CAHIERS DE TOPOLOGIE ET GEOMETRIE DIFFERENTIELLE CATEGORIQUES Vol. LVII-1 (2016)

EQUALIZERS IN KLEISLI CATEGORIES

by S. N. HOSSEINI and Y. QASEMI NEZHAD

Résumé. Dans cet article, nous donnons des conditions nécessaires et suffisantes pour qu'une paire de morphismes d'une catégorie de Kleisli, associée à une monade générale, ait un égalisateur. Nous proposons aussi, dans différents cas de monades intéressantes, un meilleur critère pour l'existence d'un égalisateur et dans ces cas nous explicitons ce qu'est l'égalisateur (lorsqu'il existe).

Abstract. In this article, we give necessary and sufficient conditions for a pair of morphisms in a Kleisli category, corresponding to a general monad, to have an equalizer. We also present a better criterion for equalizers in a number of cases of interesting monads, and in all these cases we explain what an equalizer (if it exists) of a pair of morphisms is.

Keywords. equalizer, Kleisli category, (representation, add-point or exception, M-set, power object) monad.

Mathematics Subject Classification (2010). 18A30, 18C15, 18C20.

1. Introduction

The richness of a given category depends on to what degree the category is complete and/or cocomplete. In particular this is true about the kleisli categories, that are being widely used in different areas such as, the semantics of linear logic, [2], computing, [6], Maltsev varieties, [5], extension of functors, [11], factorization-related monads, [4], and information systems, [9], to mention a few.

In [10], the completeness/cocompleteness of Kleisli categories are tackled, however as the author mentions, the results are powerless in concrete instances. In this article, we attempt the problem of the existence of equalizers in Kleisli categories. It is known that the category of sets and relations, the Kleisli category for the power set monad, does not have equalizers, [3].

So we try to answer the question of when a given pair of morphisms has equalizers. First we give some equivalent conditions for the existence of equalizers of a given pair of maps in a general Kleisli category. Then we present a better criterion for the existence of equalizers in a number of cases of interesting monads.

2. Preliminaries

A monad on a category $\mathcal{E}$, [1], is a triple $\mathbb{T} = (T, \eta, \mu)$, where $T : \mathcal{E} \longrightarrow \mathcal{E}$ is a functor, $\eta : I \longrightarrow T$ and $\mu : T^2 \longrightarrow T$ are natural transformations rendering commutative the following square and triangles.

$$\begin{array}{ccc} T^3 & \xrightarrow{T\mu} & T^2 \\ {\scriptstyle \mu T}\downarrow & & \downarrow{\scriptstyle \mu} \\ T^2 & \xrightarrow[\mu]{} & T \end{array} \qquad\qquad \begin{array}{ccccc} T & \xrightarrow{T\eta} & T^2 & \xleftarrow{\eta T} & T \\ & {\scriptstyle 1_T}\searrow & \downarrow{\scriptstyle \mu} & \swarrow{\scriptstyle 1_T} & \\ & & T & & \end{array}$$

Some examples of monads that we use in this article, are:

Example 2.1. $a)$ [1], [7], [12]. Let $\mathcal{E}$ be a category in which partial morphisms are represented, such as a quasitopos. Let $\eta_X : X \longrightarrow \tilde{X}$ represent partial morphisms to X (such a map is mono) and be universal, i.e., pullback of η_X along all morphisms exists. The functor $\sim: \mathcal{E} \longrightarrow \mathcal{E}$, taking X to $\tilde{X}$, the natural transformation $\eta : I \longrightarrow \sim$ and the natural transformation $\mu :\sim^2 \longrightarrow \sim$, where $\mu_X : \tilde{\tilde{X}} \longrightarrow \tilde{X}$ is defined by the following pullback square:

$$\begin{array}{ccc} X & \xrightarrow{1_X} & X \\ {\scriptstyle \eta_{\tilde{X}}\eta_X}\downarrow & p.b. & \downarrow{\scriptstyle \eta_X} \\ \tilde{\tilde{X}} & \xrightarrow[\exists! \mu_X]{} & \tilde{X} \end{array}$$

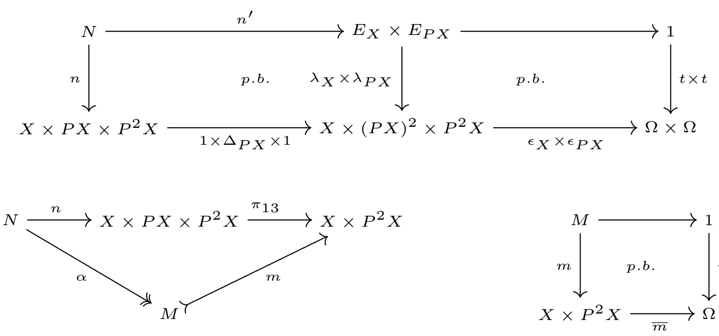
form a monad $\mathbb{R} = (\sim, \eta, \mu)$, called the representation monad.

$b)$ [3]. Let $\mathcal{E}$ be a category with a coproductive terminal object 1, i.e., a terminal object whose coproduct with any other object exists. The functor $A = - \amalg 1 : \mathcal{E} \longrightarrow \mathcal{E}$, the natural transformation $\eta : I \longrightarrow A$, with

$\eta_X = \nu_1 : X \longrightarrow X \amalg 1$, and the natural transformation $\mu : A^2 \longrightarrow A$ defined by $\mu_X = 1 \bigoplus \nu_2 : (X \amalg 1) \amalg 1 \longrightarrow X \amalg 1$, where ν_1 and ν_2 are respectively the first and the second injections of the coproduct, form a monad $\mathbb{A} = (A, \eta, \mu)$, called the add-point or exception monad.

$c)$ [3]. Let $(M, \star, 1)$ be a monoid. The functor $M : Set \longrightarrow Set$ defined on objects by $\hat{M}(X) = M \times X$, together with the natural transformations η and μ, with $\eta_X(x) = (1, x)$ and $\mu_X(m_1, (m_2, x)) = (m_1 \star m_2, x)$ form a monad $\mathbb{M} = (\hat{M}, \eta, \mu)$ called the M-Set monad. We denote $m \star n$ also by mn. Note that $\eta_X = \langle \tilde{1}, 1_X \rangle$, with $\tilde{1}$ the constant map with value $1 \in M$ and 1_X the identity function. Also $\mu_X \cong \star \times 1_X$.

$d)$ [1], [8], [12]. Let $\mathcal{E}$ be a topos. The covariant power object functor $P : \mathcal{E} \longrightarrow \mathcal{E}$, where $P(X \xrightarrow{f} Y) = PX \xrightarrow{\exists_f} PY$, the singleton natural transformation $I \xrightarrow{\eta} P$ and the monad multiplication $\mu : P^2 \longrightarrow P$, where $\mu_X = \widetilde{\overline{m}} : P^2X \longrightarrow PX$ is the transpose of $\overline{m} : X \times P^2X \longrightarrow \Omega$ and $\overline{m}$ is defined as follows:



with ϵ denoting the evaluation, Δ the diagonal and π_{13} the evident projection, form a monad $\mathbb{P} = (P, \eta, \mu)$, called the power object monad.

For naturality of μ in part (c) of 2.1, see [12]. We show μ commutes with the internal join. But first, with:

$$U : Sub(X \times (PX)^2) \times Sub(X \times (PX)^2) \longrightarrow Sub(X \times (PX)^2)$$

the external join or union, $\vee_X : (PX)^2 \longrightarrow PX$ the internal join and the map $f^{-1}(g)$ denoting the pullback of g along f, we have:

Lemma 2.2. *The map v obtained by the pullback:*

$$\begin{array}{ccc} V & \xrightarrow{v'} & E_X \\ {\scriptstyle v}\downarrow & p.b. & \downarrow{\scriptstyle \lambda_X} \\ X \times (PX)^2 & \xrightarrow[1\times\vee_X]{} & X \times PX \end{array}$$

is $v = \pi_{12}^{-1}(\lambda_X) \cup \pi_{13}^{-1}(\lambda_X) = (\lambda_X \times 1) \cup \pi_{132}(\lambda_X \times 1)$, where the maps $X \times (PX)^2 \underset{\pi_{13}}{\overset{\pi_{12}}{\rightrightarrows}} X \times PX$ and $\pi_{132} : X \times (PX)^2 \longrightarrow X \times (PX)^2$ are the evident projections.

