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THE COHERENT CATEGORY OF INVERSE SYSTEMS

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Résumé. Pour toute catégorie de modèles **C** enrichie dans la catégorie des groupoïdes **Gpd**, on définit une nouvelle catégorie $\mathbb{P}$ro **C**, dont les objets sont les systèmes inverses dans **C**; elle est isomorphe à la catégorie d'homotopie de Steenrod Ho(Pro **C**), et à la catégorie de pro-homotopie cohérente définie par Lisica and Mardešić lorsque **C** est la catégorie des espaces topologiques.

Abstract. For every model category **C** enriched over the category **Gpd** of groupoids a new category $\mathbb{P}$ro **C** is defined, with objects the inverse systems in **C**, which is isomorphic to the Steenrod homotopy category Ho(Pro **C**) and to the coherent pro-homotopy category defined by Lisica and Mardešić when **C** is the category of topological spaces

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1. Introduction

Inverse systems have been widely used in Mathematics, especially in Topology. Grothendieck (see[11]) was the first to give a good categorical definition for the category Pro **C** of inverse systems in a given category **C**. The need for a homotopy theory of Pro **C** was recognized in [1], however, the homotopy category defined there was not satisfactory for a number of reasons. Many authors were then concerned with the task of defining a Quillen model structure on Pro **C**, assuming **C** had one, in order to obtain a well behaved homotopy category. The so called Steenrod homotopy category Ho(Pro **C**) was defined by Porter in [19] (see also [20]). In the last years further work on the subject has been done notably by Isaksen, see for instance [13], [14] and the very recent paper by Descotte and Dubuc [7].

There are at hand essentially two ways to look at Ho(Pro **C**). The first one is due to Edwards-Hastings [8], who define it by localizing Pro **C** at the class of level equivalences so that in this case the morphisms are quite ugly to handle. The second one is due to Cathey-Segal [6]: given inverse systems $\mathtt{X}, \mathtt{Y}$ in **C**, they consider suitable fibrant replacements $\hat{\mathtt{X}}, \hat{\mathtt{Y}}$ for them obtaining that $\mathsf{Ho}(\mathsf{Pro}\ \mathbf{C})(\mathtt{X},\mathtt{Y}) \cong [\hat{\mathtt{X}}, \hat{\mathtt{Y}}]$, where the right member denotes the set of homotopy classes with respect to the relation generated by extending to Pro **C** a cylinder functor given on **C**. In this case morphisms are easy to manage while the constructions of the fibrant replacements is not trivial at all, see, e.g., [8], 3.2.3 and [6], 4.2. Our aim in this paper is to construct a category with objects the inverse systems in **C** having the advantages of both the points of view above.
When speaking of the category **C** we really have in mind the category **Top** of topological spaces however the construction we give works for an arbitrary ge-category **C**, that is a category enriched over groupoids, endowed with a suitable model structure. In a previous paper [22] this author has defined the ge-category $\mathbb{I}$nv **C** with objects the inverse systems in **C**, coherent maps between them and modifications of such coherent maps. The homotopy category of $\mathbb{I}$nv **C** , denoted by $\mathbb{P}$ro **C** , was used in order to redefine the strong shape category of compact metric spaces. The main result of this paper consists in showing that $\mathbb{P}$ro **C** is isomorphic to the Steenrod homotopy category Ho(Pro**Top**) as defined in [8] and then to the coherent pro-homotopy category CH(**Top**) as defined by Lisica and Mardešić, see [17].

2. Background

A groupoid is a small category whose morphisms are all invertible. **Gpd** denotes the category of groupoids and their functors.

Gpd is a complete and cocomplete category, in particular it is a symmetric, monoidal closed category, with tensor product the usual product of categories and unit object the groupoid having only one object and one morphism. **Gpd** is then suitable for enriching other categories: a category **C** is enriched over **Gpd** (hereafter called a *ge-category*) if every hom-set $Hom(X, Y)$ is the set of objects of a groupoid $\mathsf{Hom}(X, Y)$ and the composition is a functor

$$\mathsf{Hom}(X,Y) \times \mathsf{Hom}(Y,Z) \longrightarrow \mathsf{Hom}(X,Z),$$

for all $X, Y, Z \in \mathbf{C}$. A ge-category $\mathbf{C}$ has *objects* (0-cells), *maps* (1-cells) and *homotopies* (2-cells) between them, so that it is nothing but a 2-category whose 2-cells are all invertible. As for notations, we will write

