

**ADJOINTS FOR MULTIPLE CATEGORIES
(ON WEAK AND LAX MULTIPLE CATEGORIES, III)**

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Résumé. Suite aux deux premiers articles de cette série, on étudie ici les foncteurs adjoints entre catégories multiples de dimension infinie. Le cadre général est constitué par les *catégories multiples chirales* - une forme faible partiellement laxe ayant des interchangeurs dirigés entre les compositions faibles.

Abstract. Continuing our first two papers in this series, we study adjoints for infinite-dimensional multiple categories. The general setting is chiral multiple categories - a weak, partially lax form with directed interchanges between the weak composition laws.

Keywords. multiple category, double category, cubical set, adjoint functor.

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0. Introduction

This is the third paper in a series on weak and lax multiple categories, of finite or infinite dimension - an extension of weak double and weak cubical categories.

Our main framework, a *chiral multiple category*, was introduced in the first article [GP8], cited below as Part I; it is a partially lax multiple category with a strict composition $gf = f +_0 g$ in direction 0 (the *transversal direction*), weak compositions $x +_i y$ in all positive (or *geometric*) directions $i \in \mathbb{N} \setminus \{0\}$ and a directed *ij-interchanger* for the *i*- and *j*-compositions (for $0 < i < j$)

$$\chi_{ij}(x, y, z, u): (x +_i y) +_j (z +_i u) \rightarrow (x +_j z) +_i (y +_j u). \quad (1)$$

Part II [GP9] studies multiple limits in this setting. We now investigate multiple adjoints, extending the study of double adjunctions in [GP2] and cubical adjunctions in [G3].

Section 1 is an informal introduction to multiple adjunctions. After a synopsis of weak and chiral multiple categories, we describe a natural colax/lax adjunction $F \dashv G$ between the weak *double* categories of spans and cospans (already studied in [GP2]) and its extension to the corresponding infinite dimensional, weak *multiple* categories (of cubical type); the functor F is constructed with pushouts and is colax, while G is constructed with pullbacks and is lax. Then we derive from this adjunction other instances, between chiral multiple categories that are not of cubical type. Other examples are given in 1.7.

In Section 2 we introduce the strict *double* category $\mathbb{C}mc$ of chiral multiple categories (or *cm-categories*), lax and colax cm-functors and suitable double cells. Comma cm-categories are also considered. Both topics extend notions of weak double categories developed in [GP2].

Section 3 reviews the notions of companions and adjoints *in a double category*, from [GP2]. Then Sections 4 and 5 introduce and study multiple colax/lax adjunctions, as adjoint arrows *in* the double category $\mathbb{C}mc$.

Finally, Section 6 deals with the preservation of limits by right adjoints, for cm-categories.

Literature. Strict double and multiple categories were introduced and studied by C. Ehresmann and A.C. Ehresmann [Eh, BE, EE1, EE2, EE3]. Strict cubical categories can be seen as a particular case of multiple categories (as shown in Part I); their links with strict ω -categories are made clear in [BM, ABS]. Weak double categories (or pseudo double categories) were introduced and studied in our series [GP1 - GP4]; adjunctions and monads in this setting are also studied in [FGK1, FGK2, Ni]; other aspects are developed in [DPR, Fi, Ga, P2, P3]. For weak cubical categories see [G1 - G3] and [GP5]. The three-dimensional case of lax triple categories covers and combines diverse structures like duoidal categories, Gray categories, Verity double bicategories and monoidal double categories; see [GP6, GP7]. Further information on literature for higher dimensional category theory can be found in the Introduction of Part I.

Notation. We follow the notation of Parts I and II [GP8, GP9]; the reference I.2.3 or II.2.3 points to Subsection 2.3 of Part I or Part II. The symbol \subset denotes weak inclusion. Categories and 2-categories are generally denoted as $\mathbf{A}, \mathbf{B}, \dots$; weak double categories as $\mathbb{A}, \mathbb{B}, \dots$; weak or lax multiple categories

as A, B... More specific points of notation are recalled below, in 1.1.

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1. Some basic examples of adjunctions

We begin by recalling examples of adjunctions for *weak double categories*, studied in [GP2], and their extension to (infinite-dimensional) *weak cubical categories*, studied in [G3]. Then we derive from the latter some instances of adjunctions between *chiral multiple categories* that are not of a cubical type.

This section is an informal introduction to such adjunctions, precise definition will be given later.

1.1 Notation

The definitions of weak and chiral multiple categories can be found in Part I, or - briefly reviewed - in Part II, Section 1. Here we only give a sketch of them, while recalling the notation we are using.

The two-valued index α (or β) varies in the set $2 = \{0, 1\}$, also written as $\{-, +\}$.

A *multi-index* \mathbf{i} is a finite subset of \mathbb{N} , possibly empty. Writing $\mathbf{i} \subset \mathbb{N}$ it is understood that \mathbf{i} is finite; writing $\mathbf{i} = \{i_1, \dots, i_n\}$ it is understood that \mathbf{i} has n *distinct* elements, written in the natural order $i_1 < i_2 < \dots < i_n$; the integer $n \geq 0$ is called the *dimension* of \mathbf{i} . We write:

$$\begin{aligned} \mathbf{i}j = j\mathbf{i} &= \mathbf{i} \cup \{j\} && (\text{for } j \in \mathbb{N} \setminus \mathbf{i}), \\ \mathbf{i}|j &= \mathbf{i} \setminus \{j\} && (\text{for } j \in \mathbf{i}). \end{aligned} \tag{2}$$

For a weak multiple category A, the set of \mathbf{i} -cells $A_{\mathbf{i}}$ is written as A_* , A_i , A_{ij} when \mathbf{i} is \emptyset , $\{i\}$, $\{i, j\}$ respectively. Faces and degeneracies, satisfying the *multiple relations*, are denoted as

$$\partial_j^\alpha : X_{\mathbf{i}} \rightarrow X_{\mathbf{i}|j}, \quad e_j : X_{\mathbf{i}|j} \rightarrow X_{\mathbf{i}}. \tag{3}$$

The *transversal direction* $i = 0$ is set apart from the positive, or *geometric*, directions. For a *positive multi-index* $\mathbf{i} = \{i_1, \dots, i_n\} \subset \mathbb{N} \setminus \{0\}$ the *augmented multi-index* $0\mathbf{i} = \{0, i_1, \dots, i_n\}$ has dimension $n + 1$, but both \mathbf{i}

and $0i$ have *degree* n . An i -cell $x \in A_i$ of A is also called an i -cube, while a $0i$ -cell $f \in A_{0i}$ is viewed as an i -map $f: x \rightarrow_0 y$ (also written as $f: x \rightarrow y$), where $x = \partial_0^- f$ and $y = \partial_0^+ f$. Composition in direction 0 is categorical (and generally realised by ordinary composition of mappings); it is written as $gf = f +_0 g$, with identities $1_x = \text{id}(x) = e_0(x)$. The *transversal category* $\text{tv}_i(A)$ consists of the i -cubes and i -maps of A , with transversal composition and identities.

On the other hand, composition of i -cubes and i -maps in a *positive* direction $i \in \mathbf{i}$ (often realised by pullbacks, pushouts, tensor products, etc.) is written in additive notation

$$\begin{aligned} x +_i y & \qquad \qquad \qquad (\partial_i^+ x = \partial_i^- y), \\ f +_i g: x +_i y \rightarrow x' +_i y' \quad (f: x \rightarrow x', g: y \rightarrow y', \partial_i^+ f = \partial_i^- g). \end{aligned} \quad (4)$$

These operations are *categorical* and *interchangeable* up to transversally-invertible comparisons (for $0 < i < j$, see I.3.2)

$$\begin{aligned} \lambda_i x: (e_i \partial_i^- x) +_i x & \rightarrow_0 x & \text{(left } i\text{-unitor)}, \\ \rho_i x: x +_i (e_i \partial_i^+ x) & \rightarrow_0 x & \text{(right } i\text{-unitor)}, \\ \kappa_i(x, y, z): x +_i (y +_i z) & \rightarrow_0 (x +_i y) +_i z & \text{(} i\text{-associator)}, \\ \chi_{ij}(x, y, z, u): (x +_i y) +_j (z +_i u) & \rightarrow_0 (x +_j z) +_i (y +_j u) & \text{(} ij\text{-interchanger)}. \end{aligned} \quad (5)$$

The comparisons are natural with respect to transversal maps; λ_i , ρ_i and κ_i are *special in direction* i (i.e. their i -faces are transversal identities), while χ_{ij} is *special in both directions* i, j ; all of them commute with ∂_k^α for $k \neq i$ (or $k \neq i, j$ in the last case). Finally the comparisons must satisfy various conditions of coherence, listed in I.3.3 and I.3.4.

More generally for a *chiral multiple category* A the ij -interchangers χ_{ij} are not assumed to be invertible (see I.3.7).

1.2 Cubical spans and cospans

Weak multiple categories generalise *weak cubical categories* and *weak symmetric cubical categories*; the latter were introduced in [G1] for studying higher cobordism, and give our main examples of weak multiple categories

of infinite dimension. We begin by recalling two instances in an informal, incomplete way.

The weak symmetric cubical category $\text{Span}(\mathbf{C})$ of cubical spans (or $\omega\text{Span}(\mathbf{C})$) was constructed in [G1] over a category \mathbf{C} with (a fixed choice of) pullbacks. An n -cube is a functor $x: \mathbb{V}^n \rightarrow \mathbf{C}$, where \mathbb{V} is the formal-span category

$$\begin{array}{ccccc}
 0 & \longleftarrow & u & \longrightarrow & 1 & \mathbb{V} \\
 \\
 (0, 0) & \longleftarrow & (u, 0) & \longrightarrow & (1, 0) & \bullet \xrightarrow{1} \\
 \uparrow & & \uparrow & & \uparrow & \downarrow 2 \\
 (0, u) & \longleftarrow & (u, u) & \longrightarrow & (1, u) & \\
 \downarrow & & \downarrow & & \downarrow & \\
 (0, 1) & \longleftarrow & (u, 1) & \longrightarrow & (1, 1) & \mathbb{V}^2.
 \end{array} \tag{6}$$

(Identities and composites are understood.) An n -map, or transversal map of n -cubes, is a natural transformation $f: x \rightarrow y: \mathbb{V}^n \rightarrow \mathbf{C}$ of such functors; these maps form the category $\text{Span}_n(\mathbf{C}) = \mathbf{Cat}(\mathbb{V}^n, \mathbf{C})$, with composition written as gf and identities $1_x = \text{id}(x)$. There are obvious *geometric faces and degeneracies* (satisfying the cubical relations)

$$\begin{aligned}
 \partial_i^\alpha: \text{Span}_n(\mathbf{C}) &\rightarrow \text{Span}_{n-1}(\mathbf{C}), \\
 e_i: \text{Span}_{n-1}(\mathbf{C}) &\rightarrow \text{Span}_n(\mathbf{C}) \quad (i = 1, \dots, n; \alpha = \pm).
 \end{aligned} \tag{7}$$

Moreover there are *geometric* composition laws: the i -concatenation $x +_i y$ is defined for i -consecutive n -cubes ($i = 1, \dots, n; \partial_i^+ x = \partial_i^- y$), and constructed with pullbacks; it is categorical up to invertible n -maps (*unitors* and *associators*); similarly we have the i -concatenation $f +_i g$ of i -consecutive n -maps. All pairs of composition laws have a strict interchange.

Viewing $\text{Span}(\mathbf{C})$ as a weak multiple category (of cubical type), an n -cube $x: \mathbb{V}^n \rightarrow \mathbf{C}$ is viewed as an \mathbf{i} -cube, for every positive multi-index $\mathbf{i} = \{i_1, \dots, i_n\}$ of dimension $n \geq 0$; an n -map is viewed as an \mathbf{i} -map.

The 2-dimensional and 3-dimensional truncations of $\text{Span}(\mathbf{C})$ are written as:

$$\text{Span}(\mathbf{C}) = 2\text{Span}(\mathbf{C}), \quad 3\text{Span}(\mathbf{C}). \tag{8}$$

The weak double category $\text{Span}(\mathbf{C})$ was studied in our series [GP1] - [GP4]: its horizontal and vertical arrows are ordinary arrows and spans of \mathbf{C} ,

respectively, while a double cell is a morphism of spans. The 3-dimensional truncation $3\text{Span}(\mathbf{C})$ consists of \mathbf{i} -cells for $\mathbf{i} \subset \mathbf{3} = \{0, 1, 2\}$ (or i -cubes and i -maps for $i < 3$, in the cubical framework).

Similarly one can find in [G1] the construction of the weak symmetric cubical category $\text{Cosp}(\mathbf{C})$ of *cubical cospans* over a category \mathbf{C} with (a fixed choice of) pushouts. An n -cube is now a functor $x: \Lambda^n \rightarrow \mathbf{C}$, where $\Lambda = \vee^{\text{op}}$ is the formal-cospan category $0 \rightarrow u \leftarrow 1$; again, a transversal map of n -cubes is a natural transformation of such functors. $\text{Cosp}(\mathbf{C})$ is *transversally dual* to $\text{Span}(\mathbf{C}^{\text{op}})$.

The 2-dimensional and 3-dimensional truncations are written as:

$$\mathbf{Cosp}(\mathbf{C}) = \mathbf{2Cosp}(\mathbf{C}), \quad \mathbf{3Cosp}(\mathbf{C}). \quad (9)$$

1.3 The chiral case

Chiral multiple categories of *non-cubical type* are constructed in [GP7] and Part I, Section 4.

For instance, if the category \mathbf{C} has pullbacks and pushouts, the weak double category $\text{Span}(\mathbf{C})$, of arrows and spans of \mathbf{C} , can be ‘amalgamated’ with the weak double category $\text{Cosp}(\mathbf{C})$, of arrows and cospans of \mathbf{C} , to form a 3-dimensional structure: the chiral triple category $\text{SC}(\mathbf{C})$ whose arrows in direction 0, 1 and 2 are the arrows, spans and cospans of \mathbf{C} , *in this order* (as required by the 12-interchanger).

