CAHIERS DE TOPOLOGIE ET GEOMETRIE DIFFERENTIELLE CATEGORIQUES

Vol. LVIII-3&4 (2017)

CONDITION FOR AN n-PERMUTABLE CATEGORY TO BE MALTSEV

by Sean TULL

Résumé. Nous améliorons la description des catégories n-permutables introduites par Carboni, Kelly et Pedicchio [2]. Cela donne une nouvelle caractérisation des catégories régulières de Maltsev parmi celles qui sont des catégories de Goursat ou, plus généralement, des catégories n-permutables.

Abstract. We give a strengthening of the description of an n-permutable category due to Carboni, Kelly and Pedicchio [2]. This provides a new characterisation of the regular Mal'tsev categories from among those which are Goursat categories, or more generally n-permutable.

Keywords. n-permutable category, Mal'tsev category, Goursat category

Mathematics Subject Classification (2010). 18B10, 08B05

Mal'tsev categories form an important and well-known class of categories in the study of universal algebra [3, 10]. In fact, many of their interesting properties extend to the broader class of Goursat categories [4, 6]. While most examples of Goursat categories are in fact Mal'tsev categories, no simple conditions for when this is the case have yet been presented. In [2], Carboni, Kelly and Pedicchio showed that both classes belong to the more general hierarchy of n-permutable categories. In this note, we give a strengthening of their original characterisation of n-permutable categories, leading to a condition for when a Goursat or n-permutable category is a Mal'tsev category. The condition, called *positive regularity*, is mild, satisfied even in logically well-structured categories such as the category $\mathbf{Set}$ of sets and functions.

1. A Condition for n-permutability

We work in a regular category $\mathbf{C}$. The internal logic of such categories allows one to reason using elements just as in $\mathbf{Set}$ [1, Metatheorem A.5.7], and we will use this technique throughout. For objects A, B in $\mathbf{C}$, recall that a *relation* $R\colon A \to B$ is a subobject $R \rightarrowtail A \times B$. Any such relation comes with a converse which we denote $R^\circ\colon B \to A$. Every pair of relations $R\colon A \to B$ and $S\colon B \to C$ have a composite $SR\colon A \to C$ defined by

$$SR = \{(a, c) \in A \times C \mid \exists b\, R(a, b) \wedge S(b, c)\}$$

Relations are partially ordered by the usual inclusion of subobjects. We call a relation $E\colon A \to A$ *reflexive* when $\mathrm{id}_A \leq E$, *symmetric* when $E = E^\circ$, *transitive* when $EE \leq E$, and an *equivalence relation* when all of these hold. Following [2], for any pair of relations $R\colon A \to B$, $S\colon B \to C$, we define a sequence of relations

$$(S, R)_1 = S,\ (S, R)_2 = SR,\ (S, R)_3 = SRS,\ (S, R)_4 = SRSR,\ \ldots$$

A regular category $\mathbf{C}$ is then *n-permutable* whenever $(E, E')_n = (E', E)_n$ for every pair of equivalence relations E, E' on the same object. In particular, a regular category is a *Mal'tsev category* when it is 2-permutable, and a *Goursat category* when it is 3-permutable. We begin by strengthening a characterisation from [2].

Theorem 1. *Let $\mathbf{C}$ be a regular category and $n \geq 2$. The following conditions are equivalent:*

(i) $\mathbf{C}$ *is n-permutable;*

(ii) *Every relation $R\colon A \to B$ in $\mathbf{C}$ satisfies $(R^\circ, R)_{n+1} = (R^\circ, R)_{n-1}$;*

(iii) *For every reflexive relation $E\colon A \to A$ in $\mathbf{C}$, $(E, E^\circ)_{n-1}$ is an equivalence relation;*

(iv) *For every reflexive relation $E\colon A \to A$ in $\mathbf{C}$, E^{n-1} is an equivalence relation;*

(v) *For every reflexive and symmetric relation $E\colon A \to A$ in $\mathbf{C}$, E^{n-1} is an equivalence relation.*

Proof. See [2] for the equivalence of (i), (ii) and (iii). (i) $\implies$ (iv) is also well-known: in an n-permutable category any reflexive relation E has that E^{n-1} is transitive [7, Theorem 1], and every reflexive and transitive relation is symmetric [9, Theorem 1]. Clearly (v) follows from (iii) or (iv).