$$\alpha : f \Rightarrow g : X \longrightarrow Y$$

to mean that α is a homotopy connecting the maps $f, g : X \longrightarrow Y$. A map $f : X \longrightarrow Y$ in $\mathbf{C}$ is called a *homotopy equivalence* if there exists another map $g : Y \longrightarrow X$ and homotopies $g \circ f \Rightarrow 1_X$, $f \circ g \Rightarrow 1_Y$. Homotopies in $\mathbf{C}$ can be composed in two ways : vertically $(\beta \cdot \alpha)$ and horizontally $(\gamma * \alpha)$. We denote, e.g., by f both the map and the identity homotopy $1_f : f \Rightarrow f$.

The relation to be homotopic for maps in $\mathbf{C}$ is a compositive equivalence relation on each $Hom(X,Y)$. The quotient category $\mathsf{h}(\mathbf{C})$ is called the *homotopy category* of the ge-category $\mathbf{C}$. It can also be obtained by formally inverting the class $\mathtt{W}$ of all the homotopy equivalences in $\mathbf{C}$: $\mathbf{C}[\mathtt{W}^{-1}] \cong \mathsf{h}(\mathbf{C})$, [22]. A 2-functor $\mathtt{F} : \mathbf{B} \longrightarrow \mathbf{C}$ of ge-categories lifts naturally to a functor $\mathsf{h}\mathtt{F} : \mathsf{h}(\mathbf{B}) \longrightarrow \mathsf{h}(\mathbf{C})$ which acts on objects as $\mathtt{F}$ does.

Example 2.1.

(a) The category **Top** of topological spaces and continuous maps is a ge-category. Given two spaces X, Y, the continuous maps between them and the tracks (= relative homotopy classes of homotopies) [5] connecting such maps determine a groupoid.

(b) **Gpd** itself is a ge-category: the homotopies are the natural isomorphisms of functors. A functor of groupoids is a homotopy equivalence iff it is an equivalence of categories.

(c) Every ordinary category can be thought of as a ge-category having only identity homotopies.

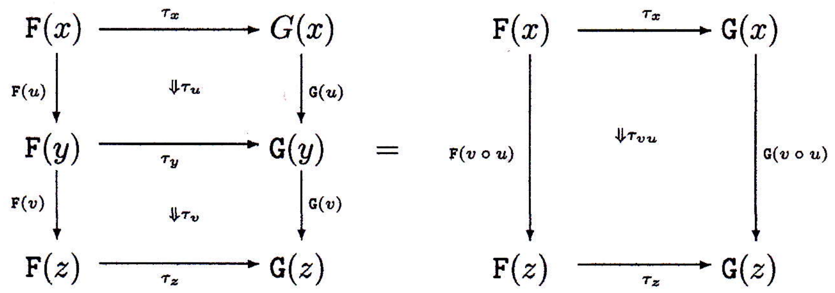
3. The ge-category of diagrams.

Let $\mathbf{C}$ be a fixed ge-category and let $\mathcal{A}$ be a small, ordinary category, also considered as a ge-category. Let us denote by $[\mathcal{A}, \mathbf{C}]$ the ge-category of diagrams in $\mathbf{C}$ of type $\mathcal{A}$, that is (2-)functors $\mathtt{F} : \mathcal{A} \to \mathbf{C}$. The maps here are the (2-)natural transformations of diagrams, while a homotopy is a modification of natural transformations [15].

3.1. Recall that, for diagrams $\mathtt{F}, \mathtt{G} : \mathcal{A} \to \mathbf{C}$, a pseudo-natural transformation (called a *psd-transformation*, for short) $\tau : \mathtt{F} \to \mathtt{G}$ consists of

- maps $\tau_x : \mathtt{F}(x) \to \mathtt{G}(x)$ in $\mathbf{C}$, for all $x \in \mathcal{A}$, together with

