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A MATHEMATICAL TRIBUTE TO REINHARD BÖRGER

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Résumé. Un résumé de la vie et du travail de Reinhard Börger (1954–2014) est presenté en mettant l'accent sur ses oeuvres premières ou non publiées.
Abstract. A synopsis of the life and work of Reinhard Börger (1954–2014) is presented, with an emphasis on his early or unpublished works.
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1. A Brief Curriculum Vitae

On June 6, 2014, Reinhard Börger passed away, after persistent heart complications. He had taught at *Fernuniversität* in Hagen, Germany, for over three decades where he had received his *Dr. rer. nat.* (Ph.D.) in 1981, with a thesis [15] on notions of connectedness, written under the direction of Dieter Pumplün. He had continued to work on mathematical problems until just hours before his death.

Born on August 19, 1954, Reinhard went to school in Gevelsberg (near Hagen) before beginning his mathematics studies at *Westfälische Wilhelms-Universität* in Münster in 1972. A year later he won a runner-up prize at the highly competitive national *Jugend forscht* competition. Quite visibly, mathematics seemed to always be on his mind, and he often seemed to appear out of nowhere at lectures, seminars or informal gatherings. These sudden appearances quickly earned him his nickname *Geist* (ghost), a name that he willingly adopted for himself as well. His trademark ability to then launch pointed and often unexpected, but always polite, questions, be it on mathematics or any other issue, quickly won him the respect of all.

Reinhard's interest in category theory started early during his studies in

Münster when, supported by a scholarship of the prestigious *Studienstiftung des Deutschen Volkes*, he took Pumplün's course on the subject that eventually led him to write his 1977 *Diplomarbeit* (M.Sc. thesis) about congruence relations on categories [3]. For his doctoral studies he accepted a scholarship from the *Cusanuswerk* and followed Pumplün from Münster to Hagen where Pumplün had accepted an inaugural chair at the newly founded Fernuniversität in 1975. After the completion of his doctoral degree in 1981 with an award-winning thesis, he assumed a number of research assistantships, at the Universities of Karlsruhe (Germany) and of Toledo (Ohio, USA), and back at Fernuniversität. For his *Habilitationsschrift* [31], which earned him the *venia legendi* in 1989, he developed a categorical approach to integration theory. Beginning from 1990 he worked as a *Hochschuldozent* at Fernuniversität, interrupted by a visiting professorship at York University in Toronto (Canada) in 1993, and in 1995 he was appointed *Außerplanmäßiger Professor* at Fernuniversität, a position that he kept until his premature death in 2014.

In what follows I give a synopsis of Reinhard's mathematical work, emphasizing early, incomplete or not easily accessible contributions. After a brief account in Section 2 of his work up to the completion of his M.Sc. thesis, I recall some of his early contributions to the development of categorical topology (Section 3), before describing in Section 4 some aspects of his Ph.D. thesis and the work that emanated from it. Section 5 sketches the work on integration theory in his *Habilitationsschrift*, and Section 6 highlights some of his more isolated mathematical contributions. For Reinhard's substantial contributions in the area of convexity theory, inspired by the Pumplün–Röhrl works on convexity (such as [86, 87]), we refer to the article [81] by his frequent coauthor on the subject, Ralf Kemper.

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2. First Steps

The earliest written mathematical work of Reinhard that I am aware of and that may still be of interest today, is the three-page mimeographed note [1] giving a sufficient condition for the non-existence of a cogenerating (also called coseparating) set of objects in a category $\mathcal{K}$. While the existence of such a set in the category of R-modules and, in particular, of abelian groups, is standard, none of the following categories can possess one: fields; skew fields; (commutative; unital) rings; groups; semigroups; monoids; small categories. Reinhard's theorem, found when he was still an undergraduate student, gives a unified reason for this, as follows.