Proof. By the above pullback diagram, the classifying map of v is $\hat{v} = \epsilon_X(1 \times \vee_X)$. Now form the following pullbacks.

$$\begin{array}{ccc} V_1 & \xrightarrow{v_1'} & E_x \\ {\scriptstyle v_1}\downarrow & p.b. & \downarrow{\scriptstyle \lambda_X} \\ X \times (PX)^2 & \xrightarrow[\pi_{12}]{} & X \times PX \end{array} \qquad \begin{array}{ccc} V_2 & \xrightarrow{v_2'} & E_x \\ {\scriptstyle v_2}\downarrow & p.b. & \downarrow{\scriptstyle \lambda_X} \\ X \times (PX)^2 & \xrightarrow[\pi_{13}]{} & X \times PX \end{array}$$

Take the epi-mono factorization of $v_1 \oplus v_2 : V_1 \amalg V_2 \longrightarrow X \times (PX)^2$ to get:

$$\begin{array}{ccc} V_1 \amalg V_2 & \xrightarrow{v_1 \oplus v_2} & X \times (PX)^2 \\ & \searrow \quad \nearrow{\scriptstyle v_1 \cup v_2} & \\ & V' & \end{array}$$

The following diagram shows $\widehat{v_1 \cup v_2} = \epsilon_X(1 \times \vee_X)$.

$$\begin{array}{ccc} (v_1, v_2) \in Sub(X \times (PX)^2) \times Sub(X \times (PX)^2) & \xrightarrow{\cup} & Sub(X \times (PX)^2) \ni v_1 \cup v_2 \\ \wr\wr & & \wr\wr \\ hom(X \times (PX)^2, \Omega) \times hom(X \times (PX)^2, \Omega) & \xrightarrow{\vee} & hom(X \times (PX)^2, \Omega) \ni \epsilon_X(1 \times \vee_X) \\ \wr\wr & & \wr\wr \\ hom((PX)^2, PX) \times hom((PX)^2, PX) & \xrightarrow{\vee} & hom((PX)^2, PX) \ni \vee_X \\ \wr\wr & & \| \\ 1 \in hom((PX)^2, (PX)^2) & \xrightarrow{\vee} & hom((PX)^2, PX) \ni \vee_X \end{array}$$

It follows that $\hat{v} = \widehat{v_1 \cup v_2}$ and so $v = v_1 \cup v_2$, proving the first equality. Since the squares:

$$\begin{array}{ccc} E_X \times PX & \xrightarrow{\pi_1} & E_X \\ {\scriptstyle \lambda_X \times 1}\downarrow & & \downarrow{\scriptstyle \lambda_X} \\ X \times (PX)^2 & \xrightarrow[\pi_{12}]{} & X \times PX \end{array} \quad \text{and} \quad \begin{array}{ccc} E_X \times PX & \xrightarrow{\pi_1} & E_X \\ {\scriptstyle \pi_{132}(\lambda_X \times 1)}\downarrow & & \downarrow{\scriptstyle \lambda_X} \\ X \times (PX)^2 & \xrightarrow[\pi_{13}]{} & X \times PX \end{array}$$

are pullbacks, we get the second equality.

□

Theorem 2.3. *The monad multiplication μ of $\mathbb{P}$ preserves the internal join, i.e., for each X the following square commutes.*

$$\begin{array}{ccc} (P^2X)^2 & \xrightarrow{\vee_{PX}} & P^2X \\ {\scriptstyle \mu_X^2 = \mu_X \times \mu_X}\downarrow & & \downarrow{\scriptstyle \mu_X} \\ (PX)^2 & \xrightarrow[\vee_X]{} & PX \end{array}$$

Proof. We have $\vee_X \mu_X^2 = \mu_X \vee_{PX}$ if and only if their transposes are equal, i.e., $\epsilon_X(1_X \times \vee_X \mu_X^2) = \epsilon_X(1_X \times (\mu_X \vee_{PX}))$ if and only if $\epsilon_X(1_X \times \vee_X)(1_X \times \mu_X^2) = \epsilon_X(1_X \times \mu_X)(1_X \times \vee_{PX})$ if and only if $w = k$, where w and k are obtained by the following pullbacks.

$$\begin{array}{ccc} W & \xrightarrow{w'} & V \\ {\scriptstyle w}\downarrow & & \downarrow{\scriptstyle v} \\ X \times (P^2X)^2 & \xrightarrow[1 \times \mu^2]{} & X \times (PX)^2 \end{array} \quad \text{and} \quad \begin{array}{ccccc} K & \xrightarrow{k'} & M & \xrightarrow{m'} & E_X \\ {\scriptstyle k}\downarrow & & \downarrow{\scriptstyle m} & & \downarrow{\scriptstyle \lambda_X} \\ X \times (P^2X)^2 & \xrightarrow[1 \times \vee]{} & X \times P^2X & \xrightarrow[1 \times \mu]{} & X \times PX \end{array}$$

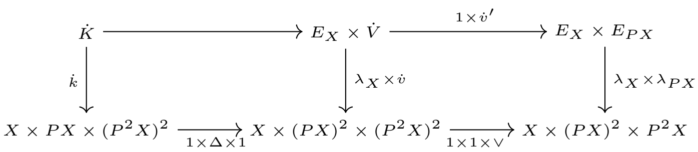
Form the pullback squares:

$$\begin{array}{ccc} \dot{K} & \xrightarrow{\dot{k}'} & N \\ {\scriptstyle \dot{k}}\downarrow & & \downarrow{\scriptstyle n} \\ X \times PX \times (P^2X)^2 & \xrightarrow[1 \times 1 \times \vee]{} & X \times PX \times P^2X \end{array} \quad \text{and} \quad \begin{array}{ccc} \dot{V} & \xrightarrow{\dot{v}'} & E_{PX} \\ {\scriptstyle \dot{v}}\downarrow & & \downarrow{\scriptstyle \lambda_{PX}} \\ PX \times (P^2X)^2 & \xrightarrow[1 \times \vee]{} & PX \times P^2X \end{array}$$

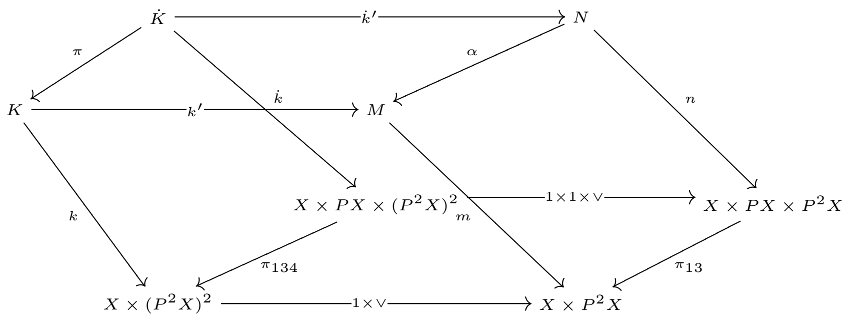
So we have the following diagram in which both squares are pullbacks.

$$\begin{array}{ccccc} \dot{K} & \xrightarrow{\dot{k}'} & N & \xrightarrow{n'} & E_X \times E_{PX} \\ {\scriptstyle \dot{k}}\downarrow & & \downarrow{\scriptstyle n} & & \downarrow{\scriptstyle \lambda_X \times \lambda_{PX}} \\ X \times PX \times (P^2X)^2 & \xrightarrow[1 \times 1 \times \vee]{} & X \times PX \times P^2X & \xrightarrow[1 \times \Delta \times 1]{} & X \times (PX)^2 \times P^2X \end{array}$$

Since $(1 \times \Delta \times 1)(1 \times 1 \times \vee) = (1 \times 1 \times \vee)(1 \times \Delta \times 1)$, we get the following pullback squares.



Therefore in the following cube, the right and left faces are commutative and all the other faces are pullbacks. Also since α is epi, so is it's pullback π.



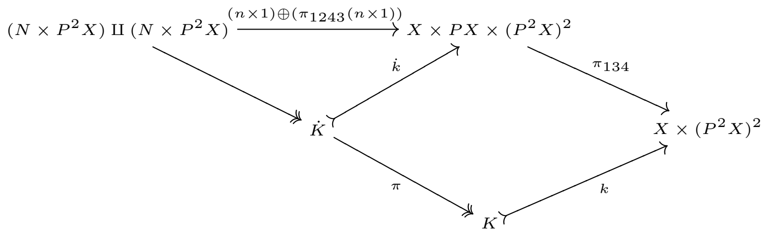
by 2.2, $\dot{v} = (\lambda_{PX} \times 1) \cup \pi_{132}(\lambda_{PX} \times 1)$ and so:

$$\lambda_X \times \dot{v} = \lambda_X \times [(\lambda_{PX} \times 1) \cup \pi_{132}(\lambda_{PX} \times 1)] =$$
$$[\lambda_X \times (\lambda_{PX} \times 1)] \cup [\lambda_X \times (\pi_{132}(\lambda_{PX} \times 1))]$$

Therefore

$$\dot{k} = (1 \times \triangle \times 1)^{-1}(\lambda_X \times \dot{v}) =$$
$$[(1 \times \triangle \times 1)^{-1}(\lambda_X \times (\lambda_{PX} \times 1))] \cup [(1 \times \triangle \times 1)^{-1}(\lambda_X \times (\pi_{132}(\lambda_{PX} \times 1)))] =$$
$$[(1 \times \triangle \times 1 \times 1)^{-1}((\lambda_X \times \lambda_{PX}) \times 1)] \cup [(1 \times \triangle \times 1 \times 1)^{-1}(\pi_{1243}((\lambda_X \times \lambda_{PX}) \times 1))] =$$
$$(n \times 1) \cup \pi_{1243}(n \times 1)$$

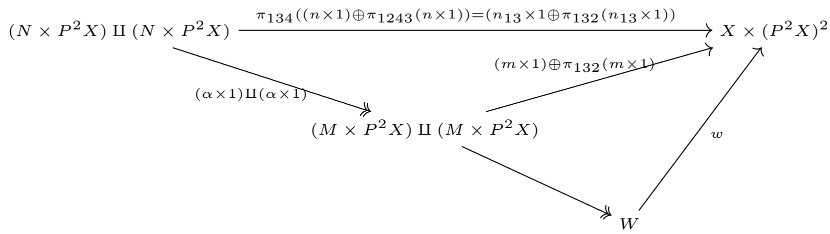
Hence taking the epi-mono factorization of $(n \times 1) \oplus (\pi_{1243}(n \times 1))$, we get the below commutative triangle; while the bottom square is the left face of the above cube.