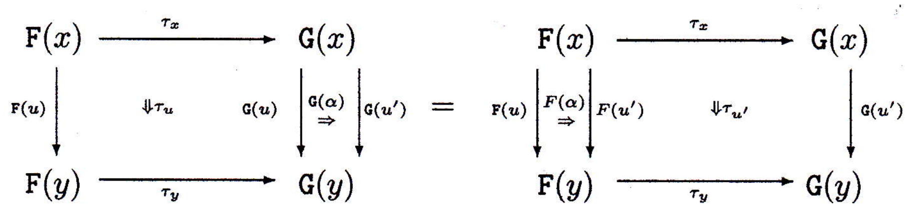
- homotopies $\tau_u : \mathtt{G}(u)\tau_x \Rightarrow \tau_y\mathtt{F}(u)$ in $\mathbf{C}$, for all $u : x \to y$ in $\mathcal{A}$, in such a way that $\tau_{1_x} = 1_{\tau_x}$ and $\tau_{vu} = [\tau_v * \mathtt{F}(u)] \cdot [\mathtt{G}(g) * \tau_u]$, for composable maps $x \xrightarrow{u} y \xrightarrow{v} z$, as in



Moreover, for a homotopy $\alpha : u \Rightarrow u' : x \to y$, one has

$$\tau_u \cdot [\mathtt{G}(\alpha) * \tau_x] = [\tau_y * \mathtt{F}(\alpha)] \cdot \tau_{u'}$$

as in

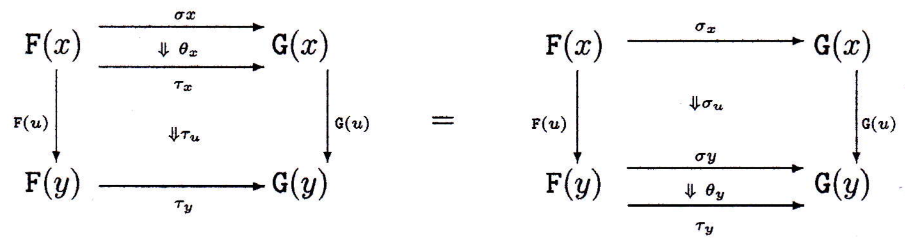


3.2. Given psd-transformations $\sigma, \tau : \mathtt{F} \to \mathtt{G}$ a homotopy (modification) $\theta : \sigma \Rightarrow \tau$ consists of homotopies $\theta_x : \sigma_x \Rightarrow \tau_x$, for $x \in \mathcal{A}$, such that, given

$u : x \longrightarrow y$, then

$$\tau_u \cdot [\mathtt{G}(u) * \theta_x] = [\theta_y * \mathtt{F}(u)] \cdot \sigma_u,$$

as in

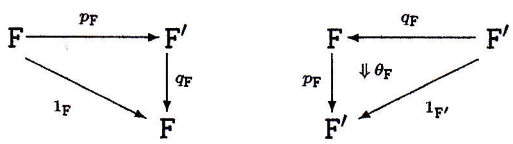


3.3. A natural transformation or psd-transformation $\tau : \mathtt{F} \longrightarrow \mathtt{G}$ of diagrams is called a *level equivalence* when, for each $x \in \mathcal{A}$, the map $\tau_x : \mathtt{F}(x) \longrightarrow \mathtt{G}(x)$ is a homotopy equivalence in $\mathbf{C}$.

3.4. Diagrams, psd-transformations and their homotopies define the ge-category $[\![\mathcal{A}, \mathbf{C}]\!]$. Since every natural transformation of diagrams is a psd-transformation, it follows that $[\mathcal{A}, \mathbf{C}]$ is a ge-subcategory of $[\![\mathcal{A}, \mathbf{C}]\!]$. The inclusion 2-functor $\mathrm{J} : [\mathcal{A}, \mathbf{C}] \longrightarrow [\![\mathcal{A}, \mathbf{C}]\!]$ has a left 2-adjoint ([4], [10]) usually denoted

$$(-)' : [\![\mathcal{A}, \mathbf{C}]\!] \longrightarrow [\mathcal{A}, \mathbf{C}], \quad \mathtt{F} \mapsto \mathtt{F}'.$$

$\mathtt{F}'$ is called the *flexible* or *cofibrant* replacement of the diagram $\mathtt{F}$. The unit p of the 2-adjunction is levelwise given by pseudo-natural transformations $p_{\mathtt{F}} : \mathtt{F} \longrightarrow \mathtt{F}'$, while the components of the counit q are natural transformations $q_{\mathtt{F}} : \mathtt{F}' \longrightarrow \mathtt{F}$. It follows from the general theory of 2-monads ([3], §4) that the pseudo-natural transformations $p_{\mathtt{F}}$ and the natural transformations $q_{\mathtt{F}}$ form an adjoint equivalence. In particular, one has $q_{\mathtt{F}} p_{\mathtt{F}} = 1_{\mathtt{F}}$ and there are homotopies $\theta_{\mathtt{F}} : p_{\mathtt{F}} q_{\mathtt{F}} \Rightarrow 1_{\mathtt{F}}$ providing the counit of the adjoint equivalence.