Theorem 2.1. *Let $\mathcal{K}$ have* $(\text{strong epi}, \text{mono})$*-factorizations and admit a functor U to* $\mathbf{Set}$ *that preserves monomorphisms. If, for every cardinal number κ, there is a simple object A in $\mathcal{K}$ with the cardinality of UA at least κ, then there is no cogenerating set in $\mathcal{K}$.*

(He defined an object A to be *simple* if the identity morphism on A is not constant while every strong epimorphism with domain A must be constant or an isomorphism; a morphism f is *constant* if for all parallel morphisms x, y composable with f one has $fx = fy$.) Reinhard returned to the theme of the existence of cogenerators repeatedly throughout his career, see [16, 34, 44, 45, 49, 50].

In 1975 Reinhard and I discussed various generalizations of the notion of right adjoint functor that had appeared in the literature at the time, in particular Kaput's [80] locally adjunctable functors. We tightened that notion to *strongly locally right adjoint* and proved, among other things, preservation of connected limits by such functors. Our paper [2] was presented at the "Categories" conference in Oberwolfach in 1976, and we discussed it with Yves Diers who was working on a slightly stricter notion for his thesis [71] that today is known under the name *multi-right adjoint* functor. Diers' only further requirement to our strong local right adjointness was that the local adjunction units of an object, known as its *spectrum*, must form a set. Without this size restriction, Reinhard and I had already given in [2] a complete characterization of the spectrum of an object, as follows.

Theorem 2.2. *For a strongly locally right adjoint functor $U : \mathcal{A} \to \mathcal{X}$ and an object $X \in \mathcal{X}$, its spectrum is the only full subcategory of the comma cat-*

egory $(X \downarrow U)$ *that is a groupoid, coreflective, and closed under monomorphisms.*

These works actually precede Reinhard's M.Sc. thesis [3] (summarized in [5]) whose starting point was a notion presented in Pumplün's categories course that went beyond the classical notion (as given in [72]) which confines equivalent morphism to the same hom set. Without that restriction, for any equivalence relation $\sim$ on the class of morphisms of a category $\mathcal{K}$ to be *(uniquely) normal* Pumplün required the existence of a (uniquely determined) composition law for the equivalence classes that makes $\mathcal{K}/\!\sim$ a category and the projection $P : \mathcal{K} \to \mathcal{K}/\!\sim$ a functor. Reinhard showed that the behaviour of a *compatible* equivalence relation $\sim$ on the morphism class of a category $\mathcal{K}$ (so that $u \sim u'$ and $v \sim v'$ implies $uv \sim u'v'$ whenever the composites are defined) requires great caution:

Theorem 2.3. *Each of the following statements on an equivalence relation* $\sim$ *on the class of morphisms of a category* $\mathcal{K}$ *implies the next, but none of these implications is reversible:*

- $\sim$ *is compatible, and* $1_A \sim 1_B$ *only if* $A = B$*, for all objects* $A, B \in \mathcal{K}$*;*
- $\sim$ *is compatible, and for all* $u : A \to B, v : C \to D$ *with* $1_B \sim 1_C$*, there are* $u' : A' \to B', v' : C' \to D'$ *with* $u \sim u', v \sim v'$ *and* $B' = C'$*;*
- $\sim$ *is uniquely normal;*
- $\sim$ *is normal;*
- *there is a functor* F *with domain* $\mathcal{K}$ *inducing* $\sim$ *(so that* $u \sim u' \iff Fu = Fu'$*);*
- $\sim$ *is compatible.*

3. Semi-topological functors and total cocompleteness

Brümmer's [68], Shukla's [90], Hoffmann's [77] and Wischnewsky's [98] theses and Wyler's [100, 99], Manes' [84] and Herrlich's [75, 76] seminal papers triggered the development of what became known as *Categorical Topology*, with various groups in Germany, South Africa, the United States and other countries working intensively throughout the 1970s on axiomatizations of "topologically behaved" functors and their generalizations and properties; see [69] for a survey. Reinhard and I, long before he started working on his doctoral dissertation, were very much part of this effort. Here are some examples of results that he has influenced the most.