The highest cubes, of type $\{1, 2\}$, are functors $x: \vee \times \Lambda \rightarrow \mathbf{C}$, the highest (3-dimensional) cells are the natural transformations of the latter

$$\begin{array}{ccccc}
 (0, 0) & \longleftarrow & (u, 0) & \longrightarrow & (1, 0) & & \bullet & \xrightarrow{1} \\
 \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
 (0, u) & \longleftarrow & (u, u) & \longrightarrow & (1, u) & & & \\
 \uparrow & & \uparrow & & \uparrow & & & \\
 (0, 1) & \longleftarrow & (u, 1) & \longrightarrow & (1, 1) & & \vee \times \Lambda &
 \end{array} \quad (10)$$

Here 0-composition works by ordinary composition in \mathbf{C} , 1-composition by composing spans (with pullbacks) and 2-composition by composing cospans (with pushouts).

Higher dimensional examples, like $S_p C_q(\mathbf{C})$, $S_p C_\infty(\mathbf{C})$ and $S_{-\infty} C_\infty(\mathbf{C})$ (and the corresponding *left-chiral* cases) can be found in I.4.4; note that

$S_{-\infty}C_{\infty}(\mathbf{C})$ is indexed by *all integers*, with spans in each negative direction, ordinary arrows in direction 0 and cospans in positive directions.

1.4 A double adjunction

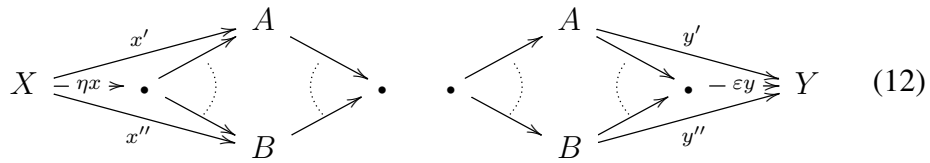
Let \mathbf{C} be a category with distinguished pullbacks and pushouts. For the sake of simplicity we assume that the distinguished pullback (resp. pushout) of an identity along any map is an identity.

The weak double categories $\mathbb{S}\text{pan}(\mathbf{C})$ and $\mathbb{C}\text{osp}(\mathbf{C})$ of spans and cospans of \mathbf{C} are linked by an obvious colax/lax adjunction

$$\begin{aligned} F: \mathbb{S}\text{pan}(\mathbf{C}) &\overset{\leftarrow}{\underset{\rightarrow}{\rightleftarrows}} \mathbb{C}\text{osp}(\mathbf{C}) : G, \\ \eta: 1 &\dashrightarrow GF, \quad \varepsilon: FG \dashrightarrow 1, \end{aligned} \tag{11}$$

that we describe here *in an informal way*. (Writing $\eta: 1 \dashrightarrow GF$ and $\varepsilon: FG \dashrightarrow 1$ is an abuse of notation, since the comparisons of F and G have conflicting directions and cannot be composed. The precise definition of a colax/lax adjunction of weak double categories can be found in [GP2]; but the reader will find here its multiple extension, in Section 4, and can easily recover the truncated notion.)

At the basic level of $\text{tv}_*(\mathbb{S}\text{pan}(\mathbf{C})) = \text{tv}_*(\mathbb{C}\text{osp}(\mathbf{C})) = \mathbf{C}$ everything is an identity. At the level 1 (of 1-cubes and 1-maps) F operates by pushout and G by pullback; the special transversal 1-maps $\eta x: x \rightarrow GFx$ and $\varepsilon y: FGy \rightarrow y$ are obvious (for a span $x = (x', x'')$ and a cospan $y = (y', y'')$ with 1-faces A and B):



The triangle identities are plainly satisfied:

$$\varepsilon(Fx).F(\eta x) = \text{id}(Fx), \quad G(\varepsilon y).\eta(Gy) = \text{id}(Gy).$$

Finally it is easy to check that F is, in a natural way, a colax double functor (while G is lax). The comparison cell

$$\underline{F}(x, y): F(x +_1 y) \rightarrow Fx +_1 Fy$$

for concatenation is given by the natural map from the pushout of $x +_1 y = (x'z', y''z'')$ to the cospan $Fx +_1 Fy$ (with vertex Z' in the diagram below)

$$\begin{array}{ccccc}
 & & & A & \\
 & & & \nearrow & \\
 & & x' & & \\
 & & X & & \bullet \\
 & & \searrow & & \nearrow \\
 & & x'' & & B \\
 & & & & \searrow \\
 & & & & \bullet \\
 & & & & \nearrow \\
 & & & & Z' \\
 Z & \nearrow & & & \\
 & z' & & & \\
 & \searrow & & & \\
 & z'' & & & \\
 & & Y & \nearrow & \\
 & & y' & & \\
 & & \searrow & & \\
 & & y'' & & C
 \end{array} \quad (13)$$

Since we agreed to follow the unitarity constraint for the choice of pull-backs and pushouts in \mathbf{C} , the adjunction is *unitary*, in the sense that this property holds for the weak double categories $\mathbb{S}\text{pan}(\mathbf{C})$, $\mathbb{C}\text{osp}(\mathbf{C})$ and the colax/lax double functors F, G . It is also interesting to note that the restricted adjunction at the \star -level

$$F_* : \mathbf{C} \rightleftarrows \mathbf{C} : G_* \quad \eta_* : 1 \rightarrow G_* F_*, \quad \varepsilon_* : F_* G_* \rightarrow 1, \quad (14)$$

is composed of identity functors and identity transformations.

We also remark that the natural transformations $F\eta, \varepsilon F, \eta G, G\varepsilon$ at level 1 are invertible (which means that the ordinary adjunction at level 1 is idempotent: see [AT, LS]).

1.5 A multiple adjunction

Following [G3], the unitary colax double functor $F : \mathbb{S}\text{pan}(\mathbf{C}) \rightarrow \mathbb{C}\text{osp}(\mathbf{C})$ can be extended to a unitary colax multiple functor of cubical type

$$F : \mathbb{S}\text{pan}(\mathbf{C}) \rightarrow \mathbb{C}\text{osp}(\mathbf{C}).$$

For instance, let us take a 2-dimensional span $x \in \mathbb{S}\text{pan}_1(\mathbf{C})$ indexed by $\mathbf{i} = \{i, j\}$, as in the left diagram below

$$\begin{array}{ccc}
 x_{00} & \leftarrow x_{u0} & \rightarrow x_{10} \\
 \uparrow & & \uparrow \\
 x_{0u} & \leftarrow x_{uu} & \rightarrow x_{1u} \\
 \downarrow & & \downarrow \\
 x_{01} & \leftarrow x_{u1} & \rightarrow x_{11}
 \end{array}
 \quad
 \begin{array}{ccccc}
 x_{00} & \longrightarrow & F(\partial_j^- x)(u) & \longleftarrow & x_{10} \\
 \downarrow & & \downarrow & & \downarrow \\
 F(\partial_i^- x)(u) & \longrightarrow & \text{colim}(x) & \longleftarrow & F(\partial_i^+ x)(u) \\
 \uparrow & & \uparrow & & \uparrow \\
 x_{01} & \longrightarrow & F(\partial_j^+ x)(u) & \longleftarrow & x_{11}
 \end{array} \quad (15)$$

The 2-dimensional cospan $F(x) = F_{\mathbf{i}}(x)$ is constructed at the right hand, with the pushouts $F(\partial_i^\alpha x)$, $F(\partial_j^\alpha x)$ of the four faces and, in the central vertex, the colimit of the whole diagram $x: \mathcal{V}^2 \rightarrow \mathbf{C}$. (The latter can be constructed in \mathbf{C} as a pushout of pushouts; a general characterisation of the dual topic, limits ‘generated’ by pullbacks, can be found in [P1].)

One proceeds in a similar way, defining $F_{\mathbf{i}}$ for a positive multi-index $\mathbf{i} = \{i_1, \dots, i_n\}$ of degree n , after all instances of degree $n - 1$

$$\begin{aligned} \partial_i^\alpha(F_{\mathbf{i}}(x)) &= F_{\mathbf{i}|i}(\partial_i^\alpha x) & \text{for } \alpha = \pm \text{ and } i \in \mathbf{i}, \\ F_{\mathbf{i}}(x)(\underline{u}) &= \text{colim}(x), & \text{where } \underline{u} = (u, \dots, u) \in \mathcal{V}^n. \end{aligned} \quad (16)$$

The definition of F on transversal \mathbf{i} -maps is obvious, as well as the comparison cells for the i -directed concatenation $\underline{F}_i(x, y): F(x +_i y) \rightarrow Fx +_i Fy$.

The unitary lax double functor $G: \mathbf{Cosp}(\mathbf{C}) \rightarrow \mathbf{Span}(\mathbf{C})$ is similarly extended, using distinguished limits instead of colimits, and gives a unitary lax multiple functor $G: \mathbf{Cosp}(\mathbf{C}) \rightarrow \mathbf{Span}(\mathbf{C})$ of cubical type.

One extends the unit $\eta: 1 \dashrightarrow GF$ by a similar inductive procedure:

$$\begin{aligned} \partial_i^\alpha(\eta_{\mathbf{i}}(x)) &= \eta_{\mathbf{i}|i}(\partial_i^\alpha x) & (\alpha = \pm, i \in \mathbf{i}), \\ (\eta_{\mathbf{i}}x)(\underline{u}): x(\underline{u}) &\rightarrow (G_{\mathbf{i}}F_{\mathbf{i}}x)(\underline{u}) = \lim(F_{\mathbf{i}}x), \end{aligned} \quad (17)$$

where the map $(\eta_{\mathbf{i}}x)(\underline{u})$ is given by the universal property of $\lim(F_{\mathbf{i}}x)$ as the limit of the cubical cospan $F_{\mathbf{i}}x: \wedge^n \rightarrow \mathbf{C}$.

Analogously for the counit $\varepsilon: FG \dashrightarrow 1$. The triangular identities hold.

1.6 Chiral examples

The colax/lax adjunction of weak triple categories of cubical type

$$F: \mathbf{3Span}(\mathbf{C}) \rightleftarrows \mathbf{3Cosp}(\mathbf{C}) : G, \quad (18)$$

can be factorised through the chiral triple category $\mathbf{SC}(\mathbf{C})$ of *spans and cospans of \mathbf{C}* , obtaining two colax/lax adjunctions of chiral triple categories (no longer of cubical type)

$$\begin{aligned} F' : \mathbf{3Span}(\mathbf{C}) &\rightleftarrows \mathbf{SC}(\mathbf{C}) : G', \\ F'' : \mathbf{SC}(\mathbf{C}) &\rightleftarrows \mathbf{3Cosp}(\mathbf{C}) : G''. \end{aligned} \quad (19)$$

Here the functor $F' : \mathbf{3Span}(\mathbf{C}) \rightarrow \mathbf{SC}(\mathbf{C})$ acts on a 12-cube x

- by pushout on the three 2-directed spans of x ,
- as the identity on the two 1-directed boundary spans $\partial_2^\alpha(x)$,
- by induced morphisms on the middle 1-directed span.

On the other hand the functor $G' : \mathbf{SC}(\mathbf{C}) \rightarrow \mathbf{3Span}(\mathbf{C})$ acts by pullback on the three (2-directed) cospans of x , as the identity on the (1-directed) boundary spans $\partial_2^\alpha(x)$ and by induced morphisms on the middle span. Similarly for F'' and G'' .

One can also factorise the adjunction (18) through the *left* chiral triple category $\mathbf{CS}(\mathbf{C})$ of *cospans and spans*, obtaining two colax/lax adjunctions of left chiral triple categories.

Similarly, the multiple adjunction constructed in 1.5 can be factorised through any right chiral multiple category $\mathbf{S}_p\mathbf{C}_\infty(\mathbf{C})$, *or* through any left chiral multiple category $\mathbf{C}_p\mathbf{S}_\infty(\mathbf{C})$.

However, in infinite dimension, one may prefer to consider a more symmetric situation, starting from a colax/lax adjunction of weak multiple categories *indexed by the ordered set* \mathbb{Z} of integers (pointed at 0)

$$F : \mathbb{Z}\mathbf{Span}(\mathbf{C}) \rightleftarrows \mathbb{Z}\mathbf{Cosp} : G. \quad (20)$$

This can be factorised through the chiral multiple category $\mathbf{S}_{-\infty}\mathbf{C}_\infty(\mathbf{C})$, obtaining two colax/lax adjunctions of ‘unbounded’ chiral multiple categories

$$\begin{aligned} F' : \mathbb{Z}\mathbf{Span}(\mathbf{C}) &\rightleftarrows \mathbf{S}_{-\infty}\mathbf{C}_\infty(\mathbf{C}) : G', \\ F'' : \mathbf{S}_{-\infty}\mathbf{C}_\infty(\mathbf{C}) &\rightleftarrows \mathbb{Z}\mathbf{Cosp}(\mathbf{C}) : G''. \end{aligned} \quad (21)$$

1.7 Other examples

Now we start from an ordinary adjunction $F \dashv G$

$$F : \mathbf{X} \rightleftarrows \mathbf{A} : G, \quad \eta : 1 \rightarrow GF, \quad \varepsilon : FG \rightarrow 1, \quad (22)$$

between categories with (a choice of) pullbacks. This can be extended in a natural way to a unitary *colax/pseudo* adjunction between the weak multiple categories of higher spans (of cubical type)

$$\mathbf{Span}(F) : \mathbf{Span}(\mathbf{X}) \rightleftarrows \mathbf{Span}(\mathbf{A}) : \mathbf{Span}(G). \quad (23)$$

In fact there is an obvious 2-functor

$$\text{Span}: \text{Cat}_{\text{pb}} \rightarrow \text{CxCmc}, \quad (24)$$

defined on the full sub-2-category of Cat containing all categories with (a choice of) pullbacks, with values in the 2-category of chiral multiple categories, colax functors and their transversal transformations (see 2.1).

It sends a category \mathbf{C} with pullbacks to the chiral multiple category $\text{Span}(\mathbf{C})$ (actually a weak multiple category of symmetric cubical type).