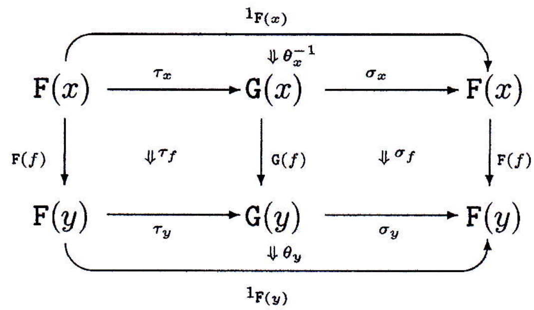
3.5. In [10], 3.2.3, its shown that, for each diagram $\mathtt{F}$, $q_{\mathtt{F}} : \mathtt{F}' \longrightarrow \mathtt{F}$ is a levelwise trivial fibration in the projective model structure on $[\mathcal{A}, \mathbf{C}]$, for $\mathbf{C}$ a model category. In particular $q_{\mathtt{F}}$ is a level homotopy equivalence in $\mathbf{C}$.

3.6. A diagram $\mathtt{F} : \mathcal{A} \to \mathbf{C}$ is flexible when $q_{\mathtt{F}} : \mathtt{F}' \to \mathtt{F}$ is a surjective equivalence in $[\mathcal{A}, \mathbf{C}]$ (see [4], and [16], 5.13). It follows from ([4], Theorem 4.7), that every psd-transformation $\mathtt{F} \to \mathtt{G}$, with $\mathtt{F}$ a flexible diagram, is homotopic to a unique natural transformation.
Given 2-categories $\mathbf{C}$ and $\mathbf{D}$, is not true in general that a pseudo-natural transformation $\tau : \mathtt{F} \to \mathtt{G} : \mathbf{C} \to \mathbf{D}$ is always homotopic to a 2-natural transformation. A nice counterexample for this fact can be found in [21].

In general, a level equivalence is not a homotopy equivalence in $[\mathcal{A}, \mathbf{C}]$ (see e.g. [8], 2.5). However the following is known.

Proposition 3.7. *Let $\mathcal{A}, \mathbf{C}$ be ge-categories. A level equivalence $\tau : \mathtt{F} \to \mathtt{G} : \mathcal{A} \to \mathbf{C}$ in $[\mathcal{A}, \mathbf{C}]$ becomes a homotopy equivalence in $[\![\mathcal{A}, \mathbf{C}]\!]$.*

Proof. Assume that each τ_x is a homotopy equivalence with homotopy inverse $\sigma_x : \mathtt{G}(x) \to \mathtt{F}(x)$ and homotopies $\eta_x : \sigma_x \tau_x \Rightarrow 1_{F(x)}$. For $f : x \to y$ in $\mathbf{C}$ define $\sigma_f : \mathtt{F}(f)\sigma_{F(x)} \Rightarrow \sigma_{F(y)}\mathtt{G}(f)$ so that the homotopy represented by the following diagram is the identity homotopy at $\mathtt{F}(f)$:



that is : $1_{\mathtt{F}(f)} = [\theta_y * \mathtt{F}(f)] \cdot [\sigma_y * \tau_f] \cdot [\sigma_f * \tau_x] \cdot [\mathtt{F}(f) * \theta_x^{-1}]$. From which it follows $\sigma_f = [\sigma_y * \tau_f]^{-1} \cdot [\theta_y * \mathtt{F}(f)]^{-1} \cdot [\mathtt{F}(f) * \theta_x^{-1}]^{-1} * \sigma_y$. The converse is clear. □

4. The ge-category of Inverse Systems.

4.1. An *inverse system* in a ge-category $\mathbf{C}$ is a diagram $\mathtt{X} : \Lambda^{op} \to \mathbf{C}$, with $(\Lambda, \leq)$ a cofinite, strongly directed set. We often write explicitly $\mathtt{X} = (X_\lambda, x_{\lambda\lambda'}, \Lambda)$, where $\mathtt{X}(\lambda) = X_\lambda$ and $\mathtt{X}(\lambda \leq \lambda') = x_{\lambda\lambda'} : X_{\lambda'} \to X_\lambda$, [8], [18].

