbf{Af\,Sys}(T)$, $E((X_1, \tau_1) \xrightarrow{f} (X_2, \tau_2)) = (X_1, e^{op}_{\tau_1}, \tau_1) \xrightarrow{(f,\varphi)} (X_2, e^{op}_{\tau_2}, \tau_2)$ is a full embedding, where e_{τ_i} is the inclusion $\tau_i \hookrightarrow TX_i$, and φ^{op} is the restriction $\tau_2 \xrightarrow{(Tf)^{op}|^{\tau_1}_{\tau_2}} \tau_1$. E has a right-adjoint-left-inverse $\mathbf{Af\,Sys}(T) \xrightarrow{Spat} \mathbf{Af\,Spc}(T)$, $Spat((X_1, \kappa_1, B_1) \xrightarrow{(f,\varphi)}$*

$(X_2, \kappa_2, B_2)) = (X_1, \kappa_1^{op}(B_1)) \xrightarrow{f} (X_2, \kappa_2^{op}(B_2))$. **Af Spc**$(T)$ *is isomorphic to a full (regular mono)-coreflective subcategory of* **Af Sys**(T).

Proof. To show that $Spat$ is a right adjoint to E, it is enough to verify that every system (X, κ, B) has an E-co-universal arrow, i.e., a T-morphism $ESpat(X, \kappa, B) \xrightarrow{\varepsilon} (X, \kappa, B)$ with the property that for every T-morphism $E(X', \tau') \xrightarrow{(f,\varphi)} (X, \kappa, B)$, there exists a unique T-morphism $(X', \tau') \xrightarrow{g} Spat(X, \kappa, B)$ with $\varepsilon \circ Eg = (f, \varphi)$.

Take a T-morphism $(ESpat(X, \kappa, B) = (X, e^{op}_{\kappa^{op}(B)}, \kappa^{op}(B))) \xrightarrow{\varepsilon=(1_X,\kappa)} (X, \kappa, B)$. Given a T-morphism $E(X', \tau') \xrightarrow{(f,\varphi)} (X, \kappa, B)$, it follows that $(Tf)^{op} \circ \kappa^{op} = e_{\tau'} \circ \varphi^{op}$, which yields the desired T-morphism $(X', \tau') \xrightarrow{f} (Spat(X, \kappa, B) = (X, \kappa^{op}(B)))$, whose uniqueness is clear.

For the last claim, it is enough to show that given a T-system (X, κ, B), the map $B \xrightarrow{\kappa^{op}} \kappa^{op}(B)$ is a regular epimorphism in **B**. Let $C = \{(b_1, b_2) \in B \times B \mid \kappa^{op}(b_1) = \kappa^{op}(b_2)\}$ (the *kernel* of κ^{op}), and let $C \xrightarrow{\pi_i} B$ be given by $\pi_i(b_1, b_2) = b_i$ for $i \in \{1, 2\}$. $(\kappa^{op}, \kappa^{op}(B))$ is a coequalizer of (π_1, π_2). □

The analogue of Theorem 2.17 for the category **Af Sys**$_s(T)$ is even better.

Theorem 2.18. *E and $Spat$ restrict to* **Af Spc**$(T) \overset{\overline{E}}{\hookrightarrow}$ **Af Sys**$_s(T)$ *and* **Af Sys**$_s(T) \xrightarrow{\overline{Spat}}$ **Af Spc**(T), *respectively, providing an equivalence between the categories* **Af Spc**(T) *and* **Af Sys**$_s(T)$ *with* $\overline{Spat}\,\overline{E} = 1_{\mathbf{Af\,Spc}(T)}$.

