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A MULTIPLE CATEGORY OF MULTIPLE LAX CATEGORIES

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Résumé. On construit une catégorie multiple, utile dans l'étude des adjunctions multiples. Les objets sont les catégories multiples 'laxes'. Les flèches transversales sont les foncteurs multiples stricts tandis que les flèches en direction positive sont des foncteurs multiples de 'laxité mixte', qui varient des foncteurs laxes (en direction 1) aux colaxes (en direction ∞).

Abstract. We construct a multiple category which occurs in the study of multiple adjunctions. The objects are all the 'lax' multiple categories. The transversal arrows are their strict multiple functors while the arrows in a positive direction are multiple functors of a 'mixed laxity', varying from the lax ones (in direction 1) to the colax ones (in direction ∞).

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0. Introduction

This note is about strict, weak and lax multiple categories, an extension of double categories that we have studied in the articles [5] – [9]. The first two of them are about the 3-dimensional case, where *intercategories* (a kind of lax triple category) cover and combine diverse structures like duoidal categories [1, 3, 12], Gray categories [10], Verity double bicategories [13] and monoidal double categories [11]. The other papers [7] – [9] are about weak and lax infinite-dimensional multiple categories, an extension of the strict case introduced by Bastiani – Ehresmann [2].

A weak multiple category has objects, i-directed arrows in each direction $i \in \mathbb{N}$, ij-cells of dimension two for all $i < j$, and so on. Composition is strict in the *transversal* direction $i = 0$ and weak in each direction $i > 0$, i.e. associative and unitary up to invertible transversal comparisons. The transversal composition has a strict interchange with all the geometric ones, while the latter have invertible ij-interchangers; more generally, *chiral multiple categories* and *intercategories* have directed ij-interchangers, for $i < j$. A (weak or lax) n-tuple category has indices in the ordinal $\mathbf{n} = \{0, 1, ..., n-1\}$.

Here we investigate the different sorts of morphisms that can link chiral multiple categories. We know, from [9], that in a general multiple adjunction $F \dashv G$ the left adjoint is a *colax* (multiple) functor, while the right adjoint is *lax*; the adjunction lives in a double category $\mathbb{C}\mathrm{mc}$ of chiral multiple categories, where the horizontal arrows are lax functors and the vertical ones are colax functors.

But we have already seen in [5] that – in dimension three – there exists an intermediate sort, called a *colax-lax morphism*, which is colax in direction 1 and lax in direction 2 (and of course strict in the transversal direction 0). Also this case is important in concrete situations, when a triple adjunction $F \dashv G$ has a *colax-pseudo* left adjoint and a *pseudo-lax* right adjoint, so that the composites GF and FG are colax-lax morphisms, forming a monad GF and a comonad FG. Higher dimensional examples present higher dimensional cases of 'mixed laxity functors'.

With these motivations, we construct here a multiple category Cmc of chiral multiple categories, indexed by the ordinal $\boldsymbol{\omega} + \mathbf{1} = \{0, 1, ..., \infty\}$. Its transversal arrows are the strict multiple functors while, in direction p (for $1 \leqslant p \leqslant \infty$), the *p-morphisms* are 'multiple functors of mixed laxity', that vary from the lax ones (in direction 1) to the colax ones (in direction ∞). The double category $\mathbb{C}\mathrm{mc}$ is embedded in Cmc, with indices in $\{1, \infty\}$. Similar frameworks are concerned with intercategories, and the n-dimensional case.

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1. Notation

We mainly follow the notation of [7] – [9]. The symbol $\subset$ denotes weak inclusion. Categories and 2-categories are generally denoted as $\mathbf{A}, \mathbf{B}, ...$; weak double categories as $\mathbb{A}, \mathbb{B}, ...$; weak or lax multiple categories as $\mathsf{A}, \mathsf{B}, ...$

The definitions of weak and chiral multiple categories can be found in [7], or – briefly reviewed – in [8], Section 1. Here we only give a sketch of them, while recalling the notation we are using.

The two-valued index α (or β) varies in the set $2 = \{0, 1\}$, also written as $\{-, +\}$.

A *multi-index* $\mathbf{i}$ is a finite subset of $\mathbb{N}$, possibly empty. Writing $\mathbf{i} \subset \mathbb{N}$ it is understood that $\mathbf{i}$ is finite; writing $\mathbf{i} = \{i_1, ..., i_n\}$ it is understood that $\mathbf{i}$ has n distinct elements, written in the natural order $i_1 < i_2 < ... < i_n$; the integer $n \geqslant 0$ is called the *dimension* of $\mathbf{i}$. We write:

$$\begin{aligned} &\mathbf{i}j = j\mathbf{i} = \mathbf{i} \cup \{j\} && (\text{for } j \in \mathbb{N} \setminus \mathbf{i}),\\ &\mathbf{i}|j = \mathbf{i} \setminus \{j\} && (\text{for } j \in \mathbf{i}). \end{aligned} \tag{1}$$

For a weak multiple category A, the set of $\mathbf{i}$-cells $A_{\mathbf{i}}$ is written as A_*, A_i, A_{ij} when $\mathbf{i}$ is $\varnothing$, $\{i\}$ or $\{i, j\}$ respectively. Faces and degeneracies, satisfying the *multiple relations* (cf. [7], Section 2.2), are denoted as

$$\partial_j^\alpha \colon A_{\mathbf{i}} \to A_{\mathbf{i}|j}, \qquad e_j \colon A_{\mathbf{i}|j} \to A_{\mathbf{i}} \qquad\qquad (\text{for } \alpha = \pm,\ j \in \mathbf{i}). \tag{2}$$

