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TWO- AND ONE-DIMENSIONAL COMBINATORIAL EXACTNESS STRUCTURES IN KUROSH–AMITSUR RADICAL THEORY, I

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Résumé. Les auteurs proposent une nouvelle version non-pointée de structure d'exactitude combinatoire pour la théorie abstraite des radicaux de type Kurosh–Amitsur introduite par les deuxième et troisième auteurs en 2003, appelée ci-dessous structure 2-dimensionnelle. Elle est motivée par la notion de catégorie semi-exacte introduite par le premier auteur en 1992 et, brièvement, elle permet de définir un triplet radical-semisimple tel que, si (R,r,S) est un tel triplet, alors (R,S) est un couple radical-semisimple par rapport à la structure d'exactitude 1-dimensionnelle sous-jacente définie dans ce qui suit.

Abstract. We propose a new, non-pointed, version of combinatorial exactness structure for the abstract theory of Kurosh–Amitsur radicals introduced by the second and third author in 2003. We call it now 2-dimensional. It is motivated by the notion of semiexact category introduced by the first author in 1992, and, briefly, it allows us to define a radical-semisimple triple in such a way that if (R,r,S) is a radical-semisimple triple, then (R,S) is a radical-semisimple pair with respect to its underlying 1-dimensional exactness structure as defined below.

Key words. Adjoint functors, Kurosh-Amitsur radical, Non-pointed combinatorial exactness, Short exact sequence, Null morphism.

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0. Introduction

Each of the papers [GrJM2013], [JM2003], and [JM2009] proposes a special combinatorial exactness structure as a framework for an abstract Kurosh–Amitsur type radical theory. We will call these three structures 1-, 2-, and 3-dimensional, respectively (although the 1-dimensional approach was, in a sense, known before: see Remark 1.3 in [GrJM2013]), and study the relationship between the resulting radical theories in a series of papers.

The structures introduced in [JM2003] and [JM2009] will be extended, in order to make them *non-pointed*. This is motivated by the following observation made in [GrJM2013]:

Surprisingly, the non-pointed context allows us to present the theory of closure operators as a special case of the theory of radicals by using semiexact categories in the sense of the first author.

In particular, in the present paper:

- In Section 1 we introduce our non-pointed counterpart of the 2-dimensional exactness structure (Definition 1.1), and its underlying 1-dimensional exactness structure (Definition 1.3). Example 1.6 explains how to associate such a structure to a semiexact category satisfying a mild additional condition.
- Section 2 briefly explains an obvious duality principle, in order to avoid various calculations that become dual to others.
- Section 3 introduces what we call radical-semisimple triples (Definition 3.1), that is, triples (R,r,S) consisting of a radical class R, its corresponding radical function r and semisimple class S; a list of counterparts of the first standard properties well known in Kurosh–Amitsur radical theory is then given.
- Section 4 is devoted to the First Comparison Theorem (Theorem 4.3), which says that if (R,r,S) is a radical-semisimple triple with respect to a given 2-dimensional exactness structure (satisfying a natural additional condition), then (R,S) is a radical-semisimple pair in the sense of [GrJM2013] with respect to the underlying 1-dimensional exactness structure.

- Section 5 briefly recalls the classical case of rings, and says a few words about the intermediate levels of generality. More about the pointed case can be found in [JM2003].
- Section 6 presents topological closure as a radical function. Unlike in [GrJM2013], we do not go to abstract-categorical closure operators here, because that would involve too much of additional material, e.g. from [DikT1995], and we are going to present this in a separate paper.
- Section 7 is devoted to a very simple example, not involving any kind of categorical exactness, showing that a 'Naive Second Comparison Theorem', converse to Theorem 4.3, would be obviously false. In fact, a Second Comparison Theorem should cover the classical result of Amitsur and Kurosh saying that the so-called Conditions (R1) and (R2) on a class R of rings characterize radical classes (see Theorem 2.15 in [GaW2004]). This will require, if not a ring-theoretic, at least a semi-abelian algebraic context.

1. 1- and 2-dimensional combinatorial exactness structures

The purpose of this section is to

- introduce (Definition 1.1) a non-pointed counterpart of pointed combinatorial exactness structure in the sense of [JM2003], which we shall call a 2-dimensional (combinatorial) exactness structure;
- define (Definition 1.3), for each such structure, its underlying 1-dimensional exactness structure in the sense of [GrJM2013];
- introduce (Definition 1.4) a new notion of a proper short exact sequence in a semiexact category in the sense of [Gr1992a], [Gr1992b], and [Gr2013], and use it to associate a 2-dimensional exactness structure to every semiexact category satisfying a certain completeness condition (Example 1.6).

Definition 1.1. A 2-dimensional (combinatorial) exactness structure is a diagram

$$
\begin{tikzcd}
X_2 \arrow[r, "d_1^1"] \arrow[r, bend left=80, "d_0^1"] \arrow[r, bend right=80, "d_2^1"'] & X_1 \arrow[l, bend right=40, "s_0^1"'] \arrow[l, bend left=40, "s_1^1"] \arrow[r, bend left=40, "d_0^0"] \arrow[r, bend right=40, "d_1^0"'] & X_0, \arrow[l, "s_0^0"']
\end{tikzcd}
\tag{1.1}
$$

in the category of sets, satisfying the simplicial identities

$$d_0^0 s_0^0 = d_1^0 s_0^0 = 1, \tag{1.2}$$
$$d_0^0 d_1^1 = d_0^0 d_0^1,\ d_0^0 d_2^1 = d_1^0 d_0^1,\ d_1^0 d_2^1 = d_1^0 d_1^1, \tag{1.3}$$
$$s_1^1 s_0^0 = s_0^1 s_0^0, \tag{1.4}$$
$$d_0^1 s_1^1 = s_0^0 d_0^0, \tag{1.5}$$
$$d_0^1 s_0^1 = d_1^1 s_0^1 = d_1^1 s_1^1 = d_2^1 s_1^1 = 1, \tag{1.6}$$
$$d_2^1 s_0^1 = s_0^0 d_1^0, \tag{1.7}$$

and equipped with a complete lattice structure on each fibre $(d_1^1)^{-1}(a)$, for $a \in X_1$, such that $s_1^1(a)$ and $s_0^1(a)$ are, respectively, the smallest and the largest element in $(d_1^1)^{-1}(a)$.