Topologicity of a functor $P : \mathcal{A} \to \mathcal{X}$ may be defined by the sole requirement that initial liftings of (arbitrarily large) so-called P-structured sources exist, without the a-priori assumption of faithfulness of P. (This is Brümmer's [68] definition, although he did not use the name *topological* for such functors in his thesis.) Herrlich realized that faithfulness is a consequence of the definition, with a proof that made essential use of the smallness of hom-sets for the categories in question. Reinhard's spontaneous idea then was to use a Cantor-type diagonal argument instead that works also for not necessarily locally small categories. In [8] we came up with a general theorem that not only proves the faithfulness of topological and, more generally, *semi-topological* functors [94, 78, 95] , but that also entails Freyd's theorem that a small category with (co)products must be, up to categorical equivalence, a complete lattice, and that in fact reproduces Cantor's original theorem about the cardinality of a a set being always exceeded by that of its power set, as follows:

Theorem 3.1. *Consider a (possibly large) family $(t_i : A_i \to C)_{i \in I}$ of morphisms and an object B in a category $\mathcal{K}$, such that any family $(h_i : A_i \to B)_{i \in I}$ factors as $h_i = ht_i$ $(i \in I)$ for some $h : C \to B$. If there is a surjection $I \to \mathcal{K}(C, B)$, then for any morphisms $f, g : C \to B$ one has $ft_i = gt_i$ for some $i \in I$.*

Semi-topological functors in their various incarnations (also called *solid* functors, at Herrlich's urging) were a topic of Reinhard's and my joint interest for considerable time, in particular in conjunction with strong completeness properties of the participating categories, as witnessed by our papers [9, 16, 34, 35, 38]. In [96] I had shown that the fundamental property of *totality* (or *total cocompletness*) introduced by Street and Walters [92] lifts from $\mathcal{X}$ to $\mathcal{A}$ along a semi-topological $P : \mathcal{A} \to \mathcal{X}$, and in [66] total categories with a (strong) generating set of objects were characterized as the categories admitting a semi-topological (and conservative) functor into some small discrete power of **Set**. For our paper [34] Reinhard constructed an incredible example:

Theorem 3.2. *There is a total category $\mathcal{A}$ with a (single-object) strong generator but no regularly generating set of objects. $\mathcal{A}$ is cowell-powered with respect to regular epimorphisms but not with respect to strong epimorphisms; $\mathcal{A}$ does not admit co-intersections of arbitrarily large families of strong epimorphisms. The colimit closure $\mathcal{B}$ of the strong generator in $\mathcal{A}$ fails to be complete since it doesn't even possess a terminal object.*

Since totality entails a very strong completeness property, called *hyper-completeness* by Reinhard (see [16]), the colimit closure $\mathcal{B}$ in the example above fails badly to inherit totality from its ambient category $\mathcal{A}$. A comparison with the following affirmative result on totality of colimit closures obtained in [38] demonstrates how "tight" this example is:

Theorem 3.3. *Let the cocomplete category $\mathcal{B}$ be the colimit closure of a small full subcategory $\mathcal{G}$, and assume that every extremal epimorphisms in $\mathcal{B}$ is the colimit of a chain of regular epimorphisms of length at most α, for some fixed ordinal α. Then $\mathcal{B}$ is total and admits large co-intersections of strong epimorphisms, and $\mathcal{G}$ is strongly generating in $\mathcal{A}$.*

4. Connectedness, coproducts, and extensive categories

Reinhard's doctoral dissertation [15] relates various categorical notions of connectedness studied throughout the 1970s with each other, adds new concepts and gives some surprising applications. Starting points for him were the notions of *component subcategory* (initiated by Herrlich [74] and developed further by Preuß [85], Strecker [91] and Tiller [97]), of *left-constant subcategory* (also initiated by Herrlich [74] in generalization of the correspondence between torsion and torsion-free classes and fully characterized within the category of topological spaces by Arhangel'skii and Wiegandt [67]), and the notion of strongly locally coreflective [2] or multi-coreflective [71] subcategory (already mentioned in Section 2 in the dual situation and applied in topology by Salicrup [88]).

