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# THE COMPLETION OF A QUANTUM B-ALGEBRA

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**Résumé.** On montre que les quantales sont les objets injectifs dans la catégorie des quantum B-algèbres, et que chaque quantum B-algèbre a une enveloppe injective. Par une construction explicite, l'enveloppe injective se révèle comme une complétion, plus générale que la complétion de Dedekind-MacNeille. Un resultat récent de Lambek et al., où des structures résiduelles surviennent de manière surprenante, est expliqué à la lumière des quantum B-algèbres, fournissant un autre exemple de leur ubiquité. Des connexions aux structures promonoïdales et aux multi-catégories sont indiquées.

**Abstract.** It is shown that quantales are the injective objects in the category of quantum B-algebras, and that every quantum B-algebra has an injective envelope. By an explicit construction, the injective envelope is revealed as a completion, more general than the Dedekind-MacNeille completion. A recent result of Lambek et al., where residual structures unexpectedly arise, is explained in the light of quantum B-algebras, which gives another instance for their ubiquity. Connections to promonoidal structures and multi-categories are indicated.

**Keywords.** Quantum B-algebra, quantale, completion, injective envelope, partially ordered monoid, promonoidal category, Day convolution, multi-poset, ternary frame.

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## 1. Introduction

Recently, J. Lambek et al. [20] proved that the injective hull of a partially ordered monoid, viewed as an object in a suitable category, is a quantale, and that quantales are injective in that category. The construction is natural, but not straightforward. For example, morphisms are submultiplicative rather than multiplicative, which appears to be natural in the presence of a partial order. Surprisingly, the construction depends on the left and right residuals of a quantale, which led to an unexpected solution, as Lambek remarks: "*mirabile dictu*, it worked!"

In this paper, we show that the reason behind this mystery is the covert presence of a quantum B-algebra. Recall that a *quantum B-algebra* [30, 31] is a partially ordered set $X$ with two binary operations $\rightarrow$ and $\rightsquigarrow$ satisfying

$$x \leqslant y \rightarrow z \iff y \leqslant x \rightsquigarrow z \tag{1}$$

$$x \rightarrow (y \rightsquigarrow z) = y \rightsquigarrow (x \rightarrow z) \tag{2}$$

$$y \leqslant z \implies x \rightarrow y \leqslant x \rightarrow z. \tag{3}$$

A certain ubiquity of quantum B-algebras was observed in [30] and [31]. To mention the two extreme cases: A group is equivalent to a quantum B-algebra with trivial partial order, while on the other hand, any partial order with a greatest element determines a quantum B-algebra. In terms of non-commutative logic, the operations $\rightarrow$ and $\rightsquigarrow$ stand for one-sided implications, while $\leqslant$ interprets the logical entailment relation. By [30], Theorem 2.3, quantum B-algebras can be characterized as systems with two operations $\rightarrow, \rightsquigarrow$ and a partial order which can be embedded into a quantale. Note that quantales can be viewed in several respects as non-commutative spaces [4, 3, 6, 5, 24, 25, 26].

We prove that quantales are the injective objects in the category of quantum B-algebras, and that every quantum B-algebra $X$ has an injective envelope (Theorem 1). Moreover, we give an explicit construction of the injective envelope, generalizing various types of completions (Theorem 2). For example, the Dedekind-MacNeille completion of a poset, or of an archimedean lattice-ordered group, occurs as a special case.

A particular instance is Lambek's above mentioned construction [20] of the injective hull of a partial ordered monoid $M$. As a first step of this construction, $M$ is embedded into the quantale $L(M)$ of lower sets in $M$. We consider the slightly more general issue where $M$ is a partially ordered semigroup. The injective hull $Q(M)$ of $M$ is then obtained as a quantalic quotient $q\colon L(M) \twoheadrightarrow Q(M)$ with a natural embedding $q|_M\colon M \hookrightarrow Q(M)$. We show that the map $q$ is determined by the quantum B-algebra $X_M \subset$

$L(M)$ generated by $M$, and that $q|_{X_M}: X_M \to Q(M)$ is nothing else than the completion of $X_M$ as a quantum B-algebra. This reveals the nature of $Q(M)$ and explains the occurrence of residuals in a context of semigroups. In particular, $q|_{X_M}$ is injective, i. e. the quantum B-algebra $X_M$ remains unaffected by passing to the quotient $L(M) \twoheadrightarrow Q(M)$.

Following a referee's suggestion who pointed out that unital quantum B-algebras form a special instance of a promonoidal category, we explain in Section 6 how quantum B-algebras $X$ and their enveloping quantales $U(X)$ fit into the much broader framework of enriched categories. In particular, we relate the multiplication in $U(X)$ to the Day convolution of $U(X)$ as a functor category. It turns out that promonoidal posets can be characterized as a special class of multicategories, enriched over the two-element quantale. Unital quantum B-algebras form a reflective full subcategory of the category of promonoidal posets (Proposition 11). In the context of multi-posets, the universal property of enveloping quantales is derived in Proposition 12.

Some examples are given in a final section. For instance, we exhibit a quantum B-algebra $X$ for which the underlying partially ordered set is a complete lattice, but where $X$ is not a quantale. This also provides an example where the completion of $X$ is strictly larger than the Dedekind-MacNeille completion. On the other hand, we show that the completion of a quantum B-algebra does not coincide with the canonical extension [14]. Another example shows that the isomorphism class of a partially ordered monoid $M$ need not be determined by the quantum B-algebra $X_M$, though $M$ can be recovered from the quantale $L(M)$.

## 2. Quantum B-algebras and quantales

Quantum B-algebras form a category **qBAlg** [31], morphisms $f: X \to Y$ being monotone and satisfying the equivalent inequalities

$$f(x \to y) \leqslant f(x) \to f(y), \qquad f(x \rightsquigarrow y) \leqslant f(x) \rightsquigarrow f(y). \tag{4}$$

If these inequalities are equations, we call $f: X \to Y$ a *strict* morphism. For example, every *quantale* [23], that is, a complete lattice with an associative multiplication that distributes over arbitrary joins, is a quantum B-algebra. More generally, every *residuated poset*, that is, a partially ordered semigroup with binary operations $\to$ and $\rightsquigarrow$ satisfying

$$x \leqslant y \to z \iff xy \leqslant z \iff y \leqslant x \rightsquigarrow z, \tag{5}$$

is a quantum B-algebra.

**Definition 1.** Let $X$ be a quantum B-algebra. We say that a product $xy$ of elements $x, y \in X$ *exists* if the set $\{z \in X \mid x \leqslant y \to z\}$ has a smallest element. This element will be denoted by $xy$.

Thus, if $xy$ exists, it is unique and satisfies (5).

**Proposition 1.** *Let $X$ be a quantum B-algebra and $x, y, z \in X$. Assume that the products $xy$ and $yz$ exist in $X$. Then $(xy)z$ extsts if and only if $x(yz)$ exists, and in case they exist, these products are equal.*

*Proof.* Assume that $(xy)z$ exists. Then $(xy)z \leqslant t \Leftrightarrow xy \leqslant z \to t \Leftrightarrow y \leqslant x \rightsquigarrow (z \to t) \Leftrightarrow y \leqslant z \to (x \rightsquigarrow t) \Leftrightarrow yz \leqslant x \rightsquigarrow t \Leftrightarrow x \leqslant yz \to t$, which shows that $x(yz)$ exists and is equal to $(xy)z$. The converse follows by symmetry. $\square$

By Proposition 1, it makes sense to speak of a *submonoid* or a *sub-semigroup* of a quantum B-algebra. The latter means a subset $M$ with existing products $xy \in M$ for each pair $x, y \in M$. As usual, we endow any subset of $X$ with the induced partial order.

**Definition 2.** We define a *morphism* $f \colon M \to N$ of partially ordered semigroups to be a monotone map satisfying $f(a)f(b) \leqslant f(ab)$ for all $a, b \in M$.

