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ON THE CATEGORY OF STRATIFOLDS

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Résumé. Nous étudions les espaces stratifiés de Kreck (stratifolds)[a] d'un point de vue catégorique. Nous montrons entre autre que la catégorie des espaces stratifiés de Kreck admet un plongement pleinement fidèle dans la catégorie des $\mathbb{R}$-algèbres tout comme la catégorie des variétés lisses. Nous établissons une variante du théorème de Serre-Swan pour les espaces stratifiés de Kreck. En particulier, nous montrons que les fibrés vectoriels sur un espace stratifié de Kreck forment une catégorie équivalente à celle formée par les fibrés vectoriels sur un schéma affine qui est canoniquement associé à, mais en général plus grand que, l'espace stratifié lui-même.

Abstract. Stratifolds are considered from a categorical point of view. We show among others that the category of stratifolds fully faithfully embeds into the category of $\mathbb{R}$-algebras as does the category of smooth manifolds. We prove that a variant of the Serre-Swan theorem holds for stratifolds. In particular, the category of vector bundles over a stratifold is shown to be equivalent to the category of vector bundles over an associated affine scheme although the latter is in general larger than the stratifold itself.

Keywords. Stratifold, differential space, ringed space, vector bundle, the Serre-Swan theorem.
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[a] The review article [7] of Kloeckner contains a nice historical review of different notions of stratified space.

1. Introduction

Stratifolds have been introduced by Kreck [8]. The new notion subsumes manifolds and algebraic varieties with isolated singularities as examples; see [3]. One of its advantages is that stratifolds give geometric counterparts of singular homology classes of a CW complex in much the same way as manifolds give geometric homology classes in the sense of Jakob [5]. More precisely, such a homology class is represented by an appropriate bordism class of stratifolds. One might therefore expect that stratifolds share some of the fascinating properties of manifolds and varieties. In this article, we focus on such properties for stratifolds and investigate them from categorical and sheaf-theoretical points of view.

Pursell [14] showed that the category of smooth manifolds fully faithfully embeds into the category of $\mathbb{R}$-algebras. We extend this result to the category of stratifolds.

Theorem 1.1. *The category of stratifolds fully faithfully embeds into the category of* $\mathbb{R}$*-algebras.*

A stratifold $(S, \mathcal{C})$ consists of a topological space S and a subalgebra $\mathcal{C}$ of the $\mathbb{R}$-algebra of continuous real-valued functions on the underlying space. Such a subalgebra defines a ringed space which is called the *structure sheaf* of the stratifold. Indeed, the subalgebra is nothing but the algebra of global sections of the sheaf. The assignment of the algebra to a stratifold, namely the forgetful functor F defined by $F(S, \mathcal{C}) = \mathcal{C}$, gives rise to the embedding in Theorem 1.1.

Let M be a smooth manifold. Then the prime spectrum of the ring $C^\infty(M)$ of real-valued smooth functions with the Zariski topology is larger than the underlying space M in general. However, the *real spectrum*, which is a subspace of the prime spectrum, is homeomorphic to M. This fact is shown to extend to stratifolds; see Propositions 2.6 and 3.3.

The results mentioned above lead us naturally to considering the affine scheme of the global sections of the structure sheaf of a stratifold. In consequence, we see that the restriction of the affine scheme to the real spectrum is isomorphic to a given stratifold as a ringed space; see Theorem 3.5. We are convinced that the result, a sheaf-theoretic description of a stratifold, enables one to consider stratifolds in the framework of derived differential geometry [6, 20] though this issue is not pursued in this manuscript; see Remark 3.6.

The category of vector bundles over a smooth manifold M is equivalent to the category of finitely generated projective modules over $C^\infty(M)$ by a classical result of Swan [21]. An analogous result for algebraic varieties has been obtained by Serre [15]. It is thus worthwhile to investigate a Serre-

Swan type theorem for stratifolds. To this end, we introduce the appropriate notion of vector bundle over stratifolds; see Definition 4.1 and Proposition 4.7. With our definition, we get the following result; see Theorem 4.9 for the precise statement.

Theorem 1.2. *The Serre-Swan theorem holds for stratifolds.*

As a consequence, we deduce that the category of vector bundles over a stratifold is equivalent to that of vector bundles over the affine scheme associated to the stratifold though the underlying prime spectrum is larger than the stratifold itself in general; see Remark 4.18.

The rest of this article is organized as follows. In Section 2, after recalling the definition of a stratifold and its important properties, we prove Theorem 1.1. We investigate stratifolds and their category from a sheaf-theoretical point of view in Section 3. In Section 4, the notion of *vector bundle* over a stratifold is introduced and the Serre-Swan theorem is shown to hold for any stratifold. In Section 5, we characterize morphisms of stratifolds by local data, and describe them inside the category of *diffeological spaces*; see [19, 4]. In Section 6, we study the product of stratifolds from a categorical perspective. This is used in Section 4 in the course of proving the Serre-Swan theorem.

We conclude this section with comments. An important device in the study of stratifolds is the existence of so-called *local retractions* near each point of the stratifolds; see [8]. These retractions are essential at several places in this article; see Sections 4, 5 and 6. Some of proofs in Sections 2 and 3 are straightforward. Yet, they are instructive for clarifying what properties of manifolds and stratifolds are responsible for the obtained results. These results are needed to set up a framework for describing the Serre-Swan theorem in our context. One of highlights in this manuscript is that a version of the Serre-Swan theorem for stratifolds is proved without using tautological bundles or the Whitney immersion theorem as is usually done for proving the theorem in the case of a manifold.

2. The real spectrum of a stratifold

This section contains a brief review of stratifolds. We begin with the definition of a differential space in the sense of Sikorski [18].

Definition 2.1. A *differential space* is a pair $(S, \mathcal{C})$ consisting of a topological space S and an $\mathbb{R}$-subalgebra $\mathcal{C}$ of the $\mathbb{R}$-algebra $C^0(S)$ of continuous real-valued functions on S, which is supposed to be *locally detectable* and *C^∞-closed.*

Local detectability means that $f \in \mathcal{C}$ if and only if for any $x \in S$, there exist an open neighborhood U of x and an element $g \in \mathcal{C}$ such that $f|_U = g|_U$.

C^∞-closedness means that for each $n \geq 1$, each n-tuple $(f_1, ..., f_n)$ of maps in $\mathcal{C}$ and each smooth map $g : \mathbb{R}^n \to \mathbb{R}$, the composite $h : S \to \mathbb{R}$ defined by $h(x) = g(f_1(x), ..., f_n(x))$ belongs to $\mathcal{C}$.

Let $(S, \mathcal{C})$ and $(S', \mathcal{C}')$ be differential spaces. We call a continuous map $f : S \to S'$ a *morphism of the differential spaces*, denoted $f : (S, \mathcal{C}) \to (S', \mathcal{C}')$, if f induces a map $f^* : \mathcal{C}' \to \mathcal{C}$; that is, $\varphi \circ f \in \mathcal{C}$ for each $\varphi \in \mathcal{C}'$. Thus we define a category Diff of differential spaces. Let Mfd denote the category of smooth manifolds. It is readily seen that the functor $i : \mathsf{Mfd} \to \mathsf{Diff}$ defined by $i(M) = (M, C^\infty(M))$ is a fully faithful embedding.

For any smooth paracompact manifold M, the defining subalgebra $C^\infty(M)$ of $C^0(M)$ has two additional properties:

(i) It extends to a sheaf of $\mathbb{R}$-algebras $U \mapsto C^\infty(U)$.

(ii) For any open cover $\mathcal{U}$ of M, there exists a *smooth* partition of unity subordinate to $\mathcal{U}$. In particular, the sheaf C^∞ is generated by global sections in the sense that the canonical map $C^\infty(M) \to (C^\infty)_x$ is surjective for any $x \in M$, where $(C^\infty)_x$ denotes the $\mathbb{R}$-algebra of the germs at x.

This in turn implies that $C^\infty(U)$ can be recovered from $C^\infty(M)$ as the set of *locally extendable functions* on U. With this in mind, we introduce such functions in the context of differential spaces.

For a differential space $(S, \mathcal{C})$ and a subspace Y of S, we call an element $g \in C^0(Y)$ a *locally extendable function* if for any $x \in Y$, there exists an open neighborhood V of x in Y and $h \in \mathcal{C}$ such that $g|_V = h|_V$. Let $\mathcal{C}_Y$ be the subalgebra of $C^0(Y)$ consisting of locally extendable functions on Y. Then it follows that $(Y, \mathcal{C}_Y)$ is a differential space; see [8, page 8]. Thus any subspace of a differential space inherits the structure of a differential space.

Let $(S, \mathcal{C})$ be a differential space and $x \in S$. The vector space consisting of derivations on the $\mathbb{R}$-algebra $\mathcal{C}_x$ of the germs at x is denoted by T_xS, which is called the *tangent space* of the differential space at x; see [8, Chapter 1, section 3].

Definition 2.2. A *stratifold* is a differential space $(S, \mathcal{C})$ such that the following four conditions hold:

(1) S is a locally compact Hausdorff space with countable basis;

(2) the *skeleta* $sk_k(S) := \{x \in S \mid \dim T_xS \leq k\}$ are closed in S;

(3) for each $x \in S$ and open neighborhood U of x in S, there exists a *bump function* at x subordinate to U; that is, a non-negative function $\rho \in \mathcal{C}$ such that $\rho(x) \neq 0$ and such that the support $\mathrm{supp}\rho := \overline{\{p \in S \mid \rho(p) \neq 0\}}$ is contained in U;

(4) the *strata* $S^k := sk_k(S) - sk_{k-1}(S)$ are k-dimensional smooth manifolds such that restriction along $i : S^k \hookrightarrow S$ induces an isomorphism of stalks
$$i^* : \mathcal{C}_x \xrightarrow{\cong} C^\infty(S^k)_x.$$
for each $x \in S^k$.

A stratifold is *finite-dimensional* if there is a non-negative integer n such that $S = sk_n(S)$. In particular, the tangent spaces of a finite-dimensional stratifold are finite-dimensional.

In what follows, we assume that all stratifolds are finite-dimensional. We may simply write S for a stratifold or differential space $(S, \mathcal{C})$ if no confusion arises. A smooth manifold $(M, C^\infty(M))$ is a typical example of a stratifold. We define the category Stfd of stratifolds as the full subcategory of Diff spanned by the stratifolds. Observe that the embedding $\mathsf{Mfd} \to \mathsf{Diff}$ mentioned above factors through Stfd.

We here recall important properties of a stratifold.

Remark 2.3. Let $(S, \mathcal{C})$ be a stratifold with strata $\{S^i\}$.
(i) Let U be an open subset of S and $\mathcal{C}_U$ the subalgebra of $\mathcal{C}$ consisting of locally extendable functions of $C^0(U)$ in $\mathcal{C}$. Then $(U, \mathcal{C}_U)$ is a stratifold with strata $\{S^i \cap U\}$; see [8, Example 5, page 22].