The *transversal direction* $i = 0$ is set apart from the positive, or *geometric*, directions. For a *positive multi-index* $\mathbf{i} = \{i_1, ..., i_n\} \subset \mathbb{N}^* = \mathbb{N} \setminus \{0\}$, the *augmented multi-index* $0\mathbf{i} = \{0, i_1, ..., i_n\}$ has dimension $n + 1$, but both $\mathbf{i}$ and $0\mathbf{i}$ are said to have *degree* n. An $\mathbf{i}$-cell $x \in A_{\mathbf{i}}$ of A is also called an $\mathbf{i}$-*cube*, while a $0\mathbf{i}$-cell $f \in A_{0\mathbf{i}}$ is viewed as an $\mathbf{i}$-*map* $f \colon x \to_0 y$, where $x = \partial_0^- f$ and $y = \partial_0^+ f$. Composition in direction 0 is categorical (and generally realised by ordinary composition of mappings); it is written as $gf = f +_0 g$, with identities $1_x = \mathrm{id}(x) = e_0(x)$.

The *transversal category* $\mathrm{tv}_{\mathbf{i}}(\mathsf{A})$ consists of the $\mathbf{i}$-cubes and $\mathbf{i}$-maps of A, with transversal composition and identities. Their family forms a multiple object in $\mathbf{Cat}$, indexed by the positive multi-indices.

Composition of $\mathbf{i}$-cubes and $\mathbf{i}$-maps in a *positive* direction $i \in \mathbf{i}$ (often realised by pullbacks, pushouts, tensor products, etc.) is written in additive notation

$$\begin{aligned} &x +_i y && (\partial_i^+ x = \partial_i^- y),\\ &f +_i g \colon x +_i y \to x' +_i y' && (f \colon x \to x',\ g \colon y \to y',\ \partial_i^+ f = \partial_i^- g). \end{aligned} \tag{3}$$

The transversal composition has a strict interchange with each of the positive operations. The latter satisfy the unitarity, associativity and interchange laws up to transversally invertible comparisons (for $0 < i < j$)

$$\begin{aligned}
&\lambda_i x \colon (e_i\partial_i^- x) +_i x \to_0 x && (\textit{left i-unitor}),\\
&\rho_i x \colon x +_i (e_i\partial_i^+ x) \to_0 x && (\textit{right i-unitor}),\\
&\kappa_i(x, y, z) \colon x +_i (y +_i z) \to_0 (x +_i y) +_i z && (\textit{i-associator}), \qquad (4)\\
&\chi_{ij}(x, y, z, u) \colon (x +_i y) +_j (z +_i u) \to_0 (x +_j z) +_i (y +_j u) &&\\
& && (\textit{ij-interchanger}).
\end{aligned}$$

The comparisons are natural with respect to transversal maps; λ_i, ρ_i and κ_i are special in direction i (i.e. their i-faces are transversal identities) while χ_{ij} is special in both directions i, j; all of them commute with ∂_k^α for $k \neq i$ (or $k \neq i, j$ in the last case). Finally the comparisons must satisfy various conditions of coherence, listed in [7], Sections 3.3 and 3.4.

More generally for a *chiral multiple category* A the ij-interchangers χ_{ij} are not assumed to be invertible (see [7], Section 3.7).

Even more generally, in an *intercategory* we also have ij-interchangers $\mu_{ij}, \delta_{ij}, \tau_{ij}$ involving the units; this extension is studied in [5, 6] for the 3-dimensional case, the really important one. Infinite dimensional intercategories have been introduced in [7], Section 5, and mentioned marginally in [8] and [9], but a further study must await good examples.

While a chiral multiple category A is a multiple object of ordinary categories $\mathrm{tv}_{\mathbf{i}}(\mathsf{A})$ indexed by positive multi-indices $\mathbf{i} = \{i, j, k...\} \subset \mathbb{N}^*$, the structure Cmc that we shall construct will be indexed by 'extended' positive multi-indices $\mathbf{p} = \{p, q, r...\} \subset \{1, 2, ..., \infty\}$.

2. Lax and colax multiple functors

We want to analyse which sorts of 'morphisms' $\mathsf{A} \to \mathsf{B}$ between chiral multiple categories are of interest.

Two main kinds stand out:

(a) a *lax* (multiple) functor $F \colon \mathsf{A} \to \mathsf{B}$ is equipped with *comparison* $\mathbf{i}$-maps $\underline{F}_{\mathbf{i}}$, for the i-directed composition (for $t \in \mathsf{A}_{\mathbf{i}|i}$ and i-consecutive cubes x, y in $A_{\mathbf{i}}$)

$$\underline{F}_i(t) \colon e_i F(t) \to_0 F(e_i t), \quad \underline{F}_i(x, y) \colon F(x) +_i F(y) \to_0 F(x +_i y), \quad (5)$$

(b) a *colax* (multiple) functor $F\colon \mathsf{A} \to \mathsf{B}$ has comparisons in the opposite direction.

The definitions of such 'functors', with the *transversal transformations* of both sorts, can be found in [7], Section 3.9 (or here, in a more general form, in Sections 4 and 5.)

A *pseudo* functor is a lax multiple functor with (transversally) invertible comparisons, and is made colax by the inverse comparisons. It is *strict* when the comparisons are identities, so that the whole structure is strictly preserved.

In a general multiple adjunction (defined and studied in [9]) these two sorts appear together: *the left adjoint* $F\colon \mathsf{A} \to \mathsf{B}$ *is colax while the right adjoint* $G\colon \mathsf{B} \to \mathsf{A}$ *is lax*; many natural situations are of this type, with non-invertible comparisons. We do not want to compose F and G, since this would destroy their comparisons; yet we must give a unit and a counit.