On the other hand using 2.2, we have:

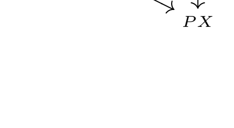
$$
\begin{gathered}
w = (1 \times \mu^2)^{-1}(v) = \\
(1 \times \mu^2)^{-1}[\pi_{12}^{-1}(\lambda_X) \cup \pi_{13}^{-1}(\lambda_X)] = \\
[(\pi_{12}(1 \times \mu^2))^{-1}(\lambda_X)] \cup [(\pi_{13}(1 \times \mu^2))^{-1}(\lambda_X)] = \\
[((1 \times \mu)\pi_{12})^{-1}(\lambda_X)] \cup [((1 \times \mu)\pi_{13})^{-1}(\lambda_X)] = \\
[\pi_{12}^{-1}(1 \times \mu)^{-1}(\lambda_X)] \cup [\pi_{13}^{-1}(1 \times \mu)^{-1}(\lambda_X)] = \\
[\pi_{12}^{-1}(m)] \cup [\pi_{13}^{-1}(m)] = \\
(m \times 1) \cup \pi_{132}(m \times 1)
\end{gathered}
$$

In the following diagram, the commutativity of the top triangle can be easily verified and that of the bottom triangle follows from the fact that $w = (m \times 1) \cup \pi_{132}(m \times 1)$.



Therefore w and k are the mono part of the morphism $\pi_{134}((n \times 1) \oplus \pi_{1243}(n \times 1))$ and so are equal. □

Lemma 2.4. *The monad multiplication μ of $\mathbb{P}$ preserves the false map, i.e., for each X the following triangle commutes.*



Proof. Since $\mathbf{f}_{PX} \leq \eta_{PX}\mathbf{f}_X$, by 2.3, $\mu_X\mathbf{f}_{PX} \leq \mu_X\eta_{PX}\mathbf{f}_X$. So $\mu_X\mathbf{f}_{PX} \leq \mathbf{f}_X$, implying $\mu_X\mathbf{f}_{PX} = \mathbf{f}_X$.
□

3. Preservation of Equalizers

Denoting the Kleisli category of a monad $\mathbb{T}$ by $\mathcal{E}_{\mathbb{T}}$ and the morphism in $\mathcal{E}_{\mathbb{T}}$ associated to $f : X \longrightarrow TY$ in $\mathcal{E}$ by $\hat{f} : X \longrightarrow Y$, we have:

Lemma 3.1. *For a monad $\mathbb{T}$ in $\mathcal{E}$, the functor $U : \mathcal{E}_{\mathbb{T}} \longrightarrow \mathcal{E}$ defined by:*

$$U(\, X \xrightarrow{\hat{f}} Y \,) = \; TX \xrightarrow{\mu_Y T(f)} TY$$

is right adjoint to the functor $I : \mathcal{E} \longrightarrow \mathcal{E}_{\mathbb{T}}$ defined by:

$$I(\, X \xrightarrow{f} Y \,) = \; X \xrightarrow{\widehat{\eta_Y f}} Y$$

Proof. See [1], [3]. □

Proposition 3.2. *The functor $U : \mathcal{E}_{\mathbb{T}} \longrightarrow \mathcal{E}$ preserves and reflects equalizers.*

Proof. The preservation of equalizers follows from 3.1. To prove U reflects equalizers, let $E \xrightarrow{\hat{e}} X \overset{\hat{f}}{\underset{\hat{g}}{\rightrightarrows}} Y$ be a diagram in $\mathcal{E}_{\mathbb{T}}$ such that the diagram $TE \xrightarrow{\mu_X T(e)} TX \overset{\mu_Y T(f)}{\underset{\mu_Y T(g)}{\rightrightarrows}} TY$ is an equalizer in $\mathcal{E}$. It follows that $\mu_Y T(f)\mu_X T(e) = \mu_Y T(g)\mu_X T(e)$. Also by monad axioms $\mu_X\eta_{TX} = 1_{TX}$ and by naturality of η, $\eta_{TX}e = T(e)\eta_E$. So $\mu_Y T(f)e = \mu_Y T(f)\mu_X\eta_{TX}e = \mu_Y T(f)\mu_X T(e)\eta_E = \mu_Y T(g)\mu_X T(e)\eta_E = \mu_Y T(g)\mu_X\eta_{TX}e = \mu_Y T(g)e$ and thus $\hat{f}\hat{e} = \hat{g}\hat{e}$.

Now if there is $\hat{k} : Z \longrightarrow X$ such that $\hat{f}\hat{k} = \hat{g}\hat{k}$, then $\mu_Y T(f)k = \mu_Y T(g)k$ and so there is a unique $\bar{k} : Z \longrightarrow TE$ such that $\mu_X T(e)\bar{k} = k$. Therefore $\hat{\bar{k}} : Z \longrightarrow E$ is in $\mathcal{E}_{\mathbb{T}}$ and $\hat{e}\hat{\bar{k}} = \hat{k}$.

If there is $\hat{k'}$ such that $\hat{e}\hat{k'} = \hat{k}$, then $\widehat{\mu_X T(e)k'} = \hat{k}$. It follows that $\mu_X Tek' = k$ and so $k' = \bar{k}$. Therefore $\hat{\bar{k}} = \hat{k'}$ and the result follows.
□

Theorem 3.3. *Let $\mathcal{E}$ be a category and $\mathbb{T} = (T, \eta, \mu)$ be a monad on $\mathcal{E}$. For morphisms $\hat{f}, \hat{g} : X \longrightarrow Y$ in $\mathcal{E}_{\mathbb{T}}$, the following conditions are equivalent.*

$a)$ In $\mathcal{E}_{\mathbb{T}}$, there is a morphism $E \xrightarrow{\hat{e}} X$ such that $E \xrightarrow{\hat{e}} X \overset{\hat{f}}{\underset{\hat{g}}{\rightrightarrows}} Y$ is an equalizer.

$b)$ In $\mathcal{E}$, there is a morphism $e : E \longrightarrow TX$ such that the diagram $TE \xrightarrow{\mu_X T(e)} TX \overset{\mu_Y T(f)}{\underset{\mu_Y T(g)}{\rightrightarrows}} TY$ is an equalizer.

$c)$ In $\mathcal{E}$, there is a morphism $m : M \longrightarrow TX$, an object E and an isomorphism $\varphi : TE \longrightarrow M$ such that $M \xrightarrow{m} TX \overset{\mu_Y T(f)}{\underset{\mu_Y T(g)}{\rightrightarrows}} TY$ is an equalizer and the following diagram is commutative.

$$\begin{array}{ccc} T^2E & \xrightarrow{\mu_E} & TE \\ {\scriptstyle T(m\varphi)}\downarrow & & \downarrow{\scriptstyle m\varphi} \\ T^2X & \xrightarrow[\mu_X]{} & TX \end{array}$$

In this case the morphism $\hat{e}$ of part (a) corresponds to the morphism e of part (b) which in turn corresponds to the morphism $m\varphi\eta_E$ of part (c).

Proof. The equivalence of (a) and (b) follows from 3.2.

$(b) \Rightarrow (c)$: Setting $M = TE$, $m = \mu_X T(e)$ and $\varphi = id_{TE}$, it is enough to show the square in (c) commutes. By naturality of μ and monad definition we have, $\mu_X T(e)\mu_E = \mu_X \mu_{TX} T^2 e = \mu_X T\mu_X T^2 e = \mu_X T(\mu_X T(e))$, as desired.

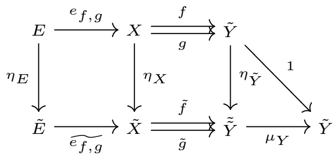
$(c) \Rightarrow (b)$: Setting $e = m\varphi\eta_E : E \longrightarrow TX$, we have, $\mu_X T(e) = \mu_X T(m\varphi\eta_E) = \mu_X T(m\varphi)T\eta_E = m\varphi\mu_E T\eta_E = m\varphi$. The result now follows from the fact that φ is an isomorphism and m is an equalizer of $\mu_Y T(f)$ and $\mu_Y T(g)$.

The last assertion holds obviously. □

Lemma 3.4. *Let partial morphisms in a category $\mathcal{E}$ be representable by universal arrows, and let $\mathbb{R}$ be the representation monad. The morphisms $X \overset{f}{\underset{g}{\rightrightarrows}} \tilde{Y}$ have an equalizer in $\mathcal{E}$ if and only if the morphisms $\tilde{X} \overset{\mu_Y \tilde{f}}{\underset{\mu_Y \tilde{g}}{\rightrightarrows}} \tilde{Y}$ do.*

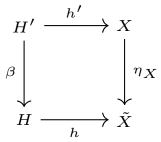
In this case an equalizer of $\mu_Y \tilde{f}$ *and* $\mu_Y \tilde{g}$ *is* $\widetilde{e_{f,g}}$*, where* $e_{f,g}$ *is an equalizer of* f *and* g*.*

Proof. Suppose an equalizer $e_{f,g} : E \longrightarrow X$ of f and g exists in $\mathcal{E}$. Consider the following diagram in which all the squares are pullbacks and the triangle commutes.

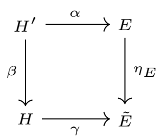


We show $\widetilde{e_{f,g}}$ is an equalizer of $\mu \tilde{f}$ and $\mu \tilde{g}$. Pulling back η_Y along $\mu_Y \tilde{f} \widetilde{e_{f,g}}$ and $\mu_Y \tilde{g} \widetilde{e_{f,g}}$, and using the fact that $f e_{f,g} = g e_{f,g}$, we get the same pullback square. Since η_Y represents partial morphisms, by uniqueness we get $\mu_Y \tilde{f} \widetilde{e_{f,g}} = \mu_Y \tilde{g} \widetilde{e_{f,g}}$.

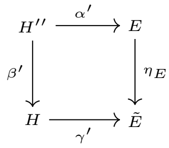
Now suppose $h : H \longrightarrow \tilde{X}$ is given such that $\mu_Y \tilde{f} h = \mu_Y \tilde{g} h$. Form the following pullback.