(ii) For any $x \in S^i$, there exist an open neighborhood U of x in S and a morphism

$$r_x : (U, \mathcal{C}_U) \to (U \cap S^i, \mathcal{C}_{U\cap S^i})$$

such that $r_x|_{U\cap S^i} = id$. Such a map is called a *local retraction* near x; see [8, Proposition 2.1]
(iii) Any locally compact Hausdorff space with countable basis is paracompact and in particular countable at infinity. This together with the other properties of a stratifold $(S, \mathcal{C})$ shows that for any open cover $\mathcal{U}$ of S, there exists a partition of unity subordinate to $\mathcal{U}$ *consisting of functions in* $\mathcal{C}$, i.e. the structure sheaf $\mathcal{O}_S$ of the stratifold $(S, \mathcal{C})$ is *fine*; see [8, Proposition 2.3] and Sections 3 and 4.

We refer the reader to the book [8] of Kreck for other fundamental properties, examples of stratifolds and fascinating results on the stratifold homology.

For an $\mathbb{R}$-algebra $\mathcal{F}$, we define $|\mathcal{F}|$ to be the set of all morphisms of $\mathbb{R}$-algebras from $\mathcal{F}$ to $\mathbb{R}$ which preserve the unit. Moreover, we define a map $\widetilde{f} : |\mathcal{F}| \to \mathbb{R}$ by $\widetilde{f}(x) = x(f)$ for any $f \in \mathcal{F}$. Let $\widetilde{\mathcal{F}}$ be the $\mathbb{R}$-algebra of maps from $|\mathcal{F}|$ to $\mathbb{R}$ of the form $\widetilde{f}$ for $f \in \mathcal{F}$. Then we consider the Gelfand topology on $|\mathcal{F}|$; that is, $|\mathcal{F}|$ is regarded as the topological space with the open basis

$$\{\widetilde{f}^{-1}(U) \mid U : \text{open in } \mathbb{R}, \widetilde{f} \in \widetilde{\mathcal{F}}\};$$

see [11, 2.1] and [12, 3.12]. Thus the assignment of a topological space to an $\mathbb{R}$-algebra gives rise to a contravariant functor

$$|\ | : \mathbb{R}\text{-}\mathsf{Alg} \to \mathsf{Top}$$

which is called the *realization functor*, where $\mathbb{R}$-Alg denotes the category of $\mathbb{R}$-algebras.

By definition, the map $\tau : \mathcal{F} \to \widetilde{\mathcal{F}}$ defined by $\tau(f) = \widetilde{f}$ is surjective. It follows that τ is an isomorphism if $\mathcal{F}$ is a subalgebra of the $\mathbb{R}$-algebra of continuous functions on a space; see [12, 3.14].

Lemma 2.4. *Let* $(S, \mathcal{C})$ *be a stratifold. Then the map* $\theta : S \to |\mathcal{C}|$ *defined by* $\theta(p)(f) = f(p)$ *is a homeomorphism.*

Proof. In virtue of Remark 2.3, the same argument as in the proof of [12, Theorem 7.2] shows that θ is a bijection.

For any open set U in $\mathbb{R}$ and $f \in \mathcal{C}$, we see that $\theta^{-1}(\widetilde{f}^{-1}(U)) = f^{-1}(U)$ since $\widetilde{f} \circ \theta = f$. This implies that θ is continuous.

Let W be an open set of S. By definition, a stratifold has a bump function for each $x \in W$; that is, there exists a non-negative function $f_x \in \mathcal{C}$ such that $\mathrm{supp} f_x \subset W$ and $f_x(x) \neq 0$. Then we see that $x \in f_x^{-1}(\mathbb{R}^+) \subset W$ for any $x \in W$ and hence $W = \bigcup_{x \in W} f_x^{-1}(\mathbb{R}^+)$. Therefore, it follows that

$$\theta(W) = \theta(\bigcup_{x \in W} f_x^{-1}(\mathbb{R}^+)) = \bigcup_{x \in W} (\theta^{-1})^{-1} f_x^{-1}(\mathbb{R}^+) = \bigcup_{x \in W} \widetilde{f_x}^{-1}(\mathbb{R}^+).$$

Observe that $\widetilde{f_x} \circ \theta = f_x$ as mentioned above. This shows that θ is open. $\square$

Let $\mathcal{F}$ be a subalgebra of $C^0(X)$ the $\mathbb{R}$-algebra of continuous maps from a space X to $\mathbb{R}$. We call the pair $(X, \mathcal{F})$ a *continuous space*. Let Csp be the category of continuous spaces. Observe that a morphism $\varphi : (X, \mathcal{F}_X) \to (Y, \mathcal{F}_Y)$ is a continuous map $\varphi : X \to Y$ which satisfies the condition that $f \circ \varphi \in \mathcal{F}_X$ for any $f \in \mathcal{F}_Y$. By definition, the category Diff of differential spaces is a full subcategory of Csp; therefore, the categories Mfd and Stfd are full subcategories of Csp as well.

Proposition 2.5. *The map $\theta : S \to |\mathcal{C}|$ gives rise to an isomorphism $\theta : (S, \mathcal{C}) \to (|\mathcal{C}|, \widetilde{\mathcal{C}})$ of continuous spaces.*

Proof. Recall the isomorphism $\tau : \mathcal{C} \to \widetilde{\mathcal{C}}$. We consider the composite $\theta^* \circ \tau : \mathcal{C} \to \widetilde{\mathcal{C}} \to \mathcal{C}$. Then it is readily seen that $(\theta^* \circ \tau)(f) = f$ for any $f \in \mathcal{C}$. This implies that $\theta^* : \widetilde{\mathcal{C}} \to \mathcal{C}$ is a well-defined isomorphism. Since $(\theta^{-1})^*\theta^*(\widetilde{f}) = \widetilde{f}$, it follows that $(\theta^{-1})^* : \mathcal{C} \to C^0(|\mathcal{C}|)$ factors through the subalgebra $\widetilde{\mathcal{C}}$ and that $(\theta^{-1})^* : \mathcal{C} \to \widetilde{\mathcal{C}}$ is an isomorphism. This completes the proof. $\square$

We call a maximal ideal m of $\mathcal{C}$ real if the quotient $\mathcal{C}/\mathsf{m}$ is isomorphic to $\mathbb{R}$ as an $\mathbb{R}$-algebra. Let $\mathrm{Spec}_r\, \mathcal{C}$ be the *real spectrum*, namely the subset of the prime spectrum $\mathrm{Spec}\, \mathcal{C}$ of $\mathcal{C}$ consisting of real ideals. We consider $\mathrm{Spec}_r\, \mathcal{C}$ the subspace of $\mathrm{Spec}\, \mathcal{C}$ with the Zariski topology. It is readily seen that a map $u : |\mathcal{C}| \to \mathrm{Spec}_r\, \mathcal{C}$ defined by $u(\varphi) = \mathrm{Ker}\, \varphi$ is bijective. Moreover, the map u is continuous. In fact, for an open base $D(f) = \{\mathsf{m} \in \mathrm{Spec}_r\, \mathcal{C} \mid f \notin \mathsf{m}\}$ for some $f \in \mathcal{C}$, we see that $u^{-1}(D(f)) = \widetilde{f}^{-1}(\mathbb{R} \backslash \{0\})$.

Proposition 2.6. (*cf. [11, Remark, page 23]*) *The bijection* $u : |\mathcal{C}| \overset{\cong}{\to} \mathrm{Spec}_r\, \mathcal{C}$ *is a homeomorphism.*

Proof. With the same notation as in Lemma 2.4, we see that $u(\widetilde{f_x}^{-1}(\mathbb{R}^+)) = D(f_x)$. We observe that $\widetilde{f_x}$ is non-negative since $\widetilde{f_x} \circ \theta = f_x$ with θ the bijection. ☐

In consequence, the space $\mathrm{Spec}_r\, \mathcal{C}$ is homeomorphic to $|\mathcal{C}|$ and hence the underlying space S:

$$S \cong |\mathcal{C}| \cong \mathrm{Spec}_r\, \mathcal{C} \subset \mathrm{Spec}\, \mathcal{C}.$$

Remark 2.7. In [6, 4.3] and [2], the spectrum for an $\mathbb{R}$-algebra corresponds to what we call the real spectrum of the $\mathbb{R}$-algebra, which in general is not the same as its prime spectrum.

The following result yields Theorem 1.1.

Theorem 2.8. *The forgetful functor* $F : \mathsf{Stfd} \to \mathbb{R}\text{-}\mathsf{Alg}$ *defined by* $F(S, \mathcal{C}) = \mathcal{C}$ *is fully faithful; that is, the induced map* $F : \mathrm{Hom}_{\mathsf{Stfd}}((S, \mathcal{C}), (S', \mathcal{C}')) \to \mathrm{Hom}_{\mathbb{R}\text{-}\mathsf{Alg}}(\mathcal{C}', \mathcal{C})$ *is a bijection.*

Proof. For a morphism $\varphi : (S, \mathcal{C}) \to (S', \mathcal{C}')$ of stratifolds, namely a morphism of continuous spaces, we have a commutative diagram

(2.1)
$$\begin{tikzcd}
S \arrow[r, "\theta", "\cong"'] \arrow[d, "\varphi"'] & {|\mathcal{C}|} \arrow[d, "|\varphi^*|"] \\
S' \arrow[r, "\theta", "\cong"'] & {|\mathcal{C}'|.}
\end{tikzcd}$$

In fact, we see that for any $f' \in \mathcal{C}'$ and $p \in S$,

$$\begin{aligned} (|\varphi^*| \circ \theta)(p)(f') &= |\varphi^*|(\theta(p))(f') = \theta(p)(\varphi^*(f')) \\ &= \theta(p)(f' \circ \varphi) = (f' \circ \varphi)(p) \end{aligned}$$

and that $(\theta \circ \varphi)(p)(f') = \theta(\varphi(p))(f') = f'(\varphi(p))$. This yields that F is injective.

For any morphism $u : \mathcal{C}' \to \mathcal{C}$ of $\mathbb{R}$-algebras, we define $\varphi : S \to S'$ to be the composite

$$S \xrightarrow[\cong]{\theta} |\mathcal{C}| \xrightarrow{|u|} |\mathcal{C}'| \xrightarrow[\cong]{\theta^{-1}} S'.$$

Observe that $|u|$ is a continuous map defined by $|u|(p) = p \circ u$; see [12, 3.19]. For any $x \in |\mathcal{C}|$, we see that $|u|^*(\widetilde{f})(x) = (\widetilde{f} \circ |u|)(x) = \widetilde{f}(x \circ u) = (x \circ u)(f) = \widetilde{u(f)}(x)$. Thus it follows that $|u|^* : \widetilde{\mathcal{C}'} \to \widetilde{\mathcal{C}}$ is well defined. Moreover, we have a commutative diagram

(2.2)
$$\begin{tikzcd}
\widetilde{\mathcal{C}'} \arrow[r, "|u|^*"] \arrow[d, "\theta^*"'] & \widetilde{\mathcal{C}} \arrow[d, "\theta^*"] \\
\mathcal{C}' \arrow[r, "u"'] & \mathcal{C}.
\end{tikzcd}$$

This follows from the fact that for any $\widetilde{f'} \in \widetilde{\mathcal{C}}$ and $x \in S$,

$$\begin{aligned}
(\theta^* \circ |u|^*)(\widetilde{f'})(x) &= (|u| \circ \theta)^*(\widetilde{f'})(x) = (\widetilde{f'} \circ (|u| \circ \theta))(x) \\
&= ((|u| \circ \theta)(x))(f') = (|u|(\theta(x)))(f') \\
&= \theta(x)(u(f')) = u(f')(x).
\end{aligned}$$

Furthermore, we see that $(u \circ \theta^*)(\widetilde{f'})(x) = u(\theta(\widetilde{f'}))(x) = (u(\widetilde{f'} \circ \theta))(x) = u(f')(x)$. This enables us to deduce that $\varphi^* = u$. It turns out that F is a bijection. □

We conclude this section with comments concerning Theorems 1.1 and 2.8.