If $f : M \to \Lambda$ is an increasing map of directed sets, then there is an inverse system $\mathtt{X}_f = \mathtt{X}f^{op} : M^{op} \to \mathbf{C}$, given by $\mathtt{X}_f = (X_{f(\mu)}, x_{f(\mu)f(\mu')}, M)$. Here $M = (M, \leq)$ and f is considered as a functor.

4.2. Let $\mathtt{X} = (X_\lambda, x_{\lambda\lambda'}, \Lambda)$ and $\mathtt{Y} = (Y_\mu, y_{\mu,\mu'}, M)$ be inverse systems in $\mathbf{C}$. A *map of systems* $\mathtt{f} = (f, f_\mu) : \mathtt{X} \to \mathtt{Y}$ consists of

- an increasing map $f : M \to \Lambda$,
- a natural transformation $(f_\mu) : \mathtt{X}_f \to \mathtt{Y}$.

If $\mathtt{Z} = (Z_\nu, r_{\nu\nu'}, N)$ is another inverse system and $\mathtt{g} = (g, g_\nu) : \mathtt{Y} \to \mathtt{Z}$ is another map of systems, the composition $\mathtt{gf} : \mathtt{X} \to \mathtt{Z}$ is the map $(fg, g_\nu f_{g(\nu)})$, while the identity map on $\mathtt{X}$ is given by $(1_\Lambda, 1_{X_\lambda})$.

Let $\mathtt{f} = (f, f_\mu) : \mathtt{X} \to \mathtt{Y}$ be a map of systems and let $F : M \to \Lambda$ be an increasing map such that $f \leq F$, that is $f(\mu) \leq F(\mu)$, for all $\mu \in M$. The *shift* of $\mathtt{f}$ by F is the map of systems $\overline{\mathtt{f}} = (F, \overline{f}_\mu) : \mathtt{X} \to \mathtt{Y}$, where $\overline{f}_\mu = f_\mu x_{F(\mu)f(\mu)}$.

4.3. Given two maps of systems $\mathtt{f}, \mathtt{f}' : \mathtt{X} \to \mathtt{Y}$, a homotopy $\chi : \mathtt{f} \Rightarrow \mathtt{f}'$ consists of an increasing map $F : \Lambda \to M$, $F \geq f, f'$, and of a usual modification of natural transformations $\chi : (\overline{f}_\mu) \rightsquigarrow (\overline{f'}_\mu)$.
Two maps of systems $\mathtt{f}, \mathtt{f}' : \mathtt{X} \to \mathtt{Y}$ are said to be *congruent* if they admit a common shift. Congruences of maps of systems are trivial modifications, so we can form the ge-category $\mathsf{Inv}\,\mathbf{C}$ whose objects, maps and homotopies are inverse systems, maps of systems and their congruences, respectively. The resulting homotopy category of $\mathsf{Inv}\,\mathbf{C}$ is Grothendieck's category $\mathsf{Pro}\,\mathbf{C}$ of inverse systems in $\mathbf{C}$ [11].

$\mathsf{Inv}\,\mathbf{C}$ is actually a ge-category whose constituent bricks are the ge-categories of diagrams $[\Lambda^{op}, \mathbf{C}]$, for $(\Lambda, \leq)$ a cofinite, strongly directed set. Changing $[\Lambda^{op}, \mathbf{C}]$ to $[\![\Lambda^{op}, \mathbf{C}]\!]$ leads to :

4.4. A *coherent map* of inverse systems $\varphi = (f, f_\mu, f_{\mu\mu}) : \mathtt{X} \to \mathtt{Y}$ consists of:

- an increasing map $f : M \to \Lambda$,
- a psd-transformation $(f_\mu, f_{\mu\mu'}) : \mathtt{X}_f \to \mathtt{Y}$.

Let $\psi = (g, g_\nu, g_{\nu\nu'}) : \mathtt{Y} \to \mathtt{Z} = (Z_\nu, z_{\nu\nu'}, N)$ be another coherent map. The composition $\psi\varphi$ is the coherent map given by $(fg, g_\nu f_{g(\nu)}, g_{\nu\nu'} * f_{g(\nu)g(\nu')})$. Such a composition is indeed associative and the identity coherent map $\mathtt{X} \to \mathtt{X}$ is given by $1_\mathtt{X} = (1_\Lambda; 1_{X_\lambda}, 1_{f_\lambda})$.