This point was solved in [9], Section 2, where we constructed a (strict) double category $\mathbb{C}\mathrm{mc}$ of chiral multiple categories. The lax and colax multiple functors form the horizontal and vertical arrows, respectively. They are not to be composed, but linked by suitable double cells.

Finally, a *colax-lax (multiple) adjunction* $F \dashv G$ is a pair of adjoint arrows in this double category. This means a colax functor $F\colon \mathsf{A} \to \mathsf{B}$, a lax functor $G\colon \mathsf{B} \to \mathsf{A}$ and two double cells of $\mathbb{C}\mathrm{mc}$, called a *unit* and a *counit*

$$\begin{array}{ccc} \mathsf{A} & = & \mathsf{A} \\ {\scriptstyle F}\downarrow & \eta & \| \\ \mathsf{B} & \xrightarrow[G]{} & \mathsf{A} \end{array} \qquad\qquad \begin{array}{ccc} \mathsf{B} & \xrightarrow{G} & \mathsf{A} \\ \| & \varepsilon & \downarrow{\scriptstyle F} \\ \mathsf{B} & = & \mathsf{B} \end{array} \tag{6}$$

They have components $\eta x\colon x \to_0 GF(x)$ and $\varepsilon y\colon FG(y) \to_0 y$ (for any cube x in A and y in B), whose coherence conditions are based – separately – on the comparisons of F and G. The triangular laws state that the composites $(\varepsilon\,|\,\eta)$ and $(\frac{\eta}{\varepsilon})$ are identities.

If F is a pseudo functor, this is the same as an adjunction in the 2-category $\mathrm{Lx}\mathbf{Cmc}$ of chiral multiple categories, lax functors and their transversal transformations (as proved in [9], Section 5). Symmetrically, *if G is a pseudo functor*, this is the same as an adjunction in the 2-category $\mathrm{Cx}\mathbf{Cmc}$, whose arrows are the colax multiple functors.

These two particular cases, a *pseudo-lax* and a *colax-pseudo* adjunction, do not cover the examples of [9]; furthermore, composing adjunctions of these two kinds we come back to the general case.

Yet the particular cases are important, since the first gives a *lax* (multiple) *monad* $GF\colon \mathsf{A} \to \mathsf{A}$ and a *lax comonad* $FG\colon \mathsf{B} \to \mathsf{B}$, while the second case gives a *colax monad* and a *colax comonad*.

3. Examples

Some examples, from [9], Section 1.7, will lead to new morphisms, intermediate between the two previous kinds, and also important in adjunctions. For the sake of simplicity, we begin by working in dimension 3.

For a category $\mathbf{C}$ with (a choice of) pullbacks, we have a weak triple category $\mathbf{3}\mathsf{Span}(\mathbf{C})$ of 'spans of spans'. A 12-cube is a functor $x\colon \vee\times\vee \to \mathbf{C}$ (where $\vee$ is the formal-span category) and a 12-map $f\colon x \to_0 y$ is a natural transformation of such functors (a 3-dimensional item $f\colon \vee\times\vee\times\mathbf{2} \to \mathbf{C}$).

Dually, if $\mathbf{C}$ has pushouts, there is a weak triple category $\mathbf{3}\mathsf{Cosp}(\mathbf{C})$ whose highest cubes are 'cospans of cospans' $x\colon \wedge\times\wedge \to \mathbf{C}$.

When $\mathbf{C}$ has both pullbacks and pushouts, we can form a chiral triple category $\mathsf{SC}(\mathbf{C}) = \mathsf{S}_1\mathsf{C}_1(\mathbf{C})$ where a 12-cube is a functor $x\colon \vee\times\wedge \to \mathbf{C}$; the 1-directed composition is by pullbacks, the 2-directed one by pushouts.

An ordinary functor $F\colon \mathbf{X} \to \mathbf{A}$ between categories with pullbacks and pushouts produces:

(a) *a colax (triple) functor* $\mathbf{3}\mathsf{Span}(F)\colon \mathbf{3}\mathsf{Span}(\mathbf{X}) \to \mathbf{3}\mathsf{Span}(\mathbf{A})$ of weak triple categories,

(b) a *lax (triple) functor* $\mathbf{3}\mathsf{Cosp}(F)\colon \mathbf{3}\mathsf{Cosp}(\mathbf{X}) \to \mathbf{3}\mathsf{Cosp}(\mathbf{A})$ of weak triple categories,

(c) a *colax-lax morphism* $\mathsf{SC}(F)\colon \mathsf{SC}(\mathbf{X}) \to \mathsf{SC}(\mathbf{A})$ of chiral triple categories.

We have thus a new morphism of an intermediate sort: $\mathsf{SC}(F)$ is colax for the 1-directed composition, realised by pullbacks, and lax for the 2-composition, realised by pushouts; the precise definition can be found in [5], Section 5. Moreover, if F preserves pushouts, $\mathbf{3}\mathsf{Cosp}(F)$ is a pseudo functor and $\mathsf{SC}(F)$ is a colax-pseudo morphism; and so on.

Now, an ordinary adjunction between categories with pullbacks and push-outs

$$F\colon\ \mathbf{X} \rightleftarrows \mathbf{A}\ :G, \qquad\qquad F \dashv G, \tag{7}$$

has three natural extensions to colax-lax triple adjunctions:

$$\mathbf{3}\mathsf{Span}(F) \dashv \mathbf{3}\mathsf{Span}(G),\ \ \mathbf{3}\mathsf{Cosp}(F) \dashv \mathbf{3}\mathsf{Cosp}(G),\ \ \mathsf{SC}(F) \dashv \mathsf{SC}(G). \tag{8}$$

The first is actually a *colax-pseudo adjunction* (because G preserves pullbacks), and gives a colax triple monad on $\mathbf{3}\mathsf{Span}(\mathbf{X})$. The second is *pseudo-lax*, and gives a lax triple monad on $\mathbf{3}\mathsf{Cosp}(\mathbf{X})$.