Remark 2.9. A stratifold $(S, \mathcal{C})$ is a differential space. Then it is readily seen that

$$\mathrm{Hom}_{\mathsf{Stfd}}((S, \mathcal{C}), (\mathbb{R}, C^\infty(\mathbb{R}))) = \mathcal{C}.$$

Remark 2.10. The result [12, 7.19] asserts that the category Mfd of manifolds is equivalent to the category of *smooth* $\mathbb{R}$-algebras, which is a full subcategory of the category $\mathbb{R}$-Alg. Moreover, we have the embedding j : Mfd $\to$ Stfd as mentioned above. However, Theorem 1.1 is not an immediate consequence of these results.

3. The structure sheaf of a stratifold

The goal of this section is to give a sheaf-theoretical extension of Theorem 1.1.

Let X be a space and $\mathcal{C}$ an $\mathbb{R}$-subalgebra of $C^0(X)$ the $\mathbb{R}$-algebra of real-valued continuous functions on X. Recall that for any open subset U of X, an element $f \in C^0(U)$ is called *locally extendable* in $\mathcal{C}$ if for any element x in U, there exist an open neighborhood V_x of x in U and a function $g \in \mathcal{C}$ such that $f|_{V_x} = g|_{V_x}$. It is readily seen that the pair $(X, \mathcal{O}_X)$ is a ringed subspace of (X, C^0) of real-valued continuous functions, where $\mathcal{O}_X(X) = \mathcal{C}$ and $\mathcal{O}_X(U)$ is the $\mathbb{R}$-subalgebra of $C^0(U)$ consisting of locally extendable elements in $\mathcal{C}$ for any open subset U of X. Such a ringed subspace $(X, \mathcal{O}_X)$ is called a *ringed continuous space*.

A map between the underlying spaces of ringed continuous spaces, which induces a well-defined map between global sections, gives rise to a morphism of ringed spaces. The proof is straightforward. More precisely, we have

Lemma 3.1. *Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be ringed continuous spaces. Let $f : X \to Y$ be a continuous map. Suppose that $f^\sharp(g) := g \circ f$ is in $\mathcal{O}_X(X)$ for any $g \in \mathcal{O}_Y(Y)$. Then $f^\sharp$ induces a well-defined morphism of sheaves $f^\sharp| : \mathcal{O}_Y \to f_*\mathcal{O}_X$.*

Let $(\mathsf{RS})_{C^0}$ be the category of ringed continuous spaces on locally compact, Hausdorff spaces with countable basis whose morphisms are continuos maps between underlying spaces satisfying the assumption on Lemma 3.1. Observe that every stratifold $(S, \mathcal{C})$ gives rise to a ringed continuous space $(S, \mathcal{O}_S)$. Its sheaf of rings $\mathcal{O}_S$ will be called the *structure sheaf* of the stratifolds $(S, \mathcal{C})$, and is explicitly given by $\mathcal{O}_S(U) = \mathcal{C}_U$ for an open set U of S; see Remark 2.3.

Definition 3.2. A ringed continuous space $(X, \mathcal{O}_X)$ is called *fine* if the sheaf of rings $\mathcal{O}_X$ is fine; that is, if for every locally finite cover $\mathcal{U}$ of X, there exists a partition of unity into a sum of global sections $s_i \in \mathcal{O}_X(X)$ whose supports are subordinate to $\mathcal{U}$.

We work with the full subcategory $\mathsf{f}(\mathsf{RS})_{C^0}$ of $(\mathsf{RS})_{C^0}$ consisting of fine ringed continuous spaces. We have seen in Remark 2.3 that the category

Stfd fully faithfully embeds into $\mathsf{f(RS)}_{C^0}$. Moreover, we have the following extensions of results in Section 2.

Proposition 3.3. (i) *The functor F which assigns global sections gives rise to fully and faithful embedding form the category $\mathsf{f(RS)}_{C^0}$ into $\mathbb{R}$-Alg.*
(ii) *Let $(X, \mathcal{O}_X)$ be in $\mathsf{f(RS)}_{C^0}$. Then there exist functional homeomorphisms*

$$X \cong |\mathcal{O}_X(X)| \cong \mathrm{Spec}_r \mathcal{O}_X(X).$$

(iii) *The category $\mathsf{f(RS)}_{C^0}$ is a full subcategory of RS the category of ringed spaces.*

Proof. The proofs of Theorem 1.1 and Proposition 2.5 yield those of (i) and (ii).

Let $(f, \varphi) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphisms of ringed spaces. In order to prove (iii), it suffices to show that $\varphi = f^\sharp$. We define a continuous map $g : X \to Y$ by $g = \theta^{-1} \circ |\varphi_Y| \circ \theta$. The commutative diagram (2.2) allows us to deduce that $g^\sharp : \mathcal{O}_Y(Y) \to (f_*\mathcal{O}_X)(Y) = \mathcal{O}_X(X)$ is nothing but the map φ_Y. For any open set U of Y, we consider a commutative diagram

$$\begin{tikzcd}
\mathcal{O}_Y(U) \arrow[r, "{\varphi_U, g^\sharp|}"] & (f_*\mathcal{O}_X)(U) \\
\mathcal{O}_Y(Y) \arrow[u, "i^\sharp"] \arrow[r, "\varphi_Y = g^\sharp"'] & (f_*\mathcal{O}_X)(Y), \arrow[u, "i^\sharp"']
\end{tikzcd}$$

where $i : U \to Y$ denotes the inclusion. By applying the realization functor $|\ |$ to the diagram above, we see that $\varphi_U = g^\sharp|$. In fact, $|i^\sharp|$ is the inclusion i up to homeomorphism θ; see the commutative diagram (2.1). Suppose that $g(x) \neq f(x)$ for some $x \in X$. Then there exists an open neighborhood $V_{f(x)}$ of $f(x)$ such that $g(x)$ is not in $V_{f(x)}$. On the other hand, since the map $g^\sharp| = \varphi_{V_{f(x)}} : \mathcal{O}_Y(V_{f(x)}) \to (f_*\mathcal{O}_X)(V_{f(x)})$ is well defined, it follows that $V_{f(x)} \supset g(f^{-1}(V_{f(x)}))$ and hence $g(x)$ is in $V_{f(x)}$, which is a contradiction. We have $(f, f^\sharp) = (g, g^\sharp) = (f, \varphi)$. This completes the proof of (iii). □

We recall the category Csp of continuous spaces; see Section 2. Let $S : \mathbb{R}\text{-}\mathsf{Alg} \to \mathsf{Csp}$ be the contravariant functor defined by $S\mathcal{F} = (|\mathcal{F}|, \widetilde{\mathcal{F}})$.

By the definition of a morphism in Csp, we see that the same maps Φ and Ψ as in Section 5 below give bijections

$$\mathrm{Hom}_{\mathsf{Csp}^{op}}(S\mathcal{F},(X,\mathcal{C})) \underset{\Psi}{\overset{\Phi}{\rightleftarrows}} \mathrm{Hom}_{\mathbb{R}\text{-}\mathsf{Alg}}(\mathcal{F},F(X,\mathcal{C})),$$

where $F : \mathsf{Csp}^{op} \to \mathbb{R}\text{-}\mathsf{Alg}$ is the forgetful functor defined by $F(X,\mathcal{C}) = \mathcal{C}$. In fact, for g in Csp, the map $\Phi(g)$ factors through $\mathcal{C}$ and hence Φ is well defined. For $\varphi : \mathcal{F} \to \mathcal{C}$ and $\widetilde{f} \in \widetilde{\mathcal{F}}$, we have $(|\varphi| \circ \theta)^*(\widetilde{f}) = \varphi(f) \in \mathcal{C}$. This implies that Ψ is well defined. Thus S is the left adjoint of F. Let $U : \mathsf{Csp} \to \mathsf{Top}$ be the forgetful functor which assigns a continuous space the underlying space. We define a functor $m : \mathsf{f(RS)}_{C^0} \to \mathsf{Csp}$ by $m(X,\mathcal{O}) = (X,\mathcal{O}(X))$ for a fine ringed continuous space $(X,\mathcal{O})$. With these functors, we have a digram

(3.1)
$$\begin{tikzcd}
 & \mathsf{Csp} \arrow[r, "U"] \arrow[dr, "F"] & \mathsf{Top} \\
(\mathsf{RS})_{C^0} \arrow[r, phantom, "\supset"] & \mathsf{f(RS)}_{C^0} \arrow[u, "m"] \arrow[r, "F"'] & \mathbb{R}\text{-}\mathsf{Alg} \arrow[ul, "S"] \arrow[u, "\mathrm{Spec}_r(\ )"'] \\
 & \mathsf{Stfd} \arrow[u, "l"] \arrow[ur, "F"'] &
\end{tikzcd}$$

in which the upper square and the triangles except for the upper right-hand side one are commutative up to isomorphism; see Propositions 2.5 and 2.6. Proposition 3.3 (i) yields that the functor $F : \mathsf{f(RS)}_{C^0} \to \mathbb{R}\text{-}\mathsf{Alg}$ gives rise to an equivalence of categories between $\mathsf{f(RS)}_{C^0}$ and its image, which is a full subcategory $\mathbb{R}\text{-}\mathsf{Alg}$. One might remember the same result in algebraic geometry as the fact that the category of affine schemes is equivalent to the category of commutative rings with the global section functor.

Remark 3.4. An object in $\mathsf{f(RS)}_{C^0}$ which comes from Stfd is a locally ringed space; that is, the ring of germs at each point is local. This follows from the definition of a stratifold and [11, Theorem 1.8].

Let A be an $\mathbb{R}$-algebra and U an open set of $\mathrm{Spec}_r A$. We put $M_U := \bigcap_{\mathfrak{m} \in U} \mathfrak{m}^c$, where $\mathfrak{m}^c$ denotes the complement of $\mathfrak{m}$. Then M_U is a multiplicative set. We denote by $M_U^{-1}A$ the localization of A with respect to M_U. Define the *structure sheaf* $\widehat{A}$ on $\mathrm{Spec}_r A$ by the sheafification of the presheaf

$U \rightsquigarrow M_U^{-1}A$. Observe that the sheaf $\widehat{A}$ is the inverse image of the affine scheme $(\mathrm{Spec}A, \widetilde{A})$ of A along the inclusion $\mathrm{Spec}_r A \hookrightarrow \mathrm{Spec}A$.

The following proposition asserts that a stratifold is indeed a restriction of an affine scheme.

