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ARITHMETIC UNIVERSES AND CLASSIFYING TOPOSES

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Résumé. Cet article utilise la structure de $\mathfrak{Con}$ (la 2-catégorie des esquisses pour les univers arithmétiques (AU) de l'auteur) pour obtenir des résultats constructifs, indépendants de la base pour les topos de Grothendieck ($\mathcal{S}$-topos bornés) comme espaces généralisés. Le principal résultat montre comment une application extension $U\colon \mathbb{T}_1 \to \mathbb{T}_0$ peut être vue comme un fibré, transformant les points de base (modèles de $\mathbb{T}_0$ dans un topos $\mathcal{S}$ avec objet des nombres naturels) en fibres (espaces généralisés au-dessus de $\mathcal{S}$). Parmi les caractéristiques de ce travail, on notera : une comparaison entre modèles stricts ou non-stricts, utilisant les propriétés des objets de $\mathfrak{Con}$; l'utilisation des produits tensoriels de Gray pour relier la transformation syntactique de modèles par des 1-cellules de $\mathfrak{Con}$ et les transformations sémantiques par des AU-foncteurs non stricts ; et l'utilisation de 2-fibrations pour indexer au-dessus d'une 2-catégorie de topos de base $\mathcal{S}$.

Abstract. The paper uses structures in $\mathfrak{Con}$, the author's 2-category of sketches for arithmetic universes (AUs), to provide constructive, base-independent results for Grothendieck toposes (bounded $\mathcal{S}$-toposes) as generalized spaces.

The main result is to show how an extension map $U\colon \mathbb{T}_1 \to \mathbb{T}_0$ can be viewed as a bundle, transforming base points (models of $\mathbb{T}_0$ in any elementary topos $\mathcal{S}$ with nno) to fibres (generalized spaces over $\mathcal{S}$).

Features of the work include comparison of strict and non-strict models, using properties of the objects of $\mathfrak{Con}$; the use of Gray tensor products to relate syntactic transformation of models by 1-cells in $\mathfrak{Con}$ and semantic transformations by non-strict AU-functors; and the use of 2-fibrations to index over a 2-category of base toposes $\mathcal{S}$.

Keywords. geometric theory, 2-fibration, sketch, Gray tensor

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1. Introduction

If $\mathbb{T}$ is a geometric theory, then the generalized topological space – in Grothendieck's sense – of models of $\mathbb{T}$ is realized mathematically as its category of sheaves, the classifying topos $\mathcal{S}[\mathbb{T}]$.

$\mathcal{S}$ here, the base, could be any elementary topos with nno that is able to support the infinite disjunctions appearing in $\mathbb{T}$, and if those disjunctions are countable then any such $\mathcal{S}$ will do. So which topos $\mathcal{S}[\mathbb{T}]$ is the true incarnation of the generalized space?

[12] developed a 2-category $\mathfrak{Con}$ whose objects are, in sketch form, such theories; and whose 1-cells are the maps got if one replaces the classifying topos $\mathcal{S}[\mathbb{T}]$ by a classifying *arithmetic universe* $\mathbf{AU}\langle\mathbb{T}\rangle$, which can thus be understood as a base-independent incarnation of the space.

The present paper shows how to recover the base-dependent topos theory, but in an indexed way, using 2-fibrations, that allows for change of base.

As a significant generalization of the indexed construction $\mathcal{S} \mapsto \mathcal{S}[\mathbb{T}]$, we also relativize by looking at certain maps $U\colon \mathbb{T}_1 \to \mathbb{T}_0$ in $\mathfrak{Con}$ considered as *bundles* – that is to say, transformations from base point M (model of $\mathbb{T}_0$) to space (fibre of U over M). If M is in an elementary topos $\mathcal{S}$, then we construct an $\mathcal{S}$-geometric theory $\mathbb{T}_1/M$ of models of $\mathbb{T}_1$ that are reduced to M by U, and then the fibre, as generalized space in the topos sense, is $\mathcal{S}[\mathbb{T}_1/M]$.

Our main result, Theorem 5.12, is that the whole construction $(\mathcal{S}, M) \mapsto \mathcal{S}[\mathbb{T}_1/M]$ is indexed over pairs $(\mathcal{S}, M)$. This is formalized 2-fibrationally using a new notion (Definition 4.4) of *local representability*.

Throughout this paper, *every elementary topos will be assumed to have a natural numbers object*. We write $\mathfrak{eTop}$ for the 2-category of elementary toposes with nno, geometric morphisms (not necessarily bounded), and natural transformations.

1.1 Generalized spaces and their categories of sheaves

Let us elaborate on the underlying question. Grothendieck discovered a huge generalization of the notions of topology and continuity, with a generalized space represented concretely by its category of sheaves (continuous set-valued maps).

This is point-free topology, analogous to representing a space X by its frame ΩX of opens, albeit on a much grander scale.

[10] made an explicit attempt to make the analogous notational distinction, writing X for the generalized space and $\mathcal{S}X$ for its category of sheaves. If $[\mathbb{T}]$ is written for the space of models of a geometric theory $\mathbb{T}$, then $\mathcal{S}[\mathbb{T}]$ can be read either as "$\mathcal{S}$heaves over the space $[\mathbb{T}]$" or as "the (geometric) mathematics generated over the category $\mathcal{S}$ of sets by adjoining a generic model of $\mathbb{T}$".

That paper was applied to domain theory, and in particular the ideal completion of information systems (the compact bases) for SFP-domains. These were studied using a generic SFP-domain, a geometric morphism $[\mathrm{IS}][\mathrm{idl}] \to [\mathrm{IS}]$, where $[\mathrm{IS}]$ classifies SFP information systems and the fibre over one of them is its ideal completion. (We shall see a more general account of such bundles in Section 5.2.)

But what is this category $\mathcal{S}$ of sets, