*CAHIERS DE TOPOLOGIE ET GEOMETRIE DIFFERENTIELLE CATEGORIQUES* *Vol. LVIII-1 (2017)*

# AUTOCATEGORIES: III. REPRESENTATIONS, AND EXPANSIONS OF PREVIOUS EXAMPLES

*by René GUITART*

**Abstract.** An *autograph* is a set $A$ with an action of the free monoïd with 2 generators, and an *autographic monad* is a monad on the topos of autographs. In previous papers we have shown that knots and $double$-categories are examples, and we proved that basic graphic algebras are *autographic algebras*. In this third paper we add three new results. We explain how to get concrete representations of autographs and conversely how to collect any representation into an autograph. We precise previous results and extend them, showing that knots and general links and grid diagrams are autographs, and that general graphic algebras are some *autographic algebras*.

**Résumé.** Un *autographe* est un ensemble $A$ équipé d'une action du monoïde libre à deux générateurs, une *algèbre autographique* est une algèbre d'une monade sur le topos des autographes. Dans deux articles précédents nous avons vu que les diagrammes de nœuds et les 2-graphes sont des exemples, et que les algèbres graphiques basiques sont autographiques.

Dans ce troisième article, nous ajoutons trois résultats nouveaux. Nous montrons comment représenter concrètement les autographes, et réciproquement comment collecter une représentation en un autographe, nous expliquons précisément comment les nœuds, les entrelacs, les diagrammes de grilles, et aussi les catégories doubles, sont des exemples d'autographes, et nous identifions les algèbres graphiques générales avec des algèbres autographiques.

**Keywords.** graph, autograph, autographic algebra, autographic monad, knot, link, double category.

**Mathematics Subject Classification (2010).** 18C, 57M25.

## 1. Category $\mathrm{Rep}(A, d, c)$ of representations of an autograph

Of course the construction in this section could work when $\mathrm{Set}$ is replaced by an arbitrary topos $\mathcal{E}$, providing $\mathrm{Auto}[\mathrm{Rel}(\mathcal{E})]$ and $\mathrm{Auto}[\mathcal{E}]$, and consequently with the topos $Agraph$ we would get Auto[Agraph], etc.

**Definition 1.1.** *1 — An* autograph *is the data $A = (\underline{A}, d_A, c_A)$ of a set $\underline{A}$ and two maps $d_A : \underline{A} \to \underline{A}$, $c_A : \underline{A} \to \underline{A}$. Abusively, often the set $\underline{A}$ will be denoted by $A$, and $d_A$ and $c_A$ are denoted by $d$ and $c$. If we denote by $\mathbb{FM}(2) = \{d, c\}^*$ the free monoïd on two generators $d$ and $c$ (and with unit $v$) then an autograph is an action $A(-)$ of $\mathbb{FM}(2)$, with $A(v) = A, A(d) = d_A, A(c) = c_A$. We represent $a \in \underline{A}$ with $d_A a = v$ and $c_A a = w$, by: $a : v \to w$, or $v \overset{a}{\to} w$.*
*2 — The category of autographs is* $\text{Agraph} = \text{Set}^{\mathbb{FM}(2)}$ *— a topos of course — a morphism in it from $A$ to $A'$ being a map $f : \underline{A} \to \underline{A'}$ satisfying*

$$d' fa = fda, \quad c' fa = fca.$$

*3 — An* autocategory *[3, Definition 6.1] is an autograph with identifier and a unitary and associative composition for consecutive arrows.*

The purpose of this section is to show how to represent concretely such autographs, and, starting from these representations, how to elaborate new "collected" autographs.

### 1.1 From Autorelations to autographs, and conversely

**Proposition 1.2.** *Considered as sets we have $\mathbb{FM}(2) = \mathbb{FA}(()) = \mathbb{FA}(\{f\})$ (the free autograph on one generator, see [3]), and they consist in words written with c and d, with maps $d(-)$ and $c(-)$ given by $m \mapsto dm$, $m \mapsto cm$. By a (binary)* Autorelation *we mean a family of sets $R = (R_m)_{m \in \mathbb{FM}(2)}$, with*

$$R_m \subseteq R_{dm} \times R_{cm},$$

*or with the induced "projections" $c_m : R_m \to R_{cm}$ and $d_m : R_m \to R_{dm}$. The set of these $R$ is denoted by* Auto[Rel].
*In such an $R$, each element $\xi$ in $R_{()}$ or in any $R_m$ generates an image $R_\xi$ of $\mathbb{FM}(2)$ which is an autograph, and the set $R_\infty$ disjoint union of the $R_m$*

$$R_\infty = \dot{\cup}_m R_m = \cup_m (R_m \times \{m\}),$$

*is itself an autograph, union of these $R_\xi$. Furthermore $\pi_R : (\xi, m) \mapsto m$ is a morphism of autographs*

$$\pi_R : R_\infty \to \mathbb{FA}(()).$$

*Conversely, given a morphism of autographs $\pi : S \to \mathbb{FA}(())$ we can reconstruct an autorelation, with $R_m = \pi^{-1}(m)$.*