We now show that (v) $\implies$ (ii). First note that by our assumption any reflexive and symmetric relation E satisfies $E^n = E^{n-1}$. Indeed, always $E^{n-1} \leq E^n$, while we now have $E^n \leq E^{2(n-1)} \leq E^{n-1}$ by transitivity. Now for any relation $R \rightarrowtail A \times B$, we always have $(R^\circ, R)_{n-1} \leq (R^\circ, R)_{n+1}$, so it suffices to show the converse holds. Define a new relation $S \rightarrowtail R \times R$ by

$$S((a,b),(a',b')) \iff R(a,b') \wedge R(a',b)$$

noting by definition that $S((a,b),(a',b'))$ also implies $R(a,b)$ and $R(a',b')$. Then S is reflexive since it is defined on R, and symmetric by definition. Hence $S^n = S^{n-1}$.

Suppose that $(R^\circ, R)_{n+1}(b,a)$, via a sequence of elements $(x_i)_{i=0}^{n+1}$ with $x_0 = a$, $x_{n+1} = b$ satisfying $R(x_i, x_{i-1})$ for even $i \geq 2$, and $R(x_i, x_{i+1})$ for even $i \leq n$. Then we have $S((x_i, x_{i+1}), (x_{i+2}, x_{i+1}))$ for all even $i \leq n-1$ and $S((x_{i+2}, x_{i+1}), (x_{i+2}, x_{i+3}))$, for all even $i \leq n-2$. So defining $(y,z) := (b, x_n)$ if n is odd, or $(y,z) := (x_n, b)$ if n is even, we have $S^n((a,x_1),(y,z))$. Hence $S^{n-1}((a,x_1),(y,z))$ also.

Letting $(y_0, z_0) = (a, x_1)$ and $(y_{n-1}, z_{n-1}) = (y,z)$, this means there is a sequence of pairs $(y_i, z_i)_{i=0}^{n-1}$ satisfying $S((y_i,z_i),(y_{i+1},z_{i+1}))$ for all $i \leq n-2$. In particular, we have $R(y_i, z_{i-1})$ for even $i \geq 2$ and $R(y_i, z_{i+1})$ for even $i \leq n-2$. Hence via the sequence $a = y_0, z_1, y_2, z_3, \ldots$ of length n ending in b, we have $(R^\circ, R)_{n-1}(b,a)$, as desired. □

2. Positively regular and Mal'tsev categories

We now turn to classifying the Mal'tsev categories as the n-permutable categories with a special property. Let us call a relation $E \colon A \to A$ *positive* when it is of the form $E = R^\circ R$ for some relation $R \colon A \to B$. The following notion first appeared in [5].

Proposition 2. *For a regular category* $\mathbf{C}$*, the following are equivalent:*

(i) *A relation* $E \colon A \to A$ *in* $\mathbf{C}$ *is positive if and only if it satisfies:*

$$E(a,b) \implies E(a,a) \wedge E(b,a) \qquad (*)$$

(ii) *Any reflexive and symmetric relation in* $\mathbf{C}$ *is positive.*

We call a regular category satisfying either of these equivalent conditions positively regular[1].

Proof. For (i) $\implies$ (ii), and the 'only if' in (ii) $\implies$ (i), note that any reflexive, symmetric relation in a regular category automatically satisfies $(*)$, as does any positive relation. Conversely, if (ii) holds and $E\colon A \to A$ satisfies $(*)$, define

$$I = \{a \in A \mid \exists b \; E(a,b)\} \rightarrowtail A$$

writing $i\colon I \to A$ for the inclusion. Then it's easy to see that $E = Eii^{\circ} = ii^{\circ}E$. Further, $i^{\circ}Ei$ is a reflexive, symmetric relation on I, and hence is positive, say equal to $R^{\circ}R$. Then we have $E = ii^{\circ}Eii^{\circ} = iR^{\circ}Ri^{\circ} = (Ri^{\circ})^{\circ}(Ri^{\circ})$ and so E is positive. □

Example 3. $\mathbf{Set}$ is positively regular. More generally so is any regular coherent category, coming with unions of subobjects. To see this, for any relation $E \rightarrowtail A \times A$ satisfying $(*)$, define

$$R = \{(a,(a,b)) \mid E(a,b)\} \vee \{(a,(b,a)) \mid E(b,a)\} \rightarrowtail A \times E$$

Then $E = R^{\circ}R$, making E positive.