Let us concentrate here on a more category-intrinsic approach to connectedness to which Reinhard greatly contributed and which led him to make significant contributions to preservation properties of coproducts in abstract and concrete categories. The starting point is the easy observation that a topological space X is (not empty and) connected if, and only if, every continuous map $X \to \coprod_{i \in I} Y_i$ into a topological sum factors uniquely through exactly one coproduct injection; in other words, if the covariant hom-functor $\mathbf{Top} \to \mathbf{Set}$ represented by X preserves coproducts. Trading **Top** for any category $\mathcal{K}$ with coproducts Hoffmann [77] called such objects X *Z-objects*, Reinhard preferred the name *coprime*, while most people will nowadays use the term *connected* in $\mathcal{K}$. More specifically, for a cardinal number α, let us call X *α-connected* in $\mathcal{K}$ if the hom-functor of X preserves coproducts indexed by a set of cardinality $\leq \alpha$.

In his thesis Reinhard was the first to explore this concept deeply in the dual category of the category **Rng** of unital (but not necessarily commutative) rings. α-connectedness of a ring R now means that every unital homomorphism $f : \prod_{\beta < \alpha} S_\beta \to R$ depends only on exactly one coordinate (so that it factors uniquely through precisely one projection of the direct product). While it is easy to see that, without loss of generality, one may assume here that every ring S_β is the ring $\mathbb{Z}$ of integers, and that the finitely-connected (i.e., α-connected, for every finite α) rings are precisely those that traditionally are called connected (i.e., those rings that have no idempotent elements other than 0 and 1), Reinhard unravelled several surprises in the infinite case. Calling a ring *ultraconnected* when it is $\aleph_0$-connected, he proved in [15] (see also [21]) that the countable case governs the arbitrary infinite case precisely when there are no uncountable measurable cardinals:

Theorem 4.1. *If there are no uncountable measurable cardinals, then the connected objects in $\mathbf{Rng}^{\mathrm{op}}$ are precisely the ultraconnected rings. If there are uncountable measurable cardinals, then there are no ultraconnected objects in $\mathbf{Rng}^{\mathrm{op}}$.*

The fields $\mathbb{R}$ and $\mathbb{C}$ of real and of numbers are ultraconnected, and so is every subring of an ultraconnected ring. But none of the following connected rings is ultraconnected: the cyclic rings of cardinality p^m (p prime, $m \geq 1$), the ring $\mathbb{Z}_p$ of p-adic integers and its field of fractions $\mathbb{Q}_p$

The Theorem remains valid if $\mathbf{Rng}$ is traded for the category of commutative unital rings. Its proof makes essential use of a general categorical result that Reinhard had first presented at a meeting on "Categorical Algebra and Its Applications” held in Arnsberg (Germany) in 1979 (see [13]):

Theorem 4.2. *For a category $\mathcal{K}$ with an initial object and α-indexed coproducts (α an infinite cardinal), a functor $F : \mathcal{K} \to \mathbf{Set}$ preserves such coproducts if, and only if, F preserves β-indexed coproducts for every measurable $\beta \leq \alpha$.*

He only subsequently learned that Trnková [93] had proved this theorem earlier in the special case that also the domain of F is $\mathbf{Set}$. In [25], keeping the general domain $\mathcal{K}$, he went on to expand it further to functors with target categories other than $\mathbf{Set}$.

The themes touched upon in, or emerging from, Reinhard’s thesis very much reverberate in today’s research. I can mention here only one example in this regard. It concerns the important notion of extensive category, a term introduced by Carboni, Lack and Walters in [70]: a category $\mathcal{K}$ with (finite) coproducts and pullbacks is *(finitely) extensive* if (finite) coproducts are universal (i.e., stable under pullback) and disjoint (i.e., the pullback of any two coproduct injections with distinct labels is the initial object). This is a typically geometric property shared by $\mathbf{Set}$ and $\mathbf{Top}$, while a pointed extensive category must be trivial. Every elementary topos is finitely extensive, and Grothendieck topoi (i.e., the localizations of presheaf categories) may be characterized as those Barr-exact categories with a generating set of objects that are extensive. In a (finitely) extensive category the (finitely) connected objects are characterized as a topologist would expect: they are precisely the coproduct-indecomposable objects, i.e., those non-initial objects X with the property that whenever X is presented as a coproduct of Y and Z, one of Y, Z must be initial.