Example 1.2. A pointed combinatorial exactness structure in the sense of Definition 2.1 of [JM2003] is nothing but a 2-dimensional exactness structure of Definition 1.1 in the case when X_0 is a one-element set. The notation we use here is, however, not the same; specifically:

- while X_1 and X_2 in the two definitions play the same role, X_0 being a one-element set is not mentioned in [JM2003], and so are the maps d_0^0, d_1^0, and s_0^0: instead, the element of X_1 corresponding to the unique element of X_0 under s_0^0 is denoted by 0 in [JM2003];
- the maps d_0^1, d_1^1, d_2^1, s_0^1, and s_1^1 of Definition 1.1 correspond, respectively, to the maps d_0, d_1, d_2, e_1, and e_0 of [JM2003].

Recall from [GJM2013] (slightly changing the notation) that a 1-*dimensional exactness structure* is a system $(A, Z, \lhd, -\!\!\rhd)$ in which A is a set, Z is a subset of A, and $\lhd$ and $-\!\!\rhd$ are binary relations on A such that, for every a in A, there exist z and z' in Z with $z \lhd a$ and $a \mathrel{-\!\!\rhd} z'$.

Definition 1.3. Given a 2-dimensional exactness structure, we define its underlying 1-dimensional exactness structure as the system $(X_1, s^0_0(X_0), \lhd, -\!\!\rhd)$ in which $u \lhd v$ when there exists $x \in X_2$ with $d^1_0(x) = u$ and $d^1_1(x) = v$, and $v \mathrel{-\!\!\rhd} w$ when there exists $x \in X_2$ with $d^1_1(x) = v$ and $d^1_2(x) = w$.

Note that $(X_1, s^0_0(X_0), \lhd, -\!\!\rhd)$ constructed as in Definition 1.3 is indeed a 1-dimensional exactness structure, since, for every $v \in X_1$, we have

$$s^0_0 d^0_0(v) \lhd v, \tag{1.8}$$

$$v \mathrel{-\!\!\rhd} s^0_0 d^0_1(v). \tag{1.9}$$

Here (1.8) follows from $d^1_0 s^1_1(v) = s^0_0 d^0_0(v)$ and $d^1_1 s^1_1(v) = v$, while (1.9) follows from $d^1_1 s^1_0(v) = v$ and $d^1_2 s^1_0(v) = s^0_0 d^0_1(v)$.

Now, let us recall from [GrJM2013]:

A *semiexact* (=ex1-exact) *category* $\mathbf{C}$ in the sense of [Gr1992a] can be described as the data

$$\begin{tikzcd} \mathbf{C}_1 \arrow[rr, bend left=50, "D"] \arrow[rr, bend right=50, "C"'] & & \mathbf{C}_0 \arrow[ll, "E"'] \end{tikzcd}, \quad C \dashv E \dashv D, \tag{1.10}$$

in which:

- $\mathbf{C}_1$ is a category, $\mathbf{C}_0$ a full replete subcategory of $\mathbf{C}_1$, and E is the inclusion functor;
- D and C are a right adjoint left inverse and a left adjoint left inverse of E, respectively;
- all the counit components $\iota_A : D(A) \to A$ are monomorphisms that admit pullbacks along arbitrary morphisms into A, and all the unit

components $\pi_A : A \to C(A)$ are epimorphisms that admit pushouts along arbitrary morphisms from A.

Next, we need some discussion that will lead us to introducing the notion of proper short exact sequence which we are going to use:
One usually says that a diagram

$$U \to V \to W \tag{1.11}$$

in a category with a zero object is a short exact sequence if $U \to V$ is a kernel of $V \to W$ and $V \to W$ is a cokernel of $U \to V$, or, equivalently, if the diagram

$$\begin{array}{ccc} U & \to & V \\ \downarrow & & \downarrow \\ 0 & \to & W \end{array} \tag{1.12}$$

is a pullback and a pushout at the same time. We shall refer to these equivalent conditions as the *kernel-cokernel condition* and the *pullback-pushout condition*.

In the semiexact context (with $U \to V \to W$ being a diagram in $\mathbf{C}_1$, where $\mathbf{C}_1$ is as in (1.10)), although the kernel-cokernel condition can be copied word for word using kernels and cokernels in the sense of [Gr1992a], there is a problem with the pullback-pushout condition, since:

- while $U \to V$ is a kernel of $V \to W$ if and only if $U \to V$ is a pullback of $D(W) \to W$ along $V \to W$,
- $V \to W$ is a cokernel of $U \to V$ if and only if $V \to W$ is a pushout of $U \to C(U)$ along $U \to V$.

That is, in order to copy the pullback-pushout condition we need $D(W)$ and $C(U)$, both of which will replace the zero object, to be canonically isomorphic.

