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ON CERTAIN TOPOLOGICAL *-AUTONOMOUS CATEGORIES

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Résumé. Etant donné une catégorie additive et equationelle, munie d'une structure fermeé monoïdale symetrique ainsi que d'un objet dualisateur potentiel, on trouve des conditions suffisantes pour que la catégorie des objets topologiques sur cette catégorie admette une bonne notion des souscatégories pleines qui contiennent des objets fortement et faiblement topologisés. On montre que chacune des souscatégories est équivalente à la catégorie chu de la catégorie originale par rapport à l'objet dualisateur.

Abstract. Given an additive equational category with a closed symmetric monoidal structure and a potential dualizing object, we find sufficient conditions that the category of topological objects over that category admits a good notion of full subcategories of strong and weakly topologized objects and show that each is equivalent to the chu category of the original category with respect to the dualizing object.

Keywords. spherically complete fields, duality, Chu construction

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1. Introduction

This paper is a continuation of [5, 6, 7]. The first reference showed that the full subcategory of the category of (real or complex) topological vector spaces that consists of the Mackey spaces (defined in 2.5 below)is $*$-autonomous and equivalent to both the full subcategory of weakly topologized topological vector spaces and to the full subcategory of topological vector spaces topologized with the strong, or Mackey topology. This means, first, that those subcategories can, in principle at least, be studied without taking the topology into consideration. Second it implies that both of those categories are $*$-autonomous.

In [6], we showed that the category of topological abelian groups had similar properties: that both the weakly and strongly topologized abelian groups formed a $*$-autonomous category.

Later, André Joyal raised the question whether the results of [5] remained valid for vector spaces over the field $\mathbf{Q}_p$ of p-adic rationals. This question was mentioned, but not answered, in [7]. Thinking about this question, I realized that there is a useful general theorem that answers this question for any locally compact field and also for locally compact abelian groups. The current paper provides a positive answer to Joyal's question.

All these results follow from a systematic use of the chu construction, see Section 3 below.

To state the main results, we need a definition. A normed field is **spherically complete** if any family of closed balls with the finite intersection property has non-empty intersection. It is known that every locally compact field is spherically complete (so this answers Joyal's question since $\mathbf{Q}_p$, as well as its finite extensions, is locally compact) and spherically complete is known to be strictly stronger than metrically complete.

Theorem 1.1. *Let K be a spherically complete field and $|K|$ its underlying discrete field. Then the following five categories are equivalent:*

1. $\mathrm{chu}(K\text{-Vect}, |K|)$ *(Section 3)*
2. *The category $\mathcal{V}_w(K)$ of topological K-spaces topologized with the weak topology for all their continuous linear functionals into K.*
3. *The category $\mathcal{V}_s(K)$ of topological K-spaces topologized with the strong topology (see Section 2) for all their continuous linear functionals into K.*
4. *The category $\mathcal{V}_w(|K|)$ of topological $|K|$-spaces topologized with the weak topology for all their continuous linear functionals into $|K|$.*
5. *The category $\mathcal{V}_s(|K|)$ of topological $|K|$-spaces topologized with the strong topology (see Section 2) for all their continuous linear functionals into $|K|$.*

and all are $*$-autonomous (see beginning of Section 3).

The methods also apply to give the results of [6].

1.1 Terminology

We assume that all topological objects are Hausdorff. As we will see, each of the categories contains an object K with special properties. It will be convenient to call a morphism $V \longrightarrow K$ a **functional** *on V. In the case of abelian groups, the word "character" would be more appropriate, but it is convenient to have one word. In a similar vein, we may refer to a mapping of topological abelian groups as "linear" to mean additive. We will be dealing with topological objects in categories of topological vector spaces and abelian groups. If V is such an object, we will denote by $|V|$ the underlying vector space or group.*

If K is a topological field, we will say that a vector space is **linearly discrete** *if it is a categorical sum of copies of the field.*

2. The strong and weak topologies

2.1 Blanket assumptions.

In this section, we deal with a certain category $\mathcal{T}$ of topological algebras and a distinguished object K, usually called the **dualizing object**. *Maps $V \longrightarrow K$ in $\mathcal{T}$ will be called* **functionals**. *A bijective map $V \longrightarrow V'$ will be called a* **weak isomorphism** *if it induces a bijection $\mathrm{Hom}(V', K) \longrightarrow \mathrm{Hom}(V, K)$. We show that for any V, there is a space τV with the finest possible topology for which $\tau V \longrightarrow V$ is a weak isomorphism and a space σV with the coarsest possible topology for which $V \longrightarrow \sigma V$ is a weak isomorphism. We show that σ and τ are functors for which the weak isomorphisms just mentioned are natural transformations.*

Throughout this section, we make the following assumptions.