**Example 1.3.** Given a data $\underline{B}$ of 3 sets $X, Y, Z$ and 6 maps $c_X : X \to Y$, $c_Y : Y \to Z$, $c_Z : Z \to X$, and $d_X : X \to Z$, $d_Y : Y \to X$, $d_Z : Z \to Y$, we get maps $X \overset{(c_X,d_X)}{\longrightarrow} Y \times Z$, $Y \overset{(c_Y,d_Y)}{\longrightarrow} Z \times X$, $Z \overset{(c_Z,d_Z)}{\longrightarrow} X \times Y$, and a finite generator of an autorelation

$$X \subset Y \times Z, \quad Y \subset Z \times X, \quad Z \subset X \times Y,$$

the associated autorelation $B$ being given by $B_{()} = X$ and

$$B_c = Y, B_d = Z, B_{cc} = Z, B_{dc} = X, B_{cd} = X, B_{dd} = Y, B_{ccc} = X, \ldots$$

**Example 1.4.** With notations from [3, Proposition 3.1.], interpreting each $R_m$ as $\mathbb{N}$, with "projections" $c(n) = t_1(n) = 3n + 1$, $d(n) = t_2(n) = 3n+2$, we get an autorelation "on" $\mathbb{N} = \mathbb{FA}(3\mathbb{N})$, of which the corresponding autograph is $\mathbb{FA}(3\mathbb{N}) \times \mathbb{FA}(\{f\})$, equipped with a morphism

$$\mathbb{FA}(3\mathbb{N}) \times \mathbb{FA}(\{f\}) \to \mathbb{FA}(\{f\}).$$

**Proposition 1.5.** *An autograph $A = (\underline{A}, d_A, c_A)$ determines $\underline{A} \overset{(d_A,c_A)}{\longrightarrow} \underline{A} \times \underline{A}$, and so we get an autorelation "on" $\underline{A}$, as in examples 1.3 and 1.4.*

### 1.2 Set $\mathcal{R}(A, d, c)$ of relational representations of an autograph

**Definition 1.6.** *A* relational representation *(or a* spanning representation*) of an autograph $(A, d, c)$ is a data $\varphi = (\Phi, \phi^d, \phi^c)$, with for each $f \in A$, the data of a set $\Phi(f)$ and of a span of functions*

$$\Phi(df) \overset{\phi^d(f)}{\longleftarrow} \Phi(f) \overset{\phi^c(f)}{\longrightarrow} \Phi(cf),$$

*which are the induced "projections" associated to a specified inclusion*

$$\Phi(f) \subset \Phi(df) \times \Phi(cf).$$

*We denote by $\mathcal{R}(A, d, c)$ the set of these relational representations.*

**Proposition 1.7.** *Given a relational representation* $\varphi = (\Phi, \phi^d, \phi^c)$ *of an autograph* $A = (\underline{A}, d_A, c_A)$*, we* collect *it over* $A$*, constructing a map of autograph*

$$\pi_\varphi : \Sigma\varphi = (S_\varphi, d_\varphi, c_\varphi) \longrightarrow (\underline{A}, d_A, c_A) = A$$

*with*

$$S_\varphi = \{(f, u); f \in A, u \in \Phi(f)\},$$

$$d_\varphi(f, u) = (\phi^d(u), d_A f), \quad c_\varphi(f, u) = (\phi^c(u), c_A f), \quad q_\varphi(f, u) = f.$$

**Example 1.8.** A relational representation of $\mathbb{FA}(())$ is exactly an autorelation as in 1.2, so the set Autorel$[\mathrm{Set}]$ of autorelations is $\mathcal{R}(\mathbb{FA})(())$, and the $\pi_R$ is a case of a $\pi_\varphi$.

### 1.3 From autorelations to automaps, and conversely

**Definition 1.9.** *We define* Auto[Set] *as the set of automaps, an* automap *being a sequence* $f = (f_m)_{m \in \mathbb{FM}(2)}$ *of maps* $f_m : G_{dm} \to G_{cm}$*, each* $G_n$ *being the graphic of* $f_n$*,*

$$G_n = \{(x, y); x \in G_{dn}, y \in G_{cn},\ y = f_n(x)\} \simeq G_{dn}.$$

*Of course such an automap is an autorelation, and* Auto[Set] $\subset$ Auto[Rel].

**Proposition 1.10.** *An autorelation* $R$ *determines an automap* $\hat{R}$ *given by maps* $\hat{R}_m : \mathcal{P}(R_{cm}) \to \mathcal{P}(R_{dm})$ *with* $\mathcal{P}(E)$ *the set of subsets of* $E$*, and*

$$\hat{R}_m(X) = \{y \in R_{dm}; \exists z \in R_m, (c_m(z) \in X \wedge d_m(z) = y)\}.$$

*So we get an injection* $\hat{-}$ : Auto[Rel] $\longrightarrow$ Auto[Set].