We have $fh' = \mu_Y \eta_{\tilde{Y}} f h' = \mu_Y \tilde{f} \eta_X h' = \mu_Y \tilde{f} h \beta = \mu_Y \tilde{g} h \beta = \mu_Y \tilde{g} \eta_X h' = \mu_Y \eta_{\tilde{Y}} g h' = gh'$. So there is a unique morphism $\alpha : H' \longrightarrow E$ such that $e_{f,g} \alpha = h'$. Now there is a unique morphism γ rendering pullback the following square.

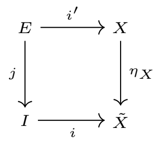


Now pullback of η_X along $\widetilde{e_{f,g}} \gamma$ and along h yields the same 2-source. So by uniqueness, $\widetilde{e_{f,g}} \gamma = h$. Now if γ' satisfies $\widetilde{e_{f,g}} \gamma' = h$, then $\widetilde{e_{f,g}} \gamma' = \widetilde{e_{f,g}} \gamma$. Form the following pullback.



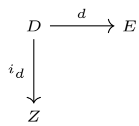
The equality $\widetilde{e_{f,g}}\gamma' = \widetilde{e_{f,g}}\gamma$, yields $H'' = H'$, $\beta' = \beta$ and $e_{f,g}\alpha = e_{f,g}\alpha'$. Since $e_{f,g}$ is mono, $\alpha = \alpha'$. Since η_E represents partial morphisms, uniqueness yields $\gamma = \gamma'$, as desired.

Conversely suppose an equalizer $i : I \longrightarrow \tilde{X}$ of $\mu_Y \tilde{f}$ and $\mu_Y \tilde{g}$ exists in $\mathcal{E}$. Form the following pullback.

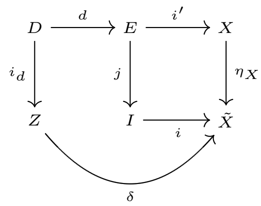


One can easily verify that $fi' = gi'$. Now suppose $h : H \longrightarrow X$ is given such that $fh = gh$. We have $\mu_Y \tilde{f}\eta_X h = \mu_Y \eta_{\tilde{Y}} fh = \mu_Y \eta_{\tilde{Y}} gh = \mu_Y \tilde{g}\eta_X h$. So there is a unique $\alpha : H \longrightarrow I$ such that $i\alpha = \eta_X h$. The above pullback square now yields a unique $k : H \longrightarrow E$ such that $i'k = h$ and $jk = \alpha$. Now if there is k' such that $i'k' = h$, then $ijk' = \eta_X i'k' = \eta_X h = i\alpha$. Since i is mono, $jk' = \alpha$. uniqueness implies $k' = k$. This proves an equalizer of f and g exists and is i'.

The last assertion holds for the direct implication. To prove it for the converse, first we show the morphism j in the above pullback square is a partial morphism classifier. Let the partial morphism:



be given. There is a unique morphism δ making the big square below a pullback.



Since $fi'd = gi'd$, the pullback of η_Y along both $\mu_Y \tilde{f}\delta$ and $\mu_Y \tilde{g}\delta$ yields the same 2-source. Uniqueness of representation gives $\mu_Y \tilde{f}\delta = \mu_Y \tilde{g}\delta$. Therefore there is a unique $\lambda : Z \longrightarrow I$ such that $i\lambda = \delta$. It can be easily shown that the square:

$$\begin{array}{ccc} D & \xrightarrow{d} & E \\ {\scriptstyle i_d}\downarrow & & \downarrow{\scriptstyle j} \\ Z & \xrightarrow[\lambda]{} & I \end{array}$$

commutes and so is a pullback.

To show uniqueness, suppose there is $\lambda' : Z \longrightarrow I$ making the above square a pullback. Then $i\lambda'$ makes the above big square a pullback and so $i\lambda' = i\lambda$. Therefore $\lambda' = \lambda$, as desired. Hence j is a partial morphism classifier.

Since η_E is also a partial morphism classifier, there is an isomorphism $\varphi : \tilde{E} \longrightarrow I$ such that $\varphi\eta_E = j$. It follows that the square:

$$\begin{array}{ccc} E & \xrightarrow{i'} & X \\ {\scriptstyle \eta_E}\downarrow & & \downarrow{\scriptstyle \eta_X} \\ \tilde{E} & \xrightarrow[i\varphi]{} & \tilde{X} \end{array}$$

is a pullback and that $i\varphi = \widetilde{i'}$ is an equalizer of $\mu_Y \tilde{f}$ and $\mu_Y \tilde{g}$. The result follows. □

Theorem 3.5. *Let partial morphisms in a category $\mathcal{E}$ be representable by universal arrows, and let $\mathbb{R}$ be the representation monad. The morphisms $X \underset{\hat{g}}{\overset{\hat{f}}{\rightrightarrows}} Y$ in the Kleisli category $\mathcal{E}_{\mathbb{R}}$ have an equalizer if and only if the morphisms $X \underset{g}{\overset{f}{\rightrightarrows}} \tilde{Y}$ in $\mathcal{E}$ have an equalizer.*

In this case an equalizer $E \xrightarrow{\hat{e}} X$ of $\hat{f}$ and $\hat{g}$ corresponds to the map $E \xrightarrow{e = \eta_X e_{f,g}} \tilde{X}$, where $E \xrightarrow{e_{f,g}} X$ is an equalizer of f and g.

Proof. The first assertion follows from 3.3 and 3.4. To prove the last assertion, by 3.3 we have $\hat{e} : E \longrightarrow X$ is an equalizer of $\hat{f}$ and $\hat{g}$ in $\mathcal{E}_{\mathbb{R}}$ if and only if $\mu_X \tilde{e} : E \longrightarrow \tilde{X}$ is an equalizer of $\mu_Y \tilde{f}$ and $\mu_Y \tilde{g}$ and by 3.4, if and only if $\mu_X \tilde{e} = \widetilde{e_{f,g}}$, with $e_{f,g}$ an equalizer of f and g. Now consider the following diagram in which pullback of η_X along e is formed to get the mono k; and all the other squares are known to be pullbacks.

$$\begin{array}{ccccccc}
K & \xrightarrow{1} & K & \xrightarrow{i'} & X & \xrightarrow{1} & X \\
{\scriptstyle 1}\downarrow & & {\scriptstyle k}\downarrow & & \downarrow{\scriptstyle \eta_X} & & \\
K & \xrightarrow{k} & E & \xrightarrow{e} & \tilde{X} & & \downarrow{\scriptstyle \eta_X} \\
{\scriptstyle \eta_K}\downarrow & & {\scriptstyle \eta_E}\downarrow & & \downarrow{\scriptstyle \eta_{\tilde{X}}} & & \\
\tilde{K} & \xrightarrow[\tilde{k}]{} & \tilde{E} & \xrightarrow[\tilde{e}]{} & \tilde{\tilde{X}} & \xrightarrow[\mu_X]{} & \tilde{X}
\end{array}$$

It follows from the above big pullback square that $\mu_X \tilde{e}\tilde{k} = \widetilde{i'}$. Now by 3.4, on the one hand i' is an equalizer of f and g and on the other hand $\widetilde{i'}$ is an equalizer of $\mu_Y \tilde{f}$ and $\mu_Y \tilde{g}$. Therefore $\mu_X \tilde{e}\tilde{k}$ and $\mu_X \tilde{e}$ are both equalizers of $\mu_Y \tilde{f}$ and $\mu_Y \tilde{g}$. Thus $\tilde{k}$ and so k are isomorphisms. Hence $e_{f,g} = i'k^{-1}$ is an equalizer of f and g; and $e = \eta_X i' k^{-1} = \eta_X e_{f,g}$, concluding the proof. $\square$

Corollary 3.6. *Let $\mathbb{R}$ be the representation monad on a topos $\mathcal{E}$. The Kleisli category $\mathcal{E}_{\mathbb{R}}$ has equalizers. Furthermore an equalizer $\hat{e} : E \longrightarrow X$ of a pair $X \underset{\hat{g}}{\overset{\hat{f}}{\rightrightarrows}} Y$ in $\mathcal{E}_{\mathbb{R}}$ corresponds to $e = \eta_X e_{f,g} : E \longrightarrow \tilde{X}$, where the map $e_{f,g} : E \longrightarrow X$ is an equalizer of f and g.*

Proof. Follows from 3.5 and the fact a topos has equalizers. $\square$

Saying $f : A \amalg 1 \longrightarrow B \amalg 1$ is point preserving if it renders commutative the following triangle,

$$\begin{array}{ccc}
1 & \xrightarrow{\nu_2} & E \amalg 1 \\
 & {\scriptstyle \nu_2}\searrow & \downarrow{\scriptstyle f} \\
 & & X \amalg 1
\end{array}$$

we have:

Theorem 3.7. *Let $\mathbb{A}$ be the add-point monad on a category $\mathcal{E}$ with a coproductive terminal object 1. The morphisms $X \underset{\hat{g}}{\overset{\hat{f}}{\rightrightarrows}} Y$ in the Kleisli category*

$\mathcal{E}_{\mathbb{A}}$ have an equalizer if and only if the morphisms $X \amalg 1 \underset{g\oplus\nu_2}{\overset{f\oplus\nu_2}{\rightrightarrows}} Y \amalg 1$ *in $\mathcal{E}$ have an equalizer* $i : I \longrightarrow X \amalg 1$ *and there exists an object E and an isomorphism* $\varphi : E \amalg 1 \longrightarrow I$ *such that $i\varphi$ is point preserving.*

In this case an equalizer $E \xrightarrow{\hat{e}} X$ *of $\hat{f}$ and $\hat{g}$ corresponds to the map* $E \xrightarrow{e=i\varphi\nu_1} X \amalg 1$.