1. *$\mathcal{A}$ is an additive equational closed symmetric monoidal category and $\mathcal{T}$ is the category of topological $\mathcal{A}$-algebras.*
2. *K is a uniformly complete object of $\mathcal{T}$.*
3. *there is a neighbourhood U of 0 in K such that*
    - *(a) U contains no non-zero subobject;*

(b) whenever $\varphi : T \longrightarrow K$ is such that $\varphi^{-1}(U)$ is open, then φ is continuous.

In connection with point 2, in the application to spherically complete fields, K will be the ground field and we have already noted that sperically complete fields are metrically complete. In the application to topological groups, K will the compact circle group.

Point 3 says that, in some sense, the neighbourhood U is small. The existence of such a neighbourhood in the circle group is well known, although we provide an argument.

Lemma 2.1. *Suppose there is an embedding $T \hookrightarrow \prod_{i\in I} T_i$ and there is a morphism $\varphi : T \longrightarrow K$. Then there is a finite subset $J \subseteq I$ and a commutative diagram*

$$\begin{tikzcd}
T \arrow[rr, hook] \arrow[dr] \arrow[dd, "\varphi"'] & & \prod_{i\in I} T_i \arrow[d] \\
& T_0 \arrow[r, hook] \arrow[dl, "\varphi_0"] & \prod_{j\in J} T_j \\
K & &
\end{tikzcd}$$

Moreover, we can take T_0 closed in $\prod_{j\in J} T_j$.

Proof. Since $\varphi^{-1}(U)$ is a neighbourhood of 0 in T, it must be the meet with T of a neighbourhood of 0 in $\prod_{i\in I} T_i$. From the definition of the product topology, we must have a finite subset $J \subseteq I$ and neighbourhoods U_j of 0 in T_j such that

$$\varphi^{-1}(U) \supseteq T \cap (\prod_{j\in J} U_j \times \prod_{i\in I-J} T_i)$$

It follows that

$$U \supseteq \varphi(T \cap (\prod_{j\in J} 0 \times \prod_{i\in I-J} T_i))$$

But the latter is a subobject of K contained in U and therefore must be 0. Now let

$$T_0 = \frac{T}{T \cap (\prod_{j\in J} 0 \times \prod_{i\in I-J} T_i)}$$

topologized as a subspace of $\prod_{j\in J} T_j$ and φ_0 be the induced map. It is immediate that $\varphi_0^{-1}(U) \supseteq \prod_{j\in J} U_j$ which is a neighbourhood of 0 in the induced topology and hence φ_0 is continuous. Finally, since K is complete, we can replace T_0 by its closure in $\prod_{j\in J} T_j$. $\square$

Theorem 2.2. *Suppose $\mathcal{S}$ is a full subcategory of $\mathcal{T}$ that is closed under finite products and closed subobjects and that $K \in \mathcal{S}$ satisfies the assumptions in 2.1. If $\mathcal{V}$ is the closure of $\mathcal{S}$ under all products and all subobjects and K is injective in $\mathcal{S}$, then it is also injective in $\mathcal{V}$.*

Proof. It is sufficient to show that if $V \subseteq \prod_{i\in I} S_i$ with each $S_i \in \mathcal{S}$, then every morphism $V \longrightarrow K$ extends to the product. But the object V_0 constructed in the preceding lemma is a closed suboject of $\prod_{j\in J} S_j$ so that $V_0 \in \mathcal{S}$ and the fact that K is injective in $\mathcal{S}$ completes the proof. $\square$

Recall that a weak isomorphism $V \longrightarrow V'$ is a bijective morphism that induces a bijection on the functionals.

Of course, a bijective morphism induces an injection so the only issue is whether the induced map is a surjection.