### 1.4 Set $\mathcal{F}(A, d, c)$ of functional representations of an autograph

**Definition 1.11.** *A* functional representation *of an autograph* $(A, d, c)$ *is a data* $(\Phi, \phi)$*, with for each* $f \in A$*, the data of sets* $\Phi(df)$ *and* $\Phi(cf)$*, and of a function*

$$\phi(f) : \Phi(df) \to \Phi(cf).$$

*The set of these functional representations is denoted by* $\mathcal{F}(A, d, c)$*, and as a functional representation is a special case of a relational representation — with* $\Phi(f) = \Phi(df)$ *— we have* $\mathcal{F}(A, d, c) \subset \mathcal{R}(A, d, c)$*.*

**Proposition 1.12.** *The definition in the construction of* Auto[Set] *in Proposition 1.9 determines each automap as a functional representation of the free autograph* $\mathbb{FA}(())$*, and* Auto[Set] *as a subset of* $\mathcal{F}(\mathbb{FA}(()))$*. A fortiori,* Auto[Rel] *being a subset of* Auto[Set]*, it is also a subset of* $\mathcal{F}(\mathbb{FA}(()))$.

**Proposition 1.13.** *As in the case of Proposition 1.10 we have an injection*

$$\hat{-} : \mathcal{R}(A, d, c) \longrightarrow \mathcal{F}(A, d, c).$$

**Proposition 1.14.** *We get a category* $\mathrm{Rep}(A, d, c)$ *of representations of an autograph, with objects the elements of* $\mathcal{F}(A, d, c)$*, a morphism from* $(\Phi, \phi)$ *to* $(\Phi', \phi')$ *being a double collection* $(t^d_f, t^c_f)_{f\in A}$ *of maps*

$$t^d_f : \Phi(df) \to \Phi'(df),\ t^c_f : \Phi(cf) \to \Phi'(cf),$$

*such that*

$$t^c_f \phi_f = \phi'_f t^d_f.$$

### 1.5 The regular representation, object of $\mathrm{Rep}(A, d, c)$

The natural representation for a category is given by Yoneda's lemma, with at first the following basic fact. For each category $\mathcal{C}$ we have a faithful representation by a functor $U_\mathcal{C} : \mathcal{C} \longrightarrow \mathrm{Set}$, given by $U_\mathcal{C}(A) = \dot{\cup}_X \mathrm{Hom}(X, A)$, and $U_\mathcal{C}(f)(u) = f.u$, when $f : A \to B$. In the special case where $\mathcal{C}$ is the free category of paths in a graph $G$, this provides the representation of $G$ by action on its paths. Similarly for an autograph $(A, d, c)$ we have:

**Proposition 1.15.** *For each autograph* $A$ *we have the following faithful* regular representation $f \mapsto (\Gamma(f), \gamma(f))$ *with:*
*1 — The set* $\Gamma(f)$ *of* $(d, c)$*-paths (cf. [3, Definition 1.4]) with end* $f$ *i.e.*

$(z_n)_{0\leq n\leq k-1}$, with $cz_0 = dz_1, cz_1 = dz_2 \ldots, cz_{k-2} = dz_{k-1}$ and $cz_{k-1} = f$.

*2 — A map* $\gamma(f) : \Gamma(df) \to \Gamma(cf)$*, given by concatenation with* $f$*, by*

$$\gamma(f)((z_n)_{0\leq n\leq k-1}) = (z'_n)_{0\leq n\leq k},$$

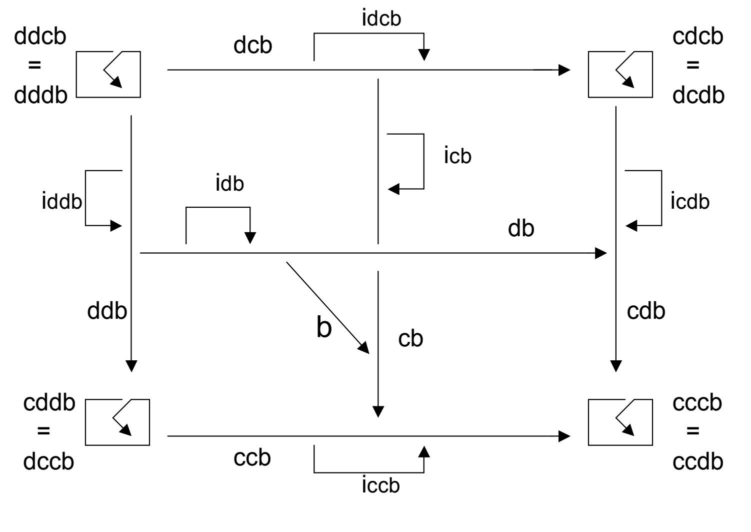
*with* $z'_n = z_n$ *if* $n < k$*, and* $z'_k = f$*.*
*Shortly if* $(z_n)_{0\leq n\leq k-1} = z$*, then* $(z'_n)_{0\leq n\leq k} = fz$*, or* $\gamma(f)(z) = fz$*.*
*So* $(A, d, c)$ *can be identified with a special element of* $\mathcal{F}(A, d, c)$ *or object of* $\mathrm{Rep}(A, d, c)$*.*