Proof. The square:

$$\begin{array}{ccc} (E \amalg 1) \amalg 1 & \xrightarrow{\mu_E = 1\oplus\nu_2} & E \amalg 1 \\ {\scriptstyle (i\varphi)\amalg 1}\downarrow & & \downarrow{\scriptstyle i\varphi} \\ (X \amalg 1) \amalg 1 & \xrightarrow[\mu_X = 1\oplus\nu_2]{} & X \amalg 1 \end{array}$$

commutes if and only if $i\varphi(1 \oplus \nu_2) = (1 \oplus \nu_2)((i\varphi) \amalg 1)$ if and only if $(i\varphi) \oplus (i\varphi\nu_2) = (i\varphi) \oplus \nu_2$ if and only if $i\varphi\nu_2 = \nu_2$ if and only if $i\varphi$ is point preserving. The result follows by 3.3. □

Definition 3.8. *A category is called well add-pointed, provided that it has a coproductive terminal object 1, in which coproducts with 1 are disjoint and universal and squares of the form:*

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle \nu_1}\downarrow & & \downarrow{\scriptstyle \nu_1} \\ A \amalg 1 & \xrightarrow{f \amalg 1} & B \amalg 1 \\ {\scriptstyle \nu_2}\uparrow & & \uparrow{\scriptstyle \nu_2} \\ 1 & \longrightarrow & 1 \end{array}$$

are pullbacks.

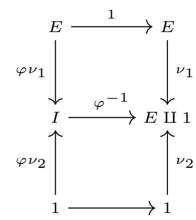
Denoting an equalizer of the pair $X \underset{g}{\overset{f}{\rightrightarrows}} Y \amalg 1$ by $E_{f,g} \overset{e_{f,g}}{\hookrightarrow} X$, we have:

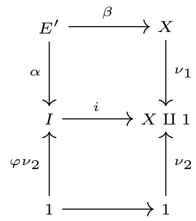
Corollary 3.9. *Let $\mathbb{A}$ be the add-point monad on a well add-pointed category $\mathcal{E}$. The morphisms* $X \underset{\hat{g}}{\overset{\hat{f}}{\rightrightarrows}} Y$ *in the Kleisli category $\mathcal{E}_{\mathbb{A}}$ have an equalizer if and only if the morphisms* $X \underset{g}{\overset{f}{\rightrightarrows}} Y \amalg 1$ *in $\mathcal{E}$ have an equalizer.*

In this case an equalizer $E_{f,g} \xrightarrow{\hat{e}} X$ *of* $\hat{f}$ *and* $\hat{g}$ *corresponds to the map* $E_{f,g} \xrightarrow{e=\nu_1 e_{f,g}} X \amalg 1$.

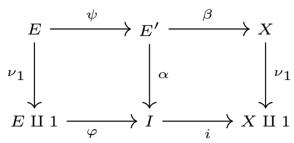
Proof. Suppose an equalizer of $X \underset{\hat{g}}{\overset{\hat{f}}{\rightrightarrows}} Y$ exists in $\mathcal{E}_{\mathbb{A}}$. By 3.7, an object E and an isomorphism $\varphi : E \amalg 1 \longrightarrow I$ exist such that $i\varphi$ is point preserving, where $i : I \longrightarrow X \amalg 1$ is an equalizer of $f \oplus \nu_2$ and $g \oplus \nu_2$.

In the below diagram, the left squares are easily seen to be pullbacks, the right top square is formed to be a pullback and the right bottom square is a pullback because $i\varphi$ is point preserving and i is a mono.



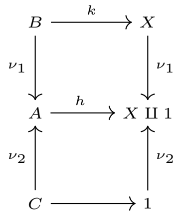


Since $\mathcal{E}$ is well add-pointed, the left vertical arrows in the above left and right diagrams are coproducts and so there is an isomorphism $\psi : E \longrightarrow E'$, such that $\alpha\psi = \varphi\nu_1$. Now we have the following pullback squares, the left of which can be verified easily.



One can now directly verify that $\beta\psi$ is an equalizer of f and g, as desired.

Conversely, suppose $E_{f,g} \overset{e_{f,g}}{\hookrightarrow} X$ is an equalizer of $X \underset{g}{\overset{f}{\rightrightarrows}} Y \amalg 1$ in $\mathcal{E}$. We show $e \amalg 1 : E \amalg 1 \longrightarrow X \amalg 1$ is an equalizer of $f \oplus \nu_2$ and $g \oplus \nu_2$. Obviously $(f \oplus \nu_2)(e \amalg 1) = (g \oplus \nu_2)(e \amalg 1)$. Now given $h : A \longrightarrow X \amalg 1$ such that $(f \oplus \nu_2)h = (g \oplus \nu_2)h$, form the following pullbacks.



Now the morphism k equalizes f and g and so there is a unique morphism $\bar{k} : B \longrightarrow E_{f,g}$ such that $e_{f,g}\bar{k} = k$. Since the left vertical arrows in the above diagram form a coproduct, $\bar{h} = (\nu_1\bar{k}) \oplus (\nu_2 !) : A \longrightarrow E \amalg 1$. We have $(e \amalg 1)\bar{h} = h$. Uniqueness follows from the fact that $e \amalg 1$ is mono, as $\mathcal{E}$ is well add-pointed. The result now follows from 3.7 by taking $\varphi = 1$.

The last assertion can be verified easily. □

Lemma 3.10. *Let $(M, \star, 1)$ be a monoid.*

$a)$ *The relation on M, defined by $m \leq n$ if there is $a \in M$ such that $n = am$, is reflexive and transitive, i.e., a preorder.*

$b)$ *The relation on M, defined by $m \sim n$ if there are $a, b \in M$ such that $am = bn$, is reflexive and symmetric.*

$c)$ *The equivalence relations $R_\leq$ induced by " $\leq$ " and $R_\sim$ induced by " $\sim$ " are equal.*

Proof. (a) and (b) are Obvious.

$c)$ Letting $k \leq\geq k'$ to mean $k \leq k'$ or $k' \leq k$, one has $nR_\leq n'$ if and only if there are $k_1, k_2, \cdots, k_i \in M$ such that $n \leq\geq k_1 \leq\geq k_2 \cdots \leq\geq k_i \leq\geq n'$. On the other hand we have $nR_\sim n'$ if and only if there are $k_1, k_2, \cdots, k_i \in M$ such that $n \sim k_1 \sim k_2 \cdots \sim k_i \sim n'$. The result follows from the facts that $k \leq\geq k'$ implies $k \sim k'$; and $k \sim k'$ implies there are $a, b \in M$ such that $ak = bk'$ so that $k \leq ak = bk' \geq k'$, and therefore $kR_\leq k'$. The result then follows. □

Setting $R = R_\leq = R_\sim$, we have:

Lemma 3.11. *Let $(M, \star, 1)$ be a monoid and X be a set. On $M \times X$,*

$a)$ *the relation $(m, x) \leq (n, y)$ if $m \leq n$ and $x = y$, is a preorder.*

$b)$ *the relation $(m, x) \sim (n, y)$ if $m \sim n$ and $x = y$, is reflexive and symmetric.*

$c)$ *the relation $(m, x)R(n, y)$ if mRn and $x = y$, is an equivalence relation.*

Proof. Follows from 3.10. □

Definition 3.12. *Let $\mathbb{M} = (\hat{M}, \eta, \mu)$ be the M-Set monad and consider the functions $X \overset{f}{\underset{g}{\rightrightarrows}} M \times Y$. We set:*

a) $I \overset{i}{\hookrightarrow} M \times X$ to be the equalizer of $M \times X \overset{f^*=(\star\times 1)(1\times f)}{\underset{g^*=(\star\times 1)(1\times g)}{\rightrightarrows}} M \times Y$.

b) $\dot{I}$ to be those $(m, x) \in I$ for which m is right cancelable, i.e. $am = bm$ implies $a = b$.

The relations on $M \times X$ given in 3.11, induce relations on I and $\dot{I}$. Denoting the quotient map to $\dot{I}/R$ by $q : \dot{I} \longrightarrow \dot{I}/R$, for sets $A \subseteq B \subseteq M \times X$, the up segment of A in B by $B \uparrow A = \{b \in B :$ there is $a \in A$ such that $b \geq a\}$ and the image of a function s by $Im(s)$, we have:

Definition 3.13. *Let $\mathbb{M} = (\hat{M}, \eta, \mu)$ be the M-Set monad and consider the functions $X \overset{f}{\underset{g}{\rightrightarrows}} M \times Y$. We say $E \subseteq I$ is invariantly (f, g)-compatible if $I \uparrow E = I$.*

Theorem 3.14. *Let $\mathbb{M} = (\hat{M}, \eta, \mu)$ be the M-Set monad. The morphisms $X \overset{\hat{f}}{\underset{\hat{g}}{\rightrightarrows}} Y$ in the Kleisli category $Set_{\mathbb{M}}$ have an equalizer if and only if there is a section $s : \dot{I}/R \longrightarrow \dot{I}$ of $q : \dot{I} \longrightarrow \dot{I}/R$ such that $Im(s)$ is invariantly (f, g)-compatible.*

In this case an equalizer $Im(s) \overset{\hat{e}}{\longrightarrow} X$ of $\hat{f}$ and $\hat{g}$ corresponds to the map $Im(s) \overset{e}{\hookrightarrow} M \times X$.