4.5. Let $\varphi = (f; f_\mu, f_{\mu\mu'}) : \mathtt{X} \to \mathtt{Y}$ and let $F : M \to \Lambda$ be an increasing map such that $f \leq F$. The *coherent shift* of φ by F is the coherent map $\overline{\varphi} = (F; \overline{f}_\mu, \overline{f}_{\mu\mu'}) : \mathtt{X} \to \mathtt{Y}$ which is given by $\overline{f}_\mu = f_\mu x_{f(\mu)F(\mu)}$ and $\overline{f}_{\mu\mu'} = f_{\mu\mu'} * x_{f(\mu')F(\mu')}$.

If $\varphi' = (f', f'_\mu, f'_{\mu\mu'})$ is another coherent map $\mathtt{X} \to \mathtt{Y}$, a *coherent homotopy* $\Phi : \varphi \Rightarrow \varphi'$ consists of:

- an increasing map $F : M \to \Lambda$ such that $f, f' \leq F$,

- a homotopy of psd-transformations $\Phi : (\overline{f}_\mu, \overline{f}_{\mu\mu'}) \Rightarrow (\overline{f}'_\mu, \overline{f}'_{\mu\mu'}) : \mathtt{X}_F \to \mathtt{Y}$, between their coherent shifts by F. It follows that Φ is family of homotopies of $\mathbf{C}$

$$\phi_\mu : f_\mu x_{f(\mu)F(\mu)} \Rightarrow g_\mu x_{g(\mu)F(\mu)}, \ \mu \in M,$$

such that $(g_{\mu\mu'} * x_{F(\mu')g(\mu')}) \cdot (y_{\mu\mu'} * \phi_{\mu'}) = (\phi_\mu * x_{F(\mu)F(\mu')}) \cdot (f_{\mu\mu'} * x_{f(\mu')F(\mu')})$.

4.6. The data above define the ge-category $\mathbb{I}\mathrm{nv}\,\mathbf{C}$ with objects the inverse systems in $\mathbf{C}$, coherent maps and their coherent homotopies. We define the *coherent category of inverse systems* in $\mathbf{C}$ to be $\mathrm{h}(\mathbb{I}\mathrm{nv}\,\mathbf{C}) = \mathbb{P}\mathrm{ro}\,\mathbf{C}$.

If $\mathtt{X}$ and $\mathtt{Y}$ are indexed over the same set Λ, then a map of systems $(1_\Lambda, f_\lambda) : \mathtt{X} \to \mathtt{Y}$ is natural transformation while a coherent map of systems $(1_\Lambda, f_\lambda, f_{\lambda\lambda'}) : \mathtt{X} \to \mathtt{Y}$ is a psd-transformation, We call such maps *level (coherent) maps* of systems.

4.7. Recall from [18] that every map of systems $(f, f_\mu) : \mathtt{X} \to \mathtt{Y}$, with $f : M \to \Lambda$, is isomorphic, in the category of maps of $\mathrm{Inv}\,\mathbf{C}$, to a level map $(1_N, f_\nu) : \mathtt{X}' \to \mathtt{Y}'$ where

$$N = \{\nu = (\lambda, \mu) \in \Lambda \times M \mid f(\mu) \leq \lambda\}$$

is directed by the relation

$$\nu = (\lambda, \mu) \leq (\lambda', \mu') = \nu' \Leftrightarrow \lambda \leq \lambda' \text{ and } \mu \leq \mu',$$

with $f_\nu = f_\mu x_{f(\mu)\lambda}$. This is the so called *Mardešić's trick*, which admits a coherent version as follows.
Starting from a coherent map of systems $(f, f_\mu, f_{\mu\mu'}) : \mathtt{X} \to \mathtt{Y}$ one obtains a level coherent map of systems

$$(1_N, f_\nu, f_{\nu\nu'}) : \mathtt{X}' \to \mathtt{Y}'$$

where $f_\nu = f_\mu \circ x_{f(\mu)\lambda}$ and $f_{\nu\nu'}$ is the homotopy represented by

$$\begin{array}{ccccc} X_{\nu'} = X_{\lambda'} & \xrightarrow{x_{f(\mu')\lambda'}} & X_{f(\mu')} & \xrightarrow{f_{\mu'}} & Y_{\mu'} = Y_{\nu'} \\ {\scriptstyle x_{\lambda\lambda'}}\downarrow & & \downarrow{\scriptstyle x_{f(\mu)f(\mu')}} & \Downarrow f_{\mu\mu'} & \downarrow{\scriptstyle y_{\mu\mu'}} \\ X_\nu = X_\lambda & \xrightarrow[x_{f(\mu)\lambda}]{} & X_{f(\mu)} & \xrightarrow[f_\mu]{} & Y_\mu = Y_\nu \end{array} .$$