In order to explain what "canonical" means, consider the commutative diagrams

$$\begin{array}{ccccccc} \mathrm{Ker}(f) & \to & U & \xrightarrow{\quad f \quad} & W & \to & \mathrm{Coker}(f) \\ & & \downarrow & & \uparrow & & \\ & & \mathrm{Coker}(\ker(f)) & \xrightarrow[\bar{f}]{} & \mathrm{Ker}(\mathrm{coker}(f)) & & \end{array} \tag{1.13}$$

$$\begin{array}{ccc} U & \xrightarrow{\quad f \quad} & W \\ \downarrow & & \uparrow \\ C(U) & \xrightarrow[\bar{f}]{\qquad} & D(W) \end{array} \tag{1.14}$$

where f is the composite $U \to V \to W$ and $\bar{f}$ is induced by f. The existence and uniqueness of such $\bar{f}$ in (1.13) follows from:

- the universal property of a kernel and the fact that f is a null morphism in the semiexact context of [Gr1992a],
- or, equivalently, from the universal property of a cokernel and the fact that f is a null morphism in the semiexact context of [Gr1992a],

while the existence and uniqueness of such $\bar{f}$ in (1.14) follows from:

- the universal property of $D(W) \to W$ and the fact that f factors as $U \to C(U) \to W$,
- the universal property of $U \to C(U)$ and the fact that f factors as $U \to D(W) \to W$.

Moreover, the square part of diagram (1.13) is in fact the same as diagram (1.14). Indeed, since f is a null morphism in the sense of [Gr1992a], we can take $\mathrm{Ker}(f) = U$ and $\mathrm{Coker}(f) = W$, and assume that $\mathrm{Ker}(f) \to U$ and $W \to \mathrm{Coker}(f)$ are the identity morphisms of U and W, respectively; this makes $U \to C(U)$ the cokernel of $\mathrm{Ker}(f) \to U$ and makes $D(W) \to W$ the kernel of $W \to \mathrm{Coker}(f)$.

It follows that there is a clear notion of the canonical morphism $C(U) \to D(W)$ for each short exact sequence $U \to V \to W$, namely, it is the morphism $\bar{f}$ above; and we introduce:

Definition 1.4. (a) A short exact sequence $U \to V \to W$ in a semiexact category (1.10) will be called proper if the canonical morphism $C(U) \to D(W)$ is an isomorphism.

(b) For two proper short exact sequences $U \to V \to W$ and $U' \to V' \to W'$, we shall write $(U \to V \to W) \le (U' \to V' \to W')$ if $V = V'$ and there exist morphisms $U \to U'$ and $W \to W'$ making the diagram

$$\begin{array}{ccccc} U & \to & V & \to & W \\ \downarrow & & \| & & \downarrow \\ U' & \to & V' & \to & W' \end{array} \tag{1.15}$$

commute.

(c) If $(U \to V \to W) \leq (U' \to V' \to W')$ and $(U' \to V' \to W') \leq (U \to V \to W)$, then we will say that $U \to V \to W$ and $U' \to V' \to W'$ are equivalent, and the equivalence class of $U \to V \to W$ will be denoted by $[U \to V \to W]$.

Remark 1.5. (a) Since a short exact sequence $U \to V \to W$ is determined, up to isomorphism, by each of the morphisms $U \to V$ and $V \to W$, Definition 1.4 also suggests us to define *proper normal monomorphisms* and *proper normal epimorphisms* as those normal monomorphisms and normal epimorphisms that appear as such $U \to V$ and $V \to W$, respectively, in proper short exact sequences.

(b) There are many situations where every short exact sequence is proper. For example, this is obviously the case if the ground semiexact category is pointed or satisfies axiom (ex3) of [Gr1992a], [Gr1992b], and [Gr2013].

Now we are ready to present our main example of a 2-dimensional exactness structure:

Example 1.6. Given a semiexact category (1.10) in which we assume $\mathbf{C}_1$ and $\mathbf{C}_0$ to be small skeletons, we would like to construct the associated 2-dimensional exactness structure (1.1) by saying that:

(a) X_0 and X_1 are the sets of objects of $\mathbf{C}_0$ and $\mathbf{C}_1$, respectively;

(b) X_2 is the set of equivalence classes of proper short exact sequences in the sense of Definition 1.4;

(b) the maps d_0^0, d_1^0, and s_0^0 are the object functions of the functors D, C, and E, respectively;

(c) the other maps involved in (1.1) are defined as follows:

$$d^1_0[U \to V \to W] = U,\ d^1_1[U \to V \to W] = V,\ d^1_2[U \to V \to W] = W,$$
$$s^1_0(U) = [U = U \to C(U)],\ s^1_1(U) = [D(U) \to U = U]; \tag{1.16}$$

(d) the order on $(d^1_1)^{-1}(V)$ is defined according to Definition 1.4.

However, to do this we need an additional assumption on the data (1.10), namely that each $(d^1_1)^{-1}(V)$ be a complete lattice. We could briefly refer to this assumption by saying that our semiexact category *admits proper intersections*. Note also that the only reason of our restriction to *proper* short exact sequences in (b) is that the second equality of (1.3) should be satisfied.

2. Duality

Any 2-dimensional exactness structure (1.1) has its *opposite*, or *dual*, 2-dimensional exactness structure, in which:

- the sets X_i (i = 1, 2, 3) and the maps d^1_1 and s^0_0 are the same as in the original structure;
- the maps d^0_0, d^1_0, and s^1_0 of the original structure play the roles of the maps d^0_1, d^1_2, and s^1_1 of the opposite structure, and vice versa;
- for each $a \in X_1$, the order on $(d^1_1)^{-1}(a)$ in the opposite structure is opposite to the order in the original structure.

This gives the obvious *duality principle*, saying that every property that holds in all 2-dimensional exactness structures has an obvious dual, which also holds in all 2-dimensional exactness structures. For example, so are properties (1.8) and (1.9), and after proving (1.8) we could simply say: "dually, we obtain (1.9)".