Proposition 2.3. *A finite product of weak isomorphisms is a weak isomorphism.*

Proof. Assume that J is a finite set and for each $j \in J$, $V_j \longrightarrow V_j'$ is a weak isomorphism. Then since finite products are the same as finite sums in an additive category, we have

$$\begin{aligned}\mathrm{Hom}(\prod V_j', K) &\cong \mathrm{Hom}(\sum V_j', K) \cong \prod \mathrm{Hom}(V_j', K)\\ &\cong \prod \mathrm{Hom}(V_j, K) \cong \mathrm{Hom}(\sum V_j, K) \cong \mathrm{Hom}(\prod V_j, k)\end{aligned} \qquad \blacksquare$$

Theorem 2.4. *Assume the conditions of Theorem 2.2 and also suppose that for every object of $\mathcal{S}$, and therefore of $\mathcal{V}$, there are enough functionals to separate points. Then for every object V of $\mathcal{V}$, there are weak isomorphisms $\tau V \longrightarrow V \longrightarrow \sigma V$ with the property that σV has the coarsest topology that has the same functionals as V and τV has the finest topology that has same functionals as V.*

Proof. The argument for σ is standard. Simply retopologize V as a subspace of $K^{\mathrm{Hom}(V,K)}$. This is the weakest topology for which all the functionals are continuous and obviously no weaker topology will admit all the functionals.

Let $\{V_i \longrightarrow V\}$ range over the isomorphism classes of weak isomorphisms to V. We define τV as the pullback in

$$\begin{tikzcd}
\tau V \arrow[r] \arrow[d] & \prod V_i \arrow[d] \\
V \arrow[r] & V^I
\end{tikzcd}$$

The bottom map is the diagonal and is a topological embedding so that the top map is also a topological embedding. We must show that every functional on τV is continuous on V. Let φ be a functional on τV. From injectivity, it extends to a functional ψ on $\prod V_i$. By Lemma 2.1, there is a finite subset $J \subseteq I$ and a functional ψ_0 on $\prod_{j\in J} V_j$ such that ψ is the composite $\prod_{i\in I} V_i \longrightarrow \prod_{j\in J} V_j \xrightarrow{\psi_0} K$. Thus we have the commutative diagram

$$\begin{tikzcd}
\tau V \arrow[r] \arrow[dd] & \prod_{i\in I} V_i \arrow[r] \arrow[dd] & \prod_{j\in I} V_j \arrow[dd] \arrow[dr, "\psi_0"] & \\
 & & & K \\
V \arrow[r] & V^I \arrow[r] & V^J \arrow[ur, dashed] &
\end{tikzcd}$$

The dashed arrow exists because of Proposition 2.3, which completes the proof. □

Remark 2.5. We will call the topologies on σV and τV the **weak** and **strong** topologies, respectively. They are the coarsest and finest topology that have the same underlying $\mathscr{A}$ structure and the same functionals as V. The strong topology is also called the **Mackey topology**.

Proposition 2.6. *Weak isomorphisms are stable under pullback.*

Proof. Suppose that

$$\begin{tikzcd} W' \arrow[r] \arrow[d, "f"'] & W \arrow[d, "f'"] \\ V' \arrow[r] & V \end{tikzcd}$$

and the bottom arrow is a weak isomorphism. Clearly, $W' \longrightarrow W$ is a bijection, so we need only show that $\mathrm{Hom}(W, K) \longrightarrow \mathrm{Hom}(W', K)$ is surjective.

I claim that $W' \subseteq W \times V'$ with the induced topology. Let us define W'' to be the subobject $W \times_V V'$ with the induced topology. Since $W' \longrightarrow W$ and $W' \longrightarrow V$ are continuous, the topology on W' is at least as fine as that of W''. On the other hand, we do have $W'' \longrightarrow W$ and $W'' \longrightarrow V'$ with the same map to V so that we have $W'' \longrightarrow W'$, so that the topology on W'' is at least as fine as that of W'. Then we have a commutative diagram

$$\begin{tikzcd} W' \arrow[r] \arrow[d, hook] & W \arrow[d, "(\mathrm{id}, f)"] \\ W \times V' \arrow[r] & W \times V \end{tikzcd}$$

Apply $\mathrm{Hom}(-, K)$ and use the injectivity of K to get:

$$\begin{tikzcd} \mathrm{Hom}(W', K) & \mathrm{Hom}(W, K) \arrow[l] \\ \mathrm{Hom}(W, K) \times \mathrm{Hom}(V', K) \arrow[u, two heads] & \mathrm{Hom}(W, K) \times \mathrm{Hom}(V, K) \arrow[l, "\cong"'] \arrow[u, two heads] \end{tikzcd}$$

The bottom arrow is a bijection and the left hand arrow is a surjection, which implies that the top arrow is a surjection. □

Proposition 2.7. *σ and τ are functors on $\mathcal{V}$.*

Proof. For σ, this is easy. If $f : W \longrightarrow V$ is a morphism, the induced $\sigma f : \sigma W \longrightarrow \sigma V$ will be continuous if and only if its composite with every functional on V is a functional on W, which obviously holds.