Reinhard started his studies of the universality and disjointness properties of coproducts years before the appearance of [70]. His initial account [26] went through a multi-year period of refinement, extension and correction before it finally got published in [46]. But his first account already contains all the ingredients to the proof of a refined analysis of the notion of (finite) extensitivity that is missing from [70]; it shows that universality almost implies disjointness, as follows:

Theorem 4.3. *A category with (finite) coproducts and pullbacks is (finitely) extensive if, and only if, non-empty (binary) coproducts are universal and pre-initial objects are initial.*

(A *pre-initial* object admits at most one morphism into any other object, while an initial object admits exactly one. A streamlined proof of the Theorem is contained in [79].) The dual of the category of commutative unital rings is finitely extensive, and Reinhard gave an example showing that commutativity is essential here, although $\mathbf{Rng}^{\mathrm{op}}$ still has the disjointness property.

5. Measure and Integration

Given the wide range of his mathematical interests, it is hardly surprising that a large part of Reinhard's work addresses analytic themes, which are also at the core of his *Habilitationsschrift* [31], titled "A categorical approach to integration theory" (written in German, with the preprint [28] giving a compressed English version of it). Before Reinhard started his work in this area, there had been only few attempts to present measure and integration theory in a categorically satisfactory fashion, with limited follow-up work; among others, see [82, 83, 73]. Of these, Reinhard's approach may be seen as a further development of Linton's early work.

The starting point in his approach is the elementary, but crucial, observation that integration of simple functions is given by a *universal property*. Specifically, for a Boolean algebra B (with top and bottom elements 1 and 0) and a real vector space A, the space $\mathrm{M}(B, A)$ of *charges* $\mu : B \to A$ (i.e., of maps μ with $\mu(u \vee v) = \mu(u) + \mu(v)$ for all $u, v \in B$ with $u \wedge v = 0$) is representable when considered as a functor in A, so that for the fixed Boolean algebra B there is a real vector space EB with $\mathrm{M}(B, -) \cong \mathrm{Hom}_{\mathbb{R}}(EB, -) : \mathbf{Vec}_{\mathbb{R}} \to \mathbf{Set}$. Hence, there is a charge $\chi_B : B \to EB$ such that any charge $\mu : B \to A$ factors as $\mu = l \cdot \chi_B$, for a uniquely determined $\mathbb{R}$-linear map $l : EB \to A$. For a set algebra B of a set Ω, EB is the space of simple functions, and χ_B assigns to a subset of Ω in B its characteristic function. In particular then, for $A = \mathbb{R}$ and a charge μ, the corresponding map l assigns to a simple function its integral with respect to μ.

Since every bounded measurable function is the uniform limit of simple functions, it is clear that one must provide for a "good" convergence setting to arrive at a satisfactory integration theory, and Reinhard formulates the following necessary steps to this end: 1. express the integration of simple functions categorically in sufficient generality; 2. provide for a "convenient convergence environment", by replacing the category of sets by a suitable category of topological spaces; 3. test the categorical theory obtained against classical approaches to, and results in, integration theory. Unfortunately, as Reinhard explains in the 18-page introduction to his *Habilitationsschrift*, this obvious roadmap is loaded with specific obstacles.