The following result shows how the inequalities (4) have to be changed in terms of existing products.

**Proposition 2.** *Let $X$ be a quantum B-algebra with a sub-semigroup $M$ such that every $a \in X$ satisfies $a = \bigvee\{z \in M \mid z \leqslant a\}$. Let $f \colon X \to Q$ be a map into a quantale $Q$ with $f(a) = \bigvee\{f(z) \mid a \geqslant z \in M\}$ for all $a \in X$. Then $f$ is a morphism of quantum B-algebras if and only if $f|_M$ is a morphism of partially ordered semigroups.*

*Proof.* Clearly, $f$ is monotone. Assume that $f$ satisfies (4). For all $x, y \in M$, we have $x \leqslant y \to xy$. Hence $f(x) \leqslant f(y \to xy) \leqslant f(y) \to f(xy)$, which gives $f(x)f(y) \leqslant f(xy)$.

Conversely, assume that this inequality holds for $x, y \in M$. For given $a, b \in X$, assume that $x \leqslant a$ and $y \leqslant a \to b$. Then (1) implies that $x \leqslant a \leqslant (a \to b) \rightsquigarrow b$, hence $y \leqslant a \to b \leqslant x \to b$. Thus $yx \leqslant b$. So we obtain $f(y)f(x) \leqslant f(yx) \leqslant f(b)$, which yields $f(a \to b)f(a) \leqslant f(b)$. Whence $f(a \to b) \leqslant f(a) \to f(b)$. $\square$

As an immediate consequence, we have

**Corollary.** *Let $Q$ and $Q'$ be quantales. A map $f\colon Q \to Q'$ is a morphism of quantum B-algebras if and only if $f$ is a morphism of partially ordered semigroups.*

Note that such a morphism $f\colon Q \to Q'$ satisfies $\bigvee f(A) \leqslant f(\bigvee A)$ for subsets $A \subset Q$. Thus, a *quantale homomorphism* is obtained if the inequalities are replaced by equations: $f(\bigvee A) = \bigvee f(A)$ and $f(ab) = f(a)f(b)$ for $A \subset Q$ and $a, b \in Q$.

Let $X$ be a quantum B-algebra, and $x, x_1, \dots, x_n \in X$ with $n > 1$. Inductively, we define

$$x_1 \cdots x_n \leqslant x \;:\Longleftrightarrow\; x_1 \cdots x_{n-1} \leqslant x_n \to x. \tag{6}$$

The fundamental rôle of this relation will become apparent in Section 4. From (5) it follows that (6) becomes a true fact if the products on both sides exist. Note that every morphism $f\colon X \to Y$ of quantum B-algebras satisfies

$$x_1 \cdots x_n \leqslant x \implies f(x_1) \cdots f(x_n) \leqslant f(x) \tag{7}$$

for all $x, x_1, \dots, x_n \in X$. Indeed, this is trivial for $n = 1$. Now $x_1 \cdots x_n \leqslant x$ gives $x_1 \cdots x_{n-1} \leqslant x_n \to x$. Hence, by induction, we can assume that $f(x_1) \cdots f(x_{n-1}) \leqslant f(x_n \to x) \leqslant f(x_n) \to f(x)$. Thus $f(x_1) \cdots f(x_n) \leqslant f(x)$.

**Definition 3.** We define an *embedding* $X \rightarrowtail Y$ of quantum B-algebras $X, Y$ to be a morphism $e\colon X \to Y$ for which the implication (7) is an equivalence. If $X \hookrightarrow Y$ is a strict embedding, we call $X$ a *quantum B-subalgebra* of $Y$.

In particular, an embedding $X \rightarrowtail Y$ is injective, and $X$ can be regarded as a subposet of $Y$. The converse holds for strict morphisms:

**Proposition 3.** *Let $e\colon X \to Y$ be a strict morphism of quantum B-algebras such that $x \leqslant y \iff e(x) \leqslant e(y)$ holds for $x, y \in X$. Then $e$ is an embedding.*

*Proof.* We have to show that $e(x_1)\cdots e(x_n) \leqslant e(x)$ implies $x_1 \cdots x_n \leqslant x$ for given $x, x_1, \ldots, x_n \in X$. For $n = 1$, this follows by the assumption. Otherwise, $e(x_1)\cdots e(x_n) \leqslant e(x)$ yields $e(x_1)\cdots e(x_{n-1}) \leqslant e(x_n) \to e(x) = e(x_n \to x)$. Thus, by induction, we can assume that $x_1 \cdots x_{n-1} \leqslant x_n \to x$. Whence $x_1 \cdots x_n \leqslant x$. □

## 3. The injective envelope

In this section, we construct an injective envelope for every quantum B-algebra. We say that an object $Q$ in **qBAlg** is *injective* if every morphism $X \to Q$ factors through any embedding $X \rightarrowtail Y$ of quantum B-algebras.

**Proposition 4.** *With respect to embeddings, quantales are injective objects in the category* **qBAlg** *of quantum B-algebras.*

*Proof.* Let $X \hookrightarrow Y$ be an embedding, and let $f\colon X \to Q$ be a morphism into a quantale $Q$. By [30], Theorem 2.3, $Y$ embeds into a quantale $Q'$. So it suffices to prove that $f$ factors through $X \hookrightarrow Q'$. Define $f'\colon Q' \to Q$ by

$$f'(a) := \bigvee\{f(x_1)\cdots f(x_n) \mid x_1, \ldots, x_n \in X \text{ and } x_1 \cdots x_n \leqslant a\}.$$

For $a, b \in Q'$, this gives

$$\begin{aligned} &f'(a)f'(b) = \\ &\bigvee\{f(x_1)\cdots f(x_n)f(y_1)\cdots f(y_m) \mid x_i, y_j \in X, x_1 \cdots x_n \leqslant a, y_1 \cdots y_m \leqslant b\} \\ &\leqslant \bigvee\{f(x_1)\cdots f(x_n)f(y_1)\cdots f(y_m) \mid x_1 \cdots x_n y_1 \cdots y_m \leqslant ab\} = f'(ab). \end{aligned}$$

Since $f'$ is monotone, the corollary of Proposition 2 shows that $f'$ is a morphism of quantum B-algebras. Furthermore, $f'|_X = f$ follows by (7) since $X \hookrightarrow Q'$ is an embedding. □

**Definition 4.** We call an embedding $e\colon X \rightarrowtail Y$ of quantum B-algebras *essential* if every morphism $f\colon Y \to Z$ in **qBAlg** for which $fe$ is an embedding is itself an embedding. If, in addition, $Y$ is injective, we call $e$ an *injective envelope* of $X$.

As usual, an injective envelope is unique, up to isomorphism.

**Proposition 5.** *Every essential embedding $e\colon X \rightarrowtail Y$ of quantum B-algebras is strict.*

*Proof.* By [30], Theorem 2.3, there is a strict embedding $i\colon X \rightarrowtail Q$ into a quantale $Q$. Therefore, Proposition 4 implies that $i = fe$ for some morphism $f\colon Y \to Q$. Since $e$ is essential, $f$ is an embedding. For $x, y \in X$, we have

$$fe(x \to y) \leqslant f(e(x) \to e(y)) \leqslant fe(x) \to fe(y) = fe(x \to y).$$

Hence $fe(x \to y) = f(e(x) \to e(y))$, and thus $e(x \to y) = e(x) \to e(y)$. □

Recall that a *nucleus* [27, 28] of a quantale $Q$ is defined to be an endomorphism $j\colon Q \to Q$ which satisfies $a \leqslant j(a) = j^2(a)$ for all $a \in Q$. There is a natural one-to-one correspondence between quantic nuclei $j\colon Q \to Q$ and congruence relations on $Q$: For any surjective quantale homomorphism $p\colon Q \twoheadrightarrow Q'$, every fiber $p^{-1}(p(a))$ of $p$ has a greatest element $j(a)$, which gives a nucleus $j$, and every nucleus arises in this way.