To see that τ is a functor, suppose $f : W \longrightarrow V$ is a morphism. Form the pullback

$$\begin{tikzcd}
W' \arrow[r] \arrow[d, "f'"'] & W \arrow[d, "f"] \\
\tau V \arrow[r] & V
\end{tikzcd}$$

Since $\tau V \longrightarrow V$ is a weak isomorphism, the preceding proposition implies that $W' \longrightarrow W$ is a weak isomorphism. But since τW has the finest topology with that property, it follows that the topology on τW is finer than that of W' and hence $\tau W \longrightarrow W$ factors through W' and the composite $\tau W \longrightarrow W' \longrightarrow \tau V$. □

Proposition 2.8. *If $V \longrightarrow V'$ is a weak isomorphism, then $\sigma V \longrightarrow \sigma V'$ and $\tau V \longrightarrow \tau V'$ are isomorphisms.*

Proof. For σ, this is obvious. Clearly, $\tau V \longrightarrow V \longrightarrow \tau V'$ is also a weak isomorphism so that τV is one of the factors in the computation of $\tau V'$ and then $\tau V' \longrightarrow \tau V$ is a continuous bijection, while the other direction is evident. □

Corollary 2.9. *Both σ and τ are idempotent, while $\sigma\tau \cong \sigma$ and $\tau\sigma \cong \tau$.* □

Proposition 2.10. *For any $V, V' \in \mathcal{V}$, we have* $\mathrm{Hom}(\sigma V, \sigma V') \cong \mathrm{Hom}(\tau V, \tau V')$.

Proof. It is easiest to assume that the underlying objects $|V| = |\sigma V| = |\tau V|$ and similarly for V'. Then for any $f : V \longrightarrow V'$, we also have that $|f| = |\sigma f| = |\tau f|$. Thus the two composition of the two maps below

$$\mathrm{Hom}(\sigma V, \sigma V') \longrightarrow \mathrm{Hom}(\tau\sigma V, \tau\sigma V') = \mathrm{Hom}(\tau V, \tau V')$$

and

$$\mathrm{Hom}(\tau V, \tau V') \longrightarrow \mathrm{Hom}(\sigma\tau V, \sigma\tau V') \cong \mathrm{Hom}(\sigma V, \sigma V')$$

give the identity in each direction. □

Let $\mathcal{V}_w \subseteq \mathcal{V}$ and $\mathcal{V}_s \subseteq \mathcal{V}$ denote the full subcategories of weak and strong objects, respectively. Then as an immediate corollary to the preceding, we have:

Theorem 2.11. *$\tau : \mathcal{V}_w \longrightarrow \mathcal{V}_s$ and $\sigma : \mathcal{V}_s \longrightarrow \mathcal{V}_w$ determine inverse equivalences of categories.*

3. Chu and chu

A $*$-autonomous is a symmetric monoidal closed category equipped with a “dualizing object” $\perp$. We will denote the monoidal structure by $\otimes$ with tensor unit $\top$ and the closed structure by $\multimap$. The basic assumption is that for every object A the canonical map $A \longrightarrow (A \multimap \perp) \multimap \perp$ is an isomorphism. We let $A^* = A \multimap \perp$. Many things follow from this, e.g. $A \multimap B \cong B^* \multimap A^*$, $A \otimes B \cong (A \multimap B^*)^*$, and $A \multimap B \cong (A \otimes B^*)^*$. See [2] for all details.

Now we add to the assumptions on $\mathcal{A}$ that it be a symmetric monoidal closed category in which the underlying set of $A \multimap B$ is $\mathrm{Hom}(A, B)$. We denote by $\mathcal{E}$ and $\mathcal{M}$ the classes of surjections and injections, respectively.