Then there is a commutative square in $\mathbb{I}\mathrm{nv}\,\mathbf{C}$

$$\begin{array}{ccc} \mathtt{X} & \xrightarrow{(f, f_\mu, f_{\mu\mu'})} & \mathtt{Y} \\ {\scriptstyle (i, i_\nu)}\downarrow & & \downarrow{\scriptstyle (j, j_\nu)} \\ \mathtt{X}' & \xrightarrow[(1_N, f_\nu, f_{\nu\nu'})]{} & \mathtt{Y}' \end{array}$$

where (i, i_ν) and (j, j_ν) are isomorphisms of systems given by $i : N \to \Lambda$, $i(\nu) = \lambda$ and $i_\nu = 1_{X_\lambda}$, $j : N \to M$, $j(\nu) = \mu$ and $j_\nu = 1_{Y_\mu}$.

4.8. Edwards-Hastings [8] consider a nicely behaved model category $\mathbf{C}$ satisfying a certain condition "N" which provides, among other things, the existence of a functorial cylinder. They define a model structure in Pro $\mathbf{C}$ where the weak equivalences and the cofibrations are defined to be retracts in the category of maps of Pro $\mathbf{C}$ of level equivalences and of level Hurewicz cofibrations from some $[\Lambda^{op}, \mathbf{C}]$, respectively. The Steenrod homotopy category of inverse systems Ho(Pro $\mathbf{C}$) is obtained by localizing Pro $\mathbf{C}$ at the class of level homotopy equivalences (see also [20]). An equivalent description of Ho(Pro $\mathbf{C}$) is given in [6], let us recall it briefly. First extend the cylinder functor given on $\mathbf{C}$ to Pro $\mathbf{C}$: for $\mathtt{X} = (X_\lambda, x_{\lambda\lambda'}, \Lambda)$, let $\mathtt{X} \times I = (X_\lambda \times I, x_{\lambda\lambda'} \times 1, \Lambda)$. Two maps of systems $\mathtt{f} = (f, f_\mu)$, $\mathtt{g} = (g, g_\mu) : \mathtt{X} \to \mathtt{Y}$ are declared *naive homotopic* if there exists a map of systems $\mathtt{F} = (F, F_\mu) : \mathtt{X} \times I \to \mathtt{Y}$, where $F : M \to \Lambda$ is an increasing map such that $F \geq f, g$ and, for each $\mu \in M$, $F_\mu : X_{F(\mu)} \times I \to Y_\mu$ is a homotopy in **C** connecting $f_\mu \circ x_{f(\mu)F(\mu)}$ and $g_\mu \circ x_{g(\mu)F(\mu)}$. The resulting quotient category is denoted $\pi(\text{Pro } \mathbf{C})$ and is called the *naive homotopy category*. We write $[\mathtt{X}, \mathtt{Y}]$ for the set of naive homotopy classes of maps $\mathtt{X} \to \mathtt{Y}$. If $\pi(\text{Pro } \mathbf{C})_f$ denotes the full subcategory of all fibrant objects in the previous model structure, then there is a reflective functor

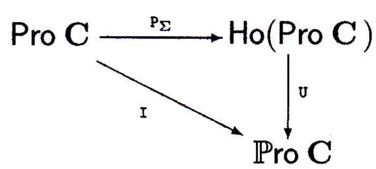
$$\mathtt{F} : \pi(\text{Pro } \mathbf{C}) \to \pi(\text{Pro } \mathbf{C})_f, \quad \mathtt{X} \mapsto \hat{\mathtt{X}},$$

with unit of adjunction $i_{\mathtt{X}} : \mathtt{X} \to \hat{\mathtt{X}}$ a level trivial cofibration. The main result is that there is a natural bijection

$$\text{Ho}(\text{Pro } \mathbf{C})(\mathtt{X}, \mathtt{Y}) \cong [\hat{\mathtt{X}}, \hat{\mathtt{Y}}],$$

which exhibits Ho(Pro **C**) as the full image of the functor $\mathtt{F}$.
We note that two coherent maps of systems that are coherently homotopic are also naive homotopic.