Theorem 3.5. *Let $(S, \mathcal{O}_S)$ be a fine ringed space which comes from a stratifold $(S, \mathcal{C})$ and $i : \mathrm{Spec}_r\mathcal{O}_S(S) \to \mathrm{Spec}\,\mathcal{O}_S(S)$ the inclusion. Then $(S, \mathcal{O}_S)$ is isomorphic to $i^*(\mathrm{Spec}\,\mathcal{O}_S(S), \widetilde{\mathcal{O}_S(S)})$ as a ringed space, where $(\mathrm{Spec}\,\mathcal{O}_S(S), \widetilde{\mathcal{O}_S(S)})$ is the affine scheme associated with the ring $\mathcal{O}_S(S)$.*

Proof. We recall the homeomorphism $\theta : S \xrightarrow{\cong} |S|$ and $u : |S| \xrightarrow{\cong} \mathrm{Spec}_r\mathcal{O}_S(S)$ in Section 2. Let m be the composite $u \circ \theta$. Then we have

$$m(p) = (u \circ \theta)(p) = \mathrm{Ker}\theta(p) = \{f \in \mathcal{C} \mid f(p) = 0\} =: \mathfrak{m}_p.$$

In order to prove the theorem, it suffices to show that $(S, \mathcal{O}_S)$ is isomorphic to the structure sheaf $(\mathrm{Spec}_r\mathcal{O}_S(S), \widehat{\mathcal{O}_S(S)})$. To this end, we construct an isomorphism from $\widehat{\mathcal{O}_S(S)}$ to $m_*\mathcal{O}_S$.

For an open set U of $\mathrm{Spec}_r\mathcal{O}_S(S)$, we define $\alpha_U : M_U^{-1}\mathcal{O}_S(S) \to (m_*\mathcal{O}_S)(U)$ by $\alpha([f/s]) = f \cdot \frac{1}{s}$. Observe that $s(p) \neq 0$ for each p in $m^{-1}(U)$. This implies that α_U is well defined. We see that α_U induces a morphism of presheaves. Moreover, the morphism of presheaves gives rise to a morphism $\widehat{\alpha} : \widehat{\mathcal{O}_S(S)} \to m_*\mathcal{O}_S$ of sheaves. The natural map

$$\alpha_p : \widehat{\mathcal{O}_S(S)}_{\mathfrak{m}_p} = \mathrm{colim}_{\mathfrak{m}_p \in V}\widehat{\mathcal{O}_S(S)}(V) = \mathcal{O}_S(S)_{\mathfrak{m}_p} \to \mathrm{colim}_{p \in U}\mathcal{O}_S(U) =: \mathcal{C}_p$$

defined by $\alpha([f/s]) = f_p \cdot (\frac{1}{s_p})$ is well defined. Here $\mathcal{O}_S(S)_{\mathfrak{m}_p}$ denotes the localization of the ring $\mathcal{O}_S(S)$ at $\mathfrak{m}_p$. In fact, if $s \in \mathfrak{m}_p^c = \mathcal{O}_S(S) \backslash \mathfrak{m}_p$, then $s(p) = \theta(p)(s) \neq 0$. Since $(S, \mathcal{O}_S(S))$ is a stratifold, it follows that $1/s \in \mathcal{O}_S(U)$ for some open set U of S. This follows from the condition (4) in Definition 2.2; see the proof of [8, Proposition 2.3]. Moreover, there exists a bump function at each $x \in S$. Thus the proof of [11, Corollary 1.6] enables us to conclude that α_p is an isomorphism. It turns out that $\widehat{\alpha} = \amalg_{p \in S}\alpha_p$ and hence $\widehat{\alpha}$ is an isomorphism. We have the result. □

Remark 3.6. For a stratifold $(S, \mathcal{C})$, we regard $\mathcal{C}$ as a C^∞-ring; see [6, 2, 9]. Theorem 3.5 asserts that the structure sheaf $(S, \mathcal{O}_S)$ of $(S, \mathcal{C})$ is a C^∞-ringed space in the sense of Joyce [6] and is isomorphic to the spectrum of the C^∞-ring $\mathcal{C}$; see [6, Definition 4.12] for example.

4. Vector bundles and the Serre-Swan theorem for stratifolds

Generalizing the notion of smooth vector bundle over a manifold, we define a vector bundle over a stratifold.

Definition 4.1. Let $(S, \mathcal{C}_S)$ be a stratifold and $(E, \mathcal{C}_E)$ a differential space. A morphism of differential spaces $\pi : (E, \mathcal{C}_E) \to (S, \mathcal{C}_S)$ is a *vector bundle* over $(S, \mathcal{C}_S)$ if the following conditions are satisfied.

(1) $E_x := \pi^{-1}(x)$ is a vector space over $\mathbb{R}$ for $x \in S$.

(2) There exist an open cover $\{U_\alpha\}_{\alpha \in J}$ of S and an isomorphism $\phi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{R}^{n_\alpha}$ of differential spaces for each $\alpha \in J$. Here $\pi^{-1}(U_\alpha)$ is regarded as a differential subspace of $(E, \mathcal{C}_E)$; see Remark 2.3, and $U_\alpha \times \mathbb{R}^{n_\alpha}$ is considered the product of the substratifold $(U_\alpha, \mathcal{C}_{U_\alpha})$ of $(S, \mathcal{C}_S)$ and the manifold $(\mathbb{R}^{n_\alpha}, C^\infty(\mathbb{R}^{n_\alpha}))$; see Section 6.

(3) The diagram

$$\begin{tikzcd}
\pi^{-1}(U_\alpha) \arrow[rr, "\phi_\alpha"] \arrow[dr, "\pi"'] & & U_\alpha \times \mathbb{R}^{n_\alpha} \arrow[dl, "pr_1"] \\
& U_\alpha &
\end{tikzcd}$$

is commutative, where pr_1 is the projection onto the first factor.

(4) The composite $pr_2 \circ \phi_\alpha|_{E_x} : E_x \to U_\alpha \times \mathbb{R}^{n_\alpha} \to \mathbb{R}^{n_\alpha}$ is a linear isomorphism, where $pr_2 : U_\alpha \times \mathbb{R}^{n_\alpha} \to \mathbb{R}^{n_\alpha}$ denotes the projection onto the second factor.

We call a vector bundle $\pi : (E, \mathcal{C}_E) \to (S, \mathcal{C}_S)$ *bounded* if for the index set J of the cover which gives the trivialization, the set of integer $\{n_\alpha\}_{\alpha \in J}$ is bounded. Observe that the integer n_α is constant on a connected component of S.

Let $\pi_E : E \to S$ and $\pi_F : F \to S$ be vector bundles over a stratifold S, We define a *morphism* $\varphi : E \to F$ *of bundles* to be a morphism of differential spaces from E to F such that $\pi_F \circ \varphi = \pi_E$ and the restrictions on each stalks $\varphi_x : E_x \to F_x$ are linear maps. We denote by $\mathsf{VBb}_{(S,\mathcal{C})}$ the category of vector bundles over $(S, \mathcal{C})$ of bounded rank.

Definition 4.2. Let $\pi : (E, \mathcal{C}_E) \to (S, \mathcal{C}_S)$ be a vector bundle over a stratifold $(S, \mathcal{C}_S)$. A morphism of differential spaces $s : (U, \mathcal{C}_U) \to (E, \mathcal{C}_E)$ is called a *section* on U if $\pi \circ s = id_{(U, \mathcal{C}_U)}$. We denote by $\Gamma(U, E)$ the set of all sections.

We observe that $\Gamma(U, E)$ is an $\mathcal{O}_S(U)$-module through the identification $\mathcal{C}_U = \mathcal{O}_S(U)$. Moreover, we have the following proposition.

Proposition 4.3. *The assignment* $\Gamma(\ , E) : U \rightsquigarrow \Gamma(U, E)$ *gives rise to an* $\mathcal{O}_S$*-module.*

Proof. We begin by showing that the assignment gives rise to a set-valued sheaf. Let $i : U \to V$ be an inclusion between open sets U and V of S. Since i is a morphism of stratifolds, restricting along i takes a section on V to a section on U.

Let $\{V_\gamma\}_\gamma$ be an open cover of an open set U of S. Suppose that $\{s_\gamma\}_\gamma$ in $\prod_\gamma \Gamma(V_\gamma, E)$ satisfies the condition that $res^{V_\gamma}_{V_\gamma \cap V_{\gamma'}}(s_\gamma) = res^{V_{\gamma'}}_{V_\gamma \cap V_{\gamma'}}(s_{\gamma'})$ for any γ and γ'. In the category Top of topological spaces, we have a section $s : U \to E$ with $res^U_{V_\gamma}(s) = s_\gamma$ for any γ. We need to verify that s is a morphism of differential spaces. For any $x \in U$, there exists an open set V_γ such that $x \in V_\gamma$. Since s_γ is a morphism of differential spaces, it follows that $(s^*(p))|_{V_\gamma} = s^*_\gamma(p) \in \mathcal{C}_{V_\gamma}$ for $p \in \mathcal{C}_E$. By the definition of $\mathcal{C}_{V_\gamma}$, we see that there exists an open neighborhood W_γ of x with $W_\gamma \subset V_\gamma$ such that $(s^*(p))|_{W_\gamma} = ((s^*(p))|_{V_\gamma})|_{W_\gamma} = p \circ s_\gamma|_{W_\gamma} = h|_{W_\gamma}$ for some $h \in \mathcal{C}_S$. This enables us to conclude that $s^*(p) \in \mathcal{C}_U$ and hence $\Gamma(\ , E)$ is a sheaf.

For s and t in $\Gamma(U, E)$, we define a section $(s+t)$ in Top by $(s+t)(x) = s(x) + t(x)$ for any $x \in U$. We show that $(s+t)$ is in $\Gamma(U, E)$. Let s_γ, $t_\gamma : V_\gamma \to \pi^{-1}(V_\gamma)$ be the restrictions of s and t to V_γ, respectively. We define $\widetilde{s_\gamma} : V_\gamma \to V_\gamma \times \mathbb{R}^n$ and $\widetilde{t_\gamma} : V_\gamma \to V_\gamma \times \mathbb{R}^n$ by $\phi_\gamma \circ s_\gamma$ and $\phi_\gamma \circ t_\gamma$, respectively. Here $\phi_\gamma : \pi^{-1}(V_\gamma) \xrightarrow{\cong} V_\gamma \times \mathbb{R}^n$ denotes a local trivialization.

Assertion 4.4. Let s_γ be in $\Gamma(V_\gamma, E)$. Then $s_\gamma : V_\gamma \to \pi^{-1}(V_\gamma)$ is a morphism of differential spaces.

Thus $\widetilde{s_\gamma}$ and $\widetilde{s_\gamma}$ are morphisms of differential spaces. The projection $pr_2 : V_\gamma \times \mathbb{R}^n \to \mathbb{R}^n$ into the second factor is a morphism of stratifolds and so are $pr_2 \circ \widetilde{s_\gamma}$ and $pr_2 \circ \widetilde{t_\gamma}$. We see that $pr_2 \circ \widetilde{s_\gamma} + pr_2 \circ \widetilde{t_\gamma}$ is a morphism of stratifolds. Proposition 6.1 below yields a morphism $(s_\gamma + t_\gamma)' : V_\gamma \to V_\gamma \times \mathbb{R}^n$ of stratifolds which fits into a commutative diagram

$$\begin{tikzcd}
 & V_\gamma \arrow[dl, "pr_2 \circ \widetilde{s_\gamma} + pr_2 \circ \widetilde{t_\gamma}"'] \arrow[d, "(s_\gamma + t_\gamma)'"] \arrow[dr, "1_{V_\gamma}"] & \\
\mathbb{R}^n & V_\gamma \times \mathbb{R}^n \arrow[l, "pr_2"] \arrow[r, "pr_1"'] & V_\gamma .
\end{tikzcd}$$

Define $s_\gamma + t_\gamma : V_\gamma \to \pi^{-1}(V_\gamma)$ to be the composite $\phi_\gamma^{-1} \circ (s_\gamma + t_\gamma)'$. Observe that $(s_\gamma + t_\gamma)(x) = (s+t)(x) = (s_{\gamma'} + t_{\gamma'})(x)$ for $x \in U_\gamma \cap U_{\gamma'}$. Since $\Gamma(, E)$ is a sheaf, it follows that there exists a unique extension $\widetilde{s+t} \in \Gamma(U, E)$ of $\{s_\gamma + t_\gamma\}_\gamma$. It is readily seen that $\widetilde{s+t} = s + t$.