Similarly, the opposite category of any semiexact category is semiexact, and the data opposite to (1.10) is

$$\begin{tikzcd} (\mathbf{C}_1)^{\mathrm{op}} \arrow[rr, bend left, "C^{\mathrm{op}}"] \arrow[rr, bend right, "D^{\mathrm{op}}"'] & & (\mathbf{C}_0)^{\mathrm{op}} \arrow[ll, "E^{\mathrm{op}}"'] \end{tikzcd}, \quad D^{\mathrm{op}} \dashv E^{\mathrm{op}} \dashv C^{\mathrm{op}}, \tag{2.1}$$

Moreover, the duality principals for the two types of data obviously agree with each other in the sense that the associated 2-dimensional exactness structure of the opposite semiexact category is opposite to the associated 2-dimensional exactness structure of the original semiexact category.

3. Radicals in terms of 2-dimensional exactness structures

The general approach to radicals developed in this section is almost a straightforward extension of the approach of Section 2 of [JM2003] from the context of a pointed combinatorial exactness structure recalled in Example 1.2 to the general context of Definition 1.1.

For a fixed 2-dimensional exactness structure (1.1) of Definition 1.3, consider the diagram

$$\begin{tikzcd} K \arrow[r, shift right, "\Sigma_f"'] & L \arrow[l, shift right, "f"'] \arrow[r, shift left, "g"] & K \arrow[l, shift left, "\Pi_g"] \end{tikzcd} \tag{3.1}$$

in which:

- $L = \{l : X_1 \to X_2 \mid d^1_1 l = 1_{X_1}\} = \prod_{a \in X_1} (d^1_1)^{-1}(a)$, considered as a complete lattice;
- K is the complete lattice of all subsets of X_1 containing the image of s^0_0;
- f and g are defined by $f(l) = d^1_0 l(X_1)$ and $g(l) = d^1_2 l(X_1)$;
- Σ_f and Π_g are defined by $\Sigma_f(k) = \vee\{l \in L \mid f(l) \leq k\}$ and $\Pi_g(k) = \wedge\{l \in L \mid g(l) \leq k\}$.

Note that, for each $a \in X_1$, since $s^1_1(a)$ and $s^1_0(a)$ are, respectively, the smallest and the largest element in $(d^1_1)^{-1}(a)$, we have:

- (1.4) implies that, for each $z \in X_0$, the lattice $(d_1^1)^{-1}(s_0^0(z))$ has only one element, namely $s_1^1 s_0^0(z) = s_0^1 s_0^0(z)$,
- and, in particular, $ls_0^0(z) = s_1^1 s_0^0(z) = s_0^1 s_0^0(z)$ for each $l \in L$;
- consequently, $d_0^1 l s_0^0(z) = s_0^0(z) = d_2^1 l s_0^0(z)$, and so $f(l)$ and $g(l)$ indeed belong to K.

Using this notation and extending Definition 2.5 of [JM2003], we introduce:

Definition 3.1. (a) A map $r \in L$ is said to be a radical function (with respect to the given 2-dimensional exactness structure) if $\Sigma_f f(r) = r = \Pi_g g(r)$.

(b) A subset R in X_1 is said to be a radical class if it corresponds to a radical function via f, that is, there exists a radical function r with $f(r) = R$.

(c) A subset S in X_1 is said to be a semisimple class if it corresponds to a radical function via g, that is, there exists a radical function r with $g(r) = S$.

(d) if (b) and (c) hold for the same radical function r, then we say that (R,r,S) is a radical-semisimple triple.

According to this definition, there are canonical bijections:

$$\text{Radical classes} \approx \text{Radical functions} \approx \text{Semisimple classes.} \qquad (3.2)$$

There is a number of standard properties of a radical-semisimple triple to be listed, to which the rest of this section is devoted.

Theorem 3.2. (R,r,S) *is a radical-semisimple triple with respect to a given 2-dimensional exactness structure if and only if* (S,r,R) *is a radical-semisimple triple with respect to the opposite 2-dimensional exactness structure.* □

In the rest of this section we are dealing with a given fixed 2-dimensional exactness structure (1.1), without further notice.

Theorem 3.3. *Let R and S be subsets of X_1, and $r : X_1 \to X_2$ be a map. Then the following conditions are equivalent:*

(a) (R,r,S) *is a radical-semisimple triple;*

(b) *for each a in X_1, $r(a)$ is the largest element x in the lattice $(d_1^1)^{-1}(a)$ with $d_0^1(x)$ in R, and, at the same time, is the smallest element y in the lattice $(d_1^1)^{-1}(a)$ with $d_2^1(y)$ in S.*

Proof. (a)⇒(b): Just note that, for each a in X_1, we have

$$\vee\{x \in (d_1^1)^{-1}(a) \mid d_0^1(x) \in R\} = r(a) = \wedge\{x \in (d_1^1)^{-1}(a) \mid d_2^1(x) \in S\}, \qquad (3.3)$$

$r(a)$ is in $(d_1^1)^{-1}(a)$, $d_0^1r(a)$ is in R (by 3.1(b) and 3.1(d)), and $d_2^1r(a)$ is in S (by 3.1(c) and 3.1(d)).