Proof. By Theorem 2.17, $\overline{Spat}$ is a right-adjoint-left-inverse to $\overline{E}$. To prove the theorem, it is enough to show that for every separated T-system (X, κ, B), the E-co-universal arrow $ESpat(X, \kappa, B) \xrightarrow{\varepsilon=(1_X,\kappa)} (X, \kappa, B)$ from the proof of Theorem 2.17 is an isomorphism. The claim follows from the definition of ε, since $B \xrightarrow{\kappa^{op}} \kappa^{op}(B)$ is always surjective, and it is injective by the property of separated T-systems. □

Corollary 2.19. *The category* **Af Spc**(T) *is the amnestic modification of the category* **Af Sys**$_s(T)$.

Proof. Follows from Theorem 2.18 and the definition of the amnestic modification of [1, Remark 5.34]. □

The following well-known results are consequences of Theorems 2.17 and 2.18, respectively.

Remark 2.20.

1. **Top** is isomorphic to a full (regular mono)-coreflective subcategory of the category **TopSys**, which provides the system spatialization procedure of S. Vickers.

2. The categories **Cls** and **SP** are equivalent [3, 4].

Moreover, from Corollary 2.19, one gets [4, Theorem 4], which states that the category **Cls** is the amnestic modification of the category **SP**.

Remark 2.21. In [4, Proposition 8], D. Aerts *et al.* showed a construction of limits in the category **SP**. Theorem 2.18 allows for an easy generalization of this result to the category $\mathbf{Af\,Sys}_s(T)$ (employing the standard technique of limits in topological categories). Let $\mathbf{I}$ be a small category such that $\mathbf{X}$ is $\mathbf{I}$-complete. Let $\mathbf{I} \xrightarrow{D} \mathbf{Af\,Sys}_s(T)$ be a functor, and let $Di = (X_i, \kappa_i, B_i)$ for every $i \in Ob(\mathbf{I})$. If $\mathcal{L} = (L \xrightarrow{l_i} X_i)_{i \in Ob(\mathbf{I})}$ is a limit of $| - |\overline{Spat}D$ in $\mathbf{X}$, then $\hat{\mathcal{L}} = ((L, \tau) \xrightarrow{l_i} (X_i, \tau_i))_{i \in Ob(\mathbf{I})}$ is a limit of $\overline{Spat}D$ in $\mathbf{Af\,Spc}(T)$, where $\tau_i = \kappa_i^{op}(B_i)$, and τ is the subalgebra of TL generated by the union $\bigcup_{i \in I} (Tl_i)^{op}(\tau_i)$. The limit of D in $\mathbf{Af\,Sys}_s(T)$ is given then by $\check{\mathcal{L}} = ((L, e_\tau^{op}, \tau) \xrightarrow{(l_i, \varphi_i)} (X_i, \kappa_i, B_i))_{i \in Ob(\mathbf{I})}$ in which $\varphi_i^{op}(b) = (\kappa_i \circ Tl_i)^{op}(b)$.

3. Localification procedure for affine systems

This section provides a localification procedure for affine systems, motivated by the above-mentioned localification procedure for topological systems of S. Vickers.

Proposition 3.1. $\mathbf{Af\,Sys}(T) \xrightarrow{Loc} \mathbf{B}^{op}$, *$Loc((X_1, \kappa_1, B_1) \xrightarrow{(f, \varphi)} (X_2, \kappa_2, B_2)) = B_1 \xrightarrow{\varphi} B_2$ is a functor.*

Unlike the affine spatialization procedure, in which the functor in the opposite direction always exists, localification procedure is more demanding.

Theorem 3.2. *Given a functor* $\mathbf{X} \xrightarrow{T} \mathbf{B}^{op}$, *the following are equivalent.*

1. *There exists an adjoint situation* $(\eta, \varepsilon) : T \dashv Pt : \mathbf{B}^{op} \to \mathbf{X}$.

2. *There exists a full embedding* $\mathbf{B}^{op} \xhookrightarrow{E} \mathbf{Af\,Sys}(T)$ *such that* Loc *is a left-adjoint-left-inverse to* E. $\mathbf{B}^{op}$ *is then isomorphic to a full reflective subcategory of* $\mathbf{Af\,Sys}(T)$.

Proof.