For a functor $F: \mathbf{X} \rightarrow \mathbf{A}$ (between categories with pullbacks), $\text{Span}(F)$, also written as F for brevity, simply acts by computing F over the diagrams of \mathbf{X} that form i -cubes and i -maps; formally, over an i -map $f: x \rightarrow y: \vee^n \rightarrow \mathbf{X}$, $F(f): F(x) \rightarrow F(y)$ is the composite

$$F.f: F.x \rightarrow F.y: \vee^n \rightarrow \mathbf{A}.$$

This extension is, in a natural way, a unitary *colax* functor, since identities of \mathbf{X} go to identities of \mathbf{A} and a composition $x +_i y$ of two spans $x, y: \vee \rightarrow \mathbf{X}$ (in any direction $i > 0$) gives rise to a diagram in \mathbf{X} and a diagram in \mathbf{A}

$$\begin{array}{ccc}
 \begin{array}{c}
 \begin{array}{ccccc}
 & & & X_1 & \\
 & & & \nearrow^{x'} & \\
 & & X & & \\
 \begin{array}{c} \nearrow^{z'} \\ \searrow^{z''} \end{array} & & & & \\
 P & & & & \\
 & & & X_2 & \\
 & & & \nearrow^{y'} & \\
 & & Y & & \\
 & & \searrow^{y''} & & \\
 & & & & X_3
 \end{array}
 \end{array}
 & \xrightarrow{F} &
 \begin{array}{c}
 \begin{array}{ccccc}
 & & & FX_1 & \\
 & & & \nearrow^{Fx'} & \\
 & & FX & & \\
 \begin{array}{c} \nearrow^{Fz'} \\ \searrow^{Fz''} \end{array} & & & & \\
 FP & \xrightarrow{a} & Q & & \\
 & & & & \\
 & & & FX_2 & \\
 & & & \nearrow^{Fy'} & \\
 & & FY & & \\
 & & \searrow^{Fy''} & & \\
 & & & & FX_3
 \end{array}
 \end{array}
 \end{array} \quad (25)$$

where the comparison $\underline{F}_i(x, y): F(x +_i y) \rightarrow F(x) +_i F(y)$ is an i -special transversal map given by the \mathbf{A} -morphism $a: FP \rightarrow Q$ determined by the universal property of the pullback Q . Similarly we define $\underline{F}_i(x, y)$ for every i -composition of i -cubes. Note that $\text{Span}(F)$ is pseudo if (and only if) the functor $F: \mathbf{X} \rightarrow \mathbf{A}$ preserves pullbacks.

Finally, a natural transformation $\varphi: F \rightarrow F': \mathbf{X} \rightarrow \mathbf{A}$ yields a transversal transformation

$$\text{Span}(\varphi): \text{Span}(F) \rightarrow \text{Span}(F'): \text{Span}(\mathbf{X}) \rightarrow \text{Span}(\mathbf{A}), \quad (26)$$

that again will often be written as φ . On an i -cube $x: \mathbb{V}^n \rightarrow \mathbf{X}$, $\varphi x: F(x) \rightarrow_0 F'(x)$ is the composite of the functor $x: \mathbb{V}^n \rightarrow \mathbf{X}$ with the natural transformation $\varphi: F \rightarrow F': \mathbf{X} \rightarrow \mathbf{A}$. Concretely, the transversal i -map $\varphi x: F(x) \rightarrow_0 F'(x)$ has components $\varphi(x(t))$, for every vertex t of \mathbb{V}^n .

Now, letting the 2-functor $\text{Span}: \text{Cat}_{\text{pb}} \rightarrow \text{CxCmc}$ act on the adjunction (22), we get an adjunction of weak multiple categories with the properties stated above: $\text{Span}(F)$ is colax, $\text{Span}(G)$ is pseudo and both are unitary.

On the other hand, if \mathbf{X} and \mathbf{A} have pushouts, the adjunction (22) yields a *pseudo/lax* adjunction of weak multiple categories

$$F: \text{Cosp}(\mathbf{X}) \rightleftarrows \text{Cosp}(\mathbf{A}) : G. \quad (27)$$

Finally, if \mathbf{X} and \mathbf{A} have pullbacks and pushouts, we can extend (22) to a colax/lax adjunction of chiral triple categories

$$F: \text{SC}(\mathbf{X}) \rightleftarrows \text{SC}(\mathbf{A}) : G, \quad (28)$$

or consider higher-dimensional extensions of ‘type’ $S_p C_q$, $S_p C_\infty$, $S_{-\infty} C_\infty$, etc. (see 1.3).

Note that, according to the analysis of [GP6], Section 5, F is a *colax-pseudo* morphism of chiral triple categories (i.e. colax for the 1-directed composition, realised by pullbacks, and pseudo for the 2-directed composition, realised by pushouts) while G is pseudo-lax.

2. The double category of lax and colax multiple functors

In the 2-dimensional case, weak double categories with lax and colax double functors and suitable double cells form a *strict* double category $\mathbb{D}bl$, a crucial structure introduced in [GP2] to define colax/lax double adjunctions - also recalled in Part I.

This construction was extended in [G3], Section 2, to form the strict *double* category $\mathbb{W}sc$ of weak symmetric cubical categories, lax and colax symmetric cubical functors (and suitable double cells), in order to define colax/lax adjunctions between weak symmetric cubical categories.

We now give a further extension, building the strict *double* category $\mathbb{C}mc$ of chiral multiple categories, lax and colax multiple functors, that will be

used below to define colax/lax adjunctions between chiral multiple categories.

Comma chiral multiple categories are also considered, extending again the cases of double and symmetric cubical categories, dealt with in [GP2, G3].

For $\mathbb{C}mc$ we follow the notation for double categories used in [GP1] - [GP4]: the horizontal and vertical compositions of cells are written as $(\pi | \rho)$ and $(\frac{\pi}{\sigma})$, or more simply as $\pi | \rho$ and $\pi \otimes \sigma$. Horizontal identities, of an object or a vertical arrow, are written as 1_A and 1_u ; vertical identities, of an object or a horizontal arrow, as 1_A^\bullet and 1_f^\bullet .

2.1 Lax cm-functors

A chiral multiple category is also called a *cm-category*, for short.

As defined in I.3.9, a *lax multiple functor* $F: A \rightarrow B$ between chiral multiple categories, or *lax cm-functor*, has components $F_i: A_i \rightarrow B_i$ for all multi-indices i (often written as F) that agree with all faces, 0-degeneracies and 0-composition. Moreover, for every positive multi-index i and $i \in i$, F is equipped with i -special comparison i -maps \underline{F}_i that agree with faces

$$\begin{aligned}
 \underline{F}_i(x) &: e_i F(x) \rightarrow_0 F(e_i x) && (\text{for } x \text{ in } A_{i|i}), \\
 \underline{F}_i(x, y) &: F(x) +_i F(y) \rightarrow_0 F(z) && (\text{for } z = x +_i y \text{ in } A_i), \\
 \partial_j^\alpha \underline{F}_i(x) &= \underline{F}_i(\partial_j^\alpha x) && (\text{for } j \neq i), \\
 \partial_j^\alpha \underline{F}_i(x, y) &= \underline{F}_i(\partial_j^\alpha x, \partial_j^\alpha y) && (\text{for } j \neq i).
 \end{aligned} \tag{29}$$

and satisfy some axioms. We write down the naturality conditions (Imf.1-2), frequently used below, while the coherence conditions (Imf.3-5) can be found in I.3.9.

(Imf.1) (*naturality of unit comparisons*) For every $i|j$ -map $f: x \rightarrow_0 y$ in A we have:

$$F(e_j f) \cdot \underline{F}_j(x) = \underline{F}_j(y) \cdot e_j(Ff): e_j F(x) \rightarrow_0 F(e_j y). \tag{30}$$

(Imf.2) (*naturality of composition comparisons*) For two j -consecutive i -maps $f: x \rightarrow_0 x'$ and $g: y \rightarrow_0 y'$ in A we have:

$$F(f +_j g) \cdot \underline{F}_j(x, y) = \underline{F}_j(x', y') \cdot (F(f) +_j F(g)). \tag{31}$$

A transversal transformation $h: F \rightarrow G: A \rightarrow B$ between lax multiple functors of chiral multiple categories consists of a face-consistent family of i -maps in B (its components), one for every positive multi-index i and every i -cube x in A

$$hx: F(x) \rightarrow_0 G(x), \quad h(\partial_j^\alpha x) = \partial_j^\alpha(hx), \quad (32)$$

under the axioms (trt.1) and (trt.2L), see I.3.9.

We have thus the 2-category $Lx\mathbf{Cmc}$ of cm -categories, lax cm -functors and their transversal transformations. Similarly one defines the 2-category $Cx\mathbf{Cmc}$, for the colax case where the comparisons of *colax cm-functors* have the opposite direction. A *pseudo cm-functor* is a lax cm -functor whose comparisons are invertible; it is made colax by the inverse comparisons.

2.2 The double category \mathbf{Cmc}

Lax and colax cm -functors do not compose well, since we cannot compose their comparisons. But they give the horizontal and vertical arrows of a *strict* double category \mathbf{Cmc} , crucial for our study, where ‘internal’ orthogonal adjunctions (recalled below, in Section 3) will provide our general notion of multiple adjunction (Section 4) while companion pairs amount to pseudo cm -functors (Section 5).

The objects of \mathbf{Cmc} are the cm -categories A, B, C, \dots ; its horizontal arrows are the *lax* cm -functors R, S, \dots ; its vertical arrows are the *colax* cm -functors F, G, \dots

A double cell $\pi: (F \begin{smallmatrix} R \\ S \end{smallmatrix} G)$

$$\begin{array}{ccc} A & \xrightarrow{R} & B \\ F \downarrow & \pi & \downarrow G \\ C & \xrightarrow{S} & D \end{array} \quad (33)$$

is - roughly speaking - a ‘transformation’ $\pi: GR \dashrightarrow SF$. But the composites GR and SF are neither lax nor colax: the coherence conditions of π require the individual knowledge of the four ‘functors’, including the comparison cells of each of them.

Precisely, the double cell π consists of the following data:

(a) two lax cm-functors $R: \mathbf{A} \rightarrow \mathbf{B}$ and $S: \mathbf{C} \rightarrow \mathbf{D}$, with comparisons as follows:

$$\begin{aligned} \underline{R}_i(x) &: e_i(Rx) \rightarrow_0 R(e_i x), & \underline{R}_i(x, y) &: Rx +_i Ry \rightarrow_0 R(x +_i y), \\ \underline{S}_i(x) &: e_i(Sx) \rightarrow_0 S(e_i x), & \underline{S}_i(x, y) &: Sx +_i Sy \rightarrow_0 S(x +_i y), \end{aligned}$$

(b) two colax cm-functors $F: \mathbf{A} \rightarrow \mathbf{C}$ and $G: \mathbf{B} \rightarrow \mathbf{D}$, with comparisons as follows:

$$\begin{aligned} \underline{F}_i(x) &: F(e_i x) \rightarrow_0 e_i(Fx), & \underline{F}_i(x, y) &: F(x +_i y) \rightarrow_0 Fx +_i Fy, \\ \underline{G}_i(x) &: G(e_i x) \rightarrow_0 e_i(Gx), & \underline{G}_i(x, y) &: G(x +_i y) \rightarrow_0 Gx +_i Gy, \end{aligned}$$

(c) a family of i -maps $\pi x: GR(x) \rightarrow_0 SF(x)$ of \mathbf{D} (for every i -cube x in \mathbf{A}), consistent with faces

$$\pi(\partial_i^\alpha x) = \partial_i^\alpha(\pi x). \quad (34)$$

These data have to satisfy the naturality condition (dc.1) and the coherence conditions (dc.2), (dc.3) (with respect to i -directed degeneracies and composition, respectively)

(dc.1) $SFf.\pi x = \pi y.GRf: GR(x) \rightarrow SF(y)$ (for $f: x \rightarrow y$ in $\text{tv}_i(\mathbf{A})$),

(dc.2) $S\underline{F}_i(x).\pi e_i(x).G\underline{R}_i(x) = \underline{S}_i F(x).e_i(\pi x).\underline{G}_i R(x)$ (for x in $A_{i|i}$),

$$\begin{array}{ccccc} Ge_i(Rx) & \xrightarrow{G\underline{R}_i(x)} & GR(e_i x) & \xrightarrow{\pi e_i(x)} & SF(e_i x) \\ \underline{G}_i R(x) \downarrow & & & & \downarrow S\underline{F}_i(x) \\ e_i GR(x) & \xrightarrow{e_i \pi(x)} & e_i SF(x) & \xrightarrow{\underline{S}_i F(x)} & S e_i F(x) \end{array}$$

(dc.3) $S\underline{F}_i(x, y).\pi z.G\underline{R}_i(x, y) = \underline{S}_i(Fx, Fy).(\pi x +_i \pi y).\underline{G}_i(Rx, Ry)$

(for $z = x +_i y$ in A_i),

$$\begin{array}{ccccc} G(Rx +_i Ry) & \xrightarrow{G\underline{R}_i(x, y)} & GR(z) & \xrightarrow{\pi z} & SF(z) \\ \underline{G}_i(Rx, Ry) \downarrow & & & & \downarrow S\underline{F}_i(x, y) \\ GRx +_i GRy & \xrightarrow{\pi x +_i \pi y} & SFx +_i SFy & \xrightarrow{\underline{S}_i(Fx, Fy)} & S(Fx +_i Fy) \end{array}$$

The horizontal composition $(\pi | \rho)$ and the vertical composition $\pi \otimes \sigma = (\frac{\pi}{\sigma})$ of double cells are both defined *via the composition of transversal maps* (in a cm-category)

$$\begin{array}{ccccc}
 A & \xrightarrow{R} & \bullet & \xrightarrow{R'} & \bullet \\
 F \downarrow & & \downarrow G & & \downarrow H \\
 & \pi & & \rho & \\
 \bullet & \xrightarrow{S} & \bullet & \xrightarrow{S'} & \bullet \\
 F' \downarrow & & \downarrow G' & & \downarrow H' \\
 & \sigma & & \tau & \\
 \bullet & \xrightarrow{T} & \bullet & \xrightarrow{T'} & \bullet
 \end{array} \tag{35}$$

$$\begin{aligned}
 (\pi | \rho)(x) &= S'\pi x.\rho R x: HR'R(x) \rightarrow_0 S'GR(x) \rightarrow_0 S'SF(x), \\
 (\frac{\pi}{\sigma})(x) &= \sigma F x.G'\pi x: G'GR(x) \rightarrow_0 G'SF(x) \rightarrow_0 TF'F(x),
 \end{aligned} \tag{36}$$

(for x in A). We verify below, in Theorem 2.3, that these compositions are well-defined and satisfy the axioms of a double category.