(b)⇒(a): According to Definition 3.1, (a) means:

$$R = f(r),\ S = g(r),\ \Sigma_f f(r) = r = \Pi_g g(r). \qquad (3.4)$$

The first two equalities of (3.4) are

$$R = d_0^1r(X_1),\ S = d_2^1r(X_1), \qquad (3.5)$$

respectively, while the last two are the same as (3.3) required for each a in X_1. We observe:

- The inclusions $d_0^1r(X_1) \subseteq R$ and $d_2^1r(X_1) \subseteq S$ follow from (b) trivially.
- For each $a \in X_1$, the largest element in the lattice $(d_1^1)^{-1}(a)$ is $s_0^1(a)$ (see Definition 1.1), and when a is in R we have $d_0^1s_0^1(a) = a \in R$ (see (1.6)). Therefore

  $$a \in R \Rightarrow r(a) = s_0^1(a) \qquad (3.6)$$

  by (b). This gives $a = d_0^1r(a)$, showing that every element a of R belongs to $d_0^1r(X_1)$. That is, $R \subseteq d_0^1r(X_1)$. The inclusion $S \subseteq d_2^1r(X_1)$ is dual to this inclusion.
- (3.3) immediately follows from (b). □

Corollary 3.4. *Let (R,r,S) be a radical-semisimple triple and a an element in X_1. Then $r(a)$ is the unique element x in $(d_1^1)^{-1}(a)$ with $d_0^1(x)$ in R and $d_2^1(x)$ in S.*

Proof. We know that $d_0^1r(a)$ is in R and $d_2^1r(a)$ is in S. On the other hand, if x is in $(d_1^1)^{-1}(a)$ with $d_0^1(x)$ in R and $d_2^1(x)$ in S, then, by 3.3(b), we have:

- $x \leq r(a)$ in $(\overset{1}{d}_1)^{-1}(a)$, since $\overset{1}{d}_0(x)$ in R;
- $r(a) \leq x$ in $(\overset{1}{d}_1)^{-1}(a)$, since $\overset{1}{d}_2(x)$ in S. □

Our next two propositions will partly use the following additional condition, which is self-dual since its parts (a) and (b) are dual to each other:

Condition 3.5. *For* $x \in X_2$,

(a) $\overset{1}{d}_0(x) = \overset{0}{s}_0\overset{0}{d}_0\overset{1}{d}_1(x) \Rightarrow x = \overset{1}{s}_1\overset{1}{d}_1(x)$;

(b) $\overset{1}{d}_2(x) = \overset{0}{s}_0\overset{0}{d}_1\overset{1}{d}_1(x) \Rightarrow x = \overset{1}{s}_0\overset{1}{d}_1(x)$.

Remark 3.6. There are several convenient equivalent ways to reformulate Condition 3.5. One of them is to replace the implications in (a) and (b) with equivalences. Indeed, $x = \overset{1}{s}_1\overset{1}{d}_1(x)$ implies $\overset{1}{d}_0(x) = \overset{1}{d}_0\overset{1}{s}_1\overset{1}{d}_1(x) = \overset{0}{s}_0\overset{0}{d}_0\overset{1}{d}_1(x)$, where the second equality follows from (1.5); and dually, $x = \overset{1}{s}_0\overset{1}{d}_1(x)$ implies $\overset{1}{d}_2(x) = \overset{0}{s}_0\overset{0}{d}_1\overset{1}{d}_1(x)$. Another equivalent way to express conditions 3.5(a) and 3.5(b), respectively, is to require:

(a) $\overset{1}{d}_0(x) \in \overset{0}{s}_0(X_0)$ if and only if x is the smallest element of the lattice $(\overset{1}{d}_1)^{-1}(\overset{1}{d}_1(x))$;

(b) $\overset{1}{d}_2(x) \in \overset{0}{s}_0(X_0)$ if and only if x is the largest element of the lattice $(\overset{1}{d}_1)^{-1}(\overset{1}{d}_1(x))$.

Proposition 3.7. *Let R be a radical class and r the corresponding radical function. Then, for $a \in X_1$, conditions (a), (b), (c) below are equivalent and imply (d), while (d) is equivalent to (e). Under Condition 3.5(b), condition (d) also implies the other conditions:*

(a) $a \in R$;

(b) $r(a) = \overset{1}{s}_0(a)$;

(c) $\overset{1}{d}_0 r(a) = a$;

(d) $\overset{1}{d}_2 r(a) = \overset{0}{s}_0\overset{0}{d}_1(a)$;

(e) $\overset{1}{d}_2 r(a) \in \overset{0}{s}_0(X_0)$.

Proof. The arguments needed to prove (c)⇒(a)⇔(b)⇒(c) are in fact contained in the proof of Theorem 3.3. Nevertheless let us present them:

Since $s_0^1(a)$ is the largest in element in $(d_1^1)^{-1}(a)$, (a)⇔(b) follows from Theorem 3.3 (cf. (3.6)).

(b)⇒(c): Assuming (b), we have: $d_0^1 r(a) = d_0^1 s_0^1(a) = a$, where the last equality follows from (1.6).

(c)⇒(a): Assuming (c) and using (1.6) again, we have: $a = d_0^1 r(a) \in d_0^1 r(X_1) = f(r) = R$.

(b)⇒(d): Assuming (b), we have: $d_2^1 r(a) = d_2^1 s_0^1(a) = s_0^0 d_1^0(a)$, where the last equality follows from (1.7).

(d)⇒(e) is trivial.

(e)⇒(d): If $d_2^1 r(a) = s_0^0(z)$ for some $z \in X_0$, then

$$\begin{aligned} d_2^1 r(a) &= s_0^0 d_1^0 d_2^1 r(a) \quad \text{(by (1.2))} \\ &= s_0^0 d_1^0 d_1^1 r(a) \quad \text{(by the third equality in (1.3))} \\ &= s_0^0 d_1^0(a) \quad (\text{since } d_1^1 r(a) = a), \end{aligned}$$

as desired.