Inv **C** is a ge-subcategory of $\mathbb{I}$nv **C** and the inclusion 2-functor Inv **C** $\to$ $\mathbb{I}$nv **C** lifts to the homotopy categories as $\mathtt{I}$: Pro **C** $\to$ $\mathbb{P}$ro **C**. Since level homotopy equivalences in Pro **C** become homotopy equivalences in $\mathbb{P}$ro **C**, then the inclusion 2-functor $\mathtt{I}$ takes level homotopy equivalences to isomorphisms. It follows [2] that there exists a unique functor $\mathtt{U}$: Ho(Pro **C**) $\to$ $\mathbb{P}$ro **C** making the following diagram



commutative, where $\mathtt{P}_\Sigma$ is the localization functor.

Theorem 4.9. *The functor* $\mathtt{U}$: Ho(Pro **C**) $\to$ $\mathbb{P}$ro **C** *is an isomorphism of categories.*

Proof. Let us note first that all functors involved in the above diagram are identical on objects. Let now $\varphi : \mathtt{X} \to \mathtt{Y}$ be a coherent map of systems.

By (4.7) we can assume that φ is actually a psd-transformation between systems indexed over the same directed set. By (3.6) there is a unique natural transformation $\varphi' : \mathtt{X}' \to \mathtt{Y}$ which is homotopic in $\mathbb{P}\text{ro}\ \mathbf{C}$ to the composition

$$\mathtt{X}' \xrightarrow{q_\mathtt{X}} \mathtt{X} \xrightarrow{\varphi} \mathtt{Y}$$

Recall (3.5) that $q_\mathtt{X}$ is a level homotopy equivalence, then in $\mathsf{Ho}(\mathsf{Pro}\ \mathbf{C})$ consider the morphism

$$[\mathtt{X} \xleftarrow{q_\mathtt{X}} \mathtt{X}' \xrightarrow{\varphi'} \mathtt{Y}]\ .$$

It follows that (see [2], A.4)

$$\mathtt{U}(\varphi'(q_\mathtt{X})^{-1}) = \mathtt{U}(\varphi')\mathtt{U}(q_\mathtt{X})^{-1} = I(\varphi')I(q_\mathtt{X})^{-1} = \varphi' p_\mathtt{X} : \mathtt{X} \to \mathtt{Y}$$

and it is clear that $\varphi' p_\mathtt{X}$ is homotopic to φ in $\mathbb{I}\text{nv}\ \mathbf{C}$, so that they give the same morphism in $\mathbb{P}\text{ro}\ \mathbf{C}$, hence the functor $\mathtt{U}$ is full. Let now $\phi, \psi : \mathtt{X} \to \mathtt{Y}$ be two morphisms in $\mathsf{Ho}(\mathsf{Pro}\ \mathbf{C})$. We may assume without loss of generality (4.8) that they correspond to homotopy classes $\phi = [\mathtt{f}]$, $\psi = [\mathtt{g}]$ of maps of systems $\mathtt{f}, \mathtt{g} : \hat{\mathtt{X}} \to \hat{\mathtt{Y}}$. Then, assuming that $\mathtt{U}(\phi) = \mathtt{U}(\psi)$ amounts to assume that $\mathtt{f}$ and $\mathtt{g}$ are coherently homotopic maps of systems. This means that there is an increasing map $F : M \to \Lambda$ and a family of homotopies

$$F_\mu : f_\mu x_{f(\mu)F(\mu)} \Rightarrow g_\mu x_{g(\mu)F(\mu)}$$

in the ge-category $\mathbf{C}$, such that $F_\mu x_{F(\mu)F(\mu')} = y_{\mu\mu'} F_{\mu'}$, for $\mu \leq \mu'$. It follows that $\mathtt{F} = (F, F_\mu) : \hat{\mathtt{X}} \times I \to \hat{\mathtt{Y}}$ is a naive homotopy connecting $\mathtt{f}$ and $\mathtt{g}$, thus $\mathtt{U}$ is also a faithful functor. □

Let us note that $\mathbb{P}\text{ro}\ \mathbf{Top}$ is also isomorphic to the coherent pro-homotopy category of Lisića and Mardešić $CH(\mathsf{Pro}\mathbf{Top})$, see [17], Theorem 4.3.8.

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