Proof. Suppose an equalizer of the pair $X \overset{\hat{f}}{\underset{\hat{g}}{\rightrightarrows}} Y$ in $Set_{\mathbb{M}}$ is the map $\hat{e} : E \longrightarrow X$. Without loss of generality we assume the corresponding map $E \overset{e}{\hookrightarrow} M \times X$ is the inclusion. By 3.3, $M \times E \xrightarrow{e^*=(\star\times 1)(1\times e)} M \times X$ is an equalizer of $M \times X \overset{f^*}{\underset{g^*}{\rightrightarrows}} M \times Y$. Since $I \overset{i}{\hookrightarrow} M \times X$ is also an equalizer, there is a bijection $M \times E \xrightarrow{\psi} I$ such that $i\psi = e^*$. It follows that $\psi(m,n,x) = (mn,x)$. Let $(n,x) \in E$. On the one hand $(n,x) = \psi(1,n,x) \in I$, thus $E \subseteq I$. On the other hand if $a, b \in M$ and $an = bn$, then since e^* is mono, $a = b$, thus $E \subseteq \dot{I}$.

Next we define $s : \dot{I}/R \longrightarrow \dot{I}$. Let $[(m,x)] \in \dot{I}/R$, with $(m,x) \in \dot{I}$. Then $\psi^{-1}(m,x) = (m', \dot{m}, x)$, with $(\dot{m}, x) \in E$ and $m = m'\dot{m}$. Set $s([(m,x)]) = (\dot{m}, x)$. If $[(m,x)] = [(n,x)]$, then mRn, i.e., there are $k_1, k_2, \cdots, k_i$ such that $m \sim k_1$, $k_1 \sim k_2$, $\cdots$, $k_i \sim n$. Also $s([(m,x)]) = (\dot{m}, x)$, with $(\dot{m}, x) \in E$, $m = m'\dot{m}$ and $s([(n,x)]) = (\dot{n}, x)$, with $(\dot{n}, x) \in E$ and $n = n'\dot{n}$. Since each $(k_j, x) \in \dot{I} \subseteq I$, $k_j = k'_j\dot{k}_i$ with $(\dot{k}_j, x) \in E$; and since $m \sim k_1$, there are $a, b \in M$ such that $am = bk_1$. Then $am'\dot{m} = bk'_1\dot{k}_1$. Monotonicity of e^* implies $\dot{m} = \dot{k}_1$. Continuing in this manner, we get $\dot{k}_1 = \dot{k}_2 = \cdots = \dot{k}_i = \dot{n}$. So $(\dot{m}, x) = (\dot{n}, x)$ and therefore s is well defined. Now $m = m'\dot{m}$ implies $m \sim \dot{m}$ and so $[(\dot{m}, x)] = [(m,x)]$. It follows that $qs([(m,x)]) = q(\dot{m}, x) = [(\dot{m}, x)] = [(m,x)]$. Hence $qs = 1$, i.e., s is a section of q. To show $E = Im(s)$, we know $Im(s) \subseteq E$. Now if $(m,x) \in E$, then $(m,x) \in I$ and so $m = m'\dot{m}$ with $(\dot{m}, x) \in E$. Since (m,x) and $(\dot{m}, x)$ are in E and $1m = m'\dot{m}$, monotonicity of e^* yields $m = \dot{m}$. Hence $(m,x) = (\dot{m}, x) = s([(m,x)]) \in Im(s)$. Thus $E \subseteq Im(s)$. Therefore $E = Im(s)$. Finally we show $I \uparrow Im(s) = I$. Let $(n,x) \in I$. We know $n = n'\dot{n}$ with $(\dot{n}, x) \in E$. It follows that $(n,x) \geq (\dot{n}, x)$ and $(\dot{n}, x) \in E = Im(s)$. Thus $(n,x) \in I \uparrow Im(s)$. Hence $I \uparrow Im(s) = I$, i.e., $Im(s)$ is invariantly (f,g)-compatible.

Conversely suppose there is a section $s : \dot{I}/R \longrightarrow \dot{I}$ of the morphism $q : \dot{I} \longrightarrow \dot{I}/R$ such that $Im(s)$ is invariantly (f,g)-compatible. Denote by $Im(s) \overset{e}{\hookrightarrow} M \times X$ the inclusion. For $(m,n,x) \in M \times Im(s)$, we have $f^*e^*(m,n,x) = (mnf_1(x), f_2(x))$, where f_1 and f_2 denote the first and the second projection of f, respectively. Since $(n,x) \in Im(s) \subseteq I$, we have $(nf_1(x), f_2(x)) = (ng_1(x), g_2(x))$. It follows that $f^*e^*(m,n,x) = g^*e^*(m,n,x)$. Hence $f^*e^* = g^*e^*$.

Now suppose $h : A \longrightarrow M \times X$ is given such that $f^*h = g^*h$. Let $a \in A$ and $h(a) = (m,x)$. Then $f^*(m,x) = g^*(m,x)$ and so $(m,x) \in I = I \uparrow Im(s)$. So $(m,x) \geq (\dot{m}, x)$ with $(\dot{m}, x) \in Im(s)$. It follows that $m = m'\dot{m}$ and $(\dot{m}, x) \in Im(s)$. Define $\bar{h} : A \longrightarrow M \times Im(s)$ by $\bar{h}(a) =$

$(m', \dot{m}, x)$. If $a = b$, then $h(a) = h(b) = (m, x)$. $m = m'\dot{m} = n'\dot{n}$ with $(\dot{m}, x)$ and $(\dot{n}, x)$ in $Im(s)$. So $(\dot{m}, x) \sim (\dot{n}, x)$, $(\dot{m}, x) = s([(k, x)])$ and $(\dot{n}, x) = s([(l, x)])$, for some k and l. It follows that $[(k, x)] = qs([(k, x)]) = [(\dot{m}, x)] = [(\dot{n}, x)] = qs([(l, x)]) = [(l, x)]$. Therefore $(\dot{m}, x) = (\dot{n}, x)$ and so $\dot{m} = \dot{n}$. Now $m'\dot{m} = n'\dot{n}$ implies $m'\dot{m} = n'\dot{m}$ and since $\dot{m}$ is right cancelable, we get $m' = n'$. So $\bar{h}(a) = (m', \dot{m}, x) = (n', \dot{n}, x) = \bar{h}(b)$. Hence $\bar{h}$ is well defined. One can easily prove $e^*\bar{h} = h$ and e^* is mono. Uniqueness of $\bar{h}$ with $e^*\bar{h} = h$ therefore follows. Hence e^* is an equalizer of f^* and g^* and by 3.3 the result follows.

Finally the last assertion obviously holds. □

Denoting an equalizer of the pair $X \underset{g}{\overset{f}{\rightrightarrows}} M \times Y$ by $E_{f,g} \overset{e_{f,g}}{\hookrightarrow} X$, we have:

Corollary 3.15. *Let $\mathbb{M} = (\hat{M}, \eta, \mu)$ be the M-Set monad with $(M, \star, 1)$ an abelian monoid. The morphisms $X \underset{\hat{g}}{\overset{\hat{f}}{\rightrightarrows}} Y$ in the Kleisli category $Set_{\mathbb{M}}$ have an equalizer if and only if for $(m, x) \in M \times X$, $mf_1(x) = mg_1(x)$ and $f_2(x) = g_2(x)$ implies $x \in E_{f,g}$.*

In this case an equalizer$E_{f,g} \overset{\hat{e}}{\longrightarrow} X$ of $\hat{f}$ and $\hat{g}$ corresponds to the map $E_{f,g} \xrightarrow{e = \langle \tilde{1}, e_{f,g} \rangle} M \times X$.

Proof. Suppose the equalizer of $\hat{f}$ and $\hat{g}$ exists. Then by 3.14, we have the existence of a section $s : \dot{I}/R \longrightarrow \dot{I}$ of $q : \dot{I} \longrightarrow \dot{I}/R$ such that $Im(s)$ is invariantly (f, g)-compatible. Let $(m, x) \in Im(s)$. Then $(m, x) \in \dot{I} \subseteq I$ and so $mf_1(x) = mg_1(x)$, $f_2(x) = g_2(x)$ and m is right cancelable. Since the monoid is abelian, m is also left cancelable. Therefore $f_1(x) = g_1(x)$. It follows that $f(x) = g(x)$, i.e., $x \in E_{f,g}$. Now suppose $x \in E_{f,g}$. Then $(1, x) \in I$ and so there is $(u_x, x) \in Im(s)$ such that $1 \geq u_x$. Therefore $1 = au_x = u_x a$, i.e., u_x is invertible. If (m, x) and (n, x) are in $Im(s)$, then since $mn = nm$, $e^*(m, n, x) = e^*(n, m, x)$. Therefore $m = n$ and so $(m, x) = (n, x)$. Putting these together we conclude that $Im(s) = \{(u_x, x) : x \in E_{f,g}\}$, where each u_x is invertible.

Now if $(m, x) \in M \times X$, $mf_1(x) = mg_1(x)$ and $f_2(x) = g_2(x)$, then $(m, x) \in I$ and therefore $(m, x) \geq (n, x)$ with $(n, x) \in Im(s)$. This implies $x \in E_{f,g}$.

Conversely suppose for $(m, x) \in M \times X$, $mf_1(x) = mg_1(x)$ and $f_2(x) = g_2(x)$ implies $x \in E_{f,g}$. It is easy to see for$E_{f,g} \xrightarrow{e=\langle \tilde{1}, e_{f,g}\rangle} M \times X$, $M \times E_{f,g} \xrightarrow{e^*} M \times X$ is the inclusion. Now direct computation shows that e^* is an equalizer of f^* and g^*. The result then follows by 3.3.