(d)⇒(b) under Condition 3.5(b): Since $r(a)$ belongs to $(d_1^1)^{-1}(a)$, (d) gives $d_2^1 r(a) = s_0^0 d_1^0 d_1^1 r(a)$, and then Condition 3.5(b) gives $r(a) = s_0^1 d_1^1 r(a)$. But $d_1^1 r(a) = a$, and so we obtain (b). □

Dually, we have:

Proposition 3.8. *Let S be a semisimple class and r the corresponding radical function. Then, for $a \in X_1$, conditions (a), (b), (c) below are equivalent to each other and imply (d), while (d) is equivalent to (e). Under Condition 3.5(a), condition (d) also implies the other conditions:*

(a) $a \in S$;

(b) $r(a) = s_1^1(a)$;

(c) $d_2^1 r(a) = a$;

(d) $d_0^1 r(a) = s_0^0 d_0^0(a)$.

(e) $d_0^1 r(a) \in s_0^0(X_0)$. □

Proposition 3.9. *Let* (R,r,S) *be a radical-semisimple triple. Then* $R \cap S = s_0^0(X_0)$.

Proof. The inclusion $s_0^0(X_0) \subseteq R \cap S$ follows from the definition of K in (3.1). If a is in $R \cap S$, then $a = d_0^1 r(a)$ by 3.7(c) and $d_0^1 r(a) \in s_0^0(X_0)$ by 3.8(e), which implies that a is in $s_0^0(X_0)$. □

4. The First Comparison Theorem

The purpose of this section is to formulate and prove Theorem 4.3, which describes a situation where every radical-semisimple triple determines a radical-semisimple pair in the sense of [GrJM2013].

Let us recall from [GrJM2013]:

Given a 1-dimensional exactness structure $(A,Z,\lhd,\mathrel{\multimap}\!\!\rhd,)$, and using the binary relations

$$\alpha = \{(a,b) \in A \times A \mid a \lhd b \Rightarrow a \in Z\}, \tag{4.1}$$
$$\beta = \{(a,b) \in A \times A \mid a \mathrel{-\!\!\rhd} b \Rightarrow b \in Z\} \tag{4.2}$$

on A, we define maps α_* and β^* from the power set $P(A)$ to itself by

$$\alpha_*(U) = \{b \in A \mid a \in U \Rightarrow a\alpha b\},\ \beta^*(U) = \{a \in A \mid b \in U \Rightarrow a\beta b\}. \tag{4.3}$$

Then a pair (R,S) of subsets of A is said to be a *radical-semisimple pair* (Definition 5.2(b) of [GrJM2013]), with respect to the given 1-dimensional exactness structure, if $R = \beta^*(S)$ and $S = \alpha_*(R)$. Accordingly, a subset U of A is said to be a radical class (semisimple class) if it occurs as the first (second) component in some radical-semisimple pair; that is, U is a radical class (semisimple class) if and only if $U = \beta^*\alpha_*(U)$ ($U = \alpha_*\beta^*(U)$).

As mentioned in [GrJM2013], the following two propositions are nothing but explicit reformulations of the definition above:

Proposition 4.1. (Proposition 5.3 of [GrJM2013]) *Let* $(A,Z,\lhd, \mathrel{-\!\!\rhd})$ *be a 1-dimensional exactness structure. A subset R in A is a radical class if and only if satisfies the following conditions:*

(a) *if a is in R, then, for every $b \in A \setminus Z$ with $a \mathrel{-\!\!\rhd} b$, there exists $c \in R \setminus Z$ with $c \lhd b$*;

(b) *given a in A, if, for every $b \in A \setminus Z$ with $a \mathrel{-\!\!\rhd} b$, there exists $c \in R \setminus Z$ with $c \lhd b$, then a is in R.* □

Proposition 4.2. (Proposition 5.4 of [GrJM2013]) *Let $(A,Z,\lhd, \mathrel{-\!\!\rhd})$ be a 1-dimensional exactness structure. A subset S in A is a semisimple class if and only if satisfies the following conditions:*

(a) *if a is in S, then, for every $b \in A \setminus Z$ with $b \lhd a$, there exists $c \in S \setminus Z$ with $b \mathrel{-\!\!\rhd} c$*;

(b) *given a in A, if, for every $b \in A \setminus Z$ with $b \lhd a$, there exists $c \in S \setminus Z$ with $b \mathrel{-\!\!\rhd} c$, then a is in S.* □

Our First Comparison Theorem, which compares radical-semisimple triples in the sense of Definition 3.1 with radical-semisimple pairs in the sense of [GrJM2013], is:

Theorem 4.3. *Let (R,r,S) be a radical-semisimple triple with respect to a given 2-dimensional exactness structure in the sense of Definition 1.1, satisfying Condition 3.5. Then (R,S) is a radical-semisimple pair in the sense of [GrJM2013] with respect to the underlying 1-dimensional exactness structure in the sense of Definition 1.3.*

Proof. First of all note that, for every $x \in X_2$, we have

$$d^1_0(x) \lhd d^1_1(x) \mathrel{-\!\!\rhd} d^1_2(x), \tag{4.4}$$

which trivially follows from the definitions of $\lhd$ and $\mathrel{-\!\!\rhd}$. In particular, for every $a \in X_1$ and every radical-semisimple triple (R,r,S), we have

$$d^1_0 r(a) \lhd a \mathrel{-\!\!\rhd} d^1_2 r(a) \text{ with } d^1_0 r(a) \text{ in } R \text{ and } d^1_2 r(a) \text{ in } S, \tag{4.5}$$

obtained from (4.4) by taking $x = r(a)$.

What we have to prove are the equalities $R = \beta^*(S)$ and $S = \alpha_*(R)$. To prove the inclusion $\beta^*(S) \subseteq R$, we take $a \in \beta^*(S)$ and observe:

- Since a is in $\beta^*(S)$ and $d^1_2 r(a)$ in S, we have $a\beta d^1_2 r(a)$ by the definition of $\beta^*(S)$.