We briefly review the categories $\mathrm{Chu}(\mathcal{A}, K)$ and $\mathrm{chu}(\mathcal{A}, K)$. See [4] for details. The first has a objects pairs (A, X) of objects of $\mathcal{A}$ equipped with a “pairing” $\langle -, - \rangle : A \otimes X \longrightarrow K$. A morphism $(f, g) : (A, X) \longrightarrow (B, Y)$ consists of a map $f : A \longrightarrow B$ and a map $g : Y \longrightarrow X$ such that

$$\begin{tikzcd}
A \otimes Y \arrow[r, "f \otimes Y"] \arrow[d, "A \otimes g"'] & B \otimes Y \arrow[d, "\langle -,- \rangle"] \\
A \otimes X \arrow[r, "\langle -,- \rangle"'] & K
\end{tikzcd}$$

commutes. This says that $\langle fa, y \rangle = \langle a, gy \rangle$ for all $a \in A$ and $y \in Y$. This can be enriched over $\mathcal{A}$ by internalizing this definition as follows. Note first that the map $A \otimes X \longrightarrow K$ induces, by exponential transpose, a map $X \longrightarrow A \multimap K$. This gives a map $Y \multimap X \longrightarrow Y \multimap (A \multimap K) \cong A \otimes Y \multimap K$. There is a similarly defined arrow $A \multimap B \longrightarrow A \otimes Y \multimap K$. Define $[(A,X),(B,Y)]$ so that

$$\begin{tikzcd}
{[(A,X),(B,Y)]} \arrow[r] \arrow[d] & A \multimap B \arrow[d] \\
Y \multimap X \arrow[r] & A \otimes Y \multimap K
\end{tikzcd}$$

is a pullback. Then define

$$(A,X) \multimap (B,Y) = ([(A,X),(B,Y)], A \otimes Y)$$

with $\langle (f,g), a \otimes y \rangle = \langle fa, y \rangle = \langle a, gy \rangle$ and

$$(A,X) \otimes (B,Y) = (A \otimes B, [(A,X),(Y,B)])$$

with pairing $\langle a \otimes b, (f,g) \rangle = \langle b, fa \rangle = \langle a, gb \rangle$. The duality is given by $(A,X)^* = (X,A) \cong (A,X) \multimap (K,\top)$ where $\top$ is the tensor unit of $\mathcal{A}$. Incidentally, the tensor unit of $\mathrm{Chu}(\mathcal{A},K)$ is $(\top,K)$.

The category $\mathrm{Chu}(\mathcal{A},K)$ is complete (and, of course, cocomplete). The limit of a diagram is calculated using the limit of the first coordinate and the colimit of the second. The full subcategory $\mathrm{chu}(\mathcal{A},K) \subseteq \mathrm{Chu}(\mathcal{A},K)$ consists of those objects (A,X) for which the two transposes of $A \otimes X \longrightarrow K$ are injective homomorphisms. When $A \rightarrowtail X \multimap K$, the pair is called separated and when $X \rightarrowtail A \multimap K$, it is called extensional. In the general case, one must choose a factorization system $(\mathcal{E},\mathcal{M})$ and assume that the arrows in $\mathcal{E}$ are epic and that $\mathcal{M}$ is stable under $\multimap$, but here these conditions are clear. Let us denote by $\mathrm{Chu}_s(\mathcal{A},K)$ the full subcategory of separated pairs and by $\mathrm{Chu}_e(\mathcal{A},K)$ the full subcategory of extensional pairs.

The inclusion $\mathrm{Chu}_s(\mathcal{A},K) \hookrightarrow \mathrm{Chu}(\mathcal{A},K)$ has a left adjoint S and the inclusion $\mathrm{Chu}_e(\mathcal{A},K) \hookrightarrow \mathrm{Chu}(\mathcal{A},K)$ has a right adjoint E. Moreover, S takes an extensional pair into an extensional one and E does the dual. In addition, when (A,X) and (B,Y) are separated and extensional, $(A,X) \multimap (B,Y)$ is separated but not necessarily extensional and, dually, $(A,X) \otimes (B,Y)$ is extensional, but not necessarily separated. Thus we must apply the reflector to the internal hom and the coreflector to the tensor, but everything works out and $\mathrm{chu}(\mathcal{A},K)$ is also $*$-autonomous. See [4] for details.

In the chu category it is evident that for any $(f, g) : (A, X) \longrightarrow (B, Y)$, f and g determine each other uniquely. So a map could just as well be described as an $f : A \longrightarrow B$ such that $x.\tilde{y} \in X$ for every $y \in Y$. Here $\tilde{y} : B \longrightarrow K$ is the evaluation at $y \in Y$ of the exponential transpose $Y \longrightarrow B \multimap K$.