A special type of nucleus is obtained as follows. For a subset $X$ of a quantale $Q$, let $X^*$ denote the sub-semigroup generated by $X$. So the subquantale generated by $X$ is $\{\bigvee A \mid A \subset X^*\}$.

**Proposition 6.** *Let $Q$ be a quantale, generated by a quantum B-subalgebra $X$. Then $j(a) := \bigwedge\{x \in X \mid a \leqslant x\}$ defines a nucleus $j\colon Q \to Q$.*

*Proof.* By definition, $j$ is a *closure operator*, that is, $j$ is monotone with $a \leqslant j(a) = j^2(a)$ for all $a \in Q$. For given $a, b \in Q$, assume that $ab \leqslant x$ for some $x \in X$. For all $y \in X^*$ with $y \leqslant b$, this gives $ay \leqslant x$, hence $a \leqslant y \to x$. Thus $j(a) \leqslant y \to x$, which gives $j(a)y \leqslant x$. Since $b = \bigvee\{y \in X^* \mid y \leqslant b\}$, we obtain $j(a)b \leqslant x$. Similarly, every $z \in X^*$ with

$z \leqslant j(a)$ satisfies $zb \leqslant x$, which gives $b \leqslant z \rightsquigarrow x$. Thus $j(b) \leqslant z \rightsquigarrow x$, which yields $zj(b) \leqslant x$. So we get $j(a)j(b) \leqslant x$ for all $x \in X$ with $ab \leqslant x$. Whence $j(a)j(b) \leqslant j(ab)$. $\square$

**Definition 5.** We say that an embedding $X \hookrightarrow Q$ of a quantum B-algebra $X$ into a quantale $Q$ is *dense* if $X$ generates the quantale $Q$ and every $a \in Q$ is of the form $a = \bigwedge A$ with $A \subset X$.

**Proposition 7.** *An embedding $X \hookrightarrow Q$ of a quantum B-algebra $X$ into a quantale $Q$ is essential if and only if it is dense.*

*Proof.* Assume that $X \hookrightarrow Q$ is dense, and let $f\colon Q \to Y$ be a morphism of quantum B-algebras such that $f|_X$ is an embedding. By [30], Theorem 2.3, there exists a strict embedding $Y \hookrightarrow Q'$ into a quantale $Q'$. Now assume that $a, a_1, \ldots, a_n \in Q$ and $f(a_1)\cdots f(a_n) \leqslant f(a)$. To verify that $X \hookrightarrow Q$ is essential, we have to prove that $a_1 \cdots a_n \leqslant a$. To this end, it is enough to show that $x_1^* \cdots x_n^* \leqslant x$ holds for all $x_1^*, \ldots, x_n^* \in X^*$ and $x \in X$ with $x_i^* \leqslant a_i$ and $a \leqslant x$. If $x_i^* = x_{i,1} \cdots x_{i,m_i}$ with $x_{i,j} \in X$, then $f(x_{i,1}) \cdots f(x_{i,m_i}) \leqslant f(x_{i,1} \cdots x_{i,m_i}) \leqslant f(a_i)$. Therefore, the inequality $f(x_{1,1}) \cdots f(x_{1,m_1}) \cdots f(x_{n,1}) \cdots f(x_{n,m_n}) \leqslant f(x)$ holds in $Q'$. Since $X \xrightarrow{f|_X} Y \hookrightarrow Q'$ is an embedding, this yields

$$x_1^* \cdots x_n^* = x_{1,1} \cdots x_{1,m_1} \cdots x_{n,1} \cdots x_{n,m_n} \leqslant x.$$

Conversely, assume that $X \hookrightarrow Q$ is essential. By Proposition 5, $X$ is a quantum B-subalgebra of $Q$. Let $Q_0$ be the subquantale of $Q$ generated by $X$. Then $Q_0 \hookrightarrow Q$ is an embedding. By Proposition 6,

$$j(a) := \bigwedge \{x \in X \mid a \leqslant x\}$$

defines a nucleus $j\colon Q_0 \to Q_0$. So the quantale homomorphism $Q_0 \twoheadrightarrow jQ_0$ factors through $Q_0 \hookrightarrow Q$, which gives a commutative diagram

$$\begin{tikzcd}
X \arrow[r, hook] \arrow[dr, "i"'] & Q_0 \arrow[r, hook] \arrow[d, two heads] & Q \arrow[dl, "f"] \\
& jQ_0 &
\end{tikzcd}$$

with an embedding $i$. Hence $f$ is an embedding, and thus $j$ is the identity map. So $f$ is an injective retraction, which shows that $Q_0 = Q$. Consequently, $X \hookrightarrow Q$ is dense. □

Now we are ready to prove

**Theorem 1.** *Every quantum B-algebra $X$ has an injective envelope.*

*Proof.* By [30], Theorem 2.3, there is a strict embedding $e\colon X \hookrightarrow Q$ into a quantale $Q$. As in the preceding proof, let $Q_0$ be the subquantale generated by $X$. So the nucleus $j\colon Q_0 \to Q_0$ on $Q_0$ yields a dense embedding $X \hookrightarrow jQ_0$ into the quantale $jQ_0$. □

**Corollary.** *For a quantum B-algebra $Q$, the following are equivalent.*

(a) *$Q$ is a quantale.*

(b) *$Q$ is injective in* **qBAlg**.

(c) *Every essential embedding $Q \rightarrowtail X$ is an isomorphism.*

*Proof.* (a) ⇒ (b) follows by Proposition 4.

(b) ⇒ (c): Let $e\colon Q \rightarrowtail X$ be an essential embedding. Then there is a morphism $f\colon X \to Q$ with $fe = 1_Q$. Since $e$ is essential, $f$ is an embedding. Hence $e$ is invertible.

(c) ⇒ (a): This follows immediately by the proof of Theorem 1. □

## 4. The completion

By [30], Theorem 2.3, every quantum B-algebra $X$ admits a strict embedding

$$X \hookrightarrow U(U(X))$$

into a quantale, where $U(X)$ denotes the quantale of upper sets of $X$, with multiplication

$$A \cdot B := \{x \in X \mid \exists\, y \in B\colon y \to x \in A\} \tag{8}$$

for $A, B \in U(X)$. Together with the proof of Theorem 1, this leads to an explicit construction of the injective envelope. In this section, we give a direct approach, without using the embedding $X \hookrightarrow U(U(X))$. As a byproduct, this yields an independent proof of the strict embeddability of $X$ into a quantale.