The last assertion follows easily. □

Definition 3.16. *Let $\mathbb{P}$ be the power object monad on a topos $\mathcal{E}$ and let* $\mathbf{f} : 1 \longrightarrow PX$ *denote the false map. Given morphisms* $X \underset{g}{\overset{f}{\rightrightarrows}} PY$*,*

$a)$ a morphism $A \xrightarrow{a} PX$ is said to be (f, g)-invariant relative to μ, if it factors through the equalizer $i : I \longrightarrow PX$ of $PX \underset{\mu P(g)}{\overset{\mu P(f)}{\rightrightarrows}} PY$, i.e., if $\mu P(f)a = \mu P(g)a$.

$b)$ a morphism $1 \xrightarrow{a} PX$ is said to be (f, g)-simple if it is a minimal element of $\mathcal{E}(1, PX)$ that is not equal to **f** *and that is (f, g)-invariant relative to μ, where $\mathcal{E}(A, PX)$ is partially ordered by $a \leq b$ if $b = a \vee b$.*

Lemma 3.17. *Let $\mathbb{P}$ be the power object monad on a topos $\mathcal{E}$ and consider the maps $X \underset{g}{\overset{f}{\rightrightarrows}} PY$ in $\mathcal{E}$. A morphism $e : E \longrightarrow PX$ is (f, g)-invariant if and only if the morphism $\mu_X P(e) : PE \longrightarrow PX$ is.*

Proof. Suppose e is (f, g)-invariant. We have:

$$\mu_Y P(f)\mu_X P(e) = \mu_Y \mu_{PY} P^2(f)P(e) = \\ \mu_Y P(\mu_Y)P^2(f)P(e) = \mu_Y P(\mu_Y P(f)e)$$

Similarly $\mu_Y P(g)\mu_X P(e) = \mu_Y P(\mu_Y P(g)e)$. The result then follows.

Now suppose $\mu_X P(e)$ is (f, g)-invariant. We have:

$$\mu_Y P(f)e = \mu_Y P(f)\mu_X \eta_{PX} e = \mu_Y P(f)\mu_X P(e)\eta_E$$

Similarly

$\mu_Y P(g)e = \mu_Y P(g)\mu_X P(e)\eta_E$. The result then follows. □

Recall that, [8], a well-pointed topos is one in which parallel morphisms are equal if they are equal when composed on the right with all the global elements; also that a well-pointed topos is two-valued.

Lemma 3.18. *A topos is well-pointed if and only if for all X, the sink $(\ 1 \xrightarrow{x} X\ )_{x \in hom(1,X)}$ is a coproduct.*

Proof. For an object X of a well-pointed topos, set $G_X = \amalg_{hom(1,X)} 1$. There is a unique φ making the following triangle commute.

$$\begin{array}{ccc} 1 & \xrightarrow{\nu_x} & G_X \\ & \searrow^{x} & \downarrow{\scriptstyle \varphi} \\ & & X \end{array}$$

If $\alpha\varphi = \beta\varphi$, then for all $x \in hom(1, X)$, $\alpha\varphi\nu_x = \beta\varphi\nu_x$ and so for all $x \in hom(1, X)$, $\alpha x = \beta x$. Therefore $\alpha = \beta$, implying φ is an epimorphism. (Note if $hom(1, X) = \emptyset$, then the assertion for all $x \in hom(1, X)$, $\alpha x = \beta x$ is vacuously true, implying $\alpha = \beta$, so that there is at most one morphism with domain X, implying φ is an epimorphism).

Now given $f : 1 \longrightarrow G_X$, form the pullback:

$$\begin{array}{ccc} A_x & \longrightarrow & 1 \\ {\scriptstyle f^{-1}(\nu_x)}\downarrow & p.b. & \downarrow{\scriptstyle \nu_x} \\ 1 & \xrightarrow[f]{} & G_X \end{array}$$

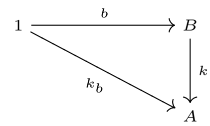
Since the topos is two-valued and A_x is a subobject of 1, it is either 0 or 1. If for all x, $A_x = 0$, then $f^{-1}(\nu_x)$ is $! : 0 \longrightarrow 1$. So $1 = f^{-1}(\oplus_x \nu_x) = \oplus_x f^{-1}(\nu_x) = \oplus_x ! = !$, which is a contradiction. Therefore there exists x such that $A_x = 1$. It follows that there is x such that $f = \nu_x$.

Now if $1 \overset{f}{\underset{g}{\rightrightarrows}} G_X$ are morphisms with $\varphi f = \varphi g$, then there are x and y such that $f = \nu_x$ and $g = \nu_y$. So $\varphi\nu_x = \varphi\nu_y$ implying $x = y$. It follows that $f = \nu_x = \nu_y = g$. So φ is a monomorphism, hence an isomorphism.

The converse follows from the fact that a coproduct is an epi sink. □

Lemma 3.19. *In a well-pointed topos a morphism that has a unique right inverse is an isomorphism.*

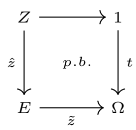
Proof. Suppose $f : A \longrightarrow B$ has a unique right inverse $g : B \longrightarrow A$. Let $a : 1 \longrightarrow A$ be a morphism. For $b \in hom(1, B)$, define $k_b \in hom(1, A)$ by $k_b = \begin{cases} gb \text{ if } b \neq fa \\ a \text{ if } b = fa \end{cases}$. By 3.18, there is a unique morphism $k : B \longrightarrow A$ rendering commutative the following triangle.



Now for all $b \in hom(1, B)$, $fkb = fk_b = \begin{cases} fgb \text{ if } b \neq fa \\ fa \text{ if } b = fa \end{cases} = b$. It follows that $fk = 1$. Uniqueness of g yields $k = g$. So $a = k_{fa} = kfa = gfa$. Therefore for all $a \in hom(1, A)$, $gfa = a$. It follows that $gf = 1$. Hence g is the inverse of f.

□

Lemma 3.20. *Let $\mathbb{P}$ be the power object monad on a topos $\mathcal{E}$. If the arrows $e : E \longrightarrow PX$ and $z : 1 \longrightarrow PE$ are given, then for all arrows $b : 1 \longrightarrow Z$, $e\hat{z}b \leq \mu_X P(e)z$, where $\hat{z}$ is the corresponding subobject of z obtained by:*



Proof. Let $b : 1 \longrightarrow Z$ be given. $\hat{z}$ is classified by $\tilde{z}$ and one can easily verify that $\hat{z}b$ is classified by $\widetilde{\eta\hat{z}b} = \epsilon_E(1 \times \eta_E)(1 \times \hat{z})\langle 1, b! \rangle$. Now since $\hat{z}b \leq \hat{z}$, $\widetilde{\eta\hat{z}b} \leq \tilde{z}$. It follows that $\eta\hat{z}b \leq z$ and so by 2.3, $\mu_X P(e)\eta\hat{z}b \leq \mu_X P(e)z$ which in turn implies $e\hat{z}b \leq \mu_X P(e)z$. □

Lemma 3.21. *Let $\mathbb{P}$ be the power object monad on a topos $\mathcal{E}$. If the map $e : E \longrightarrow PX$ in $\mathcal{E}$ is such that $\mu_X P(e)$ is a monomorphism, then the map $\mathbf{f}_X : 1 \longrightarrow PX$ does not factor through e.*

Proof. Suppose there is $f' : 1 \longrightarrow E$ such that $\mathbf{f}_X = ef'$. Then we have $\mu_X P(e)\eta_E f' = \mu_X \eta_{PX} ef' = \mathbf{f}_X$ and by 2.4 and the fact that $P(e) = \exists_e$

preserves the false map, we have $\mu_X P(e)\mathbf{f}_E = \mathbf{f}_X$. So $\mu_X P(e)\eta_E f' = \mu_X P(e)\mathbf{f}_E$. Since $\mu_X P(e)$ is a monomorphism, we get $\eta_E f' = \mathbf{f}_E$. This implies the subobjects $f' : 1 \longrightarrow E$ and $! : 0 \longrightarrow E$, corresponding to $\eta_E f'$ and $\mathbf{f}_E$ respectively, are equal; which is a contradiction. □

Lemma 3.22. *$a)$ In a topos, if $!_E : 0 \longrightarrow E$ is the unique morphism, then $P(!_E) = \mathbf{f}_E$.*

$b)$ In a well-pointed topos, if $b : 1 \longrightarrow E$ and $z : 1 \longrightarrow PE$ are morphisms such that $\mathbf{f}_E \neq z \leq \eta_E b$, then $z = \eta_E b$.

Proof. $a)$ Since $\mathbf{f}_0 : 1 \longrightarrow P(0) = 1$ is the identity morphism and $P(!_A)\mathbf{f}_0 = \exists_{!_A}\mathbf{f}_0 = \mathbf{f}_A$, we get $P(!_A) = \mathbf{f}_A$.