The same argument as above does work well to show that sk defined by $sk(x) = k(x)s(x)$ is in $\Gamma(U, E)$ for $k \in \mathcal{C}_U$ and $s \in \Gamma(U, E)$. This completes the proof. $\square$

Proof of Assertion 4.4. Let x be an element in V_γ. For any $\rho \in \mathcal{C}_{\pi^{-1}(V_\gamma)}$, by definition, there exists an open neighborhood $W_{s_\gamma(x)}$ of $s_\gamma(x)$ such that $\rho|_{W_{s_\gamma(x)}} = \overline{\rho}|_{W_{s_\gamma(x)}}$ for some $\overline{\rho} \in \mathcal{C}_E$. Thus we see that

$$\begin{aligned} \rho \circ s_\gamma|_{s_\gamma^{-1}(W_{s_\gamma(x)})} = \rho|_{W_{s_\gamma(x)}} \circ s_\gamma|_{s_\gamma^{-1}(W_{s_\gamma(x)})} &= \overline{\rho}|_{W_{s_\gamma(x))}} \circ s_\gamma|_{s_\gamma^{-1}(W_{s_\gamma(x)})} \\ &= \overline{\rho} \circ s_\gamma|_{s_\gamma^{-1}(W_{s_\gamma(x)})}. \end{aligned}$$

Since $s_\gamma : V_\gamma \to E$ is a morphism of differential spaces, it follows that $\overline{\rho} \circ s_\gamma$ is in $\mathcal{C}_{V_\gamma}$. Then $\overline{\rho} \circ s_\gamma$ is a restriction of a map in $\mathcal{C}_S$ to an appropriate open neighborhood of x and hence so is $\rho \circ s_\gamma$. We have the result. $\square$

We denote by $\mathcal{L}_E$ the $\mathcal{O}_S$-module of Proposition 4.3.

Lemma 4.5. *Let* $pr_1 : S \times \mathbb{R}^n \to S$ *be the product bundle over a stratifold* $(S, \mathcal{C})$*. The map* $e_i : S \to S \times \mathbb{R}^n$ *defined by* $e_i(x) = (x, \mathsf{e}_i)$ *is a section of this bundle for* $i = 1, ..., n$*, where* $\{\mathsf{e}_1, ..., \mathsf{e}_n\}$ *is a canonical basis for* $\mathbb{R}^n$*.*

Proof. We prove that e_i is a morphism of differential spaces. Suppose that f is in $\mathcal{C}_{S \times \mathbb{R}^n}$; see Section 6. Then there are local retractions $r_x : U_x \to U_x \cap S^j$ and $r_y = id : U_y \to U_y$ such that $f|_{U_x \times U_y} = f(r_x \times r_y)$ for $x \in S^j$ and $y = \mathsf{e}_i \in \mathbb{R}^n$. This yields that $f \circ e_i|U_x = f \circ e_i \circ r_x$. Since the restriction map $e_i|_{S^j} : S^j \to S^j \times \mathbb{R}^n$ is smooth, it follows that the composite $f \circ e_i|_{S^j} : S^j \to S^j \times \mathbb{R}^n \to \mathbb{R}$ is also smooth. In consequence, we have $f \circ e_i \in \mathcal{C}$. $\square$

Proposition 4.6. *The transition functions* $g_{\alpha\beta} : U_\alpha \cap U_\beta \to GL_n(\mathbb{R})$ *are morphisms of stratifolds.*

Proof. By the definition of the transition function, we see that $\phi_\beta\phi_\alpha^{-1}(x, v) = (x, g_{\alpha\beta}(x)v)$. It follows from Lemma 4.5 that the composite

$$\begin{tikzcd} \psi_j : U_\alpha \cap U_\beta \arrow[r, "e_j"] & U_\alpha \cap U_\beta \times \mathbb{R}^n \arrow[r, "\phi_\beta\phi_\alpha^{-1}"] & U_\alpha \cap U_\beta \times \mathbb{R}^n \arrow[r, "pr_2"] & \mathbb{R}^n \end{tikzcd}$$

is a morphism of differential spaces. Therefore, for the well-defined map $\psi_j^* : C^\infty(\mathbb{R}^n) \to \mathcal{C}_{U_\alpha \cap U_\beta}$, we see that $u_{ij} := \psi_j^*(p_i) = p_i \circ \psi_j \in \mathcal{C}_{U_\alpha \cap U_\beta}$ and $g_{\alpha\beta}(x) = (u_{ij}(x))$, where $p_i : \mathbb{R}^n \to \mathbb{R}$ is the projection onto the ith factor. It turns out that $g_{\alpha\beta}$ is a morphism of stratifolds. In fact, for any $f \in \mathcal{C}_{GL_n(\mathbb{R})}$, there exists a smooth map $\overline{f} : M_{nn}(\mathbb{R}) = \mathbb{R}^{n^2} \to \mathbb{R}$ whose restriction coincides with f. Then we have $g_{\alpha\beta}^*(f)(x) = fg_{\alpha\beta}(x) = f(u_{11}(x), u_{12}(x), ..., u_{nn}(x)) = \overline{f}(u_{11}(x), u_{12}(x), ..., u_{nn}(x))$. This yields that $g_{\alpha\beta}^*(f)$ is in $\mathcal{C}_{U_\alpha \cap U_\beta}$. □

Proposition 4.7. *Let* $\pi : (E, \mathcal{C}_E) \to (S, \mathcal{C}_S)$ *be a vector bundle in the sense of Definition 4.1. Then the differential space* $(E, \mathcal{C}_E)$ *admits a stratifold structure for which* π *is a morphism of stratifolds.*

Proof. Without loss of generality, we assume that there exists a countable trivialization. Indeed S has a countable basis. Thus the existence of a countable basis of E follows from the local triviality. Moreover, the local triviality allows us to deduce that E is a Hausdorff space.

Let S^i be a stratum of S. Observe that S^i is a manifold for each i. By virtue of Proposition 4.6, we see that $\pi^{-1}(S^i)$ is a manifolds and $\pi : \pi^{-1}(S^i) \to S^i$ is a smooth vector bundle. It remains to prove that for any $x \in S^i$, the inclusion $i : \pi^{-1}(S^i) \to E$ induces an isomorphism $i^* : C(E)_x \to C^\infty(\pi^{-1}(S^i))_x$. Suppose that x is in U_α with $\phi_\alpha : \pi^{-1}(U) \xrightarrow{\cong} U_\alpha \times \mathbb{R}^n$ a trivialization. Then we have a commutative diagram

$$\begin{tikzcd} (\mathcal{C}_E)_x \arrow[r, "i^*"] \arrow[d, "res^*"', "\cong"] & C^\infty(\pi^{-1}(S^i))_x \arrow[d, "\cong"', "res^*"] \\ (\mathcal{C}_{\pi^{-1}(U_\alpha)})_x \arrow[r, "i^*"] & C^\infty(\pi^{-1}(S^i \cap U_\alpha))_x \\ (\mathcal{C}_{U_\alpha \times \mathbb{R}^n})_{\phi_\alpha(x)} \arrow[r, "(i \times 1_{\mathbb{R}^n})^*"] \arrow[u, "\phi_\alpha^*", "\cong"'] & \mathcal{C}(S^i \cap U_\alpha \times \mathbb{R}^n)_{\phi_\alpha(x)} \arrow[u, "\cong", "\phi_\alpha^*"'] \end{tikzcd}$$

The stratifold structure on $U_\alpha \times \mathbb{R}^n$ allows us to deduce that $(i \times 1_{\mathbb{R}^n})^*$ is an isomorphism. Then we see that the upper horizontal arrow i^* is an isomorphism.

The local triviality of the bundle implies the existence of a bump function. In fact, the existence is a local property. This completes the proof. $\square$

Proposition 4.8. *Let $(S, \mathcal{C})$ be a stratifold and $(E, \pi) \in \mathsf{VBb}_{(S,\mathcal{C})}$. Then the $\mathcal{O}_S$-module $\mathcal{L}_E$ is a locally free module.*

Proof. Let $(\{U_\alpha\}, \{\phi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{R}^{n_\alpha}\})$ be a trivialization. Define sections $s_i \in \mathcal{L}_E(U_\alpha)$ by $s_i = \phi_\alpha^{-1} \circ e_i|_{U_\alpha}$ for $i = 1, ..., n_\alpha$. Then these sections are bases of $\mathcal{L}_E(U_\alpha)$, so there exists an isomorphism between $\mathcal{L}_E(U_\alpha)$ and $\mathcal{O}_S(U_\alpha)^{n_\alpha}$. This induces an isomorphism between $\mathcal{L}_E|_{U_\alpha}$ and $\mathcal{O}_S^{n_\alpha}|_{U_\alpha}$. $\square$

For $f \in \mathrm{Hom}_{\mathsf{VBb}_{(S,\mathcal{C})}}(E, F)$, we define a map $f_* : \Gamma(U, E) \to \Gamma(U, F)$ by $f_*(s) = f \circ s$. Since $f_x : E_x \to F_x$ are linear maps, it follows that f_* is a morphism of $\mathcal{O}_S(U)$-modules. Thus f_* gives rise to a morphism $\mathcal{L}_f : \mathcal{L}_E \to \mathcal{L}_F$. Let $\mathsf{Lfb}(S)$ be the full subcategory of $\mathcal{O}_S$-Mod consisting of locally free $\mathcal{O}_S$-modules of bounded rank. Proposition 4.8 enables us to define a functor $\mathcal{L} : \mathsf{VBb}_{(S,\mathcal{C})} \to \mathsf{Lfb}(S)$. Our goal of this section is to verify that the global section functor is an equivalence of categories as well as the usual result in case of smooth manifolds.

Theorem 4.9. *Let $(S, \mathcal{C})$ be a stratifold. Then the global section functor*

$$\Gamma(S, -) : \mathsf{VBb}_{(S,\mathcal{C})} \to \mathsf{Fgp}(\mathcal{C})$$

gives rise to an equivalence of categories, where $\mathsf{Fgp}(\mathcal{C})$ denotes the category of finitely generated projective modules over $\mathcal{C}$.

We shall prove Theorem 4.9 by using the result due to Morye [10], Proposition 4.3 and an equivalence between categories $\mathsf{VBb}_{(S,\mathcal{C})}$ and $\mathsf{Lfb}(S)$ for a stratifold $(S, \mathcal{C})$, which is proved below.