Let $X^f$ denote the free semigroup generated by $X$. Then (6) defines a relation $a \leqslant x$ between $a \in X^f$ and $x \in X$. For subsets $A \subset X^f$ and $Y \subset X$, we write

$$A \leqslant Y \;:\Longleftrightarrow\; \forall\, a \in A,\, y \in Y\colon a \leqslant y. \tag{9}$$

If $A$ or $Y$ is a singleton, we simply write $a \leqslant Y$ or $A \leqslant y$ instead of $\{a\} \leqslant Y$ or $A \leqslant \{y\}$, respectively. The relation (9) induces a Galois connection between the power sets $\mathfrak{P}(X^f)$ and $\mathfrak{P}(X)$, given by

$$A^{\uparrow} := \{x \in X \mid A \leqslant x\}$$
$$Y^{\downarrow} := \{a \in X^f \mid a \leqslant Y\}$$

for $A \in \mathfrak{P}(X^f)$ and $Y \in \mathfrak{P}(X)$. Thus every $Y \subset X$ has a *closure* $Y^{\downarrow\uparrow}$. We call $Y$ *closed* if $Y = Y^{\downarrow\uparrow}$ and denote the set of closed subsets of $X$ by $\widehat{X}$. For $Y, Z \in \widehat{X}$, we define

$$Y \cdot Z := \{x \in X \mid \forall\, b \leqslant Y,\, c \leqslant Z\colon bc \leqslant x\}, \tag{10}$$

and for a family of $Y_i \in \widehat{X}$, we set

$$\bigvee Y_i := \bigcap Y_i. \tag{11}$$

Note that $\bigcap Y_i$ is again closed. Finally, there is a natural injection $X \rightarrowtail \widehat{X}$ which maps $x \in X$ to the upper set $\uparrow x := \{y \in X \mid x \leqslant y\}$. We endow $\widehat{X}$ with the partial order

$$Y \leqslant Z \;:\Longleftrightarrow\; Y \supset Z.$$

**Theorem 2.** *Let $X$ be a quantum B-algebra. Then $\widehat{X}$ is a quantale, and $X \rightarrowtail \widehat{X}$ is an injective envelope of $X$.*

*Proof.* Eq. (11) makes $\widehat{X}$ into a complete lattice. For $B, C \subset X^f$, we set

$$BC := \{bc \mid b \in B,\, c \in C\}.$$

Then Eq. (10) can be written as

$$Y \cdot Z = (Y^{\downarrow} Z^{\downarrow})^{\uparrow}.$$

To prove the associativity of (10), we thus have to verify

$$((Y_1^{\downarrow} Y_2^{\downarrow})^{\uparrow\downarrow} Y_3^{\downarrow})^{\uparrow} = (Y_1^{\downarrow} (Y_2^{\downarrow} Y_3^{\downarrow})^{\uparrow\downarrow})^{\uparrow} \tag{12}$$

for $Y_1, Y_2, Y_3 \in \widehat{X}$. For $a \in X^f$ and $x \in X$, we define $a \to x$, $a \rightsquigarrow x \in X$ inductively by

$$ay \to x := a \to (x \to y), \qquad ya \rightsquigarrow x := a \rightsquigarrow (y \rightsquigarrow x),$$

for $y \in X$. Then

$$ab \leqslant x \iff a \leqslant b \to x \iff b \leqslant a \rightsquigarrow x$$

holds for $a, b \in X^f$ and $x \in X$. Now

$$\begin{aligned}
(Y_1^{\downarrow} Y_2^{\downarrow})^{\uparrow\downarrow} Y_3^{\downarrow} \leqslant x &\iff \forall\, c \in Y_3^{\downarrow} \colon (Y_1^{\downarrow} Y_2^{\downarrow})^{\uparrow\downarrow} \leqslant c \to x \\
&\iff \forall\, c \in Y_3^{\downarrow} \colon Y_1^{\downarrow} Y_2^{\downarrow} \leqslant c \to x \\
&\iff Y_1^{\downarrow} Y_2^{\downarrow} Y_3^{\downarrow} \leqslant x \iff \forall\, a \in Y_1^{\downarrow} \colon Y_2^{\downarrow} Y_3^{\downarrow} \leqslant a \rightsquigarrow x \\
&\iff \forall\, a \in Y_1^{\downarrow} \colon (Y_2^{\downarrow} Y_3^{\downarrow})^{\uparrow\downarrow} \leqslant a \rightsquigarrow x \\
&\iff Y_1^{\downarrow} (Y_2^{\downarrow} Y_3^{\downarrow})^{\uparrow\downarrow} \leqslant x
\end{aligned}$$

is valid for all $x \in X$. This proves Eq. (12).

Next assume that $Y, Y_i \in \widehat{X}$ for $i \in I \neq \varnothing$. Then $Y_j^{\downarrow} \subset \bigcup_{i \in I} Y_i^{\downarrow}$ implies $(\bigcup_{i \in I} Y_i^{\downarrow})^{\uparrow} \subset Y_j^{\downarrow\uparrow} = Y_j$ for all $j \in I$. Hence $(\bigcup_{i \in I} Y_i^{\downarrow})^{\uparrow} \subset \bigcap_{i \in I} Y_i = (\bigcap_{i \in I} Y_i)^{\downarrow\uparrow}$. For $b \in Y^{\downarrow}$ and $x \in X$, this gives

$$\forall\, i \in I \colon Y_i^{\downarrow} \leqslant b \rightsquigarrow x \implies (\bigcap_{i \in I} Y_i)^{\downarrow} \leqslant b \rightsquigarrow x. \tag{13}$$

The reverse implication is trivial. Thus

$$\forall\, i \in I \colon Y^{\downarrow} Y_i^{\downarrow} \leqslant x \iff Y^{\downarrow} (\bigcap_{i \in I} Y_i)^{\downarrow} \leqslant x$$

holds for all $x \in X$. So we obtain $\bigcap_{i\in I}(Y^{\downarrow}Y_i^{\downarrow})^{\uparrow} = (Y^{\downarrow}(\bigcap_{i\in I} Y_i)^{\downarrow})^{\uparrow}$, which proves that $\bigvee_{i\in I}(Y\cdot Y_i) = Y\cdot\bigvee_{i\in I} Y_i$. If we replace $b \rightsquigarrow x$ in (13) by $b \to x$, we obtain $\bigvee_{i\in I}(Y_i\cdot Y) = (\bigvee_{i\in I} Y_i)\cdot Y$. Thus $\widehat{X}$ is a quantale.

For $x, y \in X$ and $Y \in \widehat{X}$, we have

$$\begin{aligned} Y \leqslant {\uparrow}x \to {\uparrow}y \iff Y\cdot {\uparrow}x \leqslant {\uparrow}y \iff {\uparrow}y \subset (Y^{\downarrow}\{x\}^{\downarrow})^{\uparrow} \iff Y^{\downarrow}\{x\}^{\downarrow} \leqslant y \\ \iff Y^{\downarrow} \leqslant x \to y \iff {\uparrow}(x \to y) \subset Y \iff Y \leqslant {\uparrow}(x \to y), \end{aligned}$$

which shows that ${\uparrow}(x \to y) = {\uparrow}x \to {\uparrow}y$. Furthermore,

$$ {\uparrow}x \leqslant {\uparrow}y \iff {\uparrow}y \subset {\uparrow}x \iff x \leqslant y. $$

Hence $X \rightarrowtail \widehat{X}$ is a strict embedding. In particular,

$$ {\uparrow}x_1 \cdots {\uparrow}x_n \leqslant {\uparrow}x \iff x_1 \cdots x_n \leqslant x \tag{14} $$

holds for $x, x_1, \dots, x_n \in X$. For $Y \in \widehat{X}$ and $x \in X$,

$$ Y \leqslant {\uparrow}x \iff {\uparrow}x \subset Y \iff x \in Y. \tag{15} $$

Hence $Y = \bigwedge_{x\in Y} {\uparrow}x$. Furthermore, with the abbreviation $a^{\uparrow} := \{a\}^{\uparrow}$,

$$\begin{aligned} \bigvee\{a^{\uparrow} \mid a \in X^f,\ a \leqslant Y\} \leqslant {\uparrow}x &\iff \forall\, a \in X^f\colon (a \leqslant Y \Rightarrow x \in a^{\uparrow}) \\ &\iff \forall\, a \in X^f\colon (a \leqslant Y \Rightarrow a \leqslant x) \\ &\iff x \in Y \iff Y \leqslant {\uparrow}x. \end{aligned}$$

Hence $Y = \bigvee\{a^{\uparrow} | a \in X^f,\ a \leqslant Y\}$. For $a := x_1 \cdots x_n$ and $x_1, \dots, x_n \in X$, the equivalences (14) and (15) show that $a^{\uparrow} = {\uparrow}x_1 \cdots {\uparrow}x_n$. Therefore, $X$ is dense in $\widehat{X}$. Thus Proposition 7 completes the proof. □

Note that the construction of $\widehat{X}$ exhibits a strong similarity to the Dedekind-MacNeille completion, with the main difference that the partial order is replaced by the fundamental relation (6). Therefore, we call $\widehat{X}$ the *completion* of the quantum B-algebra $X$. This improves the same-named concept in [30], which was shown to be closely related, but not equivalent to the canonical extension [14] of $X$. The correctness of our adjustment, which makes use of the nucleus in Proposition 6 to pass to a quotient quantale, is now apparent by its affinity to the Dedekind-MacNeille completion.