Within $\mathbb{C}mc$, we have the strict 2-category $Lx\mathbb{C}mc$ of *cm-categories, lax cm-functors and transversal transformations*: namely, $Lx\mathbb{C}mc$ is the restriction of $\mathbb{C}mc$ to trivial vertical arrows. Symmetrically the strict 2-category $Cx\mathbb{C}mc$, whose arrows are the colax cm-functors, also lies in $\mathbb{C}mc$.

As in I.1.2 (for weak double categories), we can also note that a double cell $\pi: (F \begin{smallmatrix} R \\ \downarrow \\ 1 \end{smallmatrix})$ gives a notion of *transversal transformation* $\pi: R \dashrightarrow F: A \rightarrow B$ from a lax to a colax functor, while a double cell $\pi: (1 \begin{smallmatrix} \downarrow \\ S \\ G \end{smallmatrix})$ gives a notion of *transversal transformation* $\pi: G \dashrightarrow S: A \rightarrow B$ from a colax to a lax functor. Moreover, for a fixed pair A, B of chiral multiple categories, all the transversal transformations between lax and colax functors (of the four possible kinds) compose, forming a category $\{A, B\}$ whose objects are the lax *and* the colax functors $A \rightarrow B$.

2.3 Theorem

The structure $\mathbb{C}mc$, as defined above, is indeed a strict double category.

Proof. The argument is much the same as for $\mathbb{D}bl$ in [GP2] and for $\mathbb{W}sc$ in [G3].

First, to show that the double cells defined in (36) are indeed coherent, we verify the condition (dc.3) for $(\pi | \rho)$, with respect to a concatenation

$z = x +_i y$ in A. Our property amounts to the commutativity of the outer diagram below, formed of transversal maps (omitting the index i in $+_i$ and all comparisons \underline{R}_i etc.)

$$\begin{array}{ccccc}
 HR'Rz & \xrightarrow{\rho Rz} & S'GRz & \xrightarrow{S'\pi z} & S'SFz \\
 \uparrow \underline{HR'R} & & \uparrow \underline{S'GR} & & \downarrow \underline{S'SF} \\
 HR'(Rx + Ry) & \xrightarrow{\rho(Rx+Ry)} & S'G(Rx + Ry) & & S'S(Fx + Fy) \\
 \uparrow \underline{HR'R} & & \downarrow \underline{S'GR} & & \uparrow \underline{S'SF} \\
 H(R'Rx + R'Ry) & & S'(GRx + GRy) & \xrightarrow{S'(\pi x + \pi y)} & S'(SFx + SFy) \\
 \downarrow \underline{HR'R} & & \uparrow \underline{S'GR} & & \uparrow \underline{S'SF} \\
 HR'Rx + HR'Ry & \xrightarrow{\rho Rx + \rho Ry} & S'GRx + S'GRy & \xrightarrow{S'\pi x + S'\pi y} & S'SFx + S'SFy
 \end{array}$$

Indeed, the two hexagons commute applying (dc.3) to π and ρ , respectively. The upper square commutes by naturality of ρ on $\underline{R}_i(x, y)$; the lower one by axiom (Imf.2) (see 2.1) on the lax functor S' , with respect to the \mathbf{i} -maps $\pi x: GR(x) \rightarrow_0 SF(x)$ and $\pi y: GR(y) \rightarrow_0 SF(y)$

$$S'(\pi x +_i \pi y) \cdot \underline{S}'_i(GR(x), GR(y)) = \underline{S}'_i(SF(x), SF(y)) \cdot (S'(\pi x) +_i S'(\pi y)).$$

Now, both composition laws of double cells have been defined, in (36), *via the composition of transversal maps* (in a cm-category), and therefore are strictly unitary and associative.

Finally, to verify the middle-four interchange law on the four double cells of diagram (35), we compute the compositions $(\pi | \rho) \otimes (\sigma | \tau)$ and $(\pi \otimes \sigma) | (\rho \otimes \tau)$ on an \mathbf{i} -cube x , and we obtain the two transversal maps $H'HR'Rx \rightarrow_0 T'TF'Fx$ of the upper or lower path in the following diagram

$$\begin{array}{ccccc}
 H'HR'Rx & \xrightarrow{H'\rho Rx} & H'S'GRx & \xrightarrow{H'S'\pi x} & H'S'SFx \\
 & & \downarrow \tau GRx & & \downarrow \tau SFx \\
 & & T'G'GRx & \xrightarrow{T'G'\pi x} & T'G'SFx & \xrightarrow{T'\sigma Fx} & T'TF'Fx
 \end{array}$$

But these two composites coincide because the square commutes: a consequence of the naturality of τ on the transversal map $\pi x: GR(x) \rightarrow SF(x)$, by axiom (dc.1) for the double cell τ . \square

2.4 Comma cm-categories

Comma double categories [GP2] also have a natural extension to the multiple case.

Given a *colax* cm-functor $F: A \rightarrow C$ and a *lax* cm-functor $R: X \rightarrow C$ with the same codomain, we can construct the *comma cm-category* $F \downarrow R$, where the projections P and Q are strict cm-functors and π is a double cell of $\mathbb{C}mc$

$$\begin{array}{ccc}
 F \downarrow R & \xrightarrow{P} & A \\
 Q \downarrow & \pi & \downarrow F \\
 X & \xrightarrow{R} & C
 \end{array} \quad (37)$$

An *i*-cube of $F \downarrow R$ is a triple $(a, x; c: Fa \rightarrow_0 Rx)$ where a is an *i*-cube of A , x is an *i*-cube of X and c is an *i*-map of C . An *i*-map $(h, f): (a, x; c) \rightarrow_0 (a', x'; c')$ comes from a pair of *i*-maps $h: a \rightarrow_0 a'$ (in A) and $f: x \rightarrow_0 x'$ (in X) that give in C a commutative square of transversal maps

$$\begin{array}{ccc}
 Fa & \xrightarrow{c} & Rx \\
 Fh \downarrow & & \downarrow Rf \\
 Fa' & \xrightarrow{c'} & Rx'
 \end{array} \quad Rf \cdot c = c' \cdot Fh. \quad (38)$$

Faces and transversal composition are obvious. The degeneracies are defined using the fact that F is *colax* and R is *lax*:

$$e_i(a, x; c: Fa \rightarrow_0 Rx) = (e_i a, e_i x; \underline{R}_i(x) \cdot e_i c \cdot \underline{F}_i(a)). \quad (39)$$

Similarly the *i*-concatenation is defined as follows

$$\begin{aligned}
 (a, x; c) +_i (b, y; d) &= (a +_i b, x +_i y; u: F(a +_i b) \rightarrow_0 R(x +_i y)), \\
 u &= \underline{R}_i(x, y) \cdot (c +_i d) \cdot \underline{F}_i(a, b): \\
 F(a +_i b) &\rightarrow_0 Fa +_i Fb \rightarrow_0 Rx +_i Ry \rightarrow_0 R(x +_i y).
 \end{aligned} \quad (40)$$

The associativity comparison for the *i*-composition of three *i*-consecutive *i*-cubes of $F \downarrow R$

$$(a, x; c), \quad (a', x'; c'), \quad (a'', x''; c''),$$

is given by the pair $(\kappa_i(\mathbf{a}), \kappa_i(\mathbf{x}))$ of associativity \mathbf{i} -isomorphisms for our two triples of \mathbf{i} -cubes, namely $\mathbf{a} = (a, a', a'')$ in A and $\mathbf{x} = (x, x', x'')$ in X

$$\begin{aligned} (\kappa_i(\mathbf{a}), \kappa_i(\mathbf{x})) : (a, x; c) +_i ((a', x'; c') +_i (a'', x''; c'')) \\ \rightarrow_0 ((a, x; c) +_i (a', x'; c')) +_i (a'', x''; c''). \end{aligned} \quad (41)$$

The coherence of this \mathbf{i} -map, as in diagram (38) above, is proved in the lemma below.

Similarly one constructs the unitors λ_i, ρ_i and the interchangers χ_{ij} of $F \downarrow R$, using those of A and X .

Finally, the strict \mathbf{cm} -functors P and Q are projections, while the component of the ‘transformation’ π on the \mathbf{i} -cube $(a, x; c)$ of $F \downarrow R$ is the transversal map:

$$\pi(a, x; c) = c : Fa \rightarrow Rx. \quad (42)$$

2.5 Lemma

The pair $(\kappa(\mathbf{a}), \kappa(\mathbf{x}))$ is indeed an \mathbf{i} -map of $F \downarrow R$, with domain and codomain as specified in (41).

Proof. First, these two \mathbf{i} -cubes of $F \downarrow R$ can be written as:

$$\begin{aligned} (a, x; c) + ((a', x'; c') + (a'', x''; c'')) &= (a_1, x_1; c_1), \\ ((a, x; c) + (a', x'; c')) + (a'', x''; c'') &= (a_2, x_2; c_2), \end{aligned} \quad (43)$$

where $a_1, x_1, a_2, x_2, c_1, c_2$ are defined as follows (by commutative diagrams in the last two cases, and always leaving the index i understood for $+$, \underline{F} , \underline{R})

$$\begin{aligned} a_1 &= a + (a' + a''), & x_1 &= x + (x' + x''), \\ a_2 &= (a + a') + a'', & x_2 &= (x + x') + x'', \end{aligned} \quad (44)$$

$$\begin{array}{ccc} Fa_1 & \xrightarrow{c_1} & Rx_1 \\ \begin{array}{c} \downarrow \underline{F}(a, a'+a'') \\ Fa + F(a' + a'') \\ \downarrow 1 + \underline{F}(a', a'') \end{array} & & \begin{array}{c} \uparrow \underline{R}(x, x'+x'') \\ Rx + R(x' + x'') \\ \uparrow 1 + \underline{R}(x', x'') \end{array} \\ Fa + (Fa' + Fa'') & \xrightarrow{c + (c'+c'')} & Rx + (Rx' + Rx'') \end{array}$$

$$\begin{array}{ccc}
 Fa_2 & \xrightarrow{c_2} & Rx_2 \\
 \underline{F}(a+a', a'') \downarrow & & \uparrow \underline{R}(x+x', x'') \\
 F(a+a') + Fa'' & & R(x+x') + Rx'' \\
 \underline{F}(a, a')+1 \downarrow & & \uparrow \underline{R}(x, x')+1 \\
 (Fa + Fa') + Fa'' & \xrightarrow{(c+c')+c''} & (Rx + Rx') + Rx''
 \end{array}$$

Now our claim, i.e. the condition for $(\kappa(\mathbf{a}), \kappa(\mathbf{x}))$ expressed in diagram (38), amounts to

$$R\kappa(\mathbf{x}).c_1 = c_2.F\kappa(\mathbf{a}): Fa_1 \rightarrow Rx_2. \quad (45)$$

First, the coherence of the lax functor R with the associator κ gives (applying axiom (Imf.4) of I.3.9):

$$\begin{aligned}
 & R\kappa(\mathbf{x}).c_1 \\
 &= R\kappa(\mathbf{x}).\underline{R}(x, x' + x'').(1 + \underline{R}(x', x'')).(c + (c' + c'')). \\
 & \quad \quad \quad (1 + \underline{F}(a', a'')).\underline{F}(a, a' + a'') \\
 &= \underline{R}(x + x', x'').(\underline{R}(x, x') + 1).\kappa R(\mathbf{x}).(c + (c' + c'')). \\
 & \quad \quad \quad (1 + \underline{F}(a', a'')).\underline{F}(a, a' + a'').
 \end{aligned}$$

Second, the coherence of the colax functor F with κ gives (applying the corresponding axiom, with reversed comparisons \underline{F}):

$$\begin{aligned}
 & c_2.F\kappa(\mathbf{a}) \\
 &= \underline{R}(x + x', x'').(\underline{R}(x, x') + 1).((c + c') + c''). \\
 & \quad \quad \quad (\underline{F}(a, a') + 1).\underline{F}(a + a', a'').F\kappa(\mathbf{a}) \\
 &= \underline{R}(x + x', x'').(\underline{R}(x, x') + 1).((c + c') + c''). \\
 & \quad \quad \quad \kappa(F\mathbf{a}).(1 + \underline{F}(a', a'')).\underline{F}(a, a' + a'').
 \end{aligned}$$

Finally, condition (45) follows from the naturality of κ on the triple of transversal maps $(c, c', c''): Fa \rightarrow Rx$, which gives the commutative diagram

$$\begin{array}{ccc}
 Fa + (Fa' + Fa'') & \xrightarrow{\kappa(F\mathbf{a})} & (Fa + Fa') + Fa'' \\
 c+(c'+c'') \downarrow & & \downarrow (c+c')+c'' \\
 Rx + (Rx' + Rx'') & \xrightarrow{\kappa(R\mathbf{x})} & (Rx + Rx') + Rx''
 \end{array}$$

□

2.6 Theorem (Universal properties of commas)

(a) For a pair of lax cm-functors S, T and a cell φ as below (in $\mathbb{C}mc$) there is a unique lax cm-functor $L: Z \rightarrow F \downarrow R$ such that $S = PL$, $T = QL$ and $\varphi = (\psi | \pi)$ where the cell ψ is defined by the identity $1: QL \rightarrow T$ (a horizontal transformation of lax cm-functors)

$$\begin{array}{ccc}
 Z & \xrightarrow{S} & A \\
 \downarrow 1 & \varphi & \downarrow F \\
 Z & \xrightarrow{T} X \xrightarrow{R} & C
 \end{array}
 =
 \begin{array}{ccc}
 Z & \xrightarrow{L} & F \downarrow R \xrightarrow{P} & A \\
 \downarrow 1 & \psi & \downarrow Q & \pi & \downarrow F \\
 Z & \xrightarrow{T} & X & \xrightarrow{R} & C
 \end{array}
 \quad (46)$$

Moreover L is pseudo or strict if and only if both S and T are.

(b) A similar property holds for a pair of colax cm-functors G, H and a double cell $\varphi': (G \overset{1}{R} FH)$.