Lemma 4.10. *Let X be a topological space and $\{X_\alpha\}$ an open cover of X. Suppose $(X_\alpha, \mathcal{C}_\alpha)$ is a differential space for each α. Define $\mathcal{C}$ to be the subalgebra of $C^0(X)$ consisting of $f : X \to \mathbb{R}$ such that $f|_{X_\alpha} \in \mathcal{C}_\alpha$ for all α. Then the pair $(X, \mathcal{C})$ is a differential space.*

Proof. The proof is straightforward. We check that $\mathcal{C}$ is a locally detectable $\mathbb{R}$-algebra. Let f be in $C^0(X)$. Assume further that, for each $x \in X$, there are an open neighborhood U_x of x and $h_x \in \mathcal{C}$ such that $f|_{U_x} = h_x|_{U_x}$. Since $h_x|_{X_\alpha} \in \mathcal{C}_\alpha$, $f|_{U_x \cap X_\alpha} = h_x|_{U_x \cap X_\alpha}$ and $\mathcal{C}_\alpha$ is locally detectable, it follows that $f|_{X_\alpha} \in \mathcal{C}_\alpha$ and hence $f \in \mathcal{C}$ by definition. ☐

Proposition 4.11. *The functor* $\mathcal{L} : \mathsf{VBb}_{(S,\mathcal{C})} \to \mathsf{Lfb}(S)$ *is essentially surjective.*

Proof. If $\mathcal{F} \in \mathsf{Lfb}(S)$, then there is an open cover $\{U_\alpha\}$ and isomorphisms $\varphi_\alpha : \mathcal{F}|_{U_\alpha} \to \mathcal{O}_S^{n_\alpha}|_{U_\alpha}$. Put $U_{\alpha,\beta} := U_\alpha \cap U_\beta$. We have a transition function $g_{\alpha\beta} : U_{\alpha\beta} \to \mathrm{GL}_n(\mathbb{R})$ induced by the isomorphisms φ_α and φ_β. More precisely, consider the sequence of morphisms of $\mathcal{O}_{U_{\alpha\beta}}$-modules

$$\begin{tikzcd}
\mathcal{O}_S|_{U_{\alpha\beta}} \arrow[r, "in_j"] & \mathcal{O}_S^n|_{U_{\alpha\beta}} \arrow[r, "\varphi_\beta^{-1}", "\cong"'] & \mathcal{F}|_{U_{\alpha\beta}} \arrow[r, "\varphi_\alpha", "\cong"'] & \mathcal{O}_S^n|_{U_{\alpha\beta}} \arrow[r, "p_i"] & \mathcal{O}_S|_{U_{\alpha\beta}},
\end{tikzcd}$$

where in_j and p_i denote the inclusion into the jth factor and the projection onto the ith factor, respectively. We define $u_{ij} \in \mathcal{O}_S|_{U_{\alpha\beta}}(U_{\alpha\beta})$ by $u_{ij} = p_i \varphi_\alpha \varphi_\beta^{-1} in_j(\mathbf{1})$ with unit $\mathbf{1}$ in $\mathcal{O}_S|_{U_{\alpha\beta}}(U_{\alpha\beta}) = \mathcal{O}(U_{\alpha\beta})$. Then $g_{\alpha\beta}$ is defined by $g_{\alpha\beta}(x) = (u_{ij}(x))$ for $x \in U_{\alpha\beta}$.

For each $x \in U_\alpha \cap U_\beta \cap U_\gamma$, these transition functions satisfy the relation $g_{\alpha\beta}(x) g_{\beta\gamma}(x) = g_{\alpha\gamma}(x)$. This enables us to define a space E by the quotient space $(\bigsqcup_\alpha U_\alpha \times \mathbb{R}^{n_\alpha})/\sim$, where the equivalence relation $\sim$ is defined by $(x, \mathsf{v}) \sim (y, \mathsf{w})$ if $x = y \in U_{\alpha\beta}$ and $\mathsf{v} = g_{\alpha\beta}(x)\mathsf{w}$. Let $\rho : \bigsqcup_\alpha U_\alpha \times \mathbb{R}^{n_\alpha} \to E$ be the canonical projection. Then we define a continuous map $\pi : E \to S$ by $\pi(\rho(x, \mathsf{v})) = x$. Since the restriction $\rho_\alpha : U_\alpha \times \mathbb{R}^{n_\alpha} \to \pi^{-1}(U_\alpha)$ is a homeomorphism, it gives a subalgebra $\mathcal{C}_\alpha$ of $C^0(\pi^{-1}(U_\alpha))$ which is naturally isomorphic to $\mathcal{C}_{U_\alpha \times \mathbb{R}^{n_\alpha}}$. By Lemma 4.10, we have a differential space $(E, \mathcal{C}_E)$.

If $f \in \mathcal{C}_S$, then $(f \circ \pi)|_{\rho_\alpha(U_\alpha \times \mathbb{R}^{n_\alpha})} \in \mathcal{C}_\alpha$ since the projection $U_\alpha \times \mathbb{R}^{n_\alpha} \to U_\alpha$ is a morphism of differential spaces; see Section 6. This implies that $f \circ \pi \in \mathcal{C}_E$ and hence the map $\pi : (E, \mathcal{C}_E) \to (S, \mathcal{C}_S)$ is a morphism of differential spaces. Moreover, we can see that the morphism π is a vector bundle with trivializations $(\{U_\alpha\}, \{\rho_\alpha\})$.

We shall show that $\mathcal{L}_E$ is isomorphic to $\mathcal{F}$. For $s \in \mathcal{L}_E(U_\alpha)$, we define $\hat{s} \in \mathcal{O}_S(U_\alpha)^{n_\alpha}$ by the composite $pr_2 \circ \rho_\alpha^{-1} \circ s : U_\alpha \to \pi^{-1}(U_\alpha) \to U_\alpha \times$

$\mathbb{R}^n \to \mathbb{R}^n$. Since $\psi_\alpha : \mathcal{L}_E(U_\alpha) \to \mathcal{O}_S(U_\alpha)^{n_\alpha}$ defined by $\psi_\alpha(s) = \hat{s}$ is an isomorphism, it gives rise to an isomorphism $\psi_\alpha : \mathcal{L}_E|_{U_\alpha} \to \mathcal{O}_S^{n_\alpha}|_{U_\alpha}$. The definitions of ψ_α and E allow us to deduce that $\varphi_\alpha^{-1} \circ \psi_\alpha = \varphi_\beta^{-1} \circ \psi_\beta$. Therefore, we have a morphisms of equalizers

$$\begin{tikzcd}
\mathcal{L}_F(U) \arrow[r] & \prod_\alpha \mathcal{L}_F(U_\alpha \cap U) \arrow[r, shift left, "res^\alpha_{\alpha\beta}"] \arrow[r, shift right, "res^\beta_{\alpha\beta}"'] \arrow[d, "\varphi_\alpha^{-1}\psi_\alpha", "\cong"'] & \prod_{\alpha\beta} \mathcal{L}_F(U_\alpha \cap U_\beta \cap U) \arrow[d, "\varphi_\alpha^{-1}\psi_\alpha=\varphi_\beta^{-1}\psi_\beta", "\cong"'] \\
\mathcal{F}(U) \arrow[r] & \prod_\alpha \mathcal{F}(U_\alpha \cap U) \arrow[r, shift left, "res^\alpha_{\alpha\beta}"] \arrow[r, shift right, "res^\beta_{\alpha\beta}"'] & \prod_{\alpha\beta} \mathcal{F}(U_\alpha \cap U_\beta \cap U).
\end{tikzcd}$$

This yields that $\mathcal{L}_E \cong \mathcal{F}$ as an $\mathcal{O}_S$-module. Hence the functor $\mathcal{L}$ is essentially surjective. □

Proposition 4.12. *The functor* $\mathcal{L}$ *is fully faithful.*

Proof. Let f and g be morphisms from (E, π_E) to (F, π_F). Assume that $\mathcal{L}_f = \mathcal{L}_g$. Then for all sections $s \in \Gamma(U, E)$, we see that $f \circ s = g \circ s$. This implies that $f = g$.

Suppose that $f : \mathcal{L}_E \to \mathcal{L}_F$ is a morphism in $\mathsf{Lfb}(S)$ and $\varphi_\alpha : \mathcal{L}_E|_{U_\alpha} \to \mathcal{O}_S^n|_{U_\alpha}$ and $\psi_\alpha : \mathcal{L}_F|_{U_\alpha} \to \mathcal{O}_S^m|_{U_\alpha}$ are trivializations which is induced by the given trivializations of E and F; see Proposition 4.8. Then we obtain the following commutative diagram

$$\begin{tikzcd}
\mathcal{O}_S^n|_{U_\alpha} \arrow[r, "\varphi_\alpha^{-1}", "\cong"'] \arrow[d, "t_\alpha"'] & \mathcal{L}_E|_{U_\alpha} \arrow[d, "f"] \\
\mathcal{O}_S^m|_{U_\alpha} & \mathcal{L}_F|_{U_\alpha}. \arrow[l, "\cong", "\psi_\alpha"']
\end{tikzcd}$$

The morphism t_α induces a morphism $t_\alpha : U_\alpha \to \mathrm{Mat}_{m,n}(\mathbb{R})$ of stratifolds with such way of defining $g_{\alpha\beta}$ in the proof of Proposition 4.11. We define a map $\eta_\alpha : E|_{U_\alpha} \cong U_\alpha \times \mathbb{R}^n \to U_\alpha \times \mathbb{R}^m \cong F|_{U_\alpha}$ by $\eta_\alpha(x, \mathsf{v}) = (x, t_\alpha(x)\mathsf{v})$. This map is a morphism of stratifolds since the restriction on each manifold $(U_\alpha \cap S^i) \times \mathbb{R}^n$ is smooth and for $l \in \mathcal{C}_{S\times\mathbb{R}^m}$, there are local retraction $r_x : U_x \to U_x \cap S^i$ and open set V of $\mathbb{R}^n$ such that $l \circ \eta_\alpha|_{U_x \times V} = l \circ \eta_\alpha(r_x \times id_V)$. Then the maps η_α induce a morphism $\eta : E \to F$ with $\eta_\alpha = \eta|_{\pi^{-1}(U_\alpha)}$:

$\pi_E^{-1}(U_\alpha) \to \pi_F^{-1}(U_\alpha)$. In fact, we have a commutative diagram

$$\begin{tikzcd}
\mathcal{O}_S^n|_{U_{\alpha,\beta}} \arrow[rr, "\varphi_\alpha\circ\varphi_\beta^{-1}"] \arrow[dr] \arrow[ddd, "t_\beta"'] & & \mathcal{O}_S^n|_{U_{\alpha,\beta}} \arrow[ddd, "t_\alpha"] \\
& \mathcal{L}_E|_{U_{\alpha,\beta}} \arrow[ur] \arrow[d, "f"] & \\
& \mathcal{L}_F|_{U_{\alpha,\beta}} \arrow[dr] & \\
\mathcal{O}_S^m|_{U_{\alpha,\beta}} \arrow[ur] \arrow[rr, "\psi_\alpha\circ\psi_\beta^{-1}"'] & & \mathcal{O}_S^m|_{U_{\alpha,\beta}}.
\end{tikzcd}$$

By the construction of $\mathcal{L}_\eta$, it is readily seen that $\mathcal{L}_\eta = f$ and hence the functor $\mathcal{L}$ is full. □

Thanks to Propositions 4.11 and 4.12, we see that the functor $\mathcal{L}$ is an equivalence of categories. We shall prove that the category $\mathsf{Lfb}(S)$ is equivalent to the full subcategory of the category $\Gamma(S, \mathcal{O}_S)$-Mod consisting of finitely generated projective modules, which is denoted by $\mathsf{Fgp}(\Gamma(S, \mathcal{O}_S))$.