In the last, $F' = \mathsf{SC}(F)$ is a *colax-pseudo morphism* while $G' = \mathsf{SC}(G)$ is a *pseudo-lax morphism*; their composites $G'F' = \mathsf{SC}(GF)$ and $F'G' = \mathsf{SC}(FG)$ make sense: they are colax-lax morphisms, and we still have a triple monad on $\mathsf{SC}(X)$, where $T = G'F'$ is a colax-lax morphism. (Multiple monads will be studied elsewhere.)

All this can be extended to higher dimensions, for the weak multiple categories $\mathsf{Span}(\mathbf{C})$, $\mathsf{Cosp}(\mathbf{C})$ and the chiral multiple categories $\mathsf{S}_p\mathsf{C}_q(\mathbf{C})$, $\mathsf{S}_p\mathsf{C}_\infty(\mathbf{C})$, $\mathsf{S}_{-\infty}\mathsf{C}_\infty(\mathbf{C})$ (see [9], Section 1.3). We get thus morphisms of 'mixed laxity', colax up to a certain degree and lax above. (The reverse case cannot occur, as we shall see below.)

Finally we recall from [9], Sections 1.5 – 1.6, a colax-lax adjunction of weak triple categories, based on an ordinary category $\mathbf{C}$ with pullbacks and pushouts:

$$F\colon\ \mathsf{Span}(\mathbf{C}) \rightleftarrows \mathsf{Cosp}(\mathbf{C})\ :G, \qquad F \dashv G, \tag{9}$$

F works by pushouts and G by pullbacks. None of them is pseudo (in general, of course), and we do not have an associated multiple monad (nor comonad).

4. Mixed-laxity functors

We are now ready to begin the construction of a multiple category Cmc containing different morphisms in different directions, that vary from the lax case to the colax one.

In degree 0, the objects of Cmc are the (small) chiral multiple categories, and the transversal arrows (or 0-morphisms) are the strict multiple functors $F\colon \mathsf{A} \to_0 \mathsf{B}$.

In degree 1 and direction p (for $1 \leqslant p \leqslant \infty$), a *$p$-morphism* $R\colon \mathsf{A} \to_p \mathsf{B}$ between chiral multiple categories will be a *mixed-laxity functor* which is colax in all positive directions $i < p$ and lax in all directions $i \geqslant p$. In particular, this is a lax functor for $p = 1$ and a colax functor for $p = \infty$.

Basically, R has components $R_{\mathbf{i}} = \mathrm{tv}_{\mathbf{i}}(R)\colon \mathrm{tv}_{\mathbf{i}}(\mathsf{A}) \to \mathrm{tv}_{\mathbf{i}}(\mathsf{B})$, for all positive multi-indices $\mathbf{i}$, that are ordinary functors and commute with faces: $\partial_i^\alpha.R_{\mathbf{i}} = R_{\mathbf{i}|i}.\partial_i^\alpha$ (for $i \in \mathbf{i}$).

Moreover R is equipped with *comparison* $\mathbf{i}$-maps $\underline{R}_i$ (for $t \in A_{\mathbf{i}|i}$ and x, y i-consecutive in $A_{\mathbf{i}}$), either in the lax direction for $i \geqslant p$

$$\underline{R}_i(t)\colon e_iR(t) \to_0 R(e_it), \ \ \underline{R}_i(x,y)\colon R(x) +_i R(y) \to_0 R(x +_i y), \quad (10)$$

or in the colax direction for $0 < i < p$

$$\underline{R}_i(t)\colon R(e_it) \to_0 e_iR(t), \ \ \underline{R}_i(x,y)\colon R(x +_i y) \to_0 R(x) +_i R(y). \quad (11)$$

All these comparisons are i-special, i.e. their two i-faces are transversal identities, and must commute with the other faces ∂_j^α (for $j \neq i$ in $\mathbf{i}$)

$$\partial_j^\alpha \underline{R}_i(t) = \underline{R}_i(\partial_j^\alpha t), \qquad \partial_j^\alpha \underline{R}_i(x,y) = \underline{R}_i(\partial_j^\alpha x, \partial_j^\alpha y). \quad (12)$$

Then they have to satisfy the axioms of naturality and coherence (see [7], Section 3.9), either in the lax form (lmf.1 – 4) for $i \geqslant p$, or in the transversally dual form for $i < p$.

Furthermore there is an axiom of coherence with the interchanger χ_{ij} (for $0 < i < j$) which has three forms (where (a) corresponds to (lmf.5), (c) corresponds to its dual and (b) is an intermediate case):

(a) for $p \leqslant i < j$ (so that R is i- and j-lax), we have commutative diagrams of transversal maps:

$$\begin{array}{ccc}
(Rx +_i Ry) +_j (Rz +_i Ru) & \xrightarrow{\chi_{ij}R} & (Rx +_j Rz) +_i (Ry +_j Ru) \\
{\scriptstyle \underline{R}_i +_j \underline{R}_i}\downarrow & & \downarrow{\scriptstyle \underline{R}_j +_i \underline{R}_j} \\
R(x +_i y) +_j R(z +_i u) & & R(x +_j y) +_i R(z +_j u) \\
{\scriptstyle \underline{R}_j}\downarrow & & \downarrow{\scriptstyle \underline{R}_i} \\
R((x +_i y) +_j (z +_i u)) & \xrightarrow[R\chi_{ij}]{} & R((x +_j z) +_i (y +_j u))
\end{array} \quad (13)$$