Proof. (a) L is defined as follows on an i -cube z and an i -map $f: z \rightarrow z'$ of Z

$$\begin{aligned}
 L(z) &= (Sz, Tz; \varphi z: FSz \rightarrow RTz), \\
 L(f) &= (Sf, Tf).
 \end{aligned}
 \quad (47)$$

The comparison transversal maps \underline{L}_i are constructed with the laxity transversal maps \underline{S} and \underline{T} (and are invertible or degenerate if and only if the latter are)

$$\begin{aligned}
 \underline{L}_i x &= (\underline{S}_i x, \underline{T}_i x): e_i(Lx) \rightarrow Le_i(x) && \text{(for } x \text{ in } Z_{i|i}), \\
 \underline{L}_i(x, y) &= (\underline{S}_i(x, y), \underline{T}_i(x, y)): Lx +_i Ly \rightarrow L(z) && (48) \\
 &&& \text{(for } z = x +_i y \text{ in } Z_i).
 \end{aligned}$$

Here $Lx +_i Ly$ is the i -cube defined as below (applying (40) and writing $+_i$ as $+$)

$$\begin{aligned}
 Lx + Ly &= (Sx, Tx; \varphi x: FSx \rightarrow RTx) + (Sy, Ty; \varphi y: FSy \rightarrow RTy) \\
 &= (Sx + Sy, Tx + Ty; u), \\
 u &= \underline{R}_i(Tx, Ty).(\varphi x + \varphi y).F_i Sx, Sy): \\
 &F(Sx + Sy) \rightarrow FSx +_i FSy \rightarrow RTx + RTy \rightarrow R(Tx + Ty).
 \end{aligned}$$

Letting $z = x +_i y$, the coherence condition (38) on the transversal map $\underline{L}_i(x, y) = (\underline{S}_i(x, y), \underline{T}_i(x, y))$ of $F \downarrow R$ (with $z = x +_i y$)

$$RT_i(x, y).u = \varphi z.F\underline{S}_i(x, y), \quad (49)$$

$$\begin{array}{ccc} F(Sx +_i Sy) & \xrightarrow{u} & R(Tx +_i Ty) \\ F\underline{S}_i(x, y) \downarrow & & \downarrow RT_i(x, y) \\ FS(z) & \xrightarrow{\varphi z} & RT(z) \end{array}$$

follows from the coherence condition (dc.3) of φ as a double cell in $\mathbb{C}mc$ (where $\underline{RT}_i(x, y) = RT_i(x, y).\underline{R}_i(Tx, Ty)$)

$$\underline{RT}_i(x, y).(\varphi x +_i \varphi y).\underline{F}_i(Sx, Sy) = \varphi z.F\underline{S}_i(x, y), \quad (50)$$

$$\begin{array}{ccccc} F(Sx +_i Sy) & \xrightarrow{F\underline{S}_i(x, y)} & FS(z) & \xrightarrow{\varphi z} & RT(z) \\ \underline{F}_i(Sx, Sy) \downarrow & & & & \downarrow 1 \\ FSx +_i FSy & \xrightarrow{\varphi x +_i \varphi y} & RTx +_i RTy & \xrightarrow{\underline{RT}_i(x, y)} & RT(z) \end{array}$$

The uniqueness of L is obvious. □

3. Companions and adjoints in double categories

This brief section, taken from [GP2], Section 1, studies the connections between horizontal and vertical morphisms in a double category: horizontal morphisms can have vertical *companions* and vertical *adjoints* (the latter were called ‘conjoints’ in [DPR]). Such phenomena are interesting in themselves and typical of double categories.

\mathbb{D} is always a weak double category, that we assume to be *unitary* for the sake of simplicity. We shall apply these notions to $\mathbb{C}mc$, which is strict.

3.1 Orthogonal companions

In the weak double category \mathbb{D} , the horizontal morphism $f: A \rightarrow B$ and the vertical morphism $u: A \dashrightarrow B$ are made (orthogonal) *companions* by

assigning a pair (η, ε) of cells as below, called the *unit* and *counit*, that satisfy the identities $\eta|\varepsilon = 1_f^\bullet$ and $\eta \otimes \varepsilon = 1_u$

$$\begin{array}{ccc}
 \begin{array}{ccc}
 A & \xlongequal{\quad} & A \\
 \downarrow 1 & & \downarrow u \\
 & \eta & \\
 A & \xrightarrow{f} & B
 \end{array} & &
 \begin{array}{ccc}
 A & \xrightarrow{f} & B \\
 \downarrow u & & \downarrow 1 \\
 & \varepsilon & \\
 B & \xlongequal{\quad} & B
 \end{array}
 \end{array} \quad (51)$$

Given f , this is equivalent (*by unitarity*) to saying that the pair (u, ε) satisfies the following universal property:

(a) for every cell $\varepsilon' : (u' \xrightarrow{f} B)$ there is a unique cell $\lambda : (u' \xrightarrow{A} u)$ such that $\varepsilon' = \lambda|\varepsilon$

$$\begin{array}{ccc}
 \begin{array}{ccc}
 A & \xrightarrow{f} & B \\
 \downarrow u' & & \downarrow 1 \\
 & \varepsilon' & \\
 A' & \xrightarrow{g} & B
 \end{array} & &
 \begin{array}{ccc}
 A & \xlongequal{\quad} & A & \xrightarrow{f} & B \\
 \downarrow u' & & \downarrow u & & \downarrow 1 \\
 & \lambda & & \varepsilon & \\
 A' & \xrightarrow{g} & B & \xlongequal{\quad} & B
 \end{array}
 \end{array} \quad (52)$$

In fact, given (η, ε) , we can (and must) take $\lambda = \eta \otimes \varepsilon'$; on the other hand, given ε' we define $\eta : (A \xrightarrow{A} u)$ by the equation $\eta|\varepsilon = 1_f^\bullet$ and deduce that $\eta \otimes \varepsilon = 1_u$ because $(\eta \otimes \varepsilon)|\varepsilon = (\eta|\varepsilon) \otimes \varepsilon = \varepsilon = (1_u|\varepsilon)$.

Similarly, also the pair (u, η) is characterised by a universal property

(b) for every cell $\eta' : (A \xrightarrow{g} u')$ there is a unique cell $\mu : (u \xrightarrow{B} u')$ such that $\eta' = \eta|\mu$.

Therefore, if f has a vertical companion, this is determined up to a unique special isocell, *and will often be written as f_** . Companions compose in the obvious (covariant) way: if $g : B \rightarrow C$ also has a companion g_* , then the vertical arrow $g_*f_* : A \twoheadrightarrow C$ is companion to $gf : A \rightarrow C$, with unit

$$\left(\frac{\eta \mid 1}{1^\bullet \mid \eta'} \right) : (A \xrightarrow{gf} g_*f_*).$$

Companionship is preserved by *unitary* lax or colax double functors.

We say that \mathbb{D} has *vertical companions* if every horizontal arrow has a vertical companion. The weak double categories recalled in Section 1 have

vertical companions, given by the obvious embedding of horizontal arrows into the vertical ones.

Companionship is simpler for horizontal *isomorphisms*. If f is one and has a companion u , then its unit and counit are also horizontally invertible and determine each other:

$$(\varepsilon | 1_g | \eta) = \eta \otimes \varepsilon = 1_u \quad (g = f^{-1}), \quad (53)$$

as one can see rewriting $(\varepsilon | 1_g | \eta)$ as follows, and then applying middle-four interchange

$$\begin{array}{ccccccc}
 A & \xrightarrow{f} & B & \xrightarrow{g} & A & \xlongequal{\quad} & A \\
 \downarrow 1 & & \downarrow 1 & & \downarrow 1 & & \downarrow u \\
 & & 1_f & & & & \eta \\
 & & \downarrow & & \downarrow & & \downarrow \\
 A & \xrightarrow{f} & B & \xrightarrow{g} & A & \xrightarrow{f} & B \\
 \downarrow u & & \downarrow 1 & & \downarrow 1 & & \downarrow 1 \\
 & & \varepsilon & & & & 1_f \\
 & & \downarrow & & \downarrow & & \downarrow \\
 B & \xlongequal{\quad} & B & \xrightarrow{g} & A & \xrightarrow{f} & B
 \end{array}$$

Conversely, the existence of a horizontally invertible cell $\eta: (A \xrightarrow{f} B \xrightarrow{u} A)$ implies that f is horizontally invertible, with companion u and counit as above.

3.2 Orthogonal adjoints

Transforming companionship by horizontal (or vertical) duality, the arrows $f: A \rightarrow B$ and $v: B \rightarrow A$ are made *orthogonal adjoints* by a pair (α, β) of cells as below

$$\begin{array}{ccc}
 A & \xrightarrow{f} & B \\
 \downarrow 1 & & \downarrow v \\
 & & \alpha \\
 & & \downarrow \\
 A & \xlongequal{\quad} & A
 \end{array}
 \quad
 \begin{array}{ccc}
 B & \xlongequal{\quad} & B \\
 \downarrow v & & \downarrow 1 \\
 & & \beta \\
 & & \downarrow \\
 A & \xrightarrow{f} & B
 \end{array}
 \quad (54)$$

with $\alpha|\beta = 1_f$ and $\beta \otimes \alpha = 1_v$. Then, f is the *horizontal adjoint* and v the *vertical one*. (In the general case there is no reason of distinguishing ‘left’ and ‘right’, unit and counit; see the examples of [GP2], Section 1.3).

Again, given f , these relations can be described by universal properties for (v, β) or (v, α)

(a) for every cell $\beta': (v' \xrightarrow{g} B)$ there is a unique cell $\lambda: (v' \xrightarrow{g} v)$ such that $\beta' = \lambda|\beta$,

(b) for every cell $\alpha': (A \xrightarrow{f} v')$ there is a unique cell $\mu: (v \xrightarrow{g} v')$ such that $\alpha' = \alpha|\mu$.

The vertical adjoint of f is determined up to a special isocell and will often be written as f^* ; vertical adjoints compose, contravariantly: $(gf)^*$ can be constructed as f^*g^* .

We say that \mathbb{D} has *vertical adjoints* if every horizontal arrow has a vertical adjoint. Plainly, this is the case for the weak double categories recalled in Section 1.

3.3 Proposition

Let $f: A \rightarrow B$ have a vertical companion $u: A \dashrightarrow B$. Then the arrow $v: B \dashrightarrow A$ is vertical adjoint to f if and only if $u \dashv v$ in the bicategory \mathbf{VD} (of objects, vertical arrows and special cells).

Proof. Given four cells $\eta, \varepsilon, \alpha, \beta$ as above (in 3.1, 3.2), we have two special cells

$$\eta \otimes \alpha: 1^\bullet \rightarrow u \otimes v, \quad \beta \otimes \varepsilon: u \otimes v \rightarrow 1^\bullet,$$

that are easily seen to satisfy the triangle identities in \mathbf{VD} . The converse is similarly obvious. \square

4. Multiple adjunctions

We now define a colax/lax adjunction between chiral multiple categories, a notion that occurs naturally in various situations, as already seen in Section 1.

4.1 Colax/lax adjunctions

A *colax/lax cm-adjunction* $(\eta, \varepsilon): F \dashv G$, or a colax/lax adjunction between chiral multiple categories, will be an orthogonal adjunction in the double category \mathbf{Cmc} (as defined in 3.2).

The data amount thus to:

- a *colax* cm-functor $F: X \rightarrow A$, the left adjoint,
- a *lax* cm-functor $G: A \rightarrow X$, the right adjoint,
- two $\mathbb{C}mc$ -cells $\eta: 1_X \dashrightarrow GF$ and $\varepsilon: FG \dashrightarrow 1_A$ (unit and counit) that satisfy the triangle equalities:

$$\begin{array}{ccc}
 \begin{array}{ccc}
 X & \xlongequal{\quad} & X \\
 \downarrow F & \eta & \parallel \\
 A & \xrightarrow{\quad G} & X
 \end{array} &
 \begin{array}{ccc}
 A & \xrightarrow{\quad G} & X \\
 \parallel & \varepsilon & \downarrow F \\
 A & \xlongequal{\quad} & A
 \end{array} &
 \begin{array}{l}
 \eta \otimes \varepsilon = 1_F, \\
 \varepsilon \mid \eta = 1_G.
 \end{array}
 \end{array} \tag{55}$$

We speak of a *pseudo/lax* (resp. a *colax/pseudo*) adjunction when the left (resp. right) adjoint is pseudo, and of a *pseudo* (or *strict*) adjunction when both adjoints are pseudo (or strict).

From general properties (see 3.2), we already know that the left adjoint of a lax cm-functor G is determined up to transversal isomorphism (which amounts to a special invertible cell between vertical arrows in $\mathbb{C}mc$) and that left adjoints compose, contravariantly. Similarly for right adjoints.

As in 2.2, the arrow of a colax cm-functor is marked with a dot *when displayed vertically*, in a double cell of $\mathbb{C}mc$. Again, we may write unit and counit as $\eta: 1 \dashrightarrow GF$ and $\varepsilon: FG \dashrightarrow 1$, *but we should recall that the coherence conditions of such ‘transformations’ involve the comparison cells of F and G* . Therefore (as with double categories, in [GP2]), a *general colax/lax adjunction cannot be seen as an adjunction in some bicategory*; although this is possible for a *pseudo/lax* or a *colax/pseudo* adjunction, as we shall prove in the next section.

4.2 A description

To make the previous definition explicit, a *colax/lax adjunction* $(\eta, \varepsilon): F \dashv G$ between the cm-categories X and A consists of the following items.

- (a) A *colax* cm-functor $F: X \rightarrow A$, with comparison transversal maps

$$\underline{E}_i(x): F(e_i x) \rightarrow_0 e_i(Fx), \quad \underline{E}_i(x, y): F(x +_i y) \rightarrow_0 Fx +_i Fy.$$

(b) A *lax* cm-functor $G: \mathbf{A} \rightarrow \mathbf{X}$, with comparison transversal maps

$$\underline{G}_i(a): e_i(Ga) \rightarrow_0 G(e_i a), \quad \underline{G}_i(a, b): Ga +_i Gb \rightarrow_0 G(a +_i b).$$

(c) An ordinary adjunction $F_i \dashv G_i$ for every positive multi-index \mathbf{i}

$$\begin{aligned} \eta_i: 1 \rightarrow G_i F_i: \text{tv}_i(\mathbf{X}) \rightarrow \text{tv}_i(\mathbf{A}), \quad \varepsilon_i: F_i G_i \rightarrow 1: \text{tv}_i(\mathbf{A}) \rightarrow \text{tv}_i(\mathbf{X}), \\ \varepsilon_i F_i \cdot F_i \eta_i = 1_{F_i}, \quad G_i \varepsilon_i \cdot \eta_i G_i = 1_{G_i}. \end{aligned}$$

(Note that *there is an abuse of notation*: for the sake of simplicity we write as F_i the functor $\text{tv}_i(F): \text{tv}_i(\mathbf{X}) \rightarrow \text{tv}_i(\mathbf{A})$, which involves the components $F_i: X_i \rightarrow A_i$ and $F_{0i}: X_{0i} \rightarrow A_{0i}$ of F .)