## 5. The case of partially ordered semigroups

Lambek et al. [20] constructed injective hulls in the category **PoM** of partially ordered monoids and showed that they coincide with unital quantales if morphisms $f$ in **PoM** are declared to satisfy Definition 2 and $f(1) = 1$. We will show now that the construction in [20] makes implicit use of a quantum B-algebra.

Let $M$ be a partially ordered semigroup. As in [20], we embed $M$ into the quantale $L(M)$ of *lower sets* $A \subset M$, that is, $A = \downarrow A := \{x \in M \mid \exists\, y \in A \colon x \leqslant y\}$. Thus $a \in M$ is mapped to the lower set $\downarrow a := \downarrow\{a\} \in L(M)$. Let $X_M$ be the quantum B-subalgebra of $L(M)$ generated by $M$. Thus $X_M$ consists of all terms built from elements of $M$ by using the residuals

$$A \to B := \{c \in M \mid cA \subset B\}, \qquad A \rightsquigarrow B := \{c \in M \mid Ac \subset B\}$$

in $L(M)$. For example, $\downarrow a \rightsquigarrow (\downarrow b \to \downarrow c) = \{d \in M \mid adb \leqslant c\}$ is an element of $X_M$. We identify $M$ with the image of $M \rightarrowtail X_M$. Thus $X_M = M$ if and only if $M$ is a residuated poset.

Following [20], we say that a morphism $f \colon M \to N$ of partially ordered semigroups is an *embedding* if the implication

$$f(x_1) \cdots f(x_n) \leqslant f(x) \implies x_1 \cdots x_n \leqslant x$$

holds for all $x, x_1, \ldots, x_n \in M$.

**Proposition 8.** *A morphism $f \colon M \to N$ of partially ordered semigroups is an embedding if and only if every morphism $M \to Q$ into a quantale $Q$ factors through $f$.*

*Proof.* The necessity follows by the same argument as in the proof of [20], Theorem 4.1. Conversely, let $f \colon M \to N$ be a morphism of partially ordered semigroups. Assume that the embedding $i \colon M \hookrightarrow L(M)$ factors through $f$. So there is a morphism $g \colon N \to L(M)$ with $gf = i$. Suppose that $f(x_1) \cdots f(x_n) \leqslant f(x)$ holds for some $x, x_1, \ldots, x_n \in M$. Then $i(x_1 \cdots x_n) = gf(x_1) \cdots gf(x_n) \leqslant g(f(x_1) \cdots f(x_n)) \leqslant gf(x) = i(x)$. Hence $x_1 \cdots x_n \leqslant x$. $\square$

As in [20], we define injectivity with respect to embeddings. We call an embedding $e \colon M \to N$ *essential* if every morphism $f \colon N \to N'$ for which

$fe$ is an embedding is itself an embedding. An essential embedding into an injective object will be called an *injective envelope*.

**Theorem 3.** *Let $M$ be a partially ordered semigroup, and let $X_M$ be the associated quantum B-algebra. The completion of $X_M$ is an injective envelope in the category of partially ordered semigroups.*

*Proof.* By Proposition 8, The quantale $\widehat{X_M}$ is injective as a partially ordered semigroup. Thus, it remains to verify that $M \hookrightarrow X_M \hookrightarrow \widehat{X_M}$ is an essential embedding. By Proposition 2, every morphism $M \to Q$ into a quantale $Q$ extends to a morphism $X_M \to Q$ of quantum B-algebras, which further extends to a morphism $f\colon \widehat{X_M} \to Q$ in **qBAlg**. By the corollary of Proposition 2, $f$ is a morphism of partially ordered semigroups. So Proposition 8 implies that $M \hookrightarrow X_M \hookrightarrow \widehat{X_M}$ is an embedding. Now let $f\colon \widehat{X_M} \to Q$ be a morphism of partially ordered semigroups such that $f|_M$ is an embedding. If the composed map $\widehat{X_M} \xrightarrow{f} Q \hookrightarrow L(Q)$ is an embedding, $f$ is an embedding, too. So we can assume, without loss of generality, that $Q$ is a quantale.

Next we show that $f|_{X_M}$ is an embedding of quantum B-algebras. Since $X_M \hookrightarrow \widehat{X_M}$ is strict by Proposition 5, we have to verify

$$f(a_1)\cdots f(a_n) \leqslant f(a) \implies a_1\cdots a_n \leqslant a \tag{16}$$

for $a, a_1, \ldots, a_n \in X_M$, where the product $a_1 \cdots a_n$ can be taken in $\widehat{X_M}$. Thus $a_1\cdots a_n = \bigvee\{x_1\cdots x_n \mid a_i \geqslant x_i \in M\}$. Hence, without loss of generality, we can assume that $a_1, \ldots, a_n \in M$. So the implication (16) is valid for $a \in M$. If $a \notin M$, then either $a = b \to c$ or $a = b \rightsquigarrow c$, with terms $b, c \in X_M$ of smaller complexity than $a$. If $a = b \to c$, we have $f(a_1)\cdots f(a_n) \leqslant f(a) \leqslant f(b) \to f(c)$, which gives $f(a_1)\cdots f(a_n)f(b) \leqslant f(c)$. Thus, by induction, we can assume that $a_1 \cdots a_n b \leqslant c$. Whence $a_1 \cdots a_n \leqslant b \to c = a$. The case $a = b \rightsquigarrow c$ is treated similarly.

So we have proved that $f|_{X_M}$ is an embedding. Since $X_M \hookrightarrow \widehat{X_M}$ is essential, this shows that $f$ is an embedding of quantum B-algebras, hence an embedding of partially ordered semigroups. $\square$

**Remark.** The construction in [20] embeds $M$ into $L(M)$ first and than passes to some quotient quantale $p\colon L(M) \twoheadrightarrow Q(M)$ with $p|_M$ invertible.

The preceding proof shows that $p|_{X_M}$ is invertible, too, which highlights the relevance of the quantum B-algebra $X_M$ as an intermediate step toward the injective envelope of $M$. A minor point is that Lambek et al. [20] deal with monoids instead of semigroups. We briefly address this special case now.

Recall that a quantum B-algebra $X$ is said to be *unital* if there is an element $u \in X$ with

$$u \to x = u \rightsquigarrow x = x$$

for all $x \in X$. Such a *unit element* $u$ is unique [31].

**Proposition 9.** *If $M$ is a partialy ordered monoid, then $X_M$ is unital. If $X$ is a unital quantum B-algebra, $\widehat{X}$ is a unital quantale.*

*Proof.* Let $M$ be a partialy ordered monoid with unit element $u$. For $a \in X_M$ and $x \in M$, we have $x \leqslant u \to a \iff xu \leqslant a \iff x \leqslant a \iff ux \leqslant a \iff x \leqslant u \rightsquigarrow a$. Hence $u \to a = a = u \rightsquigarrow a$. Now let $X$ be a unital quantum B-algebra. For $x, y \in X$, this gives $x \leqslant u \to y \iff x \leqslant y$. Thus $xu$ exists, and $xu = x$. Similarly, $ux = x$. Since $X \hookrightarrow \widehat{X}$ is strict, $ux = xu = x$ holds in $\widehat{X}$. Now every element of $\widehat{X}$ is a join of elements from $X$. Whence $ua = au = a$ for all $a \in \widehat{X}$. □

## 6. A categorical perspective

The preceding theorems admit far-reaching generalizations in the framework of enriched categories. We follow a referee's suggestion to put the above results into that wider perspective. All of this section is based upon the referee's detailed remarks.