(b) for $0 < i < p \leqslant j$ (so that R is i-colax and j-lax), we have commutative diagrams:

$$\begin{array}{ccc} (Rx +_i Ry) +_j (Rz +_i Ru) & \xrightarrow{\chi_{ij}R} & (Rx +_j Rz) +_i (Ry +_j Ru) \\ \nearrow {\scriptstyle \underline{R}_i +_j \underline{R}_i} & & \searrow {\scriptstyle \underline{R}_j +_i \underline{R}_j} \\ R(x +_i y) +_j R(z +_i u) & & R(x +_j y) +_i R(z +_j u) \\ \searrow {\scriptstyle \underline{R}_j} & & \nearrow {\scriptstyle \underline{R}_i} \\ R((x +_i y) +_j (z +_i u)) & \xrightarrow[R\chi_{ij}]{} & R((x +_j z) +_i (y +_j u)) \end{array} \tag{14}$$

(c) for $0 < i < j < p$ (so that R is i- and j-colax), we have commutative diagrams as in (13), with all vertical arrows reversed.

The composition of p-morphisms $R'R = R +_p R'$ is easily defined: their comparisons are separately composed.

Finally, a transversal map $(F, G)\colon R \to_0 S$ of p-arrows will be a commutative square

$$\begin{array}{ccc} \bullet & \xrightarrow{F} & \bullet \\ {\scriptstyle R}\downarrow & = & \downarrow{\scriptstyle S} \\ \bullet & \xrightarrow[G]{} & \bullet \end{array} \qquad SF = GR \qquad \begin{array}{l} \bullet \xrightarrow{0} \\ \downarrow{\scriptstyle p} \end{array} \tag{15}$$

with strict functors F, G and p-morphisms R, S. Commutativity means that $SF = GR$ *as* p-morphisms, including comparisons.

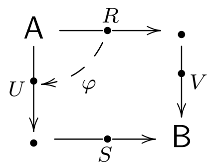
(As already remarked in [5], the 'lax-colax' case makes no sense: modifying diagram (13) by reversing all arrows $\underline{R}_j$ would lead to a diagram where no pairs of arrows compose.)

We have thus defined the double category $\mathrm{dbl}_{0p}(\mathsf{Cmc})$ of chiral multiple categories, strict functors and p-morphisms.

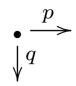
5. Two-dimensional cubes

To define a pq-cube (for $1 \leqslant p < q \leqslant \infty$) we have to adapt the axioms of transversal transformation (again in [7], Section 3.9).

A *pq-cube* $\varphi\colon (U \,{}^{R}_{S}\, V)$ will be a 'generalised quintet' consisting of two p-morphisms R, S, two q-morphisms U, V, together with – roughly speaking – a 'transversal transformation' $\varphi\colon VR \dashrightarrow SU$



$$\varphi\colon VR \dashrightarrow SU. \qquad (16)$$



This is an abuse of notation since there are no composites VR and SU in our structure: the coherence conditions of φ are based on the four morphisms R, S, U, V and all their comparison maps. Precisely, the cell φ consists of a face-consistent family of transversal maps in B

$$\begin{array}{ll} \varphi(x) = \varphi_{\mathbf{i}}(x)\colon VR(x) \to_0 SU(x), & \text{(for every } \mathbf{i}\text{-cube } x \text{ of } \mathsf{A}), \\ \partial_i^\alpha.\varphi_{\mathbf{i}} = \varphi_{\mathbf{i}|i}.\partial_i^\alpha & \text{(for } i \in \mathbf{i}), \end{array} \qquad (17)$$

so that each component $\varphi_{\mathbf{i}}\colon V_{\mathbf{i}}R_{\mathbf{i}} \to S_{\mathbf{i}}U_{\mathbf{i}}\colon \mathrm{tv}_{\mathbf{i}}(\mathsf{A}) \to \mathrm{tv}_{\mathbf{i}}(\mathsf{B})$ is a natural transformation of ordinary functors:

(nat) for all $f\colon x \to_0 y$ in A, we have a commutative diagram of transversal maps in B

$$\begin{array}{ccc} VR(x) & \xrightarrow{\ \varphi x\ } & SU(x) \\ {\scriptstyle VRf}\downarrow & = & \downarrow{\scriptstyle SUf} \\ VR(y) & \xrightarrow[\ \varphi y\ ]{} & SU(y) \end{array} \qquad (18)$$

Moreover φ has to satisfy the following coherence conditions (coh.a), (coh.b), (coh.c) with the comparisons of R, S, U, V, for a degenerate cube $e_i(t)$ (with $t \in A_{\mathbf{i}|i}$) and a composite $z = x +_i y$ in $A_{\mathbf{i}}$.

(coh.a) If $p < q \leqslant i$ (so that R, S, U, V are lax in direction i), we have commutative diagrams (with $\varphi = \varphi x +_i \varphi y$):