## 2. Double categories, Knots, Links, Grid Diagrams

In the first paper of this series [3] we obtained that the topos Agraph of autographs is a common setting for knots and 2-categories or double categories. Here this result is strengthened and extended, using 2-dimensional paths in double categories and grid diagrams.

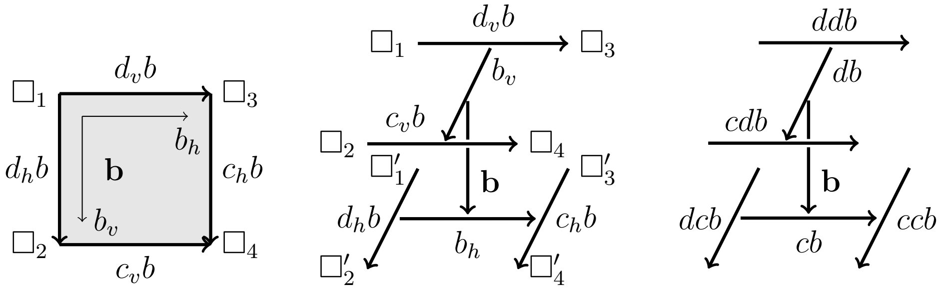
### 2.1 Double categories and knots as well formed 2-dim words

**Proposition 2.1.** *A double category* $\mathbb{C}$ *is determined by an associated autocategory* $\mathrm{Ass}(\mathbb{C})$*, according to the following picture to represent a* $2$*-block* $b$ *as an autograph:*



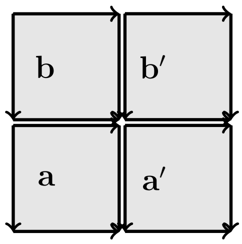
A 2-block b as an autograph

*Proof.* A 2-block $b$ (fig.[1] below) is considered as an arrow from its two oriented versions, its vertical orientation $b_v$ and its horizontal orientation $b_h$. Then $b_v$ is an arrow from $d_v b$ to $c_v b$, etc. (fig.[2]). Hence (fig.[3]) a resulting autograph, which can be completed and redrawn as in Proposition 2.1 above. The full description of $\mathrm{Ass}(\mathbb{C})$ is explained in [3, Proposition 7.1], but we have to correct a typos there: in the picture an autoarrow $I_{b_v} = (b_v)^\theta : b_v \to b_v$ should be added.




[1] [2] [3]

In particular we have shown how the two horizontal and vertical compositions, denoted by $\infty$ and $8$