First, every preordered set $A$ can be regarded as a category, enriched over the cartesian monoidal category **2** with two objects and a single non-identity morphism. Since any such category $A$ is equivalent to its skeleton, we can restrict ourselves to partially ordered sets. Then a **2**-*distributor* $\Phi\colon A \mathrel{\circ\!\!\to} B$ between posets $A$ and $B$ in the sense of Bénabou [1] is given by a monotone map $\Phi\colon B^{\mathrm{op}} \times A \to \mathbf{2}$. In other words, $\Phi^{-1}(1)$ is an upper set of $B^{\mathrm{op}} \times A$, an *ideal relation* between $A$ an $B$. By adjunction, $\Phi$ can be viewed as a functor $A \to \mathbf{2}^{B^{\mathrm{op}}}$ into the category of **2**-valued presheaves over $B$, that is, a monotone map $\widehat{\Phi}\colon A \to L(B)$ into the set of lower sets of $B$. If $I$ denotes the inclusion $B \hookrightarrow L(B)$, the composition $\Psi \otimes \Phi$ of a second distributor $\Psi\colon B \multimap C$ with $\Phi$ corresponds to $(\mathrm{Lan}_I \widehat{\Psi})\widehat{\Phi}\colon A \to L(C)$, where the left Kan-extension $\mathrm{Lan}_I \widehat{\Psi}\colon L(B) \to L(C)$ is given by

$$\mathrm{Lan}_I \widehat{\Psi}(b) := \bigvee_{b \geqslant x \in B} \widehat{\Psi}(x)$$

for $b \in L(B)$. Equivalently, $\Psi \otimes \Phi$ can be computed as a coend

$$\Psi \otimes \Phi = \int^{b \in B} \Psi(-, b) \times \Phi(b, -), \tag{17}$$

corresponding to the product of ideal relations

$$(\Psi \otimes \Phi)^{-1}(1) = \Psi^{-1}(1) \circ \Phi^{-1}(1).$$

Let **Idl** denote the category of posets with ideal relations as morphisms. For a poset $B$, we regard $L(B)$ as an object of **Sup**, the category of *sup-lattices* [19], that is, complete lattices with set-indexed join-preserving morphisms. So the morphisms $\Phi\colon A \multimap B$ in **Idl** can be viewed as morphisms $L(A) \to L(B)$ in **Sup**, which exhibits **Idl** as a reflective full subcategory of **Sup**.

Let $X$ be a quantum B-algebra. The ideal relation $P \subset (X \times X)^{\mathrm{op}} \times X$ with

$$(x, y, z) \in P \;:\Longleftrightarrow\; x \leqslant y \to z \tag{18}$$

gives a **2**-functor $X^{\mathrm{op}} \times X^{\mathrm{op}} \times X \to \mathbf{2}$. Recall that a *promonoidal category* $\mathscr{A}$ (over **2**) [9, 10, 11] is defined by a pair of **2**-functors

$$P\colon \mathscr{A}^{\mathrm{op}} \otimes \mathscr{A}^{\mathrm{op}} \otimes \mathscr{A} \to \mathbf{2} \quad \text{and} \quad J\colon \mathscr{A} \to \mathbf{2}$$

with natural isomorphisms

$$\alpha_{a,b,c,d}\colon \int^x P(a,b,x) \otimes P(x,c,d) \xrightarrow{\sim} \int^x P(b,c,x) \otimes P(a,x,d) \tag{19}$$

$$\lambda_{a,b}\colon \int^x J(x) \otimes P(x,a,b) \xrightarrow{\sim} \mathrm{Hom}_{\mathscr{A}}(a,b) \tag{20}$$

$$\rho_{a,b}\colon \int^x J(x) \otimes P(a,x,b) \xrightarrow{\sim} \mathrm{Hom}_{\mathscr{A}}(a,b) \tag{21}$$

satisfying two coherence conditions [9, 11]. Accordingly, a *promomoidal functor* [12] between promonoidal categories $\mathscr{A}, \mathscr{B}$ is a functor $\Phi\colon \mathscr{A} \to \mathscr{B}$ with two natural transformations $\varphi_{a,b,c}\colon P(a,b,c) \to P(\Phi a, \Phi b, \Phi c)$ and $\varphi_a\colon Ja \to J\Phi a$ satisfying certain relations [8, 11]. For the base category **2**, we speak of a *promonoidal poset*. Then a promomoidal **2**-functor between promonoidal posets $A, B$ is just a monotone map $\Phi\colon A \to B$ which satisfies $\Phi(J) \subset J$ and

$$(x,y,z) \in P \implies (\Phi(x), \Phi(y), \Phi(z)) \in P. \tag{22}$$

**Proposition 10.** *With respect to* (18)*, every quantum B-algebra $X$ satisfies the associativity condition* (19)*. If $X$ is unital, $X$ is a promonoidal poset.*

*Proof.* In terms of ideal relations, (19) states that for given $a, b, c, d \in X$, there exists an $x \leqslant c \to d$ with $a \leqslant b \to x$ if and only if there is an $x \in X$ with $b \leqslant c \to x$ and $a \leqslant x \to d$. The second condition is equivalent to the existence of an $x \leqslant a \rightsquigarrow d$ with $b \leqslant c \to x$. So we have to check the equivalence

$$a \leqslant b \to (c \to d) \iff b \leqslant c \to (a \rightsquigarrow d).$$

Indeed, $a \leqslant b \to (c \to d) \iff b \leqslant a \rightsquigarrow (c \to d) \iff b \leqslant c \to (a \rightsquigarrow d)$. If $X$ is unital, the upper set $\uparrow u$ defines a morphism $J\colon X \to \mathbf{2}$. Then (20) and (21) are equivalent to

$$\exists\, x \geqslant u\colon x \leqslant a \to b \iff a \leqslant b \iff \exists\, x \geqslant u\colon a \leqslant x \to b.$$

This can be rewritten as

$$a \leqslant u \rightsquigarrow b \iff a \leqslant b \iff a \leqslant u \to b,$$

which is equivalent to $u \rightsquigarrow b = b = u \to b$. $\square$

Proposition 10 sheds some light upon the enveloping quantale $U(X) = \mathbf{2}^X$. Let $X$ be a poset with a distributor $P\colon X \nrightarrow X \times X$. In **Sup** this gives a morphism $L(X) \to L(X \times X) = L(X) \otimes L(X)$, or dually, a morphism $U(X) \otimes U(X) \to U(X)$. Then (19) states that $U(X)$ is a semigroup object in **Sup**, a quantale. In terms of (18), the map $L(X) \to L(X \times X)$ is given by $z \mapsto \{(x,y) \mid x \leqslant y \to z\}$ for $z \in X$, or

$$C \mapsto \{(x,y) \mid \exists\, z \in C \colon x \leqslant y \to z\}$$

for $C \in L(X)$. The dual $f^\circ \colon U(Y) \to U(X)$ of a morphism $f \colon L(X) \to L(Y)$ is given by $f^\circ(A) := \uparrow f^{-1}(A)$. So the multiplication on $U(X)$ becomes

$$\begin{aligned} A \cdot B &= \{z \in X \mid \exists\, (x,y) \in A \times B \colon x \leqslant y \to z\} \\ &= \{z \in X \mid \exists\, y \in B \colon y \to z \in A\}, \end{aligned} \tag{23}$$

in conformity with formula (8). According to R. K. Meyer [22], the multiplication (8) is well known to logicians. Following L. Powers, he calls it *modus ponens product*. Fine [16] calls it *fusion*. If we regard $A, B \in U(X)$ as functors $X \to \mathbf{2}$, the first equation in (23) can be written as

$$A \cdot B = \int^{x,y} P(x,y,-) \otimes A(x) \otimes B(y), \tag{24}$$

which identifies the multiplication in $U(X)$ with the *Day convolution* in $\mathbf{2}^X$ [9].