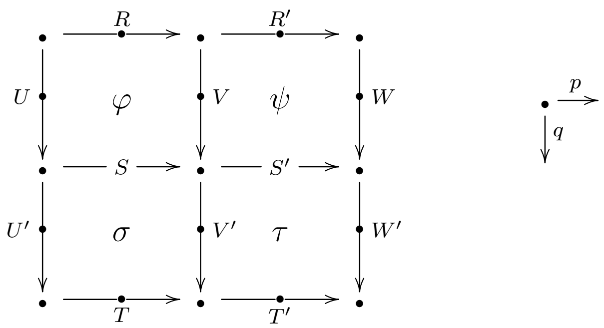
$$\begin{array}{ccccccc}
e_iVR(t) & \xrightarrow{e_i(\varphi t)} & e_iSU(t) & \quad & VRx +_i VRy & \xrightarrow{\varphi} & SUx +_i SUy \\
{\scriptstyle \underline{V}_i(Rt)}\downarrow & & \downarrow{\scriptstyle \underline{S}_i(Ut)} & & {\scriptstyle \underline{V}_i(Rx,Ry)}\downarrow & & \downarrow{\scriptstyle \underline{S}_i(Ux,Uy)} \\
V(e_iRt) & & S(e_iUt) & & V(Rx +_i Ry) & & S(Ux +_i Uy) \\
{\scriptstyle V\underline{R}_i(t)}\downarrow & & \downarrow{\scriptstyle S\underline{U}_i(t)} & & {\scriptstyle V\underline{R}_i(x,y)}\downarrow & & \downarrow{\scriptstyle S\underline{U}_i(x,y)} \\
VR(e_it) & \xrightarrow[\varphi(e_it)]{} & SU(e_it) & & VR(z) & \xrightarrow[\varphi(z)]{} & SU(z)
\end{array} \qquad (19)$$

(coh.b) If $p \leqslant i < q$ (so that R, S are lax and U, V are colax in direction i), we have commutative diagrams:

$$\begin{array}{ccccccc}
e_iVR(t) & \xrightarrow{e_i(\varphi t)} & e_iSU(t) & \quad & VRx +_i VRy & \xrightarrow{\varphi} & SUx +_i SUy \\
{\scriptstyle \underline{V}_i(Rt)}\uparrow & & \downarrow{\scriptstyle \underline{S}_i(Ut)} & & {\scriptstyle \underline{V}_i(Rx,Ry)}\uparrow & & \downarrow{\scriptstyle \underline{S}_i(Ux,Uy)} \\
V(e_iRt) & & S(e_iUt) & & V(Rx +_i Ry) & & S(Ux +_i Uy) \\
{\scriptstyle V\underline{R}_i(t)}\downarrow & & \uparrow{\scriptstyle S\underline{U}_i(t)} & & {\scriptstyle V\underline{R}_i(x,y)}\downarrow & & \uparrow{\scriptstyle S\underline{U}_i(x,y)} \\
VR(e_it) & \xrightarrow[\varphi(e_it)]{} & SU(e_it) & & VR(z) & \xrightarrow[\varphi(z)]{} & SU(z)
\end{array} \qquad (20)$$

(coh.c) If $i < p < q$ (so that R, S, U, V are colax in direction i), we have commutative diagrams as in (19), with all vertical arrows reversed.

The p- and q-composition of these cubes are both defined using componentwise the transversal composition of a chiral multiple category. Namely, for a consistent matrix of pq-cubes and $x \in \mathsf{A}$



(21)

$$\begin{aligned}(\varphi +_p \psi)(x) &= \psi(Rx) +_0 S'(\varphi x)\colon WR'Rx \to S'VRx \to S'SUx,\\ (\varphi +_q \sigma)(x) &= V'(\varphi x) +_0 \sigma(Ux)\colon V'VRx \to V'SUx \to TU'Ux.\end{aligned} \tag{22}$$

The main technical points of the whole construction of Cmc are concerned with these composition laws. We shall prove, in Theorem 10, that they are well-defined, i.e. the cells above do satisfy the previous coherence conditions. We also prove that these laws strictly satisfy unitarity, associativity and the middle-four interchange law.

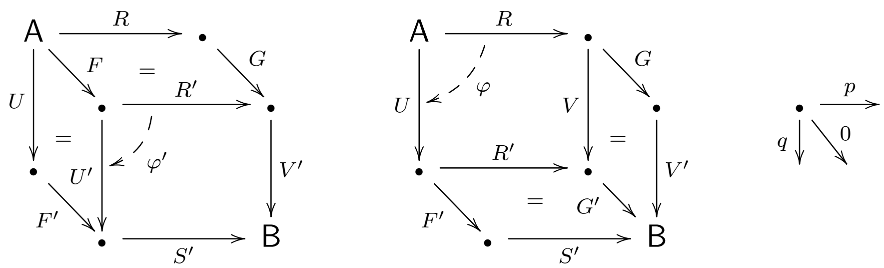
6. Transversal maps of degree two

Given two pq-cubes

$$\varphi\colon (U \, {}^{R}_{S} \, V), \qquad\qquad \varphi'\colon (U' \, {}^{R'}_{S'} \, V') \tag{23}$$

a transversal pq-map $(F, G, F', G')\colon \varphi \to_0 \varphi'$ (of degree two and dimension three) is a quadruple of strict functors forming four transversal maps of degree 1

$$\begin{aligned}(F, G)&\colon R \to_0 R', &\qquad (F', G')&\colon S \to_0 S',\\ (F, F')&\colon U \to_0 U', &\qquad (G, G')&\colon V \to_0 V',\end{aligned} \tag{24}$$



and such that 'the cube commutes', in the sense that, for every $\mathbf{i}$-cube x of A, the following transversal maps of B coincide

$$G'(\varphi x)\colon G'VR(x) \to G'SU(x), \;\; \varphi'(Fx)\colon V'R'F(x) \to S'U'F(x). \tag{25}$$

We have thus defined the triple category $\mathrm{trp}_{0pq}(\mathsf{Cmc})$ of chiral multiple categories, with strict functors and p- and q-morphisms (for $0 < p < q \leqslant \infty$). Its indices vary in the pointed ordered set $\{0, p, q\}$.