Explicitly this means that we are assigning:

- transversal maps $\eta_i x: x \rightarrow_0 G_i F_i x$ in \mathbf{X} (for x in X_i), also written as $\eta x: x \rightarrow_0 GFx$,

- transversal maps $\varepsilon_i a: F_i G_i a \rightarrow_0 a$ in \mathbf{A} (for a in A_i), also written as $\varepsilon a: FGa \rightarrow_0 a$,

satisfying the naturality conditions (c1) (for transversal maps $f: x \rightarrow_0 y$ in \mathbf{X} and $h: a \rightarrow_0 b$ in \mathbf{A}) and the triangle identities (c2),

$$(c1) \quad \eta y \cdot f = GFf \cdot \eta x, \quad \varepsilon b \cdot FGh = h \cdot \varepsilon a,$$

$$(c2) \quad \varepsilon Fx \cdot F\eta x = 1_{Fx}, \quad G\varepsilon a \cdot \eta Ga = 1_{Ga}.$$

(d) These families $\eta = (\eta_i)$ and $\varepsilon = (\varepsilon_i)$ must respect faces

$$\eta(\partial_i^\alpha x) = \partial_i^\alpha(\eta x), \quad \varepsilon(\partial_i^\alpha a) = \partial_i^\alpha(\varepsilon a), \quad (56)$$

and be coherent with the comparison cells of F and G :

(d1) (*coherence of η and ε with i -identities*) for x in \mathbf{X} and a in \mathbf{A} :

$$\begin{aligned} G\underline{F}_i(x) \cdot \eta(e_i x) &= \underline{G}_i(Fx) \cdot e_i(\eta x), \\ (\eta(e_i(x)) &= e_i(\eta x), \text{ if } F \text{ and } G \text{ are unitary}), \end{aligned} \quad (57)$$

$$\begin{aligned} \varepsilon(e_i a) \cdot F\underline{G}_i(a) &= e_i(\varepsilon a) \cdot \underline{F}_i(Ga), \\ (\varepsilon(e_i a) &= e_i(\varepsilon a), \text{ if } F \text{ and } G \text{ are unitary}), \end{aligned} \quad (58)$$

(d2) (*coherence of η and ε with i -composition*) for $z = x +_i y$ in X and $c = a +_i b$ in A :

$$GF_i(x, y) \cdot \eta z = \underline{G}_i(Fx, Fy) \cdot (\eta x +_i \eta y), \quad (59)$$

$$\varepsilon c \cdot F\underline{G}_i(a, b) = (\varepsilon a +_i \varepsilon b) \cdot \underline{F}_i(Ga, Gb). \quad (60)$$

$$\begin{array}{ccc} z & \xrightarrow{\eta x +_i \eta y} & GFx +_i GFy & & F(Ga + Gb) & \xrightarrow{F\underline{G}_i(a, b)} & FGc \\ \eta z \downarrow & & \underline{G}_i(Fx, Fy) \downarrow & & \downarrow \underline{F}_i(Ga, Gb) & & \downarrow \varepsilon c \\ GFz & \xrightarrow{GF_i(x, y)} & G(Fx +_i Fy) & & FGa + FGb & \xrightarrow{\varepsilon a +_i \varepsilon b} & c \end{array}$$

4.3 Lemma

(a) *In a colax/lax cm-adjunction $(\eta, \varepsilon): F \dashv G$, the comparison maps of G determine the comparison maps of F , through the ordinary adjunctions $F_i \dashv G_i$, as*

$$\begin{aligned} \underline{F}_i(x) &= \varepsilon e_i(Fx) \cdot F\underline{G}_i(Fx) \cdot Fe_i(\eta x): \\ & Fe_i(x) \rightarrow Fe_i(GFx) \rightarrow FG(e_i Fx) \rightarrow e_i Fx, \end{aligned} \quad (61)$$

$$\begin{aligned} \underline{F}_i(x, y) &= \varepsilon(Fx +_i Fy) \cdot F\underline{G}_i(Fx, Fy) \cdot F(\eta x +_i \eta y): \\ & F(x +_i y) \rightarrow F(GFx + GFy) \rightarrow FG(Fx + Fy) \rightarrow Fx +_i Fy. \end{aligned} \quad (62)$$

Dually, the comparison maps of F determine the comparison maps of G , through the ordinary adjunctions.

(b) *If all the components of η, ε are invertible, then G is pseudo if and only if F is.*

Note. Loosely speaking, point (a) says that a lax multiple functor *can only* have a colax left adjoint (if any), and symmetrically. This fact will be completed in Theorem 5.3, showing that if a lax functor has a *lax* adjoint, the latter is necessarily pseudo.

Proof. (a) The first equation of (d1) says that the adjoint map of $\underline{F}_i(x)$, i.e. $(\underline{F}_i(x))' = GF_i(x) \cdot \eta(e_i x)$, must be equal to $f = \underline{G}_i(Fx) \cdot e_i(\eta x)$. The adjoint map of the latter gives $\underline{F}_i(x) = f' = \varepsilon e_i(Fx) \cdot F(f)$.

In the same way the first equation of (d2) determines $\underline{F}_i(x, y)$. Point (b) is a straightforward consequence. \square

4.4 Theorem (Characterisation by transversal hom-sets)

A multiple adjunction $(\eta, \varepsilon): F \dashv G$ can equivalently be given by a colax cm-functor $F: \mathbb{X} \rightarrow \mathbb{A}$, a lax cm-functor $G: \mathbb{A} \rightarrow \mathbb{X}$ and a family $(L_{\mathbf{i}})$ of functorial isomorphisms indexed by the positive multi-indices $\mathbf{i} \subset \mathbb{N}$

$$\begin{aligned} L_{\mathbf{i}}: \text{tv}_{\mathbf{i}}(\mathbb{A})(F_{\mathbf{i}}(-), =) &\rightarrow \text{tv}_{\mathbf{i}}(\mathbb{X})(-, G_{\mathbf{i}}(=)): \\ &\text{tv}_{\mathbf{i}}(\mathbb{X})^{\text{op}} \times \text{tv}_{\mathbf{i}}(\mathbb{A}) \rightarrow \mathbf{Set}, \quad (63) \\ L_{\mathbf{i}}(x, a): \text{tv}_{\mathbf{i}}(\mathbb{A})(Fx, a) &\rightarrow \text{tv}_{\mathbf{i}}(\mathbb{X})(x, Ga). \end{aligned}$$

The components $L_{\mathbf{i}}(x, a)$, also written as $L(x, a)$ or just L , have to commute with faces and be coherent with the positive operations (through the comparison maps of F and G), i.e. must satisfy the following conditions (ad.1-3):

- (ad.1) $L_{\mathbf{i}}(\partial_i^\alpha x, \partial_i^\alpha a) = \partial_i^\alpha(L_{\mathbf{i}}(x, a))$,
 (ad.2) $L(e_i x, e_i a)(e_i(h).F(x)) = \underline{G}(a).e_i(Lh)$ (for $h: Fx \rightarrow a$ in \mathbb{A}),

$$\begin{array}{ccc} Fe_i(x) & \xrightarrow{F(x)} & e_i(Fx) \xrightarrow{e_i(h)} e_i(a) \\ e_i(x) & \xrightarrow{e_i(Lh)} & e_i(Ga) \xrightarrow{\underline{G}(a)} Ge_i(a) \end{array}$$

- (ad.3) $L((h +_i k).F_i(x, y)) = \underline{G}_i(a, b).(Lh +_i Lk)$
 (for $h: Fx \rightarrow a, k: Fy \rightarrow b$ in $\text{tv}_{\mathbf{i}}(\mathbb{A})$),

$$\begin{array}{ccc} F(x +_i y) & \xrightarrow{F_i(x, y)} & Fx +_i Fy \xrightarrow{h+ik} a +_i b \\ x +_i y & \xrightarrow{Lh+_i Lk} & Ga +_i Gb \xrightarrow{\underline{G}_i(a, b)} G(a +_i b) \end{array}$$

In this equivalence, $L_{\mathbf{i}}(x, a)$ is defined by the unit η as

$$L_{\mathbf{i}}(x, a)(h) = Gh.\eta_{\mathbf{i}}x: x \rightarrow GFx \rightarrow Ga \quad (h: Fx \rightarrow a \text{ in } \text{tv}_{\mathbf{i}}(\mathbb{A})). \quad (64)$$

The other way round, the component $\eta_{\mathbf{i}}: 1 \rightarrow G_{\mathbf{i}}F_{\mathbf{i}}: X_{\mathbf{i}} \rightarrow X_{\mathbf{i}}$ of the unit is defined by L as

$$\eta_{\mathbf{i}}(x) = L_{\mathbf{i}}(x, Fx)(\text{id}Fx): x \rightarrow GF(x) \quad (\text{for } x \text{ in } X_{\mathbf{i}}). \quad (65)$$

Proof. We have only to verify the equivalence of the conditions (56)-(60) with the conditions above.

This is straightforward. For instance, to show that (59) implies (ad.3), let $h: Fx \rightarrow a$ and $k: Fy \rightarrow b$ be i -consecutive \mathbf{i} -maps in A , and apply $L = L(x +_i y, a +_i a)$ as defined above, in (64):

$$\begin{aligned}
 L((h +_i k).F_i(x, y)) &= G(h +_i k).GF_i(x, y).\eta(x +_i y) \\
 &= G(h +_i k).\underline{G}_i(Fx, Fy).(\eta x +_i \eta y) && \text{(by (59))} \\
 &= \underline{G}_i(a, b).(Gh +_i Gk).(\eta x +_i \eta y) && \text{(by (Imf.2))} \\
 &= \underline{G}_i(a, b).(Lh +_i Lk).
 \end{aligned}$$

□

4.5 Corollary (Characterisation by commas)

With the previous notation, a multiple adjunction amounts to an isomorphism of chiral multiple categories $L: F \downarrow A \rightarrow X \downarrow G$ over the cartesian product $X \times A$

$$\begin{array}{ccc}
 F \downarrow A & \xrightarrow{L} & X \downarrow G \\
 & \searrow & \swarrow \\
 & X \times A &
 \end{array}
 \quad (66)$$

Proof. A straightforward consequence of the previous theorem. □

4.6 Theorem (Right adjoint by universal properties)

Let a colax cm -functor $F: X \rightarrow A$ be given.

The existence (and choice) of a right adjoint lax cm -functor G amounts to a family (rad.i) of conditions and choices, indexed by the positive multi-indices \mathbf{i} :

(rad.i) for every \mathbf{i} -cube a in A there is a universal arrow $(Ga, \varepsilon_{\mathbf{i}}a: F(Ga) \rightarrow a)$ from the functor $F_{\mathbf{i}}: \text{tv}_{\mathbf{i}}(X) \rightarrow \text{tv}_{\mathbf{i}}(A)$ to the object a , and we choose one, provided that these choices commute with faces.

Explicitly, the universal property means that, for each \mathbf{i} -cube x in X and \mathbf{i} -map $h: Fx \rightarrow_0 a$ in A there is a unique $f: x \rightarrow_0 Ga$ such that $h = \varepsilon_a.Ff: Fx \rightarrow_0 F(Ga) \rightarrow_0 a$.

The comparison i-maps of G

$$\underline{G}_i(a): e_i(Ga) \rightarrow_0 G(e_i a), \quad \underline{G}_i(a, b): Ga +_i Gb \rightarrow_0 G(a +_i b), \quad (67)$$

are then given by the universal property of ε , as the unique solution of the equations (58), (60), respectively.

Proof. The conditions (rad.i) are plainly necessary, including consistency with faces.

Conversely, each (rad.i) provides an ordinary adjunction $(\eta_i, \varepsilon_i): F_i \dashv G_i$ for the categories $\text{tv}_i(\mathbb{X}), \text{tv}_i(\mathbb{A})$, so that G, η and ε are correctly defined - as far as cubes, transversal maps, faces, transversal composition and transversal identities are concerned.

Now we define the comparison maps \underline{G}_i as specified in the statement, so that the coherence properties of ε are satisfied (see (58), (60)). One verifies easily, for such transversal maps, the axioms of naturality and coherence (see 2.1).

Finally, we have to prove that $\eta: 1 \dashrightarrow GF$ satisfies the coherence property (59)

$$GF\underline{F}_i(x, y) \cdot \eta z = \underline{G}_i(Fx, Fy) \cdot (\eta x +_i \eta y), \quad (68)$$

with respect to a composition $z = x +_i y$ of **i**-cubes in \mathbb{X} (similarly one proves (57)). By the universal property of ε , it will suffice to show that the composite $\varepsilon(Fx +_i Fy) \cdot F(-)$ takes the same value on both terms of (68). In fact, on the left-hand term we get $\underline{F}_i(x, y)$

$$\varepsilon(Fx +_i Fy) \cdot FGF\underline{F}_i(x, y) \cdot F\eta z = \underline{F}_i(x, y) \cdot \varepsilon Fz \cdot F\eta z = \underline{F}_i(x, y).$$

We get the same on the right-hand term of (68), using (60), the naturality of \underline{F}_i , the $0i$ -interchange in \mathbb{A} and a triangle identity

$$\begin{aligned} & \varepsilon(Fx +_i Fy) \cdot F\underline{G}_i(Fx, Fy) \cdot F(\eta x +_i \eta y) \\ &= (\varepsilon Fx +_i \varepsilon Fy) \cdot \underline{F}_i(GFx, GFy) \cdot F(\eta x +_i \eta y) \\ &= (\varepsilon Fx +_i \varepsilon Fy) \cdot (F\eta x +_i F\eta y) \cdot \underline{F}_i(x, y) \\ &= (\varepsilon Fx \cdot F\eta x +_i \varepsilon Fy \cdot F\eta y) \cdot \underline{F}_i(x, y) \\ &= (1_{Fx} +_i 1_{Fy}) \cdot \underline{F}_i(x, y) \\ &= \underline{F}_i(x, y). \end{aligned}$$

□

The coreflective adjunction $F' \dashv P$ is obvious

$$\begin{aligned} \eta'x &= 1_x: x \rightarrow PF'x, \\ \varepsilon'(x, a; f: Fx \rightarrow a) &= (1_x, f): \\ &(x, Fx; 1: Fx \rightarrow Fx) \rightarrow (x, a; f: Fx \rightarrow a), \end{aligned} \tag{72}$$

as well as the reflective adjunction $Q \dashv G'$ and the factorisation above. \square

5. Multiple adjunctions and pseudo cm-functors

We consider now cm-adjunctions where the left or right adjoint is a pseudo cm-functor. Then we introduce adjoint equivalences of chiral multiple categories.