Ad (1) $\Rightarrow$ (2). Define a functor $\mathbf{B}^{op} \xrightarrow{E} \mathbf{Af\,Sys}(T)$ by $E(B_1 \xrightarrow{\varphi} B_2) = (PtB_1, \varepsilon_{B_1}, B_1) \xrightarrow{(Pt\varphi, \varphi)} (PtB_2, \varepsilon_{B_2}, B_2)$. Correctness of E on morphisms follows from commutativity of the diagram

$$\begin{array}{ccc} TPtB_1 & \xrightarrow{TPt\varphi} & TPtB_2 \\ {\scriptstyle \varepsilon_{B_1}}\downarrow & & \downarrow{\scriptstyle \varepsilon_{B_2}} \\ B_1 & \xrightarrow[\varphi]{} & B_2. \end{array}$$

Moreover, E is clearly an embedding. To verify that E is full, we notice that given a T-morphism $(PtB_1, \varepsilon_{B_1}, B_1) \xrightarrow{(f,\varphi)} (PtB_2, \varepsilon_{B_2}, B_2)$, commutativity of the diagram

$$\begin{array}{ccc} TPtB_1 & \xrightarrow{\varepsilon_{B_1}} & B_2 \\ {\scriptstyle TPt\varphi}\downarrow\ \downarrow{\scriptstyle T\varphi} & & \downarrow{\scriptstyle \varphi} \\ TPtB_2 & \xrightarrow[\varepsilon_{B_2}]{} & B_2 \end{array}$$

implies that $\varepsilon_{B_2} \circ TPt\varphi = \varepsilon_{B_2} \circ Tf$, and thus, $Pt\varphi = f$. Given a T-system (X, κ, B), straightforward calculations show that $(X, \kappa, B) \xrightarrow{(f := Pt\kappa \circ \eta_X, 1_B)} ((PtB, \varepsilon_B, B) = ELoc(X, \kappa, B))$ is an E-universal arrow for (X, κ, B). It is also easy to see that $LocE = 1_{\mathbf{B}^{op}}$.

Ad (2) $\Rightarrow$ (1). Given an adjunction $Loc \dashv E : \mathbf{B}^{op} \to \mathbf{Af\,Sys}(T)$, $\mathbf{X} \xrightarrow{T} \mathbf{B}^{op}$ is the composition of the left adjoint functors $\mathbf{X} \to \mathbf{Af\,Spc}(T)$ (the *indiscrete functor* of, e.g., [1, Proposition 21.12 (2)], which exists by Theorem 2.7), $\mathbf{Af\,Spc}(T) \xhookrightarrow{E} \mathbf{Af\,Sys}(T)$ (the embedding of Theorem 2.17), and $\mathbf{Af\,Sys}(T) \xrightarrow{Loc} \mathbf{B}^{op}$. $\square$

The following well-known remark provides an example of the functor Pt of Theorem 3.2 (1).

Remark 3.3. Every functor **Set** $\xrightarrow{\mathcal{P}_B}$ $\mathbf{B}^{op}$ has a right adjoint $\mathbf{B}^{op}$ $\xrightarrow{Pt_B}$ **Set**, $Pt_B(B_1 \xrightarrow{\varphi} B_2) = \mathbf{B}(B_1, B) \xrightarrow{Pt_B\varphi} \mathbf{B}(B_2, B)$, where $(Pt_B\varphi)(p) = p \circ \varphi^{op}$. Given a **B**-algebra A, the map $A \xrightarrow{\varepsilon^{op}} (\mathcal{P}_B Pt_B A = B^{\mathbf{B}(A,B)})$, defined by $(\varepsilon^{op}(a))(p) = p(a)$, provides a $\mathcal{P}_B$-co-universal arrow for A.

As a consequence of Theorem 3.2 and Remark 3.3, one gets the following well-known results.

Remark 3.4.

1. **Loc** (the dual of **Frm**) is isomorphic to a full reflective subcategory of **TopSys**, which gives the system localification procedure of S. Vickers.