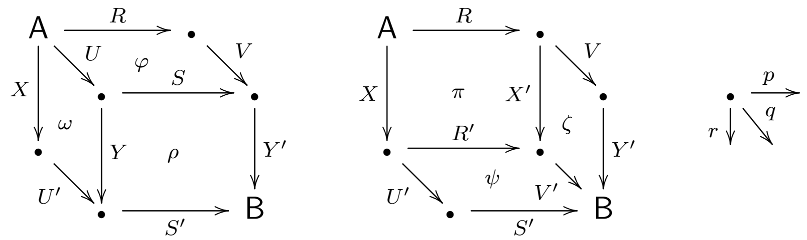
7. Three-dimensional cubes

A pqr-*cube* (for $0 < p < q < r \leqslant \infty$) will be a 'commutative cube' Π determined by its six faces:

– two pq-cubes φ, ψ (the faces $\partial_r^\alpha \Pi$),

– two pr-cubes π, ρ (the faces $\partial_q^\alpha \Pi$),

– two qr-cubes ω, ζ (the faces $\partial_p^\alpha \Pi$),



(26)

The commutativity condition means that, for every $\mathbf{i}$-cube x of A, the following composed transversal arrows in B coincide

$$S'\omega x.\rho Ux.Y'\varphi x \colon Y'VR(x) \to Y'SU(x) \to S'YU(x) \to S'U'X(x)$$
$$\psi Xx.V'\pi x.\zeta Rx : Y'VR(x) = V'X'R(x) = V'R'X(x) = S'U'X(x).$$

These cubes are composed in direction p, q, or r, by pasting cubes (with the operations of 2-dimensional cubes). Again, these operations are associative, unitary and satisfy the middle-four interchange by pairs.

8. Higher items

A transversal pqr-map $F \colon \Pi \to_0 \Pi'$ between pqr-cubes is determined by its boundary, a face-consistent family of six transversal maps of degree two (and dimension three)

$$\partial_j^\alpha F \colon \partial_j^\alpha \Pi \to_0 \partial_j^\alpha \Pi' \qquad (\alpha = \pm,\ j \in \{p, q, r\}), \tag{27}$$

under no other conditions. Their operations are computed on such faces.

We have thus defined a quadruple category of chiral multiple categories, with strict functors and p-, q-, r-morphisms (for extended positive integers $p < q < r$). The indices vary in the pointed ordered set $\{0, p, q, r\}$.

Finally, we have the multiple category Cmc (indexed by the ordinal $\boldsymbol{\omega} + \mathbf{1}$), where each cell of dimension $\geqslant 4$ (starting with the transversal maps of degree 3 considered above and the cubes of dimension 4, not yet considered) is coskeletally determined by a face-consistent family of all its iterated faces of dimension 3.

In the truncated case we have the $(n+1)$-dimensional multiple category $\mathsf{Cmc}_{\mathbf{n}}$ of (small) chiral $\mathbf{n}$-multiple categories, where the objects are indexed by the ordinal $\mathbf{n} = \{0, ..., n-1\}$, while $\mathsf{Cmc}_{\mathbf{n}}$ is indexed by $\mathbf{n} + \mathbf{1}$ (the previous ∞ being replaced by n). But one should note that $\mathsf{Cmc}_{\mathbf{n}}$ is *not* an ordinary truncation of Cmc, as its objects too are truncated.

Cmc is a substructure of the – similarly defined – multiple category Inc of small infinite dimensional intercategories, and $\mathsf{Cmc}_{\mathbf{n}}$ is a substructure of the $(n+1)$-dimensional multiple category $\mathsf{Inc}_{\mathbf{n}}$ of small $\mathbf{n}$-intercategories.

9. Comments

These multiple categories are related to various double or triple categories previously constructed.

(a) A chiral 1-multiple category is just a category, and $\mathsf{Cmc}_{\mathbf{1}}$ is the double category of small categories, with commutative squares of functors as double cells.

(b) A chiral 2-multiple category is a weak double category. We have studied in [4], Section 2, the double category $\mathbb{D}\mathrm{bl}$ of weak double categories, with lax and colax functors – where double adjunctions live. Later $\mathbb{D}\mathrm{bl}$ was extended to a triple category $\mathrm{S}\mathbb{D}\mathrm{bl}$ of weak double categories, with strict, lax and colax functors (see [7], Section 1); in the latter all 2-dimensional cells are inhabited by possibly non-trivial transformations, while in Cmc_2 the 01- and 02-cells are 'commutative squares', inhabited by identities. Thus $\mathsf{Cmc}_{\mathbf{2}}$ extends $\mathbb{D}\mathrm{bl}$ but is a triple subcategory of $\mathrm{S}\mathbb{D}\mathrm{bl}$.

(c) As we have recalled, multiple adjunctions live in the double category $\mathbb{C}\mathrm{mc}$ of chiral multiple categories, with lax and colax multiple functors ([9], Section 2). This can be extended to a triple category $\mathrm{S}\mathbb{C}\mathrm{mc}$ of chiral multiple categories, with strict, lax and colax functors, where again all 2-dimensional cells are inhabited by possibly non-trivial transformations. Then $\mathrm{S}\mathbb{C}\mathrm{mc}$ contains the triple category obtained from Cmc by restricting to the multi-indices $\mathbf{i} \subset \{0, 1, \infty\}$.

(d) The quadruple category $\mathsf{Inc}_\mathbf{3}$ of 3-dimensional intercategories is an extension of the triple category ICat of [9], Section 6, obtained by adding strict functors in the transversal direction and 'commutative transversal cells'.

10. Theorem

The structure Cmc *constructed above is indeed a strict multiple category.*

Proof. We prove the non-obvious points, listed at the end of Section 5.

(a) First we prove that the the composition law $\varphi +_p \psi$ of pq-cubes is well-defined by the formulas (22)

$$(\varphi +_p \psi)(x) = \psi(Rx) +_0 S'(\varphi x) \colon WR'Rx \to S'VRx \to S'SUx, \quad (28)$$

in the sense that this family of transversal maps does satisfy the conditions (coh.a) – (coh.c) of Section 5.