5.1 Comments

Let us recall, from 4.1, that a *pseudo/lax* cm-adjunction $F \dashv G$ is a colax/lax adjunction between cm-categories where the left adjoint F is pseudo.

Then the comparison cells of F are horizontally invertible and the composites GF and FG are lax cm-functors; it follows (from definition 2.2) that the unit and counit are horizontal transformations of such functors. Therefore a *pseudo/lax cm-adjunction gives an adjunction in the 2-category LxCmc of cm-categories, lax cm-functors and transversal transformations (see 2.2); and we shall prove that these two facts are actually equivalent (Theorem 5.3).*

Dually a *colax/pseudo* cm-adjunction, where the right adjoint G is pseudo, will amount to an adjunction in the 2-category CxCmc of cm-categories, colax cm-functors and transversal transformations. Finally a *pseudo* cm-adjunction, where both F and G are pseudo, will be the same as an adjunction in the 2-category PsCmc whose arrows are the pseudo cm-functors.

5.2 Theorem (Companions in Cmc)

A lax cm-functor G has an orthogonal companion F in the double category Cmc if and only if it is pseudo; then one can define $F = G_$ as the colax*

cm-functor which coincides with G except for comparison maps, that are transversally inverse to those of G .

Proof. We restrict to unitary cm-categories, for simplicity. If G is pseudo, it is obvious that G_* , as defined above, is an orthogonal companion.

Conversely, suppose that $G: \mathbf{A} \rightarrow \mathbf{X}$ (lax) has an orthogonal companion F (colax). There are thus two double cells η, ε in $\mathbb{C}mc$

$$\begin{array}{ccc}
 \begin{array}{ccc}
 \mathbf{A} & \xlongequal{\quad} & \mathbf{A} \\
 \parallel & \eta & \downarrow F \\
 \mathbf{A} & \xrightarrow{G} & \mathbf{X}
 \end{array} & &
 \begin{array}{ccc}
 \mathbf{A} & \xrightarrow{G} & \mathbf{X} \\
 \downarrow F & \varepsilon & \parallel \\
 \mathbf{X} & \xlongequal{\quad} & \mathbf{X}
 \end{array}
 \end{array} \quad (73)$$

which satisfy the identities $\eta|\varepsilon = 1_G^\bullet$, $\eta \otimes \varepsilon = 1_F$.

This means two ‘transformations’ $\eta: F \dashrightarrow G$, $\varepsilon: G \dashrightarrow F$, as defined in 2.2; for every i -cube a in \mathbf{A} , we have two transversal maps ηa and εa in \mathbf{X}

$$\eta a: Fa \rightarrow Ga, \quad \varepsilon a: Ga \rightarrow Fa, \quad (74)$$

consistently with faces. These maps are transversally inverse, because of the previous identities (see (36))

$$\eta a.\varepsilon a = (\eta|\varepsilon)(a) = 1_{Ga}, \quad \varepsilon a.\eta a = (\eta \otimes \varepsilon)(a) = 1_{Fa}. \quad (75)$$

Applying now the coherence condition (dc.3) (of 2.2) for the transformations η, ε and a concatenation $c = a +_i b$ in \mathbf{A} we find

$$\begin{aligned}
 \eta c &= \underline{G}_i(a, b).(\eta a +_i \eta b).\underline{F}_i(a, b): Fc \rightarrow Gc, \\
 \varepsilon a +_i \varepsilon b &= \underline{F}_i(a, b).\varepsilon c.\underline{G}_i(a, b): Ga +_i Gb \rightarrow Fa +_i Fb.
 \end{aligned} \quad (76)$$

Since all the components of η and ε are transversally invertible, this proves that $\underline{G}_i(a, b)$ has a left inverse and a right inverse transversal map, whence it is invertible. Similarly for degeneracies.

Therefore G is pseudo and F is transversally isomorphic to G_* . \square

5.3 Theorem

(a) (Pseudo/lax adjunctions) *For every adjunction $F \dashv G$ in the 2-category \mathbf{LxCmc} , the functor F is pseudo and the adjunction is pseudo/lax, in the sense of 4.1 (or 5.1).*

(b) (Colax/pseudo adjunctions) *For every adjunction $F \dashv G$ in the 2-category \mathbf{CxCmc} , the functor G is pseudo and the adjunction is colax/pseudo, in the sense of 4.1 (or 5.1).*

Note. More formally, (a) can be rewritten saying that, in the double category $\mathbb{C}mc$, if the horizontal arrow G has a ‘horizontal left adjoint’ F (within the horizontal 2-category $\mathbf{HCmc} = \mathbf{LxCmc}$), then it also has an orthogonal adjoint G^* (colax). (Then, applying Proposition 3.3, it would follow that F and G^* are companions, whence F is pseudo, by Theorem 5.2, and isomorphic to G^* .)

Proof. It suffices to prove (a); again, we only deal with the comparisons of a composition.

Let the lax structures of $F: X \rightarrow A$ and $G: A \rightarrow X$ be given by the following comparison maps, where $z = x +_i y$ and $c = a +_i b$

$$\lambda_i(x, y): Fx +_i Fy \rightarrow F(x +_i y), \quad \underline{G}_i(a, b): Ga +_i Gb \rightarrow G(a +_i b),$$

so that we have:

$$\begin{aligned} \eta z &= G\lambda_i(x, y) \cdot \underline{G}_i(Fx, Fy) \cdot (\eta x +_i \eta y): \\ &\quad z \rightarrow GFx +_i GFy \rightarrow G(Fx +_i Fy) \rightarrow GFz, \\ \varepsilon a +_i \varepsilon b &= \varepsilon c \cdot F\underline{G}_i(a, b) \cdot \lambda_i(Ga, Gb): \\ &\quad FGa +_i FGb \rightarrow F(Ga +_i Gb) \rightarrow FG(a +_i b) \rightarrow c. \end{aligned} \tag{77}$$

We construct a colax structure \underline{F} for F , letting

$$\begin{aligned} \underline{F}_i(x, y) &= \varepsilon(Fx +_i Fy) \cdot F\underline{G}_i(Fx, Fy) \cdot F(\eta x +_i \eta y): \\ &\quad Fz \rightarrow F(GFx +_i GFy) \rightarrow FG(Fx +_i Fy) \rightarrow Fx +_i Fy. \end{aligned}$$

Now it is sufficient to verify that $\underline{F}_i(x, y)$ and $\lambda_i(x, y)$ are transversally

inverse, using the naturality of ε , λ and (77):

$$\begin{aligned}
 & \lambda_i(x, y) \cdot \underline{F}_i(x, y) \\
 &= \lambda_i(x, y) \cdot \varepsilon(Fx +_i Fy) \cdot F\underline{G}_i(Fx, Fy) \cdot F(\eta x +_i \eta y) \\
 &= \varepsilon Fz \cdot FG\lambda_i(x, y) \cdot F\underline{G}_i(Fx, Fy) \cdot F(\eta x +_i \eta y) \\
 &= \varepsilon F(z) \cdot F(\eta z) = 1_{Fz},
 \end{aligned}$$

$$\begin{aligned}
 & \underline{F}_i(x, y) \cdot \lambda_i(x, y) \\
 &= \varepsilon(Fx +_i Fy) \cdot F\underline{G}_i(Fx, Fy) \cdot F(\eta x +_i \eta y) \cdot \lambda_i(x, y) \\
 &= \varepsilon(Fx +_i Fy) \cdot F\underline{G}_i(Fx, Fy) \cdot \lambda_i(GFx, GFy) \cdot (F\eta x +_i F\eta y) \\
 &= (\varepsilon Fx +_i \varepsilon Fy) \cdot (F\eta x +_i F\eta y) \\
 &= \varepsilon Fx \cdot F\eta x +_i \varepsilon Fy \cdot F\eta y = 1_{Fx} +_i 1_{Fy} = 1_{Fx+iFy}.
 \end{aligned}$$

□

5.4 Equivalences of cm-categories

An *adjoint equivalence* between two cm-categories \mathbb{X} and \mathbb{A} will be a pseudo cm-adjunction $(\eta, \varepsilon): F \dashv G$ where the transversal transformations $\eta: 1_{\mathbb{X}} \rightarrow GF$ and $\varepsilon: FG \rightarrow 1_{\mathbb{A}}$ are invertible.

The following properties of a cm-functor $F: \mathbb{X} \rightarrow \mathbb{A}$ will allow us (in the next theorem) to characterise this fact in the usual way, under the mild restriction of *transversal invariance* (see II.1.6):

(a) We say that F is *faithful* if all the ordinary functors $F_i: \text{tv}_i(\mathbb{X}) \rightarrow \text{tv}_i(\mathbb{A})$ (between the categories of \mathbf{i} -cubes and their transversal maps) are faithful: given two \mathbf{i} -maps $f, g: x \rightarrow_0 y$ of \mathbb{X} between the same \mathbf{i} -cubes, $F(f) = F(g)$ implies $f = g$.

(b) Similarly, we say that F is *full* if all the ordinary functors $F_i: \text{tv}_i(\mathbb{X}) \rightarrow \text{tv}_i(\mathbb{A})$ are: for every \mathbf{i} -map $h: F(x) \rightarrow_0 F(y)$ in \mathbb{A} there is an \mathbf{i} -map $f: x \rightarrow_0 y$ in \mathbb{X} such that $F(f) = h$.

(c) Finally, we say that F is *essentially surjective on cubes* if every F_i is: for every \mathbf{i} -cube a in \mathbb{A} there is some \mathbf{i} -cube x in \mathbb{X} and some invertible \mathbf{i} -map $h: F(x) \rightarrow_0 a$ in \mathbb{A} .

5.5 Theorem (Characterisations of equivalences)

Let $F: X \rightarrow A$ be a pseudo cm-functor between two transversally invariant cm-categories (see II.1.6). The following conditions are equivalent:

- (i) $F: X \rightarrow A$ belongs to an adjoint equivalence of cm-categories,
- (ii) F is faithful, full and essentially surjective on cubes (see 5.4),
- (iii) every ordinary functor $F_{\mathbf{i}}: \text{tv}_{\mathbf{i}}(X) \rightarrow \text{tv}_{\mathbf{i}}(A)$ is an equivalence of categories.

Moreover, if F is unitary, one can make its ‘quasi-inverse’ unitary as well.

Remark. The axiom of choice is assumed.

Proof. By our previous definitions in 5.4, conditions (ii) and (iii) are about the family of ordinary functors $(F_{\mathbf{i}})$ and are well known to be equivalent (assuming (AC)). Moreover, if F belongs to an adjoint equivalence $(\eta, \varepsilon): F \dashv G$, every $F_{\mathbf{i}}$ is obviously an equivalence of categories.

Conversely, let us assume that every $F_{\mathbf{i}}$ is an equivalence of ordinary categories and let us extend the pseudo cm-functor F to an adjoint equivalence, proceeding by induction on the degree $n \geq 0$ of the positive multi-index \mathbf{i} .

First, $F_*: \text{tv}_*(X) \rightarrow \text{tv}_*(A)$ is an equivalence of categories and we begin by choosing an adjoint quasi-inverse $G_*: \text{tv}_*(A) \rightarrow \text{tv}_*(X)$.

In other words, we choose for every \star -cube (or object) a some $G(a)$ in X and some isomorphism $\varepsilon a: FG(a) \rightarrow a$ in A ; then a transversal map $h: a \rightarrow b$ in A is sent to the unique X -map $G(h): G(a) \rightarrow G(b)$ coherent with the previous choices (since F_* is full and faithful). Finally the isomorphism $\eta x: x \rightarrow GF(x)$ is determined by the triangle equations (for every \star -cube x of X).

Assume now that the components of G, ε and η have been defined up to degree $n - 1 \geq 0$, and let us define them for a multi-index \mathbf{i} of degree n , taking care that the new choices be consistent with the previous ones.

First, for every \mathbf{i} -cube a in A we want to choose some \mathbf{i} -cube $G(a)$ in X and some \mathbf{i} -isomorphism $\varepsilon a: FG(a) \rightarrow a$ in A , consistently with all faces ∂_i^α ($i \in \mathbf{i}$). In fact there exists (and we choose) some \mathbf{i} -cube x and some \mathbf{i} -isomorphism $u: F(x) \rightarrow a$. Then, by the inductive hypothesis, we have a

family of $2n$ transversal isomorphisms of A

$$v_i^\alpha = \partial_i^\alpha u^{-1} \cdot \varepsilon(\partial_i^\alpha a): FG(\partial_i^\alpha a) \rightarrow \partial_i^\alpha a \rightarrow F(\partial_i^\alpha x) \quad (i \in \mathbf{i}, \alpha = \pm),$$

which can be uniquely lifted as transversal isomorphisms t_i^α of X , since F is full and faithful

$$t_i^\alpha: G(\partial_i^\alpha a) \rightarrow \partial_i^\alpha x, \quad v_i^\alpha = F(t_i^\alpha).$$

The family (v_i^α) has consistent positive faces (see II.1.6), because this is true of the family $(\partial_i^\alpha u^{-1})_{i,\alpha}$, by commuting faces, and of the family $(\varepsilon(\partial_i^\alpha a))_{i,\alpha}$ by inductive assumption. It follows that also the family (t_i^α) has consistent positive faces.

By transversal invariance in X we can fill this family (t_i^α) with a (chosen) transversal \mathbf{i} -isomorphism $t: y \rightarrow x$, and we define the \mathbf{i} -cube $G(a)$ and the \mathbf{i} -isomorphism εa as follows:

$$G(a) = y, \quad \varepsilon a = u \cdot Ft: FG(a) \rightarrow F(x) \rightarrow a.$$

This choice is consistent with faces:

$$\partial_i^\alpha(\varepsilon a) = (\partial_i^\alpha u) \cdot Ft_i^\alpha = (\partial_i^\alpha u) \cdot v_i^\alpha = \varepsilon(\partial_i^\alpha a).$$

Now, since F_i is full and faithful, a transversal \mathbf{i} -map $h: a \rightarrow b$ in A is sent to the unique X -map $G(h): G(a) \rightarrow G(b)$ satisfying the condition $\varepsilon b \cdot F(Gh) = h \cdot \varepsilon a$ (naturality of ε).

Again, the \mathbf{i} -isomorphism $\eta x: x \rightarrow GF(x)$ is determined by the triangle equations, for every \mathbf{i} -cube x of X .

The comparison \mathbf{i} -maps \underline{G}_i are uniquely determined by their coherence conditions (see 4.2), for an $\mathbf{i}|i$ -cube a and an i -composition of \mathbf{i} -cubes $c = a +_i b$ in A

$$\begin{aligned} \varepsilon e_i a \cdot F\underline{G}_i(a) &= e_i(\varepsilon a) \cdot \underline{F}_i(Ga), \\ \varepsilon c \cdot F\underline{G}_i(a, b) &= (\varepsilon a +_i \varepsilon b) \cdot \underline{F}_i(Ga, Gb). \end{aligned}$$

Moreover $\underline{G}_i(a)$ and $\underline{G}_i(a, b)$ are invertible, because so are their images by F , full and faithful.

The construction of G, ε and η is now achieved. One ends by proving that G is indeed a pseudo \mathbf{cm} -functor, and that ε, η are coherent with the comparison cells of F and G .

Finally, let us assume that F is unitary: $\underline{F}_i(x): F(e_i x) \rightarrow e_i(Fx)$ is always an identity. To make G unitary we assume that - in the previous inductive construction - the following constraint has been followed: for a j -degenerate \mathbf{i} -cube $a = e_j c$ we always choose the transversal isomorphism $u = e_j(\varepsilon c): F(e_j(Gc)) \rightarrow e_j c$. It follows that each

$$v_i^\alpha: FG(\partial_i^\alpha e_j c) \rightarrow F(\partial_i^\alpha e_j Gc)$$

is the identity; then $t_i^\alpha: G(\partial_i^\alpha e_j c) \rightarrow \partial_i^\alpha e_j Gc$ is the identity as well. We (choose to) fill their family with the identity $t: e_j Gc \rightarrow e_j Gc$, which gives

$$G(e_j c) = e_j Gc, \quad \varepsilon(e_j c) = u.Ft = e_j(\varepsilon c).$$

If a is also j' -degenerate, the commutativity of degeneracies ensures that both constructions give the same result. \square

6. Limits and adjoints for cm-categories

We briefly recall the definition of cones and limits from Part II, Section 3, and prove that unitary right adjoints preserve the limits of cm-functors.

6.1 Lift functors

First we recall a tool from II.1.5, II.1.8. For the *positive* integer j there is a *j -directed lift 2-functor* with values in the 2-category of chiral multiple categories indexed by the ordered set $\mathbb{N}|j = \mathbb{N} \setminus \{j\}$, pointed at 0

$$Q_j: \text{LxCmc} \rightarrow \text{LxCmc}_{\mathbb{N}|j}. \quad (78)$$

On a cm-category A the cm-category $Q_j A$ is - loosely speaking - that part of A that contains the index j , reindexed without it:

$$\begin{aligned} (Q_j A)_{\mathbf{i}} &= A_{\mathbf{i}j}, \\ (\partial_i^\alpha: (Q_j A)_{\mathbf{i}} \rightarrow (Q_j A)_{\mathbf{i}|i}) &= (\partial_i^\alpha: A_{\mathbf{i}j} \rightarrow A_{\mathbf{i}j|i}), \\ (e_i: (Q_j A)_{\mathbf{i}|i} \rightarrow (Q_j A)_{\mathbf{i}}) &= (e_i: A_{\mathbf{i}j|i} \rightarrow A_{\mathbf{i}j}) \quad (i \in \mathbf{i} \subset \mathbb{N}|j); \end{aligned} \quad (79)$$

similarly for compositions and comparisons. In the same way for a lax cm-functor $F: A \rightarrow B$ and a transversal transformation $h: F \rightarrow G: A \rightarrow B$ we let

$$(Q_j F)_{\mathbf{i}} = F_{\mathbf{i}j}, \quad (Q_j h)_{\mathbf{i}} = h_{\mathbf{i}j} \quad (\mathbf{i} \subset \mathbb{N}|j). \quad (80)$$

There is also an obvious restriction 2-functor $R_j: \mathbf{LxCmc} \rightarrow \mathbf{LxCmc}_{\mathbb{N}|j}$ where the multiple category $R_j A$ is that part of A that does not contain the index j . The j -directed faces and degeneracies of A are not used in $Q_j A$, but yield three natural transformations, also called *faces* and *degeneracy*, that act as follows for $\mathbf{i} \subset \mathbb{N}|j$

$$\begin{aligned} D_j^\alpha: Q_j \rightarrow R_j: \mathbf{LxCmc} &\rightarrow \mathbf{LxCmc}_{\mathbb{N}|j}, & (D_j^\alpha)_{\mathbf{i}} &= \partial_j^\alpha: A_{\mathbf{i}j} \rightarrow A_{\mathbf{i}}, \\ E_j: R_j \rightarrow Q_j: \mathbf{LxCmc} &\rightarrow \mathbf{LxCmc}_{\mathbb{N}|j}, & (E_j)_{\mathbf{i}} &= e_j: A_{\mathbf{i}} \rightarrow A_{\mathbf{i}j}, \end{aligned} \quad (81)$$

$$D_j^\alpha E_j = \text{id}.$$

All the functors Q_j commute. By composing n of them in any order we get an *iterated lift functor* of degree n , in a *positive direction* $\mathbf{i} = \{i_1, \dots, i_n\}$

$$\begin{aligned} Q_{\mathbf{i}}: \mathbf{LxCmc} &\rightarrow \mathbf{LxCmc}_{\mathbb{N}|\mathbf{i}}, & Q_{\mathbf{i}}(A) &= Q_{i_n} \dots Q_{i_1}(A), \\ \text{tv}_*(Q_{\mathbf{i}}(A)) &= \text{tv}_{\mathbf{i}}(A). \end{aligned} \quad (82)$$

6.2 Cones

Let \mathbf{X} and A be cm-categories, and let \mathbf{X} be small. Consider the *diagonal* functor (of ordinary categories)

$$D: \text{tv}_* A \rightarrow \text{PsCmc}(\mathbf{X}, A). \quad (83)$$

where $\text{tv}_* A$ is the ordinary category of \star -cubes (objects) of A and their transversal maps.

D takes each object A of A to a unitary pseudo functor $\mathbf{X} \rightarrow A$, ‘constant’ at A via the family of the total \mathbf{i} -degeneracies $e_{\mathbf{i}} = e_{i_1} \dots e_{i_n}: A_{\star} \rightarrow A_{\mathbf{i}}$

$$\begin{aligned} DA: \mathbf{X} &\rightarrow A \\ DA(x) &= e_{\mathbf{i}}(A), & DA(f) &= \text{id}(e_{\mathbf{i}}A) && (\text{for } x, f \text{ in } \text{tv}_{\mathbf{i}}\mathbf{X}), \\ \underline{DA}_i(x) &= \text{id}(e_{\mathbf{i}}A): e_{\mathbf{i}}(DA(x)) \rightarrow DA(e_i x) && (\text{for } x \text{ in } X_{\mathbf{i}|i}), \\ \underline{DA}_i(x, y) &= \lambda_i: e_{\mathbf{i}}(A) +_i e_{\mathbf{i}}(A) \rightarrow e_{\mathbf{i}}(A) && (\text{for } x, y \text{ in } X_{\mathbf{i}}), \end{aligned} \quad (84)$$

where $\lambda_i = \lambda_i(e_{\mathbf{i}}A) = \rho_i(e_{\mathbf{i}}A)$ is a left and right unitor of A .

Similarly, a \star -map $f: A \rightarrow B$ in A is sent to the constant transversal transformation

$$\begin{aligned} Df: DA &\rightarrow DB: \mathbf{X} \rightarrow A, \\ (Df)(x) &= e_{\mathbf{i}}(f): e_{\mathbf{i}}(A) \rightarrow e_{\mathbf{i}}(B) && (x \text{ in } \text{tv}_{\mathbf{i}}\mathbf{X}). \end{aligned} \quad (85)$$

Let $T: \mathbb{X} \rightarrow \mathbb{A}$ be a lax functor. A (*transversal*) *cone* of T will be a pair $(A, h: DA \rightarrow T)$ formed of an object A of \mathbb{A} (the *vertex* of the cone) and a transversal transformation of lax functors $h: DA \rightarrow T: \mathbb{X} \rightarrow \mathbb{A}$; in other words, it is an object of the ordinary comma category $(D \downarrow T)$, where T is viewed as an object of the category $\text{LxCmc}(\mathbb{X}, \mathbb{A})$.

By definition (cf. II.1.8), the transversal transformation h amounts to assigning the following data:

- a transversal i -map $hx: e_i(A) \rightarrow Tx$, for every i -cube x in \mathbb{X} ,

subject to the following axioms of naturality and coherence:

(tc.1) $Tf.hx = hy$ (for every i -map $f: x \rightarrow_0 y$ in \mathbb{X}),

(tc.2) h commutes with positive faces, and agrees with positive degeneracies and compositions:

$$h(\partial_i^\alpha x) = \partial_i^\alpha(hx), \quad (x \text{ in } X_i),$$

$$h(e_i x) = \underline{T}_i(x).e_i(hx): e_i(A) \rightarrow_0 T(e_i x) \quad (x \text{ in } X_{i|i}),$$

$$h(z) = \underline{T}_i(x, y).(hx +_i hy).\lambda_i^{-1}: e_i(A) \rightarrow_0 T(z) \quad (z = x +_i y \text{ in } X_i).$$

As remarked in II.3.2, a *unitary* lax functor $G: \mathbb{A} \rightarrow \mathbb{B}$ preserves diagonalisation, in the sense that $G.DA = D(GA)$; therefore G takes a cone $(A, h: DA \rightarrow T)$ of T to a cone (GA, Gh) of GT .

6.3 Limits of degree zero

As defined in II.3.3, the (*transversal*) *limit of degree zero* $\lim(T) = (L, t)$ of a lax functor $T: \mathbb{X} \rightarrow \mathbb{A}$ between chiral multiple categories is a universal cone $(L, t: DL \rightarrow T)$.

In other words:

(tl.0) L is an object of \mathbb{A} and $t: DL \rightarrow T$ is a transversal transformation of lax functors,

(tl.1) for every cone $(A, h: DA \rightarrow T)$ there is precisely one \star -map $f: A \rightarrow L$ in \mathbb{A} such that $t.Df = h$.

We say that \mathbb{A} *has limits of degree zero* on \mathbb{X} if all these exist.

Theorem II.3.6 proves that all limits of degree zero in \mathbb{A} can be constructed from products, equalisers and *tabulators* - all of degree zero; it also

gives a corresponding result for the preservation of such limits by *unitary* lax multiple functors. (Tabulators, the basic form of higher limits, were sketched in Part I and studied in Part II, Section 3.)

6.4 Multiple limits

The general definition of multiple limits in a chiral multiple category A was given in II.4.4.

- (a) For a positive multi-index $\mathbf{i} \subset \mathbb{N}$ and a chiral multiple category X we say that A has *limits of type \mathbf{i}* on X if $Q_{\mathbf{i}}A$ has limits of degree zero on X .
- (b) We say that A *has limits of type \mathbf{i}* if this happens for all small chiral multiple categories X .
- (c) We say that A *has limits of all degrees* (or *all types*) if this happens for all positive multi-indices \mathbf{i} .
- (d) We say that A *has multiple limits of all degrees* if all the previous limits exist and are preserved by the multiple functors (see 6.1)

$$D_j^\alpha : Q_{i_j}(A) \rightarrow R_j Q_{\mathbf{i}}(A), \quad E_j : R_j Q_{\mathbf{i}}(A) \rightarrow Q_{i_j}(A) \quad (j \notin \mathbf{i}). \quad (86)$$

In this case, if A is transversally invariant one can always operate a choice of multiple limits such that this preservation is strict.

The Main Theorem of Part II (II.4.5) shows that all multiple limits in A can be constructed from multiple products, multiple equalisers and multiple tabulators; again, it also gives a corresponding result for the preservation of such limits by multiple functors.

We are now ready to prove the preservation properties of unitary adjoints.

6.5 Theorem (Adjoint and limits of degree zero)

Let $(\eta, \varepsilon) : F \dashv G$ be a colax/lax cm-adjunction, where both functors are unitary.

Then $G : A \rightarrow B$ preserves all (the existing) limits of degree zero of lax cm-functors $T : X \rightarrow A$.

Proof. The argument is the usual one. Let $(A, h : D_A(A) \rightarrow T)$ be a limit of T in A . We want to prove that the pair $(GA, Gh : G.D_A(A) \rightarrow GT)$ is a limit of GT in B .

First, since G is unitary, $GD_A(A) = D_B(GA)$ and the pair (GA, Gh) is indeed a cone of the lax cm-functor $GT: X \rightarrow B$.

Moreover, given a cone $(B, k': D_B(a) \rightarrow GT)$ of GT , with transversal components $k'_x: e_i(B) \rightarrow GTx$ for every i -cube x in X , the adjunction gives a family $h'_x: Fe_i(B) \rightarrow Tx$, that is a cone $(FB, h': D_A(FB) \rightarrow T)$ in A . Therefore there is precisely one transversal map $f: FB \rightarrow A$ in A such that $h.Df = h'$. This means precisely one transversal map $g: B \rightarrow GA$ in B such that $Gh.Dg = k'$. \square

6.6 Remark

Since the lift 2-functor

$$Q_i: \text{LxCmc} \rightarrow \text{LxCmc}_{\mathbb{N}|j}$$

preserves cm-adjunctions, it follows that, if the cm-category A has *multiple limits on X* , these are preserved by a right adjoint cm-functor $G: A \rightarrow B$ (under the previous unitarity assumptions).

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