2. $\mathbf{B}^{op}$ is isomorphic to a full reflective subcategory of **Af Sys**$(\mathcal{P}_B)$.

The case of the category **TopSys** shows that in Theorem 3.2 (2), the category $\mathbf{B}^{op}$, however being a reflective subcategory of **Af Sys**(T), can be neither epi- nor mono-reflective.

In [17], S. E. Rodabaugh considered functors of the form **Set**×**S** $\xrightarrow{\mathcal{P}_\mathbf{S}}$ **Loc** and their respective categories of affine spaces, using, however, a different terminology (recall Example 2.6 (4)). The next result shows that Remark 3.3 (in general) can not be extended from the subcategory $\{B \xrightarrow{1_B} B\}$ to the whole $\mathbf{B}^{op}$.

Proposition 3.5. *Consider a functor* **Set** $\times$ $\mathbf{B}^{op}$ $\xrightarrow{T:=\mathcal{P}_{\mathbf{B}^{op}}}$ $\mathbf{B}^{op}$*, and suppose that there exists a* **B***-algebra B, whose underlying set is finite with at least two elements, e.g., has the cardinality n, $n \geqslant 2$. Then T has no right adjoint.*

Proof. If T has a right adjoint, then T preserves coproducts. Given a singleton set 1, $T((1, B) \coprod (1, B)) = T((1 \biguplus 1, B \times B)) = (B \times B)^{(1 \biguplus 1)}$ and $T(1, B) \times T(1, B) = B^1 \times B^1$. Since $T((1, B) \coprod (1, B)) \cong T(1, B) \times T(1, B)$, $n^4 = Card((B \times B)^{(1 \biguplus 1)}) = Card(B^1 \times B^1) = n^2$, which is a contradiction. $\square$

For instance, Proposition 3.5 implies that the functor $\mathbf{Set}\times\mathbf{Loc} \xrightarrow{\mathcal{P}_{\mathbf{Loc}}} \mathbf{Loc}$ has no right adjoint, i.e., Theorem 3.2 (2) is not applicable to the category **Loc**-**Top** of Example 2.6 (4).

To conclude the section, we provide a result, which is a direct consequence of Theorems 2.13, 3.2.

Proposition 3.6. *Suppose that the category* $\mathbf{X}$ *is (Epi, Mono-Source)-factorizable. If there exists a full embedding* $\mathbf{B}^{op} \overset{E}{\hookrightarrow} \mathbf{Af\,Sys}(T)$ *such that* Loc *is a left-adjoint-left-inverse to* E*, then the concrete category* $(\mathbf{Af\,Sys}(T), |-|)$ *is essentially algebraic over the ground category* $\mathbf{X}\times\mathbf{B}^{op}$.

In one word, the existence of a "proper" localification procedure for the category $\mathbf{Af\,Sys}(T)$ implies that $\mathbf{Af\,Sys}(T)$ is essentially algebraic.

4. Affine sobriety-spatiality equivalence

With the above-mentioned affine spatialization and localification procedures in hand, in this section, we provide an affine analogue of the well-known equivalence between the categories of sober topological spaces and spatial locales [14], which has already been considered by Y. Diers in the case of the category $\mathbf{Af\,Set}(A)$.

4.1 Affine sobriety and spatiality

Suppose $\mathbf{X} \xrightarrow{T} \mathbf{B}^{op}$ is a functor, which has a right adjoint. One has then the two adjoint situations $\mathbf{Af\,Spc}(T) \underset{Spat}{\overset{E_S}{\underset{\longleftarrow}{\overset{\longrightarrow}{\bot}}}} \mathbf{Af\,Sys}(T) \underset{E_L}{\overset{Loc}{\underset{\longleftarrow}{\overset{\longrightarrow}{\bot}}}} \mathbf{B}^{op}$, which provide the adjoint situation $\mathbf{Af\,Spc}(T) \underset{PT:=SpatE_L}{\overset{O:=LocE_S}{\underset{\longleftarrow}{\overset{\longrightarrow}{\bot}}}} \mathbf{B}^{op}$, or, more precisely, $(\hat{\eta},\hat{\varepsilon}) : O \dashv PT : \mathbf{B}^{op} \to \mathbf{Af\,Spc}(T)$, in which $(OPTB = O(PtB, \varepsilon_B^{op}(B)) = \varepsilon_B^{op}(B)) \xrightarrow{\hat{\varepsilon}_B} B$ with $\hat{\varepsilon}_B^{op}(b) = \varepsilon_B^{op}(b)$, for every $\mathbf{B}$-algebra B, and $(X,\tau) \xrightarrow{\hat{\eta}_{(X,\tau)}} (PTO(X,\tau) = PT\tau = (Pt\tau, \varepsilon_\tau^{op}(\tau)))$ with $X \xrightarrow{\hat{\eta}_{(X,\tau)}} Pt\tau = X \xrightarrow{\eta_X} PtTX \to Pte_\tau^{op}Pt\tau$ for the embedding $\tau \overset{e_\tau}{\hookrightarrow} TX$, for every T-space (X,τ).

Definition 4.1. **Sob** *is the full subcategory of* $\mathbf{Af\,Spc}(T)$*, which contains* T*-spaces* (X,τ) *with the property that* $(X,\tau) \xrightarrow{\hat{\eta}_{(X,\tau)}} PTO(X,\tau)$ *is an isomorphism (called* sober T-spaces*).*

Definition 4.2. **Spat** *is the full subcategory of* $\mathbf{B}^{op}$ *defined by those objects* B *for which* $OPTB \xrightarrow{\hat{\varepsilon}_B} B$ *is an isomorphism (called* spatial algebras*).*

The standard technique of getting an equivalence from an adjunction (see [12, 16]) gives the following.

Proposition 4.3. *The adjunction* $\mathbf{Af\,Spc}(T) \underset{PT}{\overset{O}{\underset{\longleftarrow}{\overset{\longrightarrow}{\bot}}}} \mathbf{B}^{op}$ *restricts to an equivalence* $\mathbf{Sob} \underset{\overline{PT}}{\overset{\overline{O}}{\underset{\longleftarrow}{\overset{\longrightarrow}{\bot}}}} \mathbf{Spat}$.

As a consequence of Proposition 4.3, one obtains the following well-known results.

Remark 4.4.

1. There exists the adjoint situation $O \dashv PT : \mathbf{Loc} \to \mathbf{Top}$ and its respective equivalence between the categories **Spat** of spatial locales and **Sob** of sober topological spaces.

2. There exists the adjoint situation $O \dashv PT : \mathbf{B}^{op} \to \mathbf{Af\,Set}(B)$ and its respective equivalence $\mathbf{Spat} \sim \mathbf{Sob}$ ("affine algebraic duality" of Y. Diers [10]).

4.2 Separated affine spaces

This subsection provides an affine analogue of the well-known result that the category of T_0 topological spaces [13] is a reflective subcategory of the category of topological spaces, which has already been extended to the category of affine sets of Y. Diers.

Definition 4.5. *A* T*-space* (X,τ) *is said to be* separated *provided that* $(X,\tau) \xrightarrow{\hat{\eta}_{(X,\tau)}} PTO(X,\tau)$ *is a monomorphism.* $\mathbf{Af\,Spc}_s(T)$ *is the full subcategory of* $\mathbf{Af\,Spc}(T)$ *of separated* T*-spaces.*

To continue, we need a simple result, which follows from the more general technique of, e.g., [12, 16].

Proposition 4.6. *Given a T-space (X,τ), the T-space $PTO(X,\tau)$ is sober, and therefore, separated.*

Theorem 4.7. *Let* $\mathbf{X} \xrightarrow{T} \mathbf{B}^{op}$ *be a functor. If* $\mathbf{X}$ *has a proper* ($\mathcal{E}$, *Mono)-factorization system (in the sense of [8]), where Mono is the class of* $\mathbf{X}$*-monomorphisms, then* $\mathbf{Af\,Spc}_s(T)$ *is an epireflective subcategory of* $\mathbf{Af\,Spc}(T)$.

Proof. Since the category $\mathbf{Af\,Spc}(T)$ is topological (recall Theorem 2.7), the proper ($\mathcal{E}$, Mono)-factorization system on $\mathbf{X}$ lifts to a proper ($\mathcal{E}_{fin}$, Mono)-factorization system on $\mathbf{Af\,Spc}(T)$, where $\mathcal{E}_{fin}$ consists of all final $\mathcal{E}$-morphisms [1, Proposition 21.14 (2)]. Given a T-space (X,τ), consider an ($\mathcal{E}_{fin}$, Mono)-factorization $(X,\tau) \xrightarrow{\hat{\eta}_{(X,\tau)}} PTO(X,\tau) = (X,\tau) \xrightarrow{e} (\overline{X},\overline{\tau}) \xrightarrow{m} PTO(X,\tau)$. The T-space $(\overline{X},\overline{\tau})$ is then separated, since it is a subobject of the separated T-space $PTO(X,\tau)$ (recall Proposition 4.6).

To show that $(X,\tau) \xrightarrow{e} (\overline{X},\overline{\tau})$ is an $\mathbf{Af\,Spc}_s(T)$-reflection arrow for (X,τ), take a T-morphism $(X,\tau) \xrightarrow{f} (X',\tau')$ with codomain in $\mathbf{Af\,Spc}_s(T)$. One has then the commutative diagram

$$\begin{array}{ccccc} (X,\tau) & \xrightarrow{e} & (\overline{X},\overline{\tau}) & \xrightarrow{m} & PTO(X,\tau) \\ & & \underset{\hat{\eta}_{(X,\tau)}}{\frown} & & \\ {\scriptstyle f}\big\downarrow & & & & \big\downarrow{\scriptstyle PTOf} \\ (X',\tau') & & \xrightarrow[\hat{\eta}_{(X',\tau')}]{} & & PTO(X',\tau'), \end{array}$$

where $\hat{\eta}_{(X',\tau')}$ is a monomorphism. Since the category $\mathbf{Af\,Spc}(T)$ has a proper ($\mathcal{E}_{fin}$, Mono)-factorization system, there is a T-morphism $(\overline{X},\overline{\tau}) \xrightarrow{d} (X',\tau')$, making the following diagram commute

$$\begin{array}{ccc} (X,\tau) & \xrightarrow{e} & (\overline{X},\overline{\tau}) \\ {\scriptstyle f}\big\downarrow & \overset{d}{\swarrow} & \big\downarrow{\scriptstyle PTOf\circ m} \\ (X',\tau') & \xrightarrow[\hat{\eta}_{(X',\tau')}]{} & PTO(X',\tau'). \end{array}$$

□

Corollary 4.8. *Let* $\mathbf{X} \xrightarrow{T} \mathbf{B}^{op}$ *be a functor. If* $\mathbf{X}$ *has a proper* $(\mathcal{E}, Mono)$*-factorization system, then* $\mathbf{Af\,Spc}_s(T)$ *is isomorphic to a reflective subcategory of the category* $\mathbf{Af\,Sys}_s(T)$*.*

Proof. Follows from Theorems 2.18, 4.7. □

The following provides a well-known example of separated T-spaces.

Example 4.9. If $\mathbf{B} = \mathbf{Frm}$, then $\mathbf{Af\,Spc}_s(\mathcal{P}_2)$ is the category $\mathbf{Top}_0$ of T_0 topological spaces.

As a consequence of Theorem 4.7, one gets the following.

Remark 4.10. Since the category **Set** has a proper (Epi, Mono)-factorization system, Theorem 4.7 is applicable to every functor $\mathbf{Set} \xrightarrow{\mathcal{P}_B} \mathbf{B}^{op}$. In particular, one gets the following well-known results.

1. $\mathbf{Top}_0$ is a reflective subcategory of **Top**.
2. The category $\mathbf{Cls}_0$ of T_0 closure spaces [3] is a reflective subcategory of **Cls**.
3. $\mathbf{Af\,Set}_s(B)$ is a reflective subcategory of $\mathbf{Af\,Set}(B)$.

Moreover, Corollary 4.8 implies now that the category of state-determined state property systems is a reflective subcategory of the category **SP** [3].

4.3 Spatial and localic affine systems

Having the embeddings of **Top** and **Loc** into **TopSys** in hand, S. Vickers restated the sobriety-spatiality equivalence in terms of topological systems. This subsection shows an affine analogue of this technique.

Let us have adjoint situations $\mathbf{B}^{op} \underset{E_L}{\overset{Loc}{\leftrightarrows}} \mathbf{Af\,Sys}(T) \underset{Spat}{\overset{E_S}{\leftrightarrows}} \mathbf{Af\,Spc}(T)$.

Definition 4.11. *A* T*-system* (X, κ, B) *is called* spatial *(resp.* localic*) provided that there exists a* T*-space* (X, τ) *(resp. a* $\mathbf{B}$*-algebra* B*) such that* (X, κ, B) *is isomorphic to* $E_S(X, \tau)$ *(resp.* $E_L B$*).*

The following two results are straightforward.

Proposition 4.12. *Given a T-system (X, κ, B), the following are equivalent:*

1. (X, κ, B) *is localic;*
2. *the T-morphism* $(X, \kappa, B) \xrightarrow{(Pt\kappa \circ \eta_X, 1_B)} (E_L Loc(X, \kappa, B) = (PtB, \varepsilon_B, B))$ *is an isomorphism;*
3. *the* **X**-*morphism* $X \xrightarrow{\eta_X} PtTX \xrightarrow{Pt\kappa} PtB$ *is an isomorphism.*

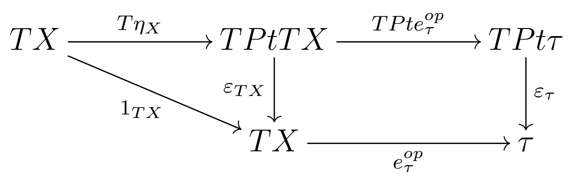
Proposition 4.13. *Given a T-system (X, κ, B), the following are equivalent:*

1. (X, κ, B) *is spatial;*
2. *the T-morphism* $(E_S Spat(X, \kappa, B) = (X, e^{op}_{\kappa^{op}(B)}, B)) \xrightarrow{(1_X, \kappa)} (X, \kappa, B)$ *is an isomorphism;*
3. *the* **B**-*homomorphism* $B \xrightarrow{\kappa^{op}} \kappa^{op}(B)$ *is an isomorphism;*
4. *the* **B**-*homomorphism* $B \xrightarrow{\kappa^{op}} TX$ *is injective.*

Given a T-space (X, τ), $E_L Loc E_S(X, \tau) = E_L Loc(X, e^{op}_\tau, \tau) = E_L \tau = (Pt\tau, \varepsilon_\tau, \tau)$ is a localic T-system.

Proposition 4.14. *The T-system $(Pt\tau, \varepsilon_\tau, \tau)$ is spatial.*

Proof. Commutativity of the diagram



gives $(T(Pte^{op}_\tau \circ \eta_X))^{op} \circ \varepsilon^{op}_\tau = e_\tau$, yielding injectivity of ε^{op}_τ. The result now follows from Proposition 4.13. □

Given a **B**-algebra B, $E_S Spat E_L B = E_S Spat(PtB, \varepsilon_B, B) = E_S(PtB, \varepsilon^{op}_B(B)) = (PtB, e^{op}_{\varepsilon^{op}_B(B)}, \varepsilon^{op}_B(B))$ is a spatial T-system.

Proposition 4.15. *The T-system $(PtB, e^{op}_{\varepsilon^{op}_B(B)}, \varepsilon^{op}_B(B))$ is localic.*

Proof. Take the factorization $B \xrightarrow{\varepsilon_B^{op}} TPtB = B \xrightarrow{\overline{\varepsilon_B^{op}}} \varepsilon_B^{op}(B) \xhookrightarrow{e_{\varepsilon_B^{op}(B)}} TPtB$. Since $\overline{\varepsilon_B^{op}}$ is an epimorphism in **B**, $\overline{\varepsilon_B^{op}}^{op}$ is a monomorphism in $\mathbf{B}^{op}$. Thus, on the one hand, $Pt\overline{\varepsilon_B^{op}}^{op} \circ Pte^{op}_{\varepsilon_B^{op}(B)} \circ \eta_{PtB} = Pt\varepsilon_B \circ \eta_{PtB} = 1_{PtB}$, and, on the other hand, $Pt\overline{\varepsilon_B^{op}}^{op} \circ Pte^{op}_{\varepsilon_B^{op}(B)} \circ \eta_{PtB} \circ Pt\overline{\varepsilon_B^{op}}^{op} = Pt\overline{\varepsilon_B^{op}}^{op} = Pt\overline{\varepsilon_B^{op}}^{op} \circ 1_{Pt\varepsilon_B^{op}(B)}$ implies $Pte^{op}_{\varepsilon_B^{op}(B)} \circ \eta_{PtB} \circ Pt\overline{\varepsilon_B^{op}}^{op} = 1_{Pt\varepsilon_B^{op}(B)}$, since Pt preserves monomorphisms (as a right adjoint). The desired result now follows from Proposition 4.12. □

Definition 4.16. SpaLoc *is the full subcategory of* **Af Sys**(T) *of* T*-systems, which are spatial and localic.*

Theorem 4.17. $\mathbf{Spat} \underset{\overline{E_L}}{\overset{\overline{Loc}}{\rightleftarrows}} \mathbf{SpaLoc} \underset{\overline{Spat}}{\overset{\overline{E_S}}{\rightleftarrows}} \mathbf{Sob}$ *are equivalences.*

Proof. Consider the case of the categories **Spat** and **SpaLoc** (the other case is similar). By Proposition 4.14, the restriction $\mathbf{Spat} \xhookrightarrow{\overline{E_L}} \mathbf{SpaLoc}$ is a full embedding. Moreover, given a T-system (X, κ, B) in **SpaLoc**, $E_S(X', \tau') \cong (X, \kappa, B) \cong E_L B'$ for some T-space (X', τ') and some algebra B'. It follows then that $\tau' \cong B'$, and therefore, B' is in **Spat**. Thus, the restriction of $\overline{E_L}$ is dense, and therefore, it is an equivalence. □

Theorem 4.17 is an internalization of the sobriety-spatiality equivalence into the category of affine systems.

5. Conclusion

Motivated by the fact that the category **TopSys** of topological systems of S. Vickers [20] includes the category **Loc** of locales (resp. **Top** of topological spaces) as a full (resp. co)reflective subcategory, this paper showed that the category **Af Sys**(T) of affine systems (motivated by affine sets of Y. Diers [9, 10, 11]) includes the category $\mathbf{B}^{op}$ of the underlying algebras of affine structures (resp. **Af Spc**(T) of affine spaces) as a full (resp. co)reflective subcategory. While the embedding of **Af Spc**(T) into **Af Sys**(T) is always possible, the embedding of $\mathbf{B}^{op}$ requires the existence of a right adjoint for the respective functor T. The obtained two embeddings allowed us to restate the equivalence between the categories of sober topological spaces and spatial locales [14] in the language of algebras and affine spaces, and, moreover, to internalize this equivalence into the category of affine systems.

Since the classical sobriety-spatiality equivalence provides a convenient framework for, e.g., the Stone representation theorems for Boolean algebras and distributive lattices [14], its affine generalization could provide a convenient setting for studying natural dualities in the sense of [7], which will be the topic of forthcoming papers.

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