The argument is an extension of a similar one for the double category $\mathbb{D}\mathrm{bl}$ in [4], Section 2, or for the double category $\mathbb{C}\mathrm{mc}$ in [9], Section 2, taking into account the mixed laxity of the present 'functors'. We prove the three coherence axioms with respect to a composed cube $z = x +_i y$ in $A_\mathbf{i}$; one would work in a similar way for a degenerate cube $e_i(t)$, with $t \in A_{\mathbf{i}|i}$.

First we prove (coh.a), letting $p < q \leqslant i$, so that all our functors R, R', S, S', U, V, W are lax in direction i. This amounts to the commutativity of the outer diagram below, formed of transversal maps (the index i being omitted in $+_i$ and in all comparisons $\underline{R}_i, \underline{R}'_i$, etc.)

$$\begin{array}{ccccc}
WR'Rz & \xrightarrow{\psi Rz} & S'VRz & \xrightarrow{S'\varphi z} & S'SUz \\
\uparrow{\scriptstyle WR'\underline{R}} & & \uparrow{\scriptstyle S'V\underline{R}} & & \uparrow{\scriptstyle S'S\underline{U}} \\
WR'(Rx + Ry) & \xrightarrow{\psi(Rx+Ry)} & S'V(Rx + Ry) & & S'S(Ux + Uy) \\
\uparrow{\scriptstyle W\underline{R}'R} & & \uparrow{\scriptstyle S'\underline{V}R} & & \uparrow{\scriptstyle S'\underline{S}U} \\
W(R'Rx + R'Ry) & & S'(VRx + VRy) & \xrightarrow{S'(\varphi x+\varphi y)} & S'(SUx + SUy) \\
\uparrow{\scriptstyle \underline{W}R'R} & & \uparrow{\scriptstyle \underline{S}'VR} & & \uparrow{\scriptstyle \underline{S}'SU} \\
WR'Rx + WR'Ry & \xrightarrow[\psi Rx+\psi Ry]{} & S'VRx + S'VRy & \xrightarrow[S'\varphi x+S'\varphi y]{} & S'SUx + S'SUy
\end{array}$$

Indeed, the two hexagons commute by (coh.a), applied to φ and ψ, respectively. The upper rectangle commutes by naturality of ψ on $\underline{R}_i(x, y)$. The lower rectangle commutes by axiom (lmf.2) (in [7], Section 3.9), on the lax functor S', with respect to the transversal **i**-maps $\varphi x \colon VR(x) \to_0 SU(x)$ and $\varphi y \colon VR(y) \to_0 SU(y)$

$$S'(\varphi x +_i \varphi y).\underline{S}'_i(VR(x), VR(y)) = \underline{S}'_i(SU(x), SU(y)).(S'(\varphi x) +_i S'(\varphi y)).$$

The proof of (coh.c) is transversally dual to the previous one. To prove (coh.b) we let $p \leqslant i < q$, so that R, R', S, S' are lax, while U, V, W are colax in direction i. We reverse the comparisons $\underline{U}_i, \underline{V}_i, \underline{W}_i$ in the diagram above

$$\begin{array}{ccccc}
WR'Rz & \xrightarrow{\psi Rz} & S'VRz & \xrightarrow{S'\varphi z} & S'SUz \\
{\scriptstyle WR'\underline{R}}\uparrow & & \uparrow{\scriptstyle S'V\underline{R}} & & \downarrow{\scriptstyle S'S\underline{U}} \\
WR'(Rx+Ry) & \xrightarrow{\psi(Rx+Ry)} & S'V(Rx+Ry) & & S'S(Ux+Uy) \\
{\scriptstyle W\underline{R}'R}\uparrow & & \downarrow{\scriptstyle S'\underline{V}R} & & \uparrow{\scriptstyle S'\underline{S}U} \\
W(R'Rx+R'Ry) & & S'(VRx+VRy) & \xrightarrow{S'(\varphi x+\varphi y)} & S'(SUx+SUy) \\
{\scriptstyle \underline{W}R'R}\downarrow & & \uparrow{\scriptstyle \underline{S}'VR} & & \uparrow{\scriptstyle \underline{S}'SU} \\
WR'Rx+WR'Ry & \xrightarrow[\psi Rx+\psi Ry]{} & S'VRx+S'VRy & \xrightarrow[S'\varphi x+S'\varphi y]{} & S'SUx+S'SUy
\end{array}$$

and note that the two hexagons commute, by (coh.b) on φ and ψ, while the rectangles are unchanged.

(b) The composition law $\varphi +_p \psi$ has been defined via the composition of transversal maps, and therefore is strictly unitary and associative.

(c) Finally, to verify the middle-four interchange law on the four double cells of diagram (21), we compute the composites $(\varphi +_p \psi) +_q (\sigma +_p \tau)$ and $(\varphi +_q \sigma) +_p (\psi +_q \tau)$ on an **i**-cube x, and we obtain the two transversal maps $W'WR'Rx \to_0 T'TU'Ux$ of the upper or lower path in the following diagram

$$\begin{array}{ccccccc}
W'WR'Rx & \xrightarrow{W'\psi Rx} & W'S'VRx & \xrightarrow{W'S'\varphi x} & W'S'SUx & & \\
& & {\scriptstyle \tau VRx}\downarrow & = & \downarrow{\scriptstyle \tau SUx} & & \\
& & T'V'VRx & \xrightarrow[T'V'\varphi x]{} & T'V'SUx & \xrightarrow[T'\sigma Ux]{} & T'TU'Ux
\end{array}$$

The square commutes, by naturality of the double cell τ (with respect to the transversal map $\varphi x\colon VR(x) \to_0 SU(x)$), so that the two composites coincide. □

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