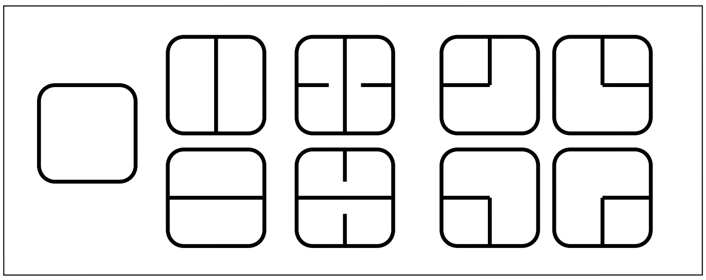



and compatible according to

$$(a'\infty a)8(b'\infty b) = (a'8b')\infty(a8b),$$

are replaced by a unique composition law. $\square$

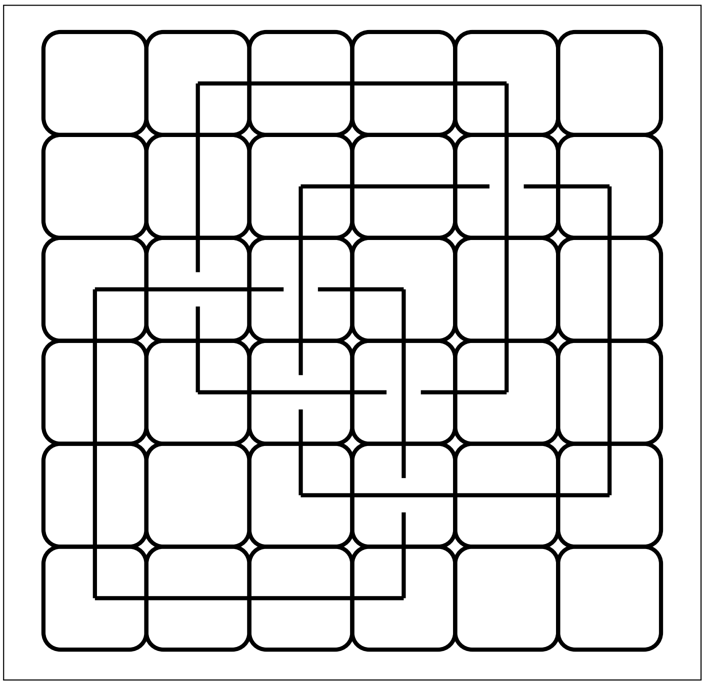
**Proposition 2.2.** *Any knot or link can be presented as a 2-dim rectangular "well formed" word, on a rectangle $\mathcal{R}_{n,m}$ of dimension $n \times m$ made with the tiles from the set $\mathbb{T}$ of the 9 following tiles (a word is well formed if each line decoration into any tile arriving on a side of this tile is pursued in the next adjacent tile). Of course it is a map $L : \mathcal{R}_{n,m} \to \mathbb{T}$, and of course such a data is representable as an autograph.*



*Consequently a link is a $2$-dim path in the double category generated by these tiles (or in the corresponding autograph according to Proposition 2.1).*

**Remark 2.3.** Hence the question of isotopy type of links becomes a question of 2-dim rewriting, as explained in [2] (This is also near from studies on mosaics [6]). There are vertical (or horizontal) dilatations: if a column consists only in empty tiles or horizontal line tiles, we can add a new similar column juxtaposed to the first one; and furthermore they are analogous to the three Reidemeister moves.

**Example 2.4.** The following 2-dim word is a borromean link.



### 2.2 Knots, from their knot diagrams

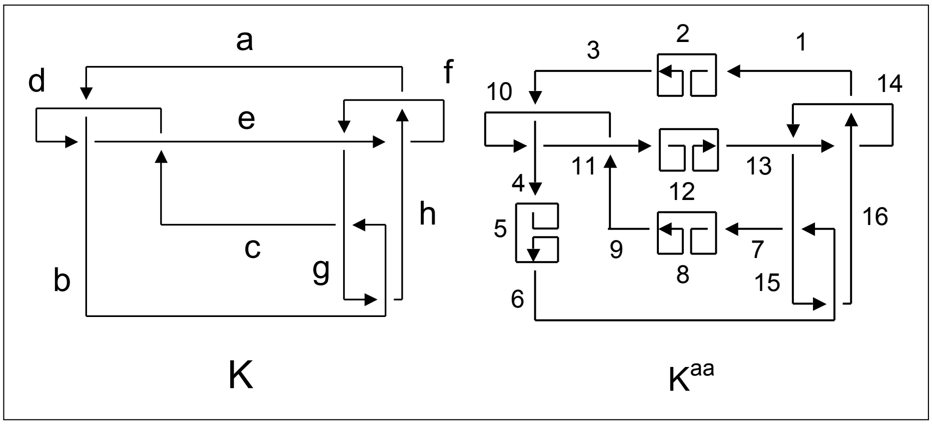
In the paper [3, section 4] for a knot $\mathbb{K}$ we introduced an associated autograph $\mathrm{As}(\mathbb{K})$, used for trefoil or borromean knot and link. The following Proposition 2.5 strengthens this result.

**Proposition 2.5.** *If $\mathbb{K}$ is an alternating knot, then from $\mathrm{As}(\mathbb{K})$ we recover the Gauss'code of $\mathbb{K}$, and so this knot is determined by its associated autograph.*

For a general knot a modification of the construction is necessary, following Proposition 2.6

**Proposition 2.6.** *If* $\mathbb{K}$ *is an arbitrary knot, then from the autograph* $\mathrm{As}(\mathbb{K}^{aa})$ *— with* $\mathbb{K}^{aa}$ *defined in the proof — we recover the Gauss'code of* $\mathbb{K}$*, and so this knot is determined by this autograph.*

*Proof.* If the knot is not alternating, then we cannot recover the Gauss'code from $\mathrm{As}(\mathbb{K})$. For example the **K** in the next picture is a not-alternating knot, and we consider an arc which is not going in an alternative way, as $e$ from $b$ to $h$, passing over in two consecutive crossings; hence we have $c$ and $f$ arriving to $e$, but in $\mathrm{As}(\mathbf{K})$ we have no information on the order in which these arrows arrived on $e$: following $e$ which one is the first met, $c$ or $f$? So before considering $\mathrm{As}(\mathbf{K})$ we decide to modify **K** into $\mathbf{K}^{aa}$ as follows. In **K** we observe arcs which are alternating, as $d$, $f$, $g$, $h$, and the others, $a$, $b$, $c$, $e$ are said to be not-alternating. Each of these not-alternating arcs (see in $\mathbf{K}^{aa}$) is now decomposed by introducing autoarrows, $2, 5, 8, 12$, and we have $a = 1.2.3$, $b = 4.5.6$, $c = 7.8.9$, $e = 11.12.13$. Now $c$ or rather $9$ arrives to $11$, whereas $14$ arrives to a different arc, namely $13$, and we can recover the Gauss'code of **K** from $\mathrm{As}(\mathbf{K}^{aa})$. For the Gauss'code see [5, p.666].