Following Morye, we say that the *Serre-Swan theorem* holds for a locally ringed space $(X, \mathcal{O}_X)$ if the global section functor induces an equivalence of categories between $\mathsf{Lfb}(X)$ and $\mathsf{Fgp}(\Gamma(X, \mathcal{O}_X))$. The following theorem then completes the proof of Theorem 4.9.

Theorem 4.13. (Morye, [10, Corollary 3.2]) *Let $(X, \mathcal{O}_X)$ be a locally ringed space such that X is a paracompact Hausdorff space of finite covering dimension, and $\mathcal{O}_X$ is a fine sheaf of rings (cf. Definition 3.2). Then the Serre-Swan theorem holds for $(X, \mathcal{O}_X)$.*

The structure sheaf $\mathcal{O}_S$ of a stratifold $(S, \mathcal{C})$ is fine and the underlying space S is paracompact; see Remark 2.3(iii). In order to prove Theorem 4.9, it is thus sufficient to show that the covering dimension $\dim S$ of S is finite.

Theorem 4.14. *[13, Proposition 5.1 in chapter 3] Any n-dimensional paracompact manifold M (without boundary) has covering dimension* $\dim M = n$.

Theorem 4.15. *[13, Proposition 5.11 in chapter 3] Let X be a normal space and A and B be subspaces of X such that $X = A \cup B$. Then, $\mathit{dim}\, X \leq \mathit{dim}\, A + \mathit{dim}\, B + 1$.*

Corollary 4.16. *Any finite-dimensional stratifold has finite covering dimension.*

Proof. By the definition of a stratifold, we see that $S = S^1 \sqcup S^2 \sqcup ... \sqcup S^n$, where S^i is a manifold of dimension i. Theorems 4.14 and 4.15 imply that $\dim S < \infty$. □

By definition, it follows that $\Gamma(S, \mathcal{O}_S) = \mathcal{O}_S(S) = \mathcal{C}$. Thus Proposition 4.13 and the results above enable us to deduce the following corollary.

Corollary 4.17. *Let $(S, \mathcal{C})$ be a stratifold and $\mathcal{O}_S$ the structure sheaf. Then the global sections functor $\Gamma(S, -) : \mathsf{Lfb}(S) \to \mathsf{Fgp}(\mathcal{C})$ is an equivalence.*

We are now ready to prove the main theorem in this section.

Proof of Theorem 4.9. Corollary 4.17, Proposition 4.11 and 4.12 yield Theorem 4.9 □

Remark 4.18. Theorem 3.5 states that a stratifold $(S, \mathcal{C})$ can be regarded as a subsheaf of an affine scheme of the form $\mathrm{Spec}\, \mathcal{O}_S(S)$. Since the space $\mathrm{Spec}\, \mathcal{O}_S(S)$ is compact, it follows that the real spectrum $\mathrm{Spec}_r \mathcal{O}_S(S)$ is a proper subspace of the prime spectrum if S is non-compact; see Proposition 3.3. Moreover, in general, there exists a point in $\mathrm{Spec}\, \mathcal{O}_S(S)$ which is a maximal ideal but not in the real spectrum. Such a point is called a *ghost*; see [12, 8.22]. However, Theorem 4.9 and the original Serre-Swan theorem yield that the category $\mathsf{VBb}_{(S,\mathcal{C})}$ is equivalent to $\mathsf{VBb}_{\mathrm{Spec}\, \mathcal{O}_S(S)}$ the category of vector bundles over the affine scheme $\mathrm{Spec}\, \mathcal{O}_S(S)$ via the category $\mathsf{Fgp}(\Gamma(S, \mathcal{O}_S))$; see [10, Corollary 3.1] and [16, Theorem 6.2] for example.

5. A local characterization of morphisms of stratifolds

In this section, we describe morphisms of stratifolds inside the category of diffeological spaces. On the way we obtain a characterization of them by local data. We use the terminology of the book [4] for diffeology.

Let $\mathsf{Diffeology}$ be the category of diffeological spaces; see [4]. We define a functor $k : \mathsf{Stfd} \to \mathsf{Diffeology}$ by $k(S, \mathcal{C}) = (S, \mathcal{D}_{\mathcal{C}})$ and $k(\phi) = \phi$ for a morphism $\phi : S \to S'$ of stratifolds, where

$$\mathcal{D}_{\mathcal{C}} = \{u : U \to S \mid U : \text{open in } \mathbb{R}^q, q \geq 0, \phi \circ u \in C^\infty(U) \text{ for any } \phi \in \mathcal{C}\}.$$

Observe that a plot in $\mathcal{D}_{\mathcal{C}}$ is a set map. The functor k is faithful, but not full; that is, for a continuous map $f : S \to S'$, it is more restrictive to be a morphism of stratifolds $(S, \mathcal{C}) \to (S', \mathcal{C}')$ than to be a morphism of diffeological spaces $(S, \mathcal{D}_{\mathcal{C}}) \to (S', \mathcal{D}_{\mathcal{C}'})$.

We recall the fully faithful functor $\ell : \mathsf{Mfd} \to \mathsf{Diffeology}$ defined in [4, 4.3]; see also [1, Theorem 2.3]. For a diffeological space $(X, \mathcal{D})$, the set X admits a topology which is referred as the D-topology. More precisely, a subset A of X is open if and only if $p^{-1}(A)$ is open for any plot $p \in \mathcal{D}$. We denote by $T(X, \mathcal{D})$ the topological space. It is readily seen that the assignment of a topological space to a diffeological space induces a functor $T : \mathsf{Diffeology} \to \mathsf{Top}$.

For a topological space Y, we define a diffeological space $D(Y) = (Y, \mathcal{D}_Y)$ in which the set of plots $\mathcal{D}_Y$ consists of all continuous maps $U \to Y$ for any open subset U of $\mathbb{R}^q$ and for $q \geq 0$.

Let $\mathcal{D}_{\mathbb{R}}$ be the standard diffeology on $\mathbb{R}$. For each diffeological space $(X, \mathcal{D})$, we have an $\mathbb{R}$-algebra $F'((X, \mathcal{D})) := \mathrm{Hom}_{\mathsf{Diffeology}}((X, \mathcal{D}), (\mathbb{R}, \mathcal{D}_{\mathbb{R}}))$ with the algebra structure defined pointwise. A usual argument enables us to conclude that F' gives rise to a contravariant functor $F' : \mathsf{Diffeology} \to \mathbb{R}\text{-}\mathsf{Alg}$.

Summarizing the functors mentioned above, we have a diagram

$$\begin{tikzcd}
\mathsf{Diffeology} \arrow[r, shift left, "T"] \arrow[ddr, "F'"] & \mathsf{Top} \arrow[l, shift left, "D"] \arrow[dd, shift left, "C^0(\ )"] \\
\mathsf{Stfd} \arrow[u, "k"'] \arrow[dr, "F"] & \\
\mathsf{Mfd} \arrow[u, "j"'] \arrow[uu, bend left=60, "\ell"] \arrow[r, "C^\infty(\ )"'] & \mathbb{R}\text{-}\mathsf{Alg} \arrow[uu, shift left, "|\ |"]
\end{tikzcd} \tag{5.1}$$

in which the lower triangle and the left-hand side diagram are commutative. We observe that the functor T is a left adjoint to D; see [17, Proposition 3.1].

Moreover, it follows that the functor C^0 and $| \ |$ are adjoints. In fact, we have bijections

$$\begin{tikzcd}
\mathrm{Hom}_{\mathsf{Top}}(X, |\mathcal{F}|) \arrow[r, shift left, "\Phi"] & \mathrm{Hom}_{\mathbb{R}\text{-}\mathsf{Alg}}(\mathcal{F}, C^0(X)) \arrow[l, shift left, "\Psi"]
\end{tikzcd}$$

which are defined by the composites $\Phi(g) : \mathcal{F} \xrightarrow{\tau} \widetilde{\mathcal{F}} \xrightarrow{l} C^0(|\mathcal{F}|) \xrightarrow{g^*} C^0(X)$ with the inclusion l and $\Psi(\varphi) : X \xrightarrow{\theta} |C^0(X)| \xrightarrow{|\varphi|} |\mathcal{F}|$, respectively. The bijectivity follows from a straightforward computation.

We give here a characterization of morphisms of stratifolds in Diffeology with local data.

Proposition 5.1. *A morphism of diffeological spaces $f : (S, \mathcal{D}_{\mathcal{C}}) \to (S', \mathcal{D}_{\mathcal{C}'})$ stems from a morphism of stratifolds $f : (S, \mathcal{C}) \to (S', \mathcal{C}')$ if and only if for any $x \in S$, there exist local retractions $r_x : U_x \to U_x \cap S^i$ and $r_{f(x)} : V_{f(x)} \to V_{f(x)} \cap S'^j$ such that $r_{f(x)} \circ f \circ r_x = r_{f(x)} \circ f$ on some neighborhood of x.*

Proof. By definition, for any $x \in S$, there exists uniquely an integer i such that x is in S^i. Let $r_x : U_x \to U_x \cap S^i$ be a local retraction. The stratum S^i is an i-dimensional manifold. Therefore, we have a local diffeomorphism $\varphi_i : V_x \to U_x \cap S^i$ for some open subset V_x of $\mathbb{R}^i$. Let $u : V_x \to S$ be the composite $l \circ \varphi_i$, where $l : U_x \cap S^i \to S$ is the inclusion.

Suppose that $f : (S, \mathcal{D}_{\mathcal{C}}) \to (S', \mathcal{D}_{\mathcal{C}'})$ is a morphism of diffeological spaces. In order to prove the "if" part, it suffices to show that for any $x \in S$, the induced morphism $f^* : \mathcal{C}'_{f(x)} \to \mathcal{S}et(S, \mathbb{R})_x$ factors through the $\mathbb{R}$-algebra $\mathcal{C}_x$ of germs, where $\mathcal{S}et(S, \mathbb{R})_x$ denotes the germ at x of set maps $S \to \mathbb{R}$ associated with open neighborhoods of x. In fact, it follows that for any $\alpha \in \mathcal{C}'$, $f^*([\alpha]_{f(x)}) = [\alpha \circ f]_x \in \mathcal{C}_x$. Then there exists $\beta \in \mathcal{C}$ such that $(\alpha \circ f)|_{W_x} = \beta|_{W_x}$ for some open subset W_x of S. Since the $\mathbb{R}$-algebra $\mathcal{C}$ is locally detectable, we see that $f : S \to S'$ is a morphism of stratifolds and hence $k(f) = f$.

Consider the following diagram

$$\begin{tikzcd}
 & & \mathcal{C}'_{f(x)} \arrow[dll, "(f \circ u)^*"'] \arrow[drr, "f^*"] & & \\
C^\infty(V_x)_{\varphi_i^{-1}(x)} \arrow[r, shift left, "\psi_i^*"] \arrow[r, phantom, "\cong"] & C^\infty(S^i)_x \arrow[l, shift left, "\varphi_i^*"] \arrow[r, shift left, "r_x^*"] \arrow[r, phantom, "\cong"] & \mathcal{C}_x \arrow[l, shift left, "l^*"] \arrow[rr, "s"'] & & \mathcal{S}et(S, \mathbb{R})_x,
\end{tikzcd}$$

where ψ_i is the local inverse of φ_i and s denotes the inclusion. Observe that $(f \circ u)^* : \mathcal{C}'_{f(x)} \to C^\infty(V_x)_{\varphi_i^{-1}(x)}$ *is well defined since* f *is a morphism of diffeological spaces.* For any $\alpha \in \mathcal{C}'$, we see that $\alpha = r^*_{f(x)}(\alpha)$ in $\mathcal{C}'_{f(x)}$; see [8, page 19]. Thus it follows that

$$\begin{aligned} r^*_x \psi^*_i (f \circ u)^* (r^*_{f(x)}(\alpha)) &= \alpha \circ r_{f(x)} \circ f \circ l \circ \varphi_i \circ \psi_i \circ r_x \\ &= \alpha \circ r_{f(x)} \circ f \circ r_x = \alpha \circ r_{f(x)} \circ f = \alpha \circ f. \end{aligned}$$

The third equality follows from the assumption. This implies that f^* factors through the algebra $\mathcal{C}_x$ since $r^*_x \psi^*_i (f \circ u)^* (r^*_{f(x)}(\alpha))$ is in $\mathcal{C}_x$.

We prove the "only if" part. Let $f : (S, \mathcal{C}) \to (S', \mathcal{C}')$ be a morphism of stratifolds and $r_x : U_x \to U_x \cap S^i$ and $r_{f(x)} : V_{f(x)} \to V_{f(x)} \cap S'^j$ are appropriate local retractions. Without loss of generalities, we may assume that the image of $r_{f(x)}$ is contained in a local coordinate $V'_{f(x)}$ of the manifold $V_{f(x)} \cap S'^j$. We have $r^*_x \circ \psi^*_i \circ (f \circ u)^* = f^*$. Observe that $(f \circ u)^* = \varphi^*_i \circ l^* \circ f^*$ and that the target of f^* is the algebra $\mathcal{C}_x$. Let π_k be an element in $\mathcal{C}^\infty(V_{f(x)} \cap S'^j)_{f(x)}$ obtained by extending the composite $V'_{f(x)} \overset{\cong}{\to} V' \xrightarrow{t} \mathbb{R}^j \xrightarrow{pr_k} \mathbb{R}$ by a bump function at $f(x)$, where $V'_{f(x)} \overset{\cong}{\to} V'$ is the homeomorphism of the local coordinate, t is the inclusion and pr_k denotes the projection onto the kth factor. Then for the element $r^*_{f(x)}(\pi_k) \in \mathcal{C}'_{f(x)}$, we have $r^*_x \circ \psi^*_i \circ (f \circ u)^*(r^*_{f(x)}(\pi_k)) = f^* \circ r^*_{f(x)}(\pi_k)$. The same argument as above enables us to deduce that

$$\pi_k \circ r_{f(x)} \circ f \circ r_x = \pi_k \circ r_{f(x)} \circ f$$

on some neighborhood W_x of x and hence $r_{f(x)} \circ f \circ r_x = r_{f(x)} \circ f$ on W_x. This completes the proof. □

Corollary 5.2. *Let* M *be a manifold and* $(S, \mathcal{C})$ *a stratifold. Then the functor* $k : \mathsf{Stfd} \to \mathsf{Diffeology}$ *induces a bijection*

$$k_* : \mathrm{Hom}_{\mathsf{Stfd}}((M, C^\infty(M)), (S, \mathcal{C})) \overset{\cong}{\to} \mathrm{Hom}_{\mathsf{Diffeology}}((M, \mathcal{D}_{C^\infty(M)}), (S, \mathcal{D}_{\mathcal{C}})).$$

In the rest of this section, we give a subcategory of Diffeology which is equivalent to Stfd as a category.

Definition 5.3. Let $(S, \mathcal{C})$ and $(S', \mathcal{C}')$ be stratifolds. A continuous map $f : S \to S'$ is $(\mathcal{C}, \mathcal{C}')$-*admissible* if for any $x \in S$, there exist local retractions

r_x and $r_{f(x)}$ near x and $f(x)$, respectively such that $r_{f(x)} \circ f \circ r_x = r_{f(x)} \circ f$ and for each $\phi \in \mathcal{C}'$, the restriction of $\phi \circ f : S \to \mathbb{R}$ to any stratum S^i of $(S, \mathcal{C})$ is smooth.

The proof of Proposition 5.1 yields the following result, which recovers as special case [8, Exercise 2.6(11)].

Proposition 5.4. *A continuos map $f : S \to S'$ induces a morphism of stratifolds $(S, \mathcal{C}) \to (S', \mathcal{C}')$ if and only if f is $(\mathcal{C}, \mathcal{C}')$-admissible.*

Let $(S, \mathcal{C})$, $(S', \mathcal{C}')$ and $(S'', \mathcal{C}'')$ be stratifolds. Proposition 5.4 immediately implies that $(\mathcal{C}, \mathcal{C}')$-admissible continuous maps compose with $(\mathcal{C}', \mathcal{C}'')$-admissible continuous maps $S' \to S''$.

Let $k : \mathsf{Stfd} \to \mathsf{Diffeology}$ be the functor in (5.1) and $\langle \mathsf{Im} k \rangle$ the full subcategory of Diffeology consisting of objects which come from Stfd by k. By the argument above, we have a wide subcategory $\langle \mathsf{Im} k \rangle_W$ of $\langle \mathsf{Im} k \rangle$ consisting of admissible maps and the same class of objects as in $\langle \mathsf{Im} k \rangle$. Then Proposition 5.1 establishes the following theorem.

Theorem 5.5. *The functor $k : \mathsf{Stfd} \to \mathsf{Diffeology}$ induces an equivalence $k : \mathsf{Stfd} \to \langle \mathsf{Im} k \rangle_W$ of categories. In particular, one has a natural bijection*

$$k_* : \mathrm{Hom}_{\mathsf{Stfd}}((S, \mathcal{C}), (S', \mathcal{C}')) \xrightarrow{\cong} \mathrm{Hom}_{\langle \mathsf{Im} k \rangle_W}((S, \mathcal{D}_{\mathcal{C}}), (S', \mathcal{D}_{\mathcal{C}'})).$$

6. Cartesian product of stratifolds

We recall the product of stratifolds defined in [8]. Let $(S, \mathcal{C}_S)$ and $(S', \mathcal{C}_{S'})$ be stratifolds. We define a stratifold with the underlying topological space $S \times S'$. Let $\mathcal{C}_{S \times S'}$ be the $\mathbb{R}$-algebra consisting of functions $f : S \times S' \to \mathbb{R}$ which are smooth on every products $S^i \times (S')^j$ and for each $(x, y) \in S^i \times (S')^j$, there are local retractions $r_x : U_x \to S^i \cap U_x$ and $r_y : V_y \to (S')^j \cap V_y$ for which $f|_{U_x \times V_y} = f(r_x \times r_y)$. Then $(S \times S', \mathcal{C}_{S \times S'})$ is a stratifold and the projections into first and second factors are morphisms of stratifolds; see [8, Appendix A].

Proposition 6.1. *The product of stratifolds mentioned above is the cartesian product in the category* Stfd.

We use lemmas to prove Proposition 6.1.

Lemma 6.2. *Let* $f_1 : (S_1, \mathcal{C}_1) \to (S'_1, \mathcal{C}'_1)$ *and* $f_2 : (S_2, \mathcal{C}_2) \to (S'_2, \mathcal{C}'_2)$ *be morphisms of stratifolds. Then the product of maps*

$$f_1 \times f_2 : (S_1 \times S_2, \mathcal{C}_{S_1 \times S_2}) \to (S'_1 \times S'_2, \mathcal{C}_{S'_1 \times S'_2})$$

is a morphism of stratifolds.

Proof. For $x \in S_i$, assume that $x \in S_i^{k_i}$ and $f_i(x) \in S_i'^{j_i}$. Then we have a diagram

$$\begin{tikzcd}
\mathcal{C}'_{f_i(x)} \arrow[r, "\cong"] \arrow[d, "f_i^*"'] & C^\infty(S_i'^{j_i})_{f_i(x)} \\
\mathcal{C}_x \arrow[r, "\cong"] & C^\infty(S_i^{k_i})_x
\end{tikzcd}$$

in which horizontal maps induced by the inclusions are isomorphisms. Therefore, for a smooth map φ defined on an appropriate neighborhood of $f_i(x)$ in $S_i'^{j_i}$, we see that $\varphi \circ r_{f_i(x)} \circ f_i$ is a smooth map on some neighborhood of x in $S_i^{k_i}$, where $r_{f_i(x)}$ denotes a local retraction near $f_i(x)$. Thus, we infer that for any h in $\mathcal{C}_{S'_1 \times S'_2}$,

$$\begin{aligned}
h \circ (f_1 \times f_2)|_{S_1^{k_1} \times S_2^{k_2}} &= h \circ (r_{f_1(x_1)} \times r_{f_2(x_2)}) \circ (f_1 \times f_2) \\
&= (h \circ \varphi_\alpha^{-1}) \circ (\varphi_\alpha \circ (r_{f_1(x_1)} \times r_{f_2(x_2)}) \circ (f_1 \times f_2)
\end{aligned}$$

on some neighborhood of (x_1, x_2) in $S_1^{k_1} \times S_2^{k_2}$, where φ_α is a local coordinate around $(f_1(x_1), f_2(x_2))$ of the manifold $S_1'^{j_1} \times S_2'^{j_2}$. This implies that $h \circ (f_1 \times f_2)|_{S_1^{k_1} \times S_2^{k_2}}$ is smooth. Since f_1 and f_2 are admissible, it follows that for $h \in \mathcal{C}_{S'_1 \times S'_2}$,

$$\begin{aligned}
h \circ (f \times f_2) \circ (r_{x_1} \times r_{x_2}) &= h \circ (r_{f_1(x_1)} \times r_{f_2(x_2)}) \circ (f \times f_2) \circ (r_{x_1} \times r_{x_2}) \\
&= h \circ (r_{f_1(x_1)} \times r_{f_2(x_2)}) \circ (f \times f_2) \\
&= h \circ (f \times f_2)
\end{aligned}$$

on an appropriate neighborhood of (x_1, x_2) in $S_1 \times S_2$. This completes the proof. □

By the same argument as in the proof of Lemma 6.2, we have the following lemma.

Lemma 6.3. *The diagonal map $\Delta : S \to S \times S$ is a morphism of stratifolds.*

Proof of Proposition 6.1. Let $(S, \mathcal{C})$, $(S', \mathcal{C}')$ and $(Z, \mathcal{C}_1)$ be stratifolds. Let $f_1 : (Z, \mathcal{C}_1) \to (S, \mathcal{C})$ and $f_2 : (Z, \mathcal{C}_1) \to (S', \mathcal{C}')$ be morphisms of stratifolds. It suffices to show that $(f_1 \times f_2) \circ \Delta$ is a morphism of stratifolds. This follows from Lemmas 6.2 and 6.3 □

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