$b)$ Since the corresponding subobjects of z and $\eta_E b$ are respectively the maps $\hat{z} : Z \longrightarrow E$ and $b : 1 \longrightarrow E$, we get $\hat{z} \leq b$, i.e., there is a morphism $\alpha : 1 \longrightarrow Z$ such that $b\alpha = \hat{z}$. But then α is a monomorphism and since the topos is well-pointed, $Z = 0$ or $Z = 1$. If $Z = 0$, then $z = \mathbf{f}_E$, a contradiction. So $Z = 1$, implying $\hat{z} = b$. It then follows that $z = \eta_E b$. □

For a pair of morphisms $X \overset{f}{\underset{g}{\rightrightarrows}} PY$, denoting an equalizer of the pair $PX \overset{\mu P(f)}{\underset{\mu P(g)}{\rightrightarrows}} PY$ by $i : I \longrightarrow PX$, and setting:

$$\mathcal{S} = \{\, a : 1 \longrightarrow PX \mid a \text{ is } (f,g)\text{-simple}\}$$

we have:

Theorem 3.23. *Let $\mathbb{P}$ be the power object monad on a well-pointed topos $\mathcal{E}$. The morphisms $X \overset{\hat{f}}{\underset{\hat{g}}{\rightrightarrows}} Y$ in the Kleisli category $\mathcal{E}_{\mathbb{P}}$ have an equalizer if and only if there is a unique morphism $I \xrightarrow{s} P(\amalg_{\mathcal{S}} 1)$ such that $\mu_X P(\oplus_{a\in\mathcal{S}} a)s = i$.*

In this case an equalizer$\amalg_{\mathcal{S}} 1 \xrightarrow{\hat{e}} X$ of $\hat{f}$ and $\hat{g}$ corresponds to the map $\amalg_{\mathcal{S}} 1 \xrightarrow{e = \oplus_{a\in\mathcal{S}} a} PX$.

Proof. Let $E \xrightarrow{\hat{e}} X$ be an equalizer of $\hat{f}$ and $\hat{g}$. By 3.3, $PE \xrightarrow{\mu_X P(e)} PX$ is an equalizer of $\mu P(f)$ and $\mu P(g)$. Since i is also an equalizer of this pair, there is a unique isomorphism $s : I \longrightarrow PE$ such that $\mu P(e)s = i$. We are done as soon as we show $E = \amalg_{\mathcal{S}} 1$ and $e = \oplus_{a\in\mathcal{S}} a$.

To prove $E = \amalg_{\mathcal{S}} 1$, we show $\mathcal{S} \cong \mathcal{E}(1, E)$. The result then follows by 3.18.

In case $E \cong 0$, then $PE \cong 1$ and so there is a unique morphism from 1 to PE. It follows that the only morphism from 1 to PE is $\mathbf{f}_E$. Now if $a \in \mathcal{S}$, then there is a unique $a' : 1 \longrightarrow PE$ such that $a = \mu_X P(e)a'$. But then $a' = \mathbf{f}_E$ and so by 2.4, $a = \mathbf{f}_X$, a contradiction. Hence $\mathcal{S} = \emptyset$ and so $\mathcal{S} = \mathcal{E}(1, E) = \emptyset$.

Now assume $E \ncong 0$. Let $a \in \mathcal{S}$. Since a is (f, g)-invariant, there is a unique $1 \xrightarrow{z} PE$ such that $\mu_X P(e)z = a$. If $Z = 0$, then $\tilde{z} = \mathbf{f}!$ and $z = \mathbf{f}_E$. But then by 2.4, $a = \mathbf{f}_X$, a contradiction. So $Z \neq 0$ and therefore by 3.18, $hom(1, Z) \neq \emptyset$. Now let $b : 1 \longrightarrow Z$ be any morphism. By 3.20, $e\hat{z}b \leq a$. Now since $\mu P(e)$ is (f, g)-invariant, so is e by 3.17. It follows that $e\hat{z}b$ is (f, g)-invariant and by 3.21, $e\hat{z}b \neq \mathbf{f}_X$. Since a is (f, g)-simple, $e\hat{z}b = a$. Define $\kappa : \mathcal{S} \longrightarrow \mathcal{E}(1, E)$ to take a to $\hat{z}b$.

Now let $b \in \mathcal{E}(1, E)$. We show eb is (f, g)-simple. Since e is (f, g)-invariant, so is eb. Also by 3.21, $eb \neq \mathbf{f}_X$. Now if $a : 1 \longrightarrow PX$ is (f, g)-invariant, $a \neq \mathbf{f}_X$ and $a \leq eb$, then there is $z : 1 \longrightarrow PE$ such that $a = \mu_X P(e)z$. By 2.3, we have $\mu_X Pe(z \vee \eta_E b) = (\mu_X P(e)z) \vee (\mu_X P(e)\eta_E b) = a \vee eb = eb = \mu_X P(e)\eta_E b$. So $z \vee \eta_E b = \eta_E b$, i.e., $z \leq \eta_E b$. Since $a \neq \mathbf{f}_X$, by 2.4, $z \neq \mathbf{f}_E$. Now by 3.22, $z = \eta_E b$ and so $a = \mu_X P(e)z = \mu_X P(e)\eta_E b = eb$ as desired. Define $\kappa' : \mathcal{E}(1, E) \longrightarrow \mathcal{S}$ to take b to eb.

One can easily verify that $\mathcal{S} \underset{\kappa'}{\overset{\kappa}{\rightleftarrows}} \mathcal{E}(1, E)$ establishes an isomorphism.

Using the isomorphisms κ and κ', one could get the isomorphisms between $\amalg_{\mathcal{S}} 1$ and $\amalg_{\mathcal{E}(1,E)} 1 \cong E$. Some computations then show $e = \oplus_{a\in\mathcal{S}} a$.

Now suppose there is a unique morphism $I \xrightarrow{s} P(\amalg_{\mathcal{S}} 1)$ such that $\mu_X P(\oplus_{a\in\mathcal{S}} a)s = i$. Set $E = \amalg_{\mathcal{S}} 1$ and $e = \oplus_{a\in\mathcal{S}} a$. Since each $a \in \mathcal{S}$ is (f, g)-invariant, so is e and by 3.17, so is $\mu_X P(e)$. Therefore there exists a unique $r : PE \longrightarrow I$ such that $ir = \mu_X P(e)$. Now $irs = \mu_X P(e)s = i$, implying $rs = 1$. Now if there is s' such that $rs' = 1$, then $\mu_X P(e)s' = irs' = i$. Uniqueness implies $s' = s$. Therefore r has a unique right inverse

s. By 3.19, r is an isomorphism. Thus $PE \xrightarrow{\mu_X P(e)} PX \overset{\mu_Y P(f)}{\underset{\mu_Y P(g)}{\rightrightarrows}} PY$ is an equalizer and by 3.3, we are done.

The last assertion is easily seen to be valid.

□

Let $\mathbb{P}$ be the power set monad and $X \overset{f}{\underset{g}{\rightrightarrows}} PY$ be functions in the category Set. Let's call a subset A of X, (f, g)-invariant if $\bigcup_{a \in A} f(a) = \bigcup_{a \in A} g(a)$ and (f, g)-simple if it is minimal non-empty (f, g)-invariant. Setting $S = \{A \subseteq X : A \text{ is } (f, g)\text{-simple}\}$, we have:

Corollary 3.24. *Let $\mathbb{P}$ be the power set monad. The morphisms $X \overset{\hat{f}}{\underset{\hat{g}}{\rightrightarrows}} Y$ in the Kleisli category $Set_{\mathbb{P}}$ have an equalizer if and only if every (f, g)-invariant subset of X can be uniquely written as a union of (f, g)-simple subsets.*

In this case an equalizer $S \xrightarrow{\hat{e}} X$ of $\hat{f}$ and $\hat{g}$ corresponds to the map $S \overset{e}{\hookrightarrow} PX$.

Proof. Follows from 3.23 and the existence of a unique s satisfying the equality $\mu_X P(\oplus_{a \in \mathcal{S}} a)s = i$. □

References

[1] Adamek J., Herrlich H., Strecker G.E., Abstract and Concrete Categories, John Wiley & Sons, Inc., 1990. http://www.tac.mta.ca/tac/reprints/articles/17/tr17.pdf.

[2] Asperti A., Longo G., Categories, Types and Structures, An Introduction to Category Theory for the Working Computer Scientist, Foundations of Computing Series, M.I.T. Press, 1991.

[3] Cockett J. R. B., DRAFT: Category Theory for Computer Science, Department of Computer Science, University of Calgary, Calgary,T2N 1N4, Alberta, Canada, October 16, 2009.

[4] Fiore M., Menni M., Reflective Kleisli Subcategories of the Category of Eilenberg-Moore Algebras for Factorization Monads, Theory and Applications of Categories, Vol. 15, No. 2, 2005, pp. 4065.

[5] Gran M., Vitale E.M., Localizations, of Maltsev Varieties, Theory and Applications of Categories, Vol. 5, No. 12, 1999, pp. 281291.

[6] Jay C.B., An Introduction to Categories in Computing, University of Technology, Sydney, School of Computing Sciences, P.O. Box 123 Broadway, Australia, 2007.

[7] Johnstone P.T., Topos Theory, Academic Press, London, New York, San Francisco, 1977.

[8] Maclane S., Moerdijk I., Sheaves in Geometry and Logic, A First Introduction to Topos Theory, Springer-Verlag, New York, 1992.

[9] Pultr A., Tozzi A., Some Categorical Aspects of Information Systems and Domains, Applied Categorical Structures (2006) 14: 135150.

[10] Szigeti J., On Limits and Colimits in the Kleisli Category, Cahiers de topologie et geometrie differentielle categoriques, tome 24, no 4 (1983), p. 381-391.

[11] Teleiko A., On Extension of Functors to the Kleisli Category of some Weakly Normal Monads, Commentationes Mathematicae Universitatis Carolinae, Vol. 37 (1996), No. 2, 405–410.

[12] Wyler O., Lecture Note on Topoi and Quasitopoi, Department of Mathematics, Carnegie-Mellon University, World Scientific Publishing Co. Pte. Ltd., 1991.

S. N. Hosseini, Y. Qasemi Nezhad
Department of Pure Mathematics
Faculty of Mathematics and Computer
Shahid Bahonar University of Kerman
Kerman, Iran
e-mail: nhoseini@uk.ac.ir