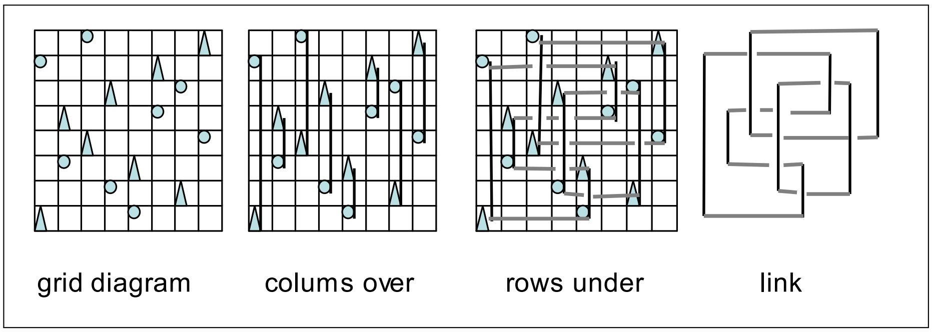



∎

## 2.3 Grid diagrams and links isotopy types

**Proposition 2.7.** *Any link can be associated to an autograph determining its isotopy type.*

*Proof.* A *Grid diagram* [7], is an $n \times n$ square with $n$ triangles and $n$ circles placed in distinct places, such that each row and each column contains exactly one triangle and one circle. In the next picture the first left drawing is an example with $n = 8$. Given such a grid, we join the triangle and the circle in each column by continuous straight vertical lines (second step in the picture), and then in each row we join the triangle and the circle by a straight line passing under the previously vertical straight lines it meets (third step). And finally (fourth step) we look at a link. In this example it is a borromean link (but presented differently from the picture given in [3, Example 4.5]). Another borromean example is furnished by Example 2.4.




Now, as any isotopic type of link can be obtained in this way [1], we conclude if we can show that any grid can be determined by an autograph, and this is obvious since a grid is a graph. □

## 3. Graphic monads among autographic monads

### 3.1 The topos Agraph of autographs, between Graph and Set

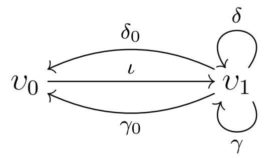
#### 3.1.1 Autographs and graphs

According to [3, def.1.1., p.66], [4, def.1.1.-1.2, p.152], we have:

**Definition 3.1** (Graphs)**.** *Let* $\mathbb{G}(2)$ *be the category with objects* $\upsilon_0$ *and* $\upsilon_1$*, five non-identity arrows*

$$\gamma_0, \delta_0 : \upsilon_1 \to \upsilon_0, \;\; \iota : \upsilon_0 \to \upsilon_1, \;\; \delta, \gamma : \upsilon_1 \to \upsilon_1,$$

*identities on* $\upsilon_1$, $\upsilon_0$, *equations:* $\delta_0.\iota = 1_{\upsilon_0},\ \gamma_0.\iota = 1_{\upsilon_0},\ \gamma = \iota.\gamma_0,\ \delta = \iota.\delta_0.$




*A presheaf* $G$ *on* $\mathbb{G}(2)$, *i.e. an object of* Graph $= \mathrm{Set}^{\mathbb{G}(2)}$ *is named a* graph. *Any* $c \in G(\upsilon_0)$ *is named a* vertex *or a* carfour, *and if* $f \in G(\upsilon_1)$, $f$ *is named an* arrow; *the fact that* $G(\delta_0)(f) = c$ *and* $G(\gamma_0)(f) = c'$ *is written:* $f : c \to c'$.

### 3.1.2 The comparison V and its equivalent W

With [4, Prop.1.4 p.153, Prop.2.2. p.154] the comparison between autographs and graphs is given by a functor V : Graph → Agraph. :

**Proposition 3.2.** *The categories* Agraph *and* Graph *are toposes, inscribed in the sequence*

$$\mathrm{Graph} \overset{\mathrm{V}}{\to} \mathrm{Agraph} \overset{\mathrm{U}}{\to} \mathrm{Set},$$

*where* $\mathrm{U} = {eva_v}^{\mathbb{FM}(2)}$ *is the monadic forgetful functor given by evaluation at* $v$, $(A, (d_A, c_A)) \mapsto A$, *and* $\mathrm{V} = \Phi = (-).\phi$ *is the monadic functor induced by the map* $\phi : \mathbb{FM}(2) \to \mathbb{G}(2); v, c, d \mapsto \upsilon_1, \gamma, \delta$.