Although the situation in the category of abelian groups is as described, in the case of vector spaces over a field, the hom and tensor of two separated extensional pairs turns out to be separated and extensional already ([3]).

4. The main theorem

Theorem 4.1. *Assume the hypotheses of Theorem 2.4 and also assume that the canonical map* $\top \longrightarrow K \multimap K$ *is an isomorphism. Then the categories of weak spaces and strong spaces are equivalent to each other and to chu*$(\mathcal{A}, K)$ *and are thus* $*$*-autonomous.*

Proof. The first claim is just Theorem 2.11. Now define $F : \mathcal{V} \longrightarrow \mathrm{chu}$ by $F(V) = (|V|, \mathrm{Hom}(V, K))$ with evaluation as pairing. We first define the right adjoint R of F. Let $R(A, X)$ be the object A, topologized as a subobject of K^X. Since it is already inside a power of K, it has the weak topology. Let $f : |V| \longrightarrow A$ be a homomorphism such that for all $x \in X$, $\tilde{x}.f \in \mathrm{Hom}(V, K)$. This just means that the composite $V \longrightarrow R(A, X) \longrightarrow K^X \xrightarrow{\pi_x} K$ is continuous for all $x \in X$, exactly what is required for the map into $R(A, X)$ to be continuous. The uniqueness of f is clear and this establishes the right adjunction.

We next claim that $FR \cong \mathrm{Id}$. That is equivalent to showing that $\mathrm{Hom}(R(A, X), K) = X$. Suppose $\varphi : R(A, X) \longrightarrow K$ is a functional. By injectivity, it extends to a $\psi : K^X \longrightarrow K$. It follows from 2.1, there is a finite set of elements $x_1, \ldots, x_n \in X$ and morphisms $\theta_1, \ldots, \theta_n$ such that ψ factors as $K^X \longrightarrow K^n \xrightarrow{(\theta_1, \ldots, \theta_n)} K$. Applied to $R(A, X)$, this means that $\varphi(a) = \langle \theta_1 x_1, a \rangle + \cdots + \langle \theta_n x_n, a \rangle$. But the $\theta_i \in I$ and the tensor products are over I so that the pairing is a homomorphism $A \otimes_I X \longrightarrow K$. This means that $\varphi(a) = \langle \theta_1 x_1 + \cdots \theta_n x_n, a \rangle$ and $\theta_n x_1 + \cdots + \theta_n x_n \in X$.

Finally, we claim that $RF = S$, the left adjoint of the inclusion $\mathcal{V}_w \subseteq \mathcal{V}$. If $V \in \mathcal{V}$, then $RFV = R(|V|, \mathrm{Hom}(V, K))$ which is just V with the weak topology it inherits from $K^{\mathrm{Hom}(V,K)}$, exactly the definition of SV. It follows that $F|\mathcal{V}_w$ is an equivalence.

Since $\mathcal{V}_w$ and $\mathcal{V}_s$ are equivalent to a $*$-autonomous category, they are $*$-autonomous. ☐

The fact that the categories of weak and Mackey spaces are equivalent was shown, for the case of B (Banach) spaces in [8, Theorem 15, p. 422]. Presumably, the general case has also been long known, but I am not aware of a reference.

5. Examples.

Example 1. Vector spaces over a spherically complete field

Let K be a spherically complete field. Let $U = \{\times \in K \mid \|x\| < 1\}$. As a ranges over the non-zero elements of K, the sets of the form aU are a neighbourhood base at 0. If V is a topological K-vector space and $\varphi : |V| \longrightarrow K$ is a linear mapping such that $\varphi^{-1}(U)$ is open in V, then $\varphi^{-1}(aU) = a\varphi^{-1}(U)$ which is open by continuity of division and thus φ is actually continuous on V. That U contains no K-subspace of K and that $K \longrightarrow K \multimap K$ is an isomorphism are obvious.

This example includes all locally compact fields, see [15, Corollary 20.3(i)].

We take for $\mathcal{S}$ the category of normed linear K-spaces, except in the case that K is discrete, we require also that the spaces have the discrete norm. We know that K is injective in the discrete case. The injectivity of K in the real or complex case is just the Hahn-Banach theorem, which has been generalized to ultrametric fields according to the theorem following the definition:

An ultrametric is a metric for which the ultratriangle inequality, $\|x + y\| \le \|x\| \vee \|y\|$, holds. This is obviously true for p-adic and t-adic norms. Spherically complete means that the meet of any descending sequence of closed balls is non-empty. This is known to be satisfied by locally compact ultrametric spaces.

