THE COHERENT CATEGORY OF INVERSE SYSTEMS

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Résumé. Pour toute catégorie de modèles $C$ enrichie dans la catégorie des groupoïdes $\text{Gpd}$, on définit une nouvelle catégorie $\text{Pro} \ C$, dont les objets sont les systèmes inverses dans $C$; elle est isomorphe à la catégorie d'homotopie de Steenrod $\text{Ho}(\text{Pro} \ C)$, et à la catégorie de pro-homotopie cohérente définie par Lisica et Mardešić lorsque $C$ est la catégorie des espaces topologiques.

Abstract. For every model category $C$ enriched over the category $\text{Gpd}$ of groupoids a new category $\text{Pro} \ C$ is defined, with objects the inverse systems in $C$, which is isomorphic to the Steenrod homotopy category $\text{Ho}(\text{Pro} \ C)$ and to the coherent pro-homotopy category defined by Lisica and Mardešić when $C$ is the category of topological spaces.

Keywords. inverse system, groupoid enriched category, pseudo-natural transformation, model category, category of fractions.

Mathematics Subject Classification (2010). 55U35, 55P55, 18D20, 18E35

1. Introduction

Inverse systems have been widely used in Mathematics, especially in Topology. Grothendieck (see[11]) was the first to give a good categorical definition for the category $\text{Pro} \ C$ of inverse systems in a given category $C$. The need for a homotopy theory of $\text{Pro} \ C$ was recognized in [1], however, the homotopy category defined there was not satisfactory for a number of reasons. Many authors were then concerned with the task of defining a Quillen model structure on $\text{Pro} \ C$, assuming $C$ had one, in order to obtain a well behaved homotopy category. The so called Steenrod homotopy category $\text{Ho} (\text{Pro} \ C)$ was defined by Porter in [19] (see also [20]). In the last years further work on the subject has been done notably by Isaksen, see for instance [13], [14] and the very recent paper by Descotte and Dubuc [7].
There are at hand essentially two ways to look at \( \text{Ho(Pro } C \text{)} \). The first one is due to Edwards-Hastings [8], who define it by localizing \( \text{Pro } C \) at the class of level equivalences so that in this case the morphisms are quite ugly to handle. The second one is due to Cathey-Segal [6]: given inverse systems \( \mathcal{X}, \mathcal{Y} \) in \( C \), they consider suitable fibrant replacements \( \hat{\mathcal{X}}, \hat{\mathcal{Y}} \) for them obtaining that \( \text{Ho(Pro } C \text{)}(\mathcal{X}, \mathcal{Y}) \simeq [\hat{\mathcal{X}}, \hat{\mathcal{Y}}] \), where the right member denotes the set of homotopy classes with respect to the relation generated by extending to \( \text{Pro } C \) a cylinder functor given on \( C \). In this case morphisms are easy to manage while the constructions of the fibrant replacements is not trivial at all, see, e.g., [8], 3.2.3 and [6], 4.2. Our aim in this paper is to construct a category with objects the inverse systems in \( C \) having the advantages of both the points of view above.

When speaking of the category \( C \) we really have in mind the category \( \text{Top} \) of topological spaces however the construction we give works for an arbitrary ge-category \( C \), that is a category enriched over groupoids, endowed with a suitable model structure. In a previous paper [22] this author has defined the ge-category \( \text{IInv } C \) with objects the inverse systems in \( C \), coherent maps between them and modifications of such coherent maps. The homotopy category of \( \text{IInv } C \), denoted by \( \text{Pro } C \), was used in order to redefine the strong shape category of compact metric spaces. The main result of this paper consists in showing that \( \text{Pro } C \) is isomorphic to the Steenrod homotopy category \( \text{Ho}(\text{ProTop}) \) as defined in [8] and then to the coherent pro-homotopy category \( \text{CH(Top)} \) as defined by Lisica and Mardešić, see [17].

2. Background

A groupoid is a small category whose morphisms are all invertible. \( \text{Gpd} \) denotes the category of groupoids and their functors.

\( \text{Gpd} \) is a complete and cocomplete category, in particular it is a symmetric, monoidal closed category, with tensor product the usual product of categories and unit object the groupoid having only one object and one morphism. \( \text{Gpd} \) is then suitable for enriching other categories: a category \( C \) is enriched over \( \text{Gpd} \) (hereafter called a ge-category) if every hom-set \( \text{Hom}(X, Y) \) is the set of objects of a groupoid \( \text{Hom}(X, Y) \) and the compo-
inition is a functor

\[ \text{Hom}(X, Y) \times \text{Hom}(Y, Z) \rightarrow \text{Hom}(X, Z), \]

for all \( X, Y, Z \in C \). A ge-category \( C \) has objects (0-cells), maps (1-cells) and homotopies (2-cells) between them, so that it is nothing but a 2-category whose 2-cells are all invertible. As for notations, we will write

\[ \alpha : f \Rightarrow g : X \rightarrow Y \]

to mean that \( \alpha \) is a homotopy connecting the maps \( f, g : X \rightarrow Y \). A map \( f : X \rightarrow Y \) in \( C \) is called a homotopy equivalence if there exists another map \( g : Y \rightarrow X \) and homotopies \( g \circ f \Rightarrow 1_X, f \circ g \Rightarrow 1_Y \). Homotopies in \( C \) can be composed in two ways: vertically \( (\beta \cdot \alpha) \) and horizontally \( (\gamma * \alpha) \). We denote, e.g., by \( f \) both the map and the identity homotopy \( 1_f : f \Rightarrow f \).

The relation to be homotopic for maps in \( C \) is a compositive equivalence relation on each \( \text{Hom}(X, Y) \). The quotient category \( h(C) \) is called the homotopy category of the ge-category \( C \). It can also be obtained by formally inverting the class \( W \) of all the homotopy equivalences in \( C : C[W^{-1}] \cong h(C) \), [22]. A 2-functor \( F : B \rightarrow C \) of ge-categories lifts naturally to a functor \( hF : h(B) \rightarrow h(C) \) which acts on objects as \( F \) does.

**Example 2.1.**

(a) The category Top of topological spaces and continuous maps is a ge-category. Given two spaces \( X, Y \), the continuous maps between them and the tracks (= relative homotopy classes of homotopies) \([5]\) connecting such maps determine a groupoid.

(b) Gpd itself is a ge-category: the homotopies are the natural isomorphisms of functors. A functor of groupoids is a homotopy equivalence iff it is an equivalence of categories.

(c) Every ordinary category can be thought of as a ge-category having only identity homotopies.
3. The ge-category of diagrams.

Let $C$ be a fixed ge-category and let $A$ be a small, ordinary category, also considered as a ge-category. Let us denote by $[A, C]$ the ge-category of diagrams in $C$ of type $A$, that is $(2)$-functors $F : A \to C$. The maps here are the $(2)$-natural transformations of diagrams, while a homotopy is a modification of natural transformations [15].

3.1. Recall that, for diagrams $F, G : A \to C$, a pseudo-natural transformation (called a psd-transformation, for short) $\tau : F \to G$ consists of

- maps $\tau_x : F(x) \to G(x)$ in $C$, for all $x \in A$, together with
- homotopies $\tau_u : G(u)\tau_x \Rightarrow \tau_y F(u)$ in $C$, for all $u : x \to y$ in $A$, in such a way that $\tau_{1x} = 1_{\tau_x}$ and $\tau_{uv} = [\tau_u * F(u)] \cdot [G(g) * \tau_y]$, for composable maps $x \xrightarrow{u} y \xrightarrow{v} z$, as in

\[
\begin{array}{c}
F(x) \xrightarrow{\tau_x} G(x) \\
F(y) \xrightarrow{\tau_y} G(y) \\
F(z) \xrightarrow{\tau_z} G(z)
\end{array}
\]

Moreover, for a homotopy $\alpha : u \Rightarrow u' : x \to y$, one has

$\tau_u \cdot [G(\alpha) * \tau_x] = [\tau_y * F(\alpha)] \cdot \tau_{u'}$

as in

\[
\begin{array}{c}
F(x) \xrightarrow{\tau_x} G(x) \\
F(y) \xrightarrow{\tau_y} G(y) \\
F(z) \xrightarrow{\tau_z} G(z)
\end{array}
\]

3.2. Given psd-transformations $\sigma, \tau : F \to G$ a homotopy (modification) $\theta : \sigma \Rightarrow \tau$ consists of homotopies $\theta_x : \sigma_x \Rightarrow \tau_x$, for $x \in A$, such that, given
$u : x \to y$, then

$$\tau_u \cdot [G(u) \ast \theta_x] = [\theta_y \ast F(u)] \cdot \sigma_u,$$

as in

\[
\begin{array}{ccc}
F(x) & \xrightarrow{\sigma_x} & G(x) \\
\downarrow \tau_u & & \downarrow \omega_u \\
F(y) & \xrightarrow{\tau_y} & G(y)
\end{array}
\quad \begin{array}{ccc}
F(x) & \xrightarrow{\sigma_x} & G(x) \\
\downarrow \tau_u & & \downarrow \omega_u \\
F(y) & \xrightarrow{\tau_y} & G(y)
\end{array}
\]

3.3. A natural transformation or psd-transformation $\tau : F \to G$ of diagrams is called a *level equivalence* when, for each $x \in A$, the map $\tau_x : F(x) \to G(x)$ is a homotopy equivalence in $C$.

3.4. Diagrams, psd-transformations and their homotopies define the ge-category $[A, C]$. Since every natural transformation of diagrams is a psd-transformation, it follows that $[A, C]$ is a ge-subcategory of $[A, C]$. The inclusion 2-functor $J : [A, C] \to [A, C]$ has a left 2-adjoint ([4], [10]) usually denoted $(-)' : [A, C] \to [A, C], \ F \mapsto F'$.

$F'$ is called the *flexible* or cofibrant replacement of the diagram $F$. The unit $p$ of the 2-adjunction is levelwise given by pseudo-natural transformations $p_r : F \to F'$, while the components of the counit $q$ are natural transformations $q_r : F' \to F$. It follows from the general theory of 2-monads ([3], §4) that the pseudo-natural transformations $p_r$ and the natural transformations $q_r$ form an adjoint equivalence. In particular, one has $q_r p_r = 1_r$ and there are homotopies $\theta_r : p_r q_r \Rightarrow 1_r$ providing the counit of the adjoint equivalence.

3.5. In [10], 3.2.3, it is shown that, for each diagram $F$, $q_r : F' \to F$ is a levelwise trivial fibration in the projective model structure on $[A, C]$, for $C$ a model category. In particular $q_r$ is a level homotopy equivalence in $C$. 

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3.6. A diagram $F : A \to C$ is flexible when $q_F : F' \to F$ is a surjective equivalence in $[A, C]$ (see [4], and [16], 5.13). It follows from ([4], Theorem 4.7), that every psd-transformation $F \to G$, with $F$ a flexible diagram, is homotopic to a unique natural transformation.

Given 2-categories $C$ and $D$, it is not true in general that a pseudo-natural transformation $\tau : F \to G : C \to D$ is always homotopic to a 2-natural transformation. A nice counterexample for this fact can be found in [21].

In general, a level equivalence is not a homotopy equivalence in $[A, C]$ (see e.g. [8], 2.5). However the following is known.

**Proposition 3.7.** Let $A$, $C$ be ge-categories. A level equivalence $\tau : F \to G : A \to C$ in $[A, C]$ becomes a homotopy equivalence in $[A, C]$.

**Proof.** Assume that each $\tau_x$ is a homotopy equivalence with homotopy inverse $\sigma_x : G(x) \to F(x)$ and homotopies $\eta_x : \sigma_x \tau_x \Rightarrow 1_{F(x)}$. For $f : x \to y$ in $C$ define $\sigma_f : F(f) \sigma_{F(x)} \Rightarrow \sigma_{F(y)} G(f)$ so that the homotopy represented by the following diagram is the identity homotopy at $F(f)$:

\[
\begin{array}{ccc}
F(x) & \xrightarrow{\tau_x} & G(x) \\
\downarrow F(f) & & \downarrow G(f) \\
F(y) & \xrightarrow{\tau_y} & G(y)
\end{array}
\]

That is $1_{F(f)} = [\theta_y * F(f)] \cdot [\sigma_y * \tau_f] \cdot [\sigma_f * \tau_x] \cdot [F(f) * \theta_x^{-1}]$. From which it follows $\sigma_f = [\sigma_y * \tau_f]^{-1} \cdot [\theta_y * F(f)]^{-1} \cdot [F(f) * \theta_x^{-1}]^{-1} \cdot \sigma_y$. The converse is clear.

4. The ge-category of Inverse Systems.

4.1. An inverse system in a ge-category $C$ is a diagram $X : \Lambda^{op} \to C$, with $(\Lambda, \leq)$ a cofinite, strongly directed set. We often write explicitly $X = (X_\lambda, x_{\lambda\lambda'}, \Lambda)$, where $X(\lambda) = X_\lambda$ and $X(\lambda \leq \lambda') = x_{\lambda\lambda'} : X_\lambda \to X_{\lambda'}$, [8], [18].
If \( f : M \to \Lambda \) is an increasing map of directed sets, then there is an inverse system \( X_f = Xf^{\text{op}} : M^{\text{op}} \to C \), given by \( X_f = (X_f(\mu), x_f(\mu) f(\mu'), M) \). Here \( M = (M, \leq) \) and \( f \) is considered as a functor.

4.2. Let \( X = (X_\lambda, x_{\lambda\lambda'}, \Lambda) \) and \( Y = (Y_\mu, y_{\mu\mu'}, M) \) be inverse systems in \( C \).

A map of systems \( \xi = (f, f_\mu) : X \to Y \) consists of
- an increasing map \( f : M \to \Lambda \),
- a natural transformation \( (f_\mu) : X_f \to Y_f \).

If \( Z = (Z_\nu, r_{\nu\nu'}, N) \) is another inverse system and \( g = (g, g_\nu) : Y \to Z \) is another map of systems, the composition \( g \xi : X \to Z \) is the map \( (fg, g_\nu f(g(\nu)) \), while the identity map on \( X \) is given by \( (1_\Lambda, 1_{X_\lambda}) \).

Let \( f = (f, f_\mu) : X \to Y \) be a map of systems and let \( F : M \to \Lambda \) be an increasing map such that \( f \leq F \), that is \( f(\mu) \leq F(\mu) \), for all \( \mu \in M \).

The shift of \( f \) by \( F \) is the map of systems \( \xi F = (F, f_\mu) : X \to Y \), where \( f_\mu = \xi F f(\mu) \).

4.3. Given two maps of systems \( \xi, \xi' : X \to Y \), a homotopy \( \chi : \xi \Rightarrow \xi' \) consists of an increasing map \( F : \Lambda \to M \), \( F \geq f, f' \), and of a usual modification of natural transformations \( \chi : (\xi F) \Rightarrow (\xi' F) \).

Two maps of systems \( \xi, \xi' : X \to Y \) are said to be congruent if they admit a common shift. Congruences of maps of systems are trivial modifications, so we can form the ge-category \( \text{Inv} C \) whose objects, maps and homotopies are inverse systems, maps of systems and their congruences, respectively. The resulting homotopy category of \( \text{Inv} C \) is Grothendieck’s category \( \text{Pro} C \) of inverse systems in \( C \) [11].

\( \text{Inv} C \) is actually a ge-category whose constituent bricks are the ge-categories of diagrams [\( \Lambda^{\text{op}}, C \)], for \( (\Lambda, \leq) \) a cofinite, strongly directed set.

Changing \( [\Lambda^{\text{op}}, C] \) to \( [\Lambda^{\text{op}}, C] \) leads to :

4.4. A coherent map of inverse systems \( \varphi = (f, f_\mu, f_{\mu\mu}) : X \to Y \) consists of:
- an increasing map \( f : M \to \Lambda \),
- a psd-transformation \( (f_\mu, f_{\mu\mu}) : X_f \to Y \).
Let $\psi = (g, g', g''') : Y \to Z = (Z, z_{\nu'}, N)$ be another coherent map. The composition $\psi \varphi$ is the coherent map given by $(f g, g_{\nu} f(g(\nu)), g_{\nu'} * f(g(\nu')))$. Such a composition is indeed associative and the identity coherent map $X \to X$ is given by $1_X = (1_\Lambda; 1_{X_{\lambda}}, 1_{f_{\lambda}})$.

4.5. Let $\varphi = (f; f_\mu, f_{\mu'}) : X \to Y$ and let $F : M \to \Lambda$ be an increasing map such that $f \leq F$. The coherent shift of $\varphi$ by $F$ is the coherent map $\bar{\varphi} = (F; \bar{f}_\mu, \bar{f}_{\mu'}) : X \to Y$ which is given by $\bar{f}_\mu = f_\mu f(f(\mu))F(\mu)$ and $\bar{f}_{\mu'} = f_{\mu'} * f(f(\mu'))F(\mu')$.

If $\varphi' = (f', f'_\mu, f'_{\mu'})$ is another coherent map $X \to Y$, a coherent homotopy $\Phi : \varphi \Rightarrow \varphi'$ consists of:

- an increasing map $F : M \to \Lambda$ such that $f, f' \leq F$;
- a homotopy of psd-transformations $\Phi : (\bar{f}_\mu, \bar{f}_{\mu'}) \Rightarrow (\bar{f}'_\mu, \bar{f}'_{\mu'}) : X_F \to Y$, between their coherent shifts by $F$. It follows that $\Phi$ is family of homotopies of $C$

$$\phi_{\mu} : \bar{f}_\mu x(f(\mu))F(\mu) \Rightarrow g_\mu x(g(\mu))F(\mu), \mu \in M,$$

such that $(g_{\mu\mu'} * x(f(\mu))g(\mu')) \cdot (y_{\mu\mu'} * \phi_{\mu'}) = (\phi_{\mu} * x(f(\mu))F(\mu')) \cdot (f_{\mu'} * f(f(\mu'))F(\mu')).$

4.6. The data above define the ge-category $\text{Inv} \ C$ with objects the inverse systems in $C$, coherent maps and their coherent homotopies. We define the coherent category of inverse systems in $C$ to be $h(\text{Inv} \ C) = \text{Pro} \ C$.

If $X$ and $Y$ are indexed over the same set $\Lambda$, then a map of systems $(1_\Lambda, f_\lambda) : X \to Y$ is natural transformation while a coherent map of systems $(1_\Lambda, f_\lambda, f_{\lambda'}) : X \to Y$ is a psd-transformation, We call such maps level (coherent) maps of systems.

4.7. Recall from [18] that every map of systems $(f, f_\mu) : X \to Y$, with $f : M \to \Lambda$, is isomorphic, in the category of maps of $\text{Inv} \ C$, to a level map $(1_N, f_\nu) : X' \to Y'$ where

$$N = \{\nu = (\lambda, \mu) \in \Lambda \times M \mid f(\mu) \leq \lambda\}$$

is directed by the relation

$$\nu = (\lambda, \mu) \leq (\lambda', \mu') \iff \lambda \leq \lambda' \text{ and } \mu \leq \mu',$$

is directed by the relation
with $f_\nu = f_\mu x(f(\mu)_\lambda)$. This is the so called Mardešić's trick, which admits a coherent version as follows.

Starting from a coherent map of systems $(f, f_\mu, f_{\nu\nu'}) : X \to Y$ one obtains a level coherent map of systems

$$(1, f_\nu, f_{\nu\nu'}) : X' \to Y'$$

where $f_\nu = f_\mu \circ x(f(\mu)_\lambda)$ and $f_{\nu\nu'}$ is the homotopy represented by

$$
\begin{align*}
X_{\nu'} = X_{\lambda'} x(f(\nu'))_{\lambda'} & \to X_{f(\mu')} f_{\nu'} \\
\downarrow x_{\lambda'} & \downarrow x_{(f(\nu))(\nu')} \\
X_{\nu} = X_{\lambda} x(f(\nu)) & \to X_{f(\mu)} f_{\mu}
\end{align*}
$$

Then there is a commutative square in $\text{Inv } C$

$$
\begin{array}{ccc}
X & (f, f_\mu, f_{\nu\nu'}) & Y \\
\downarrow (i, i_\nu) & \downarrow (j, j_\nu) & \downarrow \\
X' & (1, f_\nu, f_{\nu\nu'}) & Y'
\end{array}
$$

where $(i, i_\nu)$ and $(j, j_\nu)$ are isomorphisms of systems given by $i : N \to \Lambda$, $i(\nu) = \lambda$ and $i_\nu = 1_{X_{\lambda}}$, $j : N \to M$, $j(\nu) = \mu$ and $j_\nu = 1_{Y_{\mu}}$.

4.8. Edwards-Hastings [8] consider a nicely behaved model category $C$ satisfying a certain condition "N" which provides, among other things, the existence of a functorial cylinder. They define a model structure in $\text{Pro } C$ where the weak equivalences and the cofibrations are defined to be retracts in the category of maps of $\text{Pro } C$ of level equivalences and of level Hurewicz cofibrations from some $[\Lambda^\infty, C]$, respectively. The Steenrod homotopy category of inverse systems $\text{Ho}(\text{Pro } C)$ is obtained by localizing $\text{Pro } C$ at the class of level homotopy equivalences (see also [20]). An equivalent description of $\text{Ho}(\text{Pro } C)$ is given in [6], let us recall it briefly. First extend the cylinder functor given on $C$ to $\text{Pro } C$ : for $X = (X_\lambda, x_{\lambda\lambda'}, \Lambda)$, let $X \times I = (X_\lambda \times I, x_{\lambda\lambda'} \times 1, \Lambda)$. Two maps of systems $f = (f, f_\mu)$, $g = (g, g_\mu) : X \to Y$ are declared naive homotopic if there exists a map of systems $F = (F, F_\mu) : X \times I \to Y$, where $F : M \to \Lambda$ is an increasing map.
such that $F \geq f, g$ and, for each $\mu \in M$, $F_\mu : X_{f(\mu)} \times I \to Y_{g(\mu)}$ is a homotopy in $\mathcal{C}$ connecting $f_\mu \circ x_{f(\mu)} F(\mu)$ and $g_\mu \circ x_{g(\mu)} F(\mu)$. The resulting quotient category is denoted $\pi(\text{Pro } \mathcal{C})$ and is called the naive homotopy category. We write $[X, Y]$ for the set of naive homotopy classes of maps $X \to Y$. If $\pi(\text{Pro } \mathcal{C})_f$ denotes the full subcategory of all fibrant objects in the previous model structure, then there is a reflective functor

$$F : \pi(\text{Pro } \mathcal{C}) \to \pi(\text{Pro } \mathcal{C})_f, \quad X \mapsto \hat{X},$$

with unit of adjunction $i_X : X \to \hat{X}$ a level trivial cofibration. The main result is that there is a natural bijection

$$\text{Ho}(\text{Pro } \mathcal{C})(X, Y) \cong [\hat{X}, \hat{Y}],$$

which exhibits $\text{Ho}(\text{Pro } \mathcal{C})$ as the full image of the functor $F$.

We note that two coherent maps of systems that are coherently homotopic are also naive homotopic.

$\text{Inv } \mathcal{C}$ is a ge-subcategory of $\text{Inv } \mathcal{C}$ and the inclusion 2-functor $\text{Inv } \mathcal{C} \to \text{Inv } \mathcal{C}$ lifts to the homotopy categories as $I : \text{Pro } \mathcal{C} \to \text{Pro } \mathcal{C}$. Since level homotopy equivalences in $\text{Pro } \mathcal{C}$ become homotopy equivalences in $\text{Pro } \mathcal{C}$, then the inclusion 2-functor $I$ takes level homotopy equivalences to isomorphisms. It follows [2] that there exists a unique functor $U : \text{Ho}(\text{Pro } \mathcal{C}) \to \text{Pro } \mathcal{C}$ making the following diagram

$$\text{Pro } \mathcal{C} \xrightarrow{p_E} \text{Ho}(\text{Pro } \mathcal{C}) \xrightarrow{U} \text{Pro } \mathcal{C},$$

commutative, where $p_E$ is the localization functor.

**Theorem 4.9.** The functor $U : \text{Ho}(\text{Pro } \mathcal{C}) \to \text{Pro } \mathcal{C}$ is an isomorphism of categories.

**Proof.** Let us note first that all functors involved in the above diagram are identical on objects. Let now $\varphi : X \to Y$ be a coherent map of systems.
By (4.7) we can assume that \( \varphi \) is actually a psd-transformation between systems indexed over the same directed set. By (3.6) there is a unique natural transformation \( \varphi' : X' \to Y \) which is homotopic in \( \text{Pro } C \) to the composition

\[
X' \xrightarrow{q} X \xrightarrow{\varphi} Y
\]

Recall (3.5) that \( q_2 \) is a level homotopy equivalence, then in \( \text{Ho}(\text{Pro } C) \) consider the morphism

\[
[X \xleftarrow{q} X' \xrightarrow{\varphi'} Y]
\]

It follows that (see [2], A.4)

\[
U(\varphi'(q_2)^{-1}) = U(\varphi')U(q_2)^{-1} = I(\varphi')I(q_2)^{-1} = \varphi'p_2 : X \to Y
\]

and it is clear that \( \varphi'p_2 \) is homotopic to \( \varphi \) in \( \text{Inv } C \), so that they give the same morphism in \( \text{Pro } C \), hence the functor \( U \) is full. Let now \( \phi, \psi : X \to Y \) be two morphisms in \( \text{Ho}(\text{Pro } C) \). We may assume without loss of generality (4.8) that they correspond to homotopy classes \( \phi = [f] \), \( \psi = [g] \) of maps of systems \( f, g : X \to Y \). Then, assuming that \( U(\phi) = U(\psi) \) amounts to assume that \( f \) and \( g \) are coherently homotopic maps of systems. This means that there is an increasing map \( F : M \to \Lambda \) and a family of homotopies

\[
F_\mu : f_\mu x_{f(\mu)F(\mu)} \Rightarrow g_\mu x_{g(\mu)F(\mu)}
\]

in the ge-category \( C \), such that \( F_\mu x_{F(\mu)F(\mu')} = y_{\mu \mu'}F_{\mu'} \), for \( \mu \leq \mu' \). It follows that \( F = (F, F_\mu) : X \times I \to Y \) is a naive homotopy connecting \( f \) and \( g \), thus \( U \) is also a faithful functor.

Let us note that \( \text{Pro } \text{Top} \) is also isomorphic to the coherent pro-homotopy category of Lisića and Mardešić \( CH(\text{Pro } \text{Top}) \), see [17], Theorem 4.3.8.

References


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