The "simple integration theory" sketched above relies crucially on the fact that the symmetric monoidal-closed category $\mathbf{Vec}_{\mathbb{R}}$ lives over the Cartesian closed category $\mathbf{Set}$, with the left adjoint L to the forgetful functor $V : \mathbf{Vec}_{\mathbb{R}} \to \mathbf{Set}$ preserving the monoidal structure: $L(X \times Y) \cong L(X) \otimes L(Y)$ for all sets X, Y. Since the category $\mathbf{Top}$ fails to be Cartesian closed and can therefore not replace $\mathbf{Set}$, the first question then is which subtype of topological or analytic structure one should add on both sides of the adjunction without losing its "monoidal well-behavedness". A good replacement candidate for $\mathbf{Set}$ is the Cartesian closed category $\mathbf{SeqHaus}$ of *sequential Hausdorff spaces* (in which every sequentially closed subset is actually closed). However, since even its finite (categorical) products generally carry a finer topology than the product topology, vector space objects in !$\mathbf{SeqHaus}$ may fail to be topological vector spaces. To overcome this and other "technical" obstacles, Reinhard restricts himself to considering only vector spaces in which convergence to 0 may be tested with *convex* neighbourhoods of 0, thus replacing the functor V above by the forgetful functor $\mathbf{SCS} \to \mathbf{SeqHaus}$ of *sequentially convex spaces*. Reassuringly, $\mathbf{SCS}$ is still big enough to contain all Banach spaces (real or complex), even all locally convex Fréchet spaces.

His general categorical setting and theory is centred around a right-adjoint functor $V : \mathcal{A} \to \mathcal{X}$ with a (semi-)additive category $\mathcal{A}$ where, for simplicity, I assume here that both $\mathcal{A}$ and $\mathcal{X}$ be finitely complete and cocomplete. For every Boolean algebra object B in $\mathcal{X}$ and every A in $\mathcal{A}$ he gives a categorical construction of the set $\mathrm{M}(B, A)$ of A-valued measures on B. As described in the elementary case of set-based charges, a representation of $\mathrm{M}(B, -) : \mathcal{A} \to \mathbf{Set}$ defines a *universal* measure $\chi_B : B \to EB$, where EB plays the role of $\mathrm{L}_{(\infty)}(B)$ in concrete situations, and the factorization of an arbitrary measure μ through χ_B defines the integral with respect to μ. *Multiplicativity* of measures, a property that Reinhard defines in this abstract setting, requires a symmetric monoidal structure on $\mathcal{A}$ and the well-behavedness of the left adjoint L of V with re! spect to that structure on $\mathcal{A}$ and the Cartesian structure of $\mathcal{X}$. Under mild hypotheses he then shows that the universal measure is automatically multiplicative and that E, considered as a functor $\mathcal{B} \to \mathcal{R}$ to the category $\mathcal{R}$ of commutative monoid objects in the additive category $\mathcal{A}$, is left adjoint. As a particular consequence then, E preserves binary coproducts, a fact that may be interpreted as *Fubini's Theorem*, as one may explain for the specific categories considered earlier.

Indeed, for $\mathcal{A} = \mathbf{SCS}, \mathcal{X} = \mathbf{SeqHaus}$, a Boolean algebra object B in $\mathcal{X}$ is now called a *sequential Hausdorff Boolean algebra*, and a commutative monoid object R in $\mathcal{A}$ gives a *commutative sequentially convex algebra*. The fact that the functor $E : \mathbf{SHBool} \to \mathbf{SCA}$ preserves binary coproducts implies that, for B_0, B_1 in $\mathbf{SHBool}$, an element in $E(B_0 \otimes B_1)$, i.e., an *integrable functionoid* on the coproduct $B_0 \otimes B_1$ in $\mathbf{SHBool}$, may be considered a "functionoid in two variables", and its "iterated integral" with respect to measures μ_0, μ_1 on B_0, B_1 respectively, coincides with its integral with respect to the (real-valued) "product measure" on the coproduct $B_0 \otimes B_1$ in $\mathbf{SHBool}$ determined by μ_0, μ_1.

This is only a coarse and partial sketch of the work presented in his *Habilitationsschrift*. Reinhard kept working on refining and extending his integration theory till the end of his life. Beyond his published article [61] there are preliminary versions of a planned monograph on categorical integration theory of 2006 (see [57]) and 2010 (see [62]) which await some editorial work before they will hopefully be made available to a wider audience.