Theorem 5.1 (Ingelton)**.** *Let K be a spherically complete ultrametric field. E a K-normed space and v a subspace of E. For every bounded linear* *functional φ defined on V, there exists a bounded linear functional ψ defined on E whose restriction to V is φ and such that $\|\varphi\| = \|\psi\|$.*

The proof is found in [14]

Notice that if K is non-discrete, then what we have established is that both $\mathcal{V}_s$ and $\mathcal{V}_w$ are equivalent to $\mathrm{chu}(\mathrm{Vect}\text{-}|K|, |K|)$. But exactly the same considerations show that the same is true if we ignore the topology on K and use the discrete norm. The category $\mathcal{S}$ will now be the category of discrete finite-dimensional $|K|$-vector spaces. Its product and subobject closure will consist of spaces that are mostly not discrete, but there are still full subcategories of weakly and strongly topologized spaces within this category and they are also equivalent to $\mathrm{chu}(\mathrm{Vect}\text{-}|K|, |K|)$.

Thus, these categories really do not depend on the topologies. Another interpretation is that this demonstrates that, for these spaces, the space of functionals replaces the topology, which was arguably Mackey's original intention.

Example 2. Locally compact abelian groups.

For the abelian groups, we take for $\mathcal{V}$ the category of those abelian that are subgroups (with the induced topology) of products of locally compact abelian groups. The object K in this case is the circle group $\mathbf{R}/\mathbf{Z}$. A simple representation of this group is as the closed interval $[-1/2, 1/2]$ with the endpoints identified and addition mod 1. The group is compact. Let U be the open interval $(-1/3, 1/3)$. It is easy to see that any non-zero point in that interval, added to itself sufficiently often, eventually escapes that neighborhood so that U contains no non-zero subgroup. It is well-known that the endomorphism group of the circle is $\mathbf{Z}$.

If $f : G \longrightarrow K$ is a homomorphism such that $T = f^{-1}(U)$ is open in G, let $T = T_1, T_2, \ldots, T_n, \ldots$ be a sequence of open sets in G such that $T_{i+1} + T_{i+1} \subseteq T_i$ for all i. Let $U_i = (-2^{-i}/3, 2^{-i}/3) \subseteq K$. Then the $\{U_i\}$ form a neighborhood base in K and one readily sees that $f^{-1}(U_i) \subseteq T_i$ which implies that f is continuous.

We take for $\mathcal{S}$ the category of locally compact abelian groups. The fact that K is an injective follows directly from the Pontrjagin duality theorem. A result [9, Theorem 1.1] says that every locally compact group is strongly topologized. Thus both categories of weakly topologized and strongly topologized groups that are subobjects of products of locally compact abelian groups are equivalent to $\mathrm{chu}(\mathcal{Ab}, |K|)$ and thus are *-autonomous.

We can ask if the same trick of replacing $K = \mathbf{R}/\mathbf{Z}$ by $|K|$, as in the first example, can work. It doesn't appear so. While $\mathrm{Hom}(K, K) = \mathbf{Z}$, the endomorphism ring of $|K|$ has cardinality 2^c and is non-commutative, so we cannot draw no useful inference about maps from $|K|^n \longrightarrow |K|$, even for finite n.

Example 3. Modules over a self injective cogenerator.

If we examine the considerations that are used in vector spaces over a field, it is clear that what is used is that a field is both an injective module over itself and a cogenerator in the category of vector spaces. Then if K is a such a commutative ring, we can let $\mathcal{T}$ be the category of topological K-modules, $\mathcal{S}$ be the full subcategory of submodules of finite powers of K with the discrete topology and $\mathcal{V}$ the limit closure of $\mathcal{S}$. Then $\mathrm{chu}(\mathrm{Mod}_K, K)$ is equivalent to each of the categories $\mathcal{V}_s$ and $\mathcal{V}_w$ of topological K-modules that are strongly and weakly topologized, respectively, with respect to their continuous linear functionals into K.

We now show that there is a class of commutative rings with that property. Let k be a field and $K = k[x]/(x^n)$. When $n = 2$, this is called the ring of dual numbers over k.