- Since $a\beta d_2^1 r(a)$ and $a \twoheadrightarrow d_2^1 r(a)$, we know that $d_2^1 r(a)$ is in $s_0^0(X_0)$ by the definition of β.
- Since $d_2^1 r(a)$ is in $s_0^0(X_0)$ and Condition 3.5(b) holds, a is in R by the implication (e)⇒(a) in Proposition 3.7.

To prove the inclusion $R \subseteq \beta^*(S)$, we take $a \in R$ and $b \in S$ with $a \twoheadrightarrow b$, and we need to show that b is in $s_0^0(X_0)$. Indeed, $a \twoheadrightarrow b$ means that $d_1^1(x) = a$ and $d_2^1(x) = b$ for some $x \in X_2$, and we observe:

- By Theorem 3.3(b), $r(a)$ is the smallest element y in the lattice $(d_1^1)^{-1}(a)$ with $d_2^1(y)$ in S. By our assumptions on x, this gives $r(a) \leq x$.
- On the other hand, by the implication (a)⇒(b) in Proposition 3.7, we have $r(a) = s_0^1(a)$, which is the largest element in the lattice $(d_1^1)^{-1}(a)$. Together with the previous observation, this gives $x = r(a) = s_0^1(a)$.
- Since $x = r(a) = s_0^1(a)$, we have $b = d_2^1 s_0^1(a) = s_0^0 d_1^0(a) \in s_0^0(X_0)$, using (1.7).

This proves the equality $R = \beta^*(S)$, and the equality $S = \alpha_*(R)$ is dual to it. □

5. Classical contexts for Kurosh−Amitsur radicals

Ignoring the *problem of size* and the *difference between a category and its skeleton*, we take the ground 2-dimensional exactness structure (1.1) to be constructed as in Example 1.6 out of the category **Rings** of rings. The rings here are required to be associative but not required to be unital; in particular, the category **Rings** is pointed.

What are the radical-semisimple triples with respect to this structure and what are the radical-semisimple pairs with respect to its underlying 1-dimensional exactness structure?

The answers, as explained in [JM2003] and [GrJM2013], immediately come out of well-known results in the Kurosh−Amitsur radical theory, and they can be stated as:

Theorem 5.1. (a) (R,r,S) *is a radical-semisimple triple if and only if R, r, and S are a radical class, a radical function, and a semisimple class corresponding to each other in the classical sense.*

(b) (R,S) *is a radical-semisimple pair if and only if R and S are a radical class and a semisimple class corresponding to each other in the classical sense.* □

In particular:

- The assertion "if (R,r,S) is a radical-semisimple triple, then (R,S) is a radical-semisimple pair" of our Theorem 4.3 should be considered as well known in the present case.
- The converse assertion, namely "if (R,S) is a radical-semisimple pair, then (R,r,S) is a radical-semisimple triple for some r" should also be considered as well known in this case, although it is false in general, as a counter-example given in the next section will show.

Of course, Theorem 5.1 can be stated more generally, depending on what we mean by "classical sense". For instance, the category of rings can surely be replaced with any *semi-abelian* variety of universal algebras (in the sense of [JMT2002]; see also [BJ2003]), but even that would be far from the most general case. Various remarks on (more abstract) categorical contexts are made in [JM2003] and [GrJM2013], some referring to [MW1982]. However, full details can be found only in the case of rings: see [GaW2004] and [W1983], and references therein, especially [Div1973] and Section 2 in [FW1975].

Notice that a variant of Kurosh–Amitsur type radical theory, called connectednesses and disconnectednesses, has been developed for topological spaces and graphs and then for abstract relational structures in [AW1975], [FW1975] and [FW1982], respectively, also in a non-pointed setting. What we do here, however, is very different from their setting: we still have kernels while they have inverse images of all points ('connected components').

6. The topological closure operator

In this section, ignoring the problem of size, we take the ground 2-dimensional exactness structure (1.1) to be constructed as in Example 1.6 out of the semiexact category (1.10) in which:

- $\mathbf{C}_0$ is the category of topological spaces and inclusion maps of subspaces;
- $\mathbf{C}_1$ is the category of morphisms of $\mathbf{C}_0$ whose objects will be written as pairs (A,A'), where A' is a subspace of A.
- E is the inclusion functor and therefore C and D are defined by $C(A,A') = A$ and $D(A,A') = A'$, respectively.

In this context every short exact sequence is proper and it is just a diagram of the form

$$(A',A'') \to (A,A'') \to (A,A'), \tag{6.1}$$

where A' is any subspace of A and A'' is any subspace of A'. Using Definition 3.1 directly it is easy to prove:

Theorem 6.1. *Let $r : X_1 \to X_2$ be the map defined by*

$$r(A,A') = ((\bar{A}',A') \to (A,A') \to (A,\bar{A}')), \tag{6.2}$$

where $\bar{A}'$ denotes the closure of A' in A. Then r is a radical function in the radical-semisimple triple (R,r,S) where

$$R = \{(A,A') \in X_1 \mid A' \textit{ is dense in } A\}, \tag{6.3}$$
$$S = \{(A,A') \in X_1 \mid A' \textit{ is closed in } A\}. \square \tag{6.4}$$

This theorem obviously indicates the relationship between radicals and closure operators – a natural counterpart of what is done in [GJM2013] with radicals defined with respect to 1-dimensional exactness structures.