Theorem 4. *For a regular category* $\mathbf{C}$*, the following are equivalent:*

(i) $\mathbf{C}$ *is a Mal'tsev category;*

(ii) *Every reflexive relation in* $\mathbf{C}$ *is an equivalence relation;*

(iii) *Every reflexive and symmetric relation in* $\mathbf{C}$ *is an equivalence relation;*

(iv) $\mathbf{C}$ *is a Goursat category and every reflexive relation in* $\mathbf{C}$ *is positive;*

(v) $\mathbf{C}$ *is a Goursat category and positively regular;*

(vi) $\mathbf{C}$ *is* n*-permutable, for some* $n \geq 2$*, and positively regular.*

[1] Not to be confused with the notion of a positive coherent category [8].

Proof. The equivalence of (i), (ii) and (iii) is in Theorem 1, and clearly we have (iv) $\implies$ (v) $\implies$ (vi). For (iii) $\implies$ (iv), any reflexive relation E in $\mathbf{C}$ is an equivalence relation, and therefore positive since $E = E^\circ E$. Hence by Proposition 2, $\mathbf{C}$ is positively regular. Further, since $\mathbf{C}$ is a Mal'tsev category, it is a Goursat category.

It remains to show that (vi) $\implies$ (i). Let $\mathbf{C}$ be positively regular. First suppose $\mathbf{C}$ is $(2m+1)$-permutable, for some $m \geq 1$. Let $E\colon A \to A$ be a reflexive and symmetric relation. By positive regularity, $E = R^\circ R$ for some relation $R\colon A \to B$, and so:

$$E^{2m} = (R^\circ, R)_{4m} = (R^\circ, R)_{(2m+2)+2(m-1)} = (R^\circ, R)_{2m} = E^m$$

where we repeatedly applied $(R^\circ, R)_{2m+2} = (R^\circ, R)_{2m}$ from Theorem 1, condition (ii). Hence E^m is an equivalence relation. By condition (v) of Theorem 1, $\mathbf{C}$ is then in fact $(m+1)$-permutable.

Now if $\mathbf{C}$ is n-permutable, there is some k with $n \leq 2^k + 1$ so that $\mathbf{C}$ is (2^k+1)-permutable. Then the above argument shows that $\mathbf{C}$ is in fact $(2^{k-1}+1)$-permutable, and hence inductively that $\mathbf{C}$ is 2-permutable, *i.e.* a Mal'tsev category. □

Acknowledgements

These results grew out of collaborative work with Chris Heunen and Marino Gran. I would especially like to thank Marino for his advice on the preparation of this note. Thanks also to an anonymous referee for helpful suggestions and extra equivalent conditions in our results. This research was supported by EPSRC Studentship OUCL/2014/SET.

References

[1] F. Borceux and D. Bourn. *Mal'cev, protomodular, homological and semi-abelian categories*, volume 566. Springer Science & Business Media, 2004.

[2] A. Carboni, G. M. Kelly, and M. C. Pedicchio. Some remarks on Maltsev and Goursat categories. *Applied Categorical Structures*, 1(4):385–421, 1993.

[3] A. Carboni, J. Lambek, and M. C. Pedicchio. Diagram chasing in Mal'cev categories. *Journal of Pure and Applied Algebra*, 69(3):271–284, 1991.

[4] M. Gran and D. Rodelo. A new characterisation of Goursat categories. *Applied Categorical Structures*, 20(3):229–238, 2012.

[5] C. Heunen and S. Tull. Categories of relations as models of quantum theory. In *Quantum Physics and Logic*, volume 195 of *Electronic Proceedings in Theoretical Computer Science*, pages 247–261, 2015.

[6] G. Janelidze and G. M. Kelly. Galois theory and a general notion of central extension. *Journal of Pure and Applied Algebra*, 97(2):135–161, 1994.

[7] Z. Janelidze, D. Rodelo, and T. Van der Linden. Hagemanns theorem for regular categories. *Journal of Homotopy and Related Structures*, 9(1):55–66, 2014.

[8] P. T. Johnstone. *Sketches of an elephant: A topos theory compendium*, volume 2. Oxford University Press, 2002.

[9] N. Martins-Ferreira, D. Rodelo, and T. Van der Linden. An observation on n-permutability. *Bulletin of the Belgian Mathematical Society-Simon Stevin*, 21(2):223–230, 2014.

[10] M. C. Pedicchio. A categorical approach to commutator theory. *Journal of Algebra*, 177(3):647–657, 1995.

Sean Tull
University of Oxford, Department of Computer Science
Wolfson Building, Parks Road, Oxford OX1 3QD (United Kingdom)
sean.tull@cs.ox.ac.uk