Proposition 5.2. *K is self injective.*

We base this proof on the following well-known fact:

Lemma 5.3. *Let k be a commutative ring, K is a k-algebra, Q an injective k-module, and P a flat right K-module then $\mathrm{Hom}_k(K, Q)$ is an injective K-module.*

The K-module structure on the Hom set is given by $(rf)(a) = f(ar)$ for $r \in K$ and $a \in P$.

Proof. Suppose $A \rightarrowtail B$ is an injective homomorphism of K-modules.

Then we have

$$\begin{tikzcd}
\mathrm{Hom}_R(B, \mathrm{Hom}_k(P, Q)) \arrow[r] \arrow[d, "\cong"'] & \mathrm{Hom}_R(A, \mathrm{Hom}_k(P, Q)) \arrow[d, "\cong"] \\
\mathrm{Hom}_k(P \otimes_R B, Q) \arrow[r, tail] & \mathrm{Hom}_k(P \otimes_R A, Q)
\end{tikzcd}$$

and the flatness of P, combined with the injectivity of Q force the bottom arrow to be a surjection. □

of 5.2. From the lemma it follows that $\mathrm{Hom}_k(K, k)$ is a K-injective. We claim that, as K-modules, $\mathrm{Hom}_k(K, k) \cong K$. To see this, we map $f : K \longrightarrow \mathrm{Hom}_k(K, k)$. Since these are vector spaces over k, we begin with a k-linear map and show it is K-linear. A k-basis for K is given by $1, x, \ldots, x^{n-1}$. We define $f(x^i) : K \longrightarrow k$ for $0 \leq\leq n-1$ by $f(x^i)(x^j) = \delta_{i+j,n}$ (the Kronecker δ). For this to be K-linear, we must show that $f(xx^i) = xf(x^i)$. But

$$f(xx^i)(x^j) = f(x^{i+1}(x^j)) = \delta_{i+1+j,n} = f(x^i)(x^{j+1}) = (xf(x^i))(x^j)$$

Clearly, the $f(x^i)$, for $0 \leq i \leq n$ are linearly independent and so f is an isomorphism. □

Proposition 5.4. *K is a cogenerator in the category of K-modules.*

Proof. Using the injectivity, it suffices to show that every cyclic module can be embedded into K. Suppose M is a cyclic module with generator m. Let i be the first power for which $x^i m = 0$. I claim that $m, xm, \ldots, x^{i-1}m$ are linearly independent over k. If not, suppose that $\lambda_0 m + \lambda_1 xm + \cdots \lambda_{i-1}x^{i-1}m = 0$ and not all coefficients 0. Let λ_j be the first non-zero coefficient, so that $\lambda_j x^j + \cdots + \lambda_{i-1}x^{i-1}m = 0$. Multiply this by x^{i-j-1} and use that $x^l m = 0$ for $l \geq i$ to get $\lambda_j x^{i-1}m = 0$. But by assumption, $x^{i-1}m \neq 0$ so that this would imply that $\lambda_j = 0$, contrary to hypothesis. Thus there is a k-linear map $f : M \longrightarrow K$ given by $f(x^j m) = x^{n-i+j}$. Since the x^j are linearly independent, this is k-linear and then it is clearly K-linear. □

6. Interpretation of the dual of an internal hom

These remarks are especially relevant to the vector spaces, although they are appropriate to the other examples. The fact that $(U \multimap V)^* \cong U \multimap V^*$ can be interpreted that the dual of $U \multimap V$ is a subspace of $V \multimap U$, namely those linear transformations of finite rank. An element of the form $u \otimes v^*$ acts as a linear transformation by the formula $(u \otimes v^*)(v) = \langle v, v^* \rangle u$. This is a transformation of row rank 1. Sums of these elements is similarly an element of finite rank.

This observation generalizes the fact that in the category of finite dimensional vector spaces, we have that $(U \multimap V)^* \cong V \multimap U$ (such a category is called a compact $*$-autonomous category). In fact, Halmos avoids the complications of the definition of tensor products in that case by *defining* $U \otimes V$ as the dual of the space of bilinear forms on $U \oplus V$, which is quite clearly equivalent to the dual of $U \multimap V^* \cong V \multimap U^*$ ([10, Page 40]). (Incidentally, it might be somewhat pedantic to point out that Halmos's definition makes no sense since $U \oplus V$ is a vector space in its own right and a bilinear form on a vector space is absurd. It would have been better to use the equivalent form above or to define Bilin(U, V).)

Since linear transformations of finite rank are probably not of much interest in the theory of topological vector spaces, this may explain why the internal hom was not pursued.

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