6. Across Mathematics

In the previous sections I have tried to give an impression of Reinhard's contributions to category theory and its applications to algebra, topology and analysis. But I haven't touched upon many of his other contributions (as listed in the References) that have no apparent connection to the type of work mentioned so far, for example in number theory (algebraic or analytic) and topology (general or algebraic), of which I can mention here only very few examples. They should underline his fascination with "concrete" objects and problems, his mastery of which was as strong as that of "abstract" mathematical theories. Take, for example, the intricate proof of his solution [39] to the problem of *"How to make a path injective"* that cleverly utilizes the order of the unit interval $I = [0, 1]$:

Theorem 6.1. *Let $\varphi : I \to X$ be a continuous path from a to b in a Hausdorff space $X, a \neq b$. Then there exist an injective continuous path $\psi : I \to X$ from a to b, a closed subset $A \subseteq I$ and a continuous order-preserving map $p : I \to I$ with $p(A) = I$ and $\psi \cdot p|_A = \varphi$.*

In [53] he constructs *"A non-Jordan measurable regularly open subset of the unit interval"*, and in [33] he exploits the role of rational numbers in $\mathbb{R}$ to give a surprisingly easy example of a "reasonable" connected Hausdorff space in which every point has a hereditarily disconnected neighbourhood. In fact, he proves the following theorem.

Theorem 6.2. *There is a topology on the set of real numbers finer than the Euclidean topology, making it a connected Hausdorff space that is the union of two hereditarily disconnected open subspaces.*

His proof takes less than a page and "adds" just a little elementary number theory to everybody's knowledge of the topology of the real line. Quite a different side of number theory is displayed in Reinhard's informal discussion note [30] that was sparked by the observation $6! \cdot 7! = 10!$ and the quest for other integer solutions x, y, z of $x! \cdot y! = z!$ with $1 \leq x \leq y$. Hence, after discarding the "trivial" solutions $1, y, y$ with $y \geq 1$ and $x, x! - 1, x!$ with $x \geq 3$ he asked whether the set S of *non-trivial* solutions is finite or, in fact, contains any triple other than $6, 7, 10$. His note, which asks for input from specialist number theorists, does not settle this question, but it does provide the following constraint on members of S that he obtained with analytic methods:

Theorem 6.3. *Any non-trivial integer solution to $x! \cdot y! = z!$ with $1 \leq x \leq y$ must satisfy $2^{\sqrt{\frac{x}{2}}} - x < y$. As a consequence, there is no non-trivial integer solution to that equation with $x = y$.*

7. Farewell

As a former colleague and frequent coauthor I belong to the many privileged people with whom Reinhard generously shared the depth and breadth of his mathematical knowledge and ideas. They include his teachers as much as his students and the accidental acquaintance at a conference, all of whom may have experienced his initial shyness that, however, could quickly give way to a spark in his eyes when confronted with an interesting mathematical question, usually followed by a rapid flow of pointed remarks that were often difficult to comprehend at first. Reinhard's premature death is surely a great loss to all of us.

Despite his superior talents Reinhard was a fundamentally modest person, with firm beliefs in Christian values. He saw no conflict between science and his religion, the principles of which he consistently upheld as a letter writer to papers and author of non-mathematical articles. His life-long dedicated engagement in local parish work as well as his contributions to national organizations addressing social and environmental issues, especially regarding the impact of individual car traffic, may not have been as visible to the people around him as they deserved to be. For example, in spite of having known him since his early university student times, it took me years to understand that his passion for railways and especially the use of local trains and public transport were rooted in much more than just a hobby.

Reinhard hardly ever talked much about himself, neither about his accomplishments nor his problems. His mathematical coworkers would rarely hear from him about his engagements outside mathematics, even when these were professionally related to his mathematical activities, such as his ambition to learn the Czech language. And only when asked directly would one hear the proud father speak about his three sons Lukas, Simon and Jonas. He fought hard to overcome the consequences of a devastating stroke some seven years before his death, especially as he was looking forward to celebrate later in 2014 his sixtieth birthday and the thirtieth anniversary of his wedding to Andrea Börger. Sadly, he lost that battle.

In what follows I first list, in approximate chronological order and typeset in *italics*, Reinhard Börger's written mathematical contributions, including unpublished or incomplete works, to the extent I was able to trace them, followed by an alphabetical list of references to other works cited in this article.

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