It remains to clarify the difference between a promonoidal poset and a unital quantum B-algebra. By [31], Theorem 1, the category of quantum B-algebras is equivalent to the category of *logical quantales*, that is, quantales of the form $U(X)$ for some poset $X$ with

$$x \cdot \big(\bigwedge_{i \in I} a_i\big) = \bigwedge_{i \in I} (x \cdot a_i), \qquad \big(\bigwedge_{i \in I} a_i\big) \cdot x = \bigwedge_{i \in I} (a_i \cdot x)$$

for all $x \in X$ and $a_i \in U(X)$. Thus, in comparison with a promonoidal poset, a unital quantum B-algebras satisfies this extra condition. The point is that the promonoidal structure does not guarantee that $X \subset U(X)$ is closed under $\to$ and $\rightsquigarrow$. In other words, a promonoidal poset is an implicational algebra without implicational operations.

To make this precise, let us interpret the relation (6) in the framework of multicategories [21]. Let $X^f$ denote the free semigroup generated by a set $X$. We define a *multi-poset* to be a set $X$ with a binary relation $a \leqslant x$ for $x \in X$ and $a \in X^f$, such that the following are satisfied for $x, y, x_i \in X$ and $a_i \in X^f$:

(a) $(a_1 \leqslant x_1, \dots, a_n \leqslant x_n$ and $x_1 \cdots x_n \leqslant x) \implies a_1 \cdots a_n \leqslant x$.

(b) $x \leqslant y \leqslant x \iff x = y$.

Morphisms of multi-posets are multi-functors, that is, maps $f\colon X \to Y$ satisfying (7). For quantum B-algebras, the case $n = 2$ is equivalent to the first inequality of (4). Thus quantum B-algebras form a full subcategory **qBAlg** of the category **mPos** of multi-posets.

Just as in (18), we can define a distributor $X \multimap X \times X$ for any multi-poset $X$ by the corresponding relation $P \subset X^{\mathrm{op}} \times X^{\mathrm{op}} \times X$ with

$$(x, y, z) \in P \;:\Longleftrightarrow\; xy \leqslant z. \tag{25}$$

The convolution formula (24) then makes $U(X)$ into a quantale with multiplication

$$A \cdot B = \{z \in X \mid \exists\,(x, y) \in A \times B\colon xy \leqslant z\}.$$

Define a *truth set* of a multi-poset $X$ to be an upper set $U \subset X$ such that for all $x, y \in X$,

$$x \leqslant y \iff \exists\, t \in U\colon tx \leqslant y \iff \exists\, t \in U\colon xt \leqslant y.$$

If $X$ admits a truth set, we call $X$ *unital*. Let us call a multi-poset $X$ *coherent* if the implication

$$axy \leqslant z \implies \exists\, t \in X\colon xy \leqslant t,\ at \leqslant z \tag{26}$$

holds for $x, y, z \in X$ and $a \in X^f$. Not every multi-poset is coherent. For example, let $\{x\}$ be a singleton with $x \cdots x \leqslant x$ if and only if the length of $x \cdots x$ is odd. Then $X$ is not coherent.

**Proposition 11.** *The category of promonoidal posets is equivalent to the category of unital coherent multi-posets. The category of unital quantum B-algebras admits a full embedding into each of these categories.*

*Proof.* For a promonoidal poset $X$, note first that with (25), condition (19) turns into the equivalence

$$(ab)c \leqslant d \iff a(bc) \leqslant d \tag{27}$$

which defines a unique relation $abc \leqslant d$ for $a, b, c, d \in X$. As the reverse implication in (26) holds for multi-posets, we use (26) to define $x_1 \cdots x_n \leqslant x$ via induction. By (27), this gives a coherent multi-poset $X$. Moreover, (20) and (21) state that $X$ is unital. Therefore, promonoidal posets are equivalent to coherent unital multi-posets. For a map $\Phi\colon X \to Y$ between multi-posets, the implication (22) states that $xy \leqslant z$ implies $\Phi(x)\Phi(y) \leqslant \Phi(z)$. By induction, this proves the first statement of the proposition. By (6), quantum B-algebras are coherent as multi-posets. This gives the second statement. □

To determine the full subcategory of unital quantum B-algebras, let us denote the one-element poset by **1**. Then a promonoidal poset $X$ is given by a pair of distributors $\mathbf{1} \overset{J}{\leftarrow\!\circ\!-} X \overset{P}{-\!\circ\!\to} X \times X$, that is, monotone functions

$$J\colon X \to \mathbf{2}, \qquad P\colon (X \times X)^{\mathrm{op}} \times X \to \mathbf{2}, \tag{28}$$

satisfying (19)-(21). Let us call $X$ *representable* if $J$ and the presheaves $P(-, y, z)$ and $P(x, -, z)$ are representable for all $x, y, z \in X$. Then we have

**Corollary.** *A promonoidal poset* (28) *is representable if and only if it is a unital quantum B-algebra.*

*Proof.* Representability of $J$ means that there is an element $u \in X$ with $J^{-1}(1) = {\uparrow}u$. Similarly, the presheaves $P(-, y, z)$ and $P(x, -, z)$ are representable if and only if there are binary operations $\to$ and $\rightsquigarrow$ on $X$ with

$$P(-, y, z)^{-1}(1) = {\downarrow}(y \to z), \qquad P(x, -, z)^{-1}(1) = {\downarrow}(x \rightsquigarrow z)$$

for all $x, y, z \in X$. Thus, as a ternary relation, $P$ is given by

$$(x, y, z) \in P \iff x \leqslant y \to z \iff y \leqslant x \rightsquigarrow z,$$

in accordance with (18). As shown in the proof of Proposition 10, condition (19) is equivalent to Eq. (2), while (20) and (21) state that $u \to x = u \rightsquigarrow x = x$ holds for all $x \in X$. As the monotonicity condition (3) holds for every promonoidal poset, the proof is complete. □

**Remark.** Note that with the above notation, a quantum B-algebra $X$ admits a product $xy$ for $x, y \in X$ (Definition 1) if and only if the presheaf $P(x, y, -)$ is representable.

For a complete and cocomplete symmetric monoidal closed base category $\mathscr{V}$, the main theorem of [18] gives a universal property for the category $\mathscr{V}^{\mathscr{A}^{\mathrm{op}}}$ of presheaves over a monoidal category $\mathscr{A}$. For $\mathscr{V} = \mathbf{2}$, this implies the obvious universal property of $L(X) \cong \mathbf{2}^{X^{\mathrm{op}}}$ for a partially ordered semigroup $X$ via the Yoneda embedding $X \hookrightarrow L(X)$. The next result shows that a multiplication in $X$ is not needed if $L(X)$ is replaced by $U(X)$.

Let **Qu** be the subcategory of **mPos** consisting of the quantales with set-indexed join-preserving morphisms, and let

$$M\colon \mathbf{Qu} \to \mathbf{mPos} \tag{29}$$

be the functor which associates the multi-poset $MQ := Q^{\mathrm{op}}$ to a quantale $Q$. So the defining relation in $MQ$ is $x_1 \cdots x_n \geqslant x$.