**Remark 3.3.** In a graph $G$ the $G(\iota) = \phi$ associates to each vertex an arrow. Hence we have the more general situation of *flexigraph* in [3, Defg. 5.4.]. An autograph $A$ appears as a special case of a flexigraph, when $\phi = 1_A$. Let us recall also from [4] that with graphs we have 2 types of arities (vertices and arrows), wheras with autographs only 1 type (arrows) is considered. Next Proposition 3.4 clarifies thist point.

**Proposition 3.4.** *The category* $\mathbb{G}(2)$ *is the Karoubian envelope of the monoïd* $\mathbb{M}(2) = \{1, c, d\}$, *with equations* $c^2 = c, d^2 = d, cd = d, dc = c$, *and* Graph *is equivalent to* $\mathrm{Set}^{\mathbb{M}(2)}$. *Up to this equivalence, the functor* V *is the functor* $\mathrm{W} = \mathrm{Set}^{\bar{\phi}}$ *induced by the composition with the monoïd quotient homomorphism* $\bar{\phi}$ *given by* $v \mapsto 1$, $c \mapsto c, d \mapsto d$:

$$\bar{\phi} : \mathbb{FM}(2) \to \mathbb{M}(2),$$

$$\mathrm{W} = \mathrm{Set}^{\bar{\phi}} : \mathrm{Set}^{\mathbb{M}(2)} \to \mathrm{Set}^{\mathbb{FM}(2)}.$$

### 3.2 From graphic monads to autographic monads

In the second paper of this series [4], via graphic monoïds of Lawvere we have shown that basic Albert Burroni's graphic algebras are autographic algebras; and especially as autographic algebras we get categories as well as autocategories. Now we have to precise the study for graphic algebras which are not necessarily basic. Albert Burroni defined a *graphic algebra* as an algebra of a monad on Graph (a graphic monad); similarly we defined an *autographic algebra* as an algebra of a monad on Agraph (an autographic monad). So we have to complete the result in the general case, for algebras of arbitrary monads on Graph $= \mathrm{Set}^{\mathbb{G}(2)} \simeq \mathrm{Set}^{\mathbb{M}(2)}$ and their transport via the W in Proposition 3.4.

Generally speaking, for any homomorphism of monoïds $\varphi : F \to M$ the induced functor

$$\mathrm{Set}^{\varphi} : \mathrm{Set}^{M} \to \mathrm{Set}^{F}$$

is monadic, This works for our $\mathrm{W} = \mathrm{Set}^{\bar{\phi}} : \mathrm{Set}^{\mathbb{M}(2)} \to \mathrm{Set}^{\mathbb{FM}(2)}$ from Proposition 3.4, as well as for any quotient map of monoïds

$$q_M : \mathbb{FM}(2) \to M,$$

the corresponding $\mathrm{W}_{q_M} = \mathrm{Set}^{q_M}$, its left adjoint $\mathrm{Lan}_{q_M}$, $T_M = (-)^{q_M}\mathrm{Lan}_{q_M}$ and $\mathbb{T}_M = (T_M, r)$ the associated idempotent monad on $\mathrm{Set}^{\mathbb{FM}(2)} =$ Agraph. So, for a given $q_M$ — see [4, Proposition 2.6]) — the topos $\mathrm{Set}^M$ is a reflexive subcategory of Agraph, with for any $E = (A, d, c)$ a reflexion

$$r_E : E \to T_M(E),$$

given by a quotient set $T_M(E) = T_M(A, d, c) = A/[q_M]$, quotient of $A$ by the smallest congruence $[q_M]$ on $E$ compatible with $q_M$.

**Proposition 3.5.** *With the notations above for a given $q_M : \mathbb{FM}(2) \to M$ (and the associated monad $\mathbb{T}_M$) on Agraph, we consider another monad $\mathbb{T} = (T, \eta, \mu)$ on $\mathrm{Set}^M$. Then the functor*

$$(\mathrm{Set}^M]^{\mathbb{T}} \xrightarrow{U_{\mathbb{T}}} \mathrm{Set}^M \xrightarrow{\mathrm{W}_{q_M}} \mathrm{Set}^{\mathbb{FM}(2)} = \text{Agraph}$$

*determines a monad $\bar{\mathbb{T}} = (\bar{T}, \bar{\eta}, \bar{\mu})$ on* Agraph*, of which an algebra $\bar{\theta}$ on $E$ is a composition $\bar{\theta} = \lambda\theta$ of an algebra $\theta = r_E\bar{\theta}$ of $\mathbb{T}$ on $T_M(E)$ and of a*