7. A simplified framework

Intuitively, the relations $\lhd$ and $\twoheadrightarrow$ are "almost order relations": for example, in the usual radical theory of rings, $a \lhd b$ means that a is (isomorphic to) an ideal in a, while $a \twoheadrightarrow b$ means that b is (isomorphic to) a quotient ring of a. However, even in that example, both antisymmetry (of $\lhd$

and $\mathrel{-\!\!\rhd}$) and transitivity (of $\lhd$) fail. This suggests us to consider a simplified version of a 1-dimensional exactness structure of the form $(A,\{0\},\leq,\geq)$, in which $(A,\leq)$ is an ordered set with smallest element 0 (cf. Section 2 of [FW1975]). This will also give us a very simple counterexample (see Example 7.4) to the assertion "if (R,S) is a radical-semisimple pair, then (R,r,S) is a radical-semisimple triple for some r", as mentioned in Section 5.

The following two propositions should be considered obvious after reading Section 2 of [FW1975], but since our proofs are very short and easy anyway, we do not discuss this connection.

Proposition 7.1. *If* $(A,\{0\},\leq,\geq)$ *is as above, then the following conditions on a subset U of A are equivalent:*

(a) *U is a radical class with respect to* $(A,\{0\},\leq,\geq)$;

(b) *U is a semisimple class with respect to* $(A,\{0\},\leq,\geq)$;

(c) *an element a of A is in U if and only if, for every non-zero* $b \leq a$, *there exists a non-zero* $c \leq b$ *which is in* U;

(d) *U is a down-closed subset of A such that an element a of A is in U whenever for every non-zero* $b \leq a$, *there exists a non-zero* $c \leq b$ *which is in* U.

Proof. The implications (a)$\Leftrightarrow$(b)$\Leftrightarrow$(c)$\Leftarrow$(d) immediately follow from the definitions, while (c)$\Rightarrow$(d) easily follows from the transitivity of $\leq$. □

Proposition 7.2. *If* $(A,\{0\},\leq,\geq)$ *is as above, then a pair* (R,S) *of subsets of A is a radical-semisimple pair if and only if*

$$R = \{b \in A \mid (a \in S \;\&\; a \leq b) \Rightarrow a = 0\}, \tag{7.1}$$
$$S = \{b \in A \mid (a \in R \;\&\; a \leq b) \Rightarrow a = 0\}. \tag{7.2}$$

Proof. Just note that ($\alpha_* = \beta^*$ and) the equalities above are nothing but $R = \beta^*(S)$ and $S = \alpha_*(R)$, respectively, where α and β are as in (4.3) in the case of $(A,\{0\},\leq,\geq)$. □

Continuing to develop our simplified counterpart of usual radical theory, what would be a reasonable 2-dimensional exactness structure whose underlying 1-dimensional exactness structure is $(A,\{0\},\leq,\geq)$? We propose the following one, requiring an additional condition on A; then, its underlying 1-dimensional exactness structure is indeed $(A,\{0\},\leq,\geq)$ under a further additional condition mentioned in Example 7.4(b) below.

Definition 7.3. Let A be an ordered set with smallest element 0 and such that, for every $b \in A$, the set

$$\{(a,c) \in A{\times}A \mid a \wedge c = 0 \;\&\; a \vee c = b\} \tag{7.3}$$

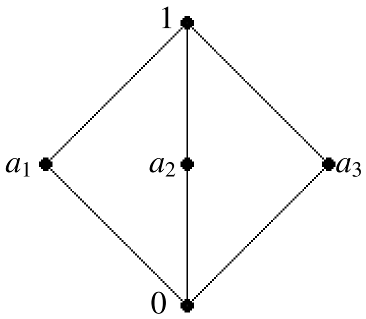
forms a complete lattice under the order defined by $(a,c) \leq (a',c') \Leftrightarrow (a \leq a'$ & $c' \leq c)$. The 2-dimensional exactness structure associated to A is

$$\begin{tikzcd}[column sep=huge] A' \arrow[r, "d^1_1"] \arrow[r, bend left=80, "d^1_0"] \arrow[r, bend right=80, "d^1_2"'] & A \arrow[l, bend right=40, "s^1_0"'] \arrow[l, bend left=40, "s^1_1"] \arrow[r, bend left=40, "d^0_0"] \arrow[r, bend right=40, "d^0_1"'] & \{0\}, \arrow[l, "s^0_0"'] \end{tikzcd} \tag{7.4}$$

where $A' = \{(a,b,c) \in A{\times}A{\times}A \mid a \wedge c = 0 \;\&\; a \vee c = b\}$, $s^0_0(0) = 0$, $s^1_0(a) = (a,a,0)$, $s^1_1(a) = (0,a,a)$, $d^1_0(a,b,c) = a$, $d^1_1(a,b,c) = b$, $d^1_2(a,b,c) = c$, and the complete lattice structure on $(d^1_1)^{-1}(b)$ is defined via $(a,b,c) \leq (a',b,c')$ $\Leftrightarrow (a \leq a'$ & $c' \leq c)$.

Although a further analysis of this 2-dimensional exactness structure, which always satisfies Condition 3.5, would be interesting, we will use it only in

Example 7.4. Consider the 2-dimensional exactness structure of Definition 7.3 where A is the lattice



(7.5)

and observe:

(a) this *non-distributive* lattice indeed satisfies the conditions required in Definition 4.3;

(b) the underlying 1-dimensional exactness structure is $(A,\{0\},\leq,\geq)$; more generally, this is true in the situation of Definition 7.3 whenever, for all $a \leq b$ in A, there exists c in A with (a,c) in the set (7.3);

(c) as follows from (b) and Proposition 7.2, $(\{0,a_1,a_2,\},\{0,a_3\})$ is a radical-semisimple pair.

Nevertheless there is no radical function r making $(\{0,a_1\},r,\{0,a_2,a_3\})$ a radical-semisimple triple. Indeed, having such an r, consider $r(1)$: by Theorem 3.3, it should be the largest element x in the lattice $(d_1^1)^{-1}(1)$ with $d_0^1(x)$ in $\{a_1\}$ – but such an element does not exist.

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