**Proposition 12.** *The functor* (29) *makes* **Qu** *into a reflective subcategory of* **mPos** *with reflector* $U$.

*Proof.* For a multi-poset $X$, we show that the morphism $\eta_X\colon X \hookrightarrow MU(X)$ with $\eta_X(x) := \uparrow x$ is a unit of an adjunction $U \dashv M$. For $A \in U(X)$, we have $A = \bigcup_{x\in A} \uparrow x$. Hence, if $f\colon X \to MQ$ is a morphism in **mPos** and $f'\colon U(X) \to Q$ a morphism in **Qu** with $Mf' \circ \eta_X = f$, we necessarily have $f'(A) = \bigvee_{x\in A} f(x)$. For $A, B \in U(X)$, this gives

$$\begin{aligned} f'(A)f'(B) &= \bigvee\{f(x)f(y) \mid x \in A,\ y \in B\} \\ &\geqslant \bigvee\{f(z) \mid x \in A,\ y \in B,\ xy \leqslant z\} = f'(AB). \qquad \square \end{aligned}$$

At first glance, the replacement of $L(X)$ by $U(X)$ appears to be counter-intuitive. However, it allows to embed arbitrary multi-posets $X$ into a quantale $U(U(X))$. A similar switch led to the invention of quantum B-algebras [31].

Finally, we remark that a **2**-promonoidal structure gives rise to a *ternary frame* [29, 13, 22], that is, a ternary relation $R$ on a poset $X$, with a compatibility condition which states that $R$ is an upper set in $X^{\mathrm{op}} \times X^{\mathrm{op}} \times X$. The logical connectives can then be realized as operations on $U(X)$. For example, the *linear implication* is given by

$$A \to B := \{x \in X \mid \forall\, y \in A\, \forall\, z \in X\colon (x, y, z) \in R \Rightarrow z \in B\}.$$

Associativity of $R$ is given by the relational analogue to (19). For details, we refer to [13, 15, 17, 22].

## 7. Further examples

In the introduction, partially ordered sets with a greatest element, and groups (with no partial order) were mentioned as two extreme types of quantum B-algebras. Let us discuss these two cases first.

**Example 1.** Every partially ordered set $\Omega$ with greatest element 1 is a quantum B-algebra with

$$x \to y = x \rightsquigarrow y := \begin{cases} 1 & \text{for } x \leqslant y \\ y & \text{for } x \nleqslant y \end{cases}$$

for $x, y \in \Omega$. The fundamental relation (6) is given by

$$x_1 \cdots x_n \leqslant x \iff x_i \leqslant x \text{ for some } i \in \{1, \ldots, n\}.$$

We introduce a topology on $\Omega$ by taking the sets

$$D(x) := \{y \in \Omega \mid x \nleqslant y\}$$

as a subbasis of open sets. Then $\widehat{\Omega}$ consists of the closed sets, with reverse inclusion as partial order. The natural embedding $\Omega \hookrightarrow \widehat{\Omega}$ is given by

$$x \mapsto \overline{\{x\}} = \uparrow x.$$

Thus $\widehat{\Omega}$ is a locale. If $\Omega$ is totally ordered, $\widehat{\Omega}$ coincides with the Dedekind-MacNeille completion of $\Omega$.

As a special case, consider the poset

$$\Omega := \omega + \omega^* = \{0, 1, 2, 3, \ldots, 3^*, 2^*, 1^*, 0^*\}.$$

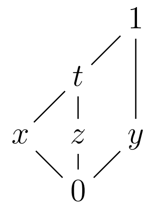
Here $\widehat{\Omega}$ has exactly one additional element, represented by the upper set $\omega^*$. By contrast, the canonical extension [14] of $\Omega$ has two additional elements between $\omega$ and $\omega^*$.

**Example 2.** A quantum B-algebra $G$ with trivial partial order is equivalent to a group (see [30], Theorem 4.2). More generally, every semigroup $M$ determines a quantum B-algebra $X_M$. For example, consider the commutative semigroup $M = \{x, y, z\}$ with multiplication table

| $\cdot$ | $x$ | $y$ | $z$ |
|---|---|---|---|
| $x$ | $y$ | $z$ | $y$ |
| $y$ | $z$ | $y$ | $z$ |
| $z$ | $y$ | $z$ | $y$ |

Then $X_M$ has six elements, and its residuals coincide since $M$ is commutative. Precisely, $X_M = \{0, x, y, z, t, 1\}$ with table for $\to$ and Hasse diagram

| $\to$ | $0$ | $x$ | $y$ | $z$ | $t$ | $1$ |
|---|---|---|---|---|---|---|
| $0$ | $1$ | $1$ | $1$ | $1$ | $1$ | $1$ |
| $x$ | $0$ | $0$ | $t$ | $y$ | $y$ | $1$ |
| $y$ | $0$ | $0$ | $y$ | $t$ | $t$ | $1$ |
| $z$ | $0$ | $0$ | $t$ | $y$ | $y$ | $1$ |
| $t$ | $0$ | $0$ | $t$ | $y$ | $y$ | $1$ |
| $1$ | $0$ | $0$ | $0$ | $0$ | $0$ | $1$ |




Here $X_M = \widehat{X_M}$, but $X_M$ is not a submonoid of $L(M)$.

**Example 3.** For a cancellative semigroup $M$ with $|M| > 1$ which is not a group, the quantum B-algebra $X_M$ is obtained from $M$ by adjoining a greatest element 1 and a smallest element 0. For $x, y \in M$,

$$x \to y = \begin{cases} z & \text{if } zx = y \text{ for some } z \in M \\ 0 & \text{otherwise,} \end{cases}$$

and similarly for $x \rightsquigarrow y$. Furthermore,

$$0 \to x = 0 \rightsquigarrow x = x \to 1 = x \rightsquigarrow 1 = 1$$

for all $x \in X_M$, and $x \to 0 = x \rightsquigarrow 0 = 0$ for $x \neq 0$, and $1 \to x = 1 \rightsquigarrow x = 0$ for $x \neq 1$. Here $X_M$ is the injective envelope of $M$. In particular, $N := X_M$ satisfies $X_N \cong X_M$, which shows that in general, a partially ordered semigroup $M$ cannot be recovered from the quantum B-algebra $X_M$.

**Example 4.** Between the two extreme cases, every partially ordered group $G$ is a unital quantum B-algebra with residuals $a \to b := ba^{-1}$ and $a \rightsquigarrow b := a^{-1}b$. As a partially ordered set, $\widehat{G}$ coincides with the Dedekind-MacNeille completion. If $G$ is lattice-ordered and archimedean, $\widehat{G} \smallsetminus \{0, 1\}$ is an $\ell$-group. This group is usually called the *completion* of $G$ (see [2, 7]).

**Example 5.** If a quantum B-algebra is a complete lattice, it need not be a quantale. For example, the quantum B-algebra $X := \{0, \ldots, \frac{1}{3}, \frac{1}{2}, 1\}$ with the natural order and

$$x \to y = x \rightsquigarrow y := \begin{cases} 0 & \text{for } x \neq 0 \text{ and } y = 0 \\ 1 & \text{otherwise} \end{cases}$$

is a complete lattice. However, $1 \leqslant 1 \to \frac{1}{n}$ for all positive integers $n$. Suppose that the product $1 \cdot 1$ exists. Then $1 \cdot 1 \leqslant \frac{1}{n}$ for all $n$. Hence $1 \cdot 1 = 0$, and thus $1 \leqslant 1 \to 0 = 0$, a contradiction. So the product $1 \cdot 1$ does not exist in $X$.

The completion of $X$ is obtained by adjoining an element $\varepsilon > 0$ with $\varepsilon \leqslant \frac{1}{n}$ for all $n$. Indeed, the multiplication

$$ab := \begin{cases} \varepsilon & \text{for } a, b \neq 0 \\ 0 & \text{otherwise} \end{cases}$$

makes $\widehat{X} := X \sqcup \{\varepsilon\}$ into a quantale. Moreover, $X$ is dense in $\widehat{X}$ since $\varepsilon = \bigwedge \frac{1}{n}$, and it is easily checked that $X$ is a quantum B-subalgebra of $\widehat{X}$. Note, however, that $\varepsilon$ is not a join of elements from $X$.

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