*special section $\lambda = \bar{\theta}\eta_{E/[q]}$ of the reflexion $r_E : E \to T_M(E)$.*
*Consequently we have*

$$(\mathrm{Set}^M]^{\mathbb{T}} \simeq \mathrm{Agraph}^{\bar{\mathbb{T}}} \cap \mathrm{Agraph}^{\mathbb{T}_M}.$$

*In particular this is true for $M = \mathbb{M}(2)$ and the corresponding $W$, and algebras of graphic monad are such special algebras of autographic monads.*

*Proof.* With $(\bar{\mathbb{T}} = (\bar{T}, \bar{\eta}, \bar{\mu})$ the monad associated to $\mathrm{W}_{q_M} U_{\mathbb{T}}$ we have, with $T_M(E) = E/[q] = A/[q_M]$ and $r_E : E \to E/[q]$, the following formula: $\bar{T}(E) = T(E/[q])$, $\bar{\eta}_E = \eta_{E/[q]} r_E$, $\bar{\mu}_E = \mu_{E/[q]}$, $\bar{T}^2(E) = T^2(E/[q])$.

If $(E, \bar{\theta})$ is a $\bar{\mathbb{T}}$-algebra on $E$, then we introduce $\theta = r_E\bar{\theta}$, and so $T\theta = T(r_E\bar{\theta})$. The $\bar{\mathbb{T}}$-associativity $\bar{\theta}\bar{T}\bar{\theta} = \bar{\theta}\bar{\mu_E}$ implies, by composition on the left with $r_E$, $r_E\bar{\theta}T(r_E\bar{\theta}) = r_E\bar{\theta}\bar{\mu}_E$ i.e. the $\mathbb{T}$-associativity: $\theta.T\theta = \theta\mu_{E/[q]}$. Also from $\bar{\mathbb{T}}$-unitarity we obtain $\mathbb{T}$-unitarity, $\theta\eta_{E/[q]} = 1_{E/[q]}$, from $\bar{\theta}\bar{\eta}_E = 1_{E]}$ by composition on the left with $r_E$: $\theta\eta_{E/[q]}r_E = r_E$. So we obtain $(E/[q], \theta)$ a $\mathbb{T}$-algebra on $E/[q]$.

In fact introducing $\lambda = \bar{\theta}\eta_{E/[q]}$, we obtain $\lambda\theta = \bar{\theta}$, and $r_E\lambda = 1_{E/[q]}$. For the first we have $\bar{\theta}\eta_{E/[q]}r_E\bar{\theta} = \bar{\theta}$, i.e. $\bar{\theta}\bar{\eta_E}\bar{\theta} = \bar{\theta}$. For the second, by composition on the right with the epimorphism $\theta$ we get $r_E\lambda\theta = \theta$, or $r_E\bar{\theta}\eta_{E/[q]}\theta = \theta$, $\theta\eta_{E/[q]}\theta = \theta$.

So any $\bar{\theta}$, $\bar{\mathbb{T}}$-algebra on $E$, determines two things: $\theta$, an algebra on $E/[q]$, and $\lambda$, a section of $r_E : E \to E/[q]$. Conversely, given $\theta$ and $\lambda$, we recover $\bar{\theta} = \lambda\theta$.
Especially, a $\mathbb{T}$-algebra is a $\bar{\mathbb{T}}$-algebra on a $E$ such that $E \simeq E/[q]$, i.e. a $E$ equipped with a $\mathbb{T}_M$-algebra structure ($\lambda = 1_E = r_E$). □

# References


[1] P. R. Cromwell, 'Embedding knots and links in an open book I: Basic properties', *Topology Appl.* 64, (1995), no 1, 37-58.

[2] R. Guitart, *Taquins, spineurs, fibrés*, Conférence à Tours, le 8 avril 1993.

[3] R. Guitart, Autocategories: I. A common setting for knots and 2-categories, *Cahiers Top. Géo. Diff. Cat.* LV-1 (2014), 66-80.

[4] R. Guitart, Autocategories: II. Autographic Algebras, *Cahiers Top. Géo. Diff. Cat.* LV-2 (2014), 151– 160.

[5] L. H. Kauffman, 'Virtual Knot Theory', *Europ. J. Combinatorics* (1999), 20, 663-691.

[6] S. Lomonaco and L. N. Kauffman, *Quantum Knots and Mosaics*, arXiv:0805.0339v1 [quant-ph] 3 may 2008, 32 p.

[7] L. Ng and D. Thurston, 'Grid Diagrams, Braids, and Contact Geometry', in *Proceedings of 13th Gökova Geometry-Topology Conference 2006*, edited by S. Akbulut, T. Onder, R.J. Stern, Int. Press of Boston Inc., Somerville, (2007), pp. 1-17.

René Guitart
Université Paris Diderot Paris 7. IMJ-PRG. UMR 7586
Bâtiment Sophie Germain. Case 7012
75205 Paris Cedex 13
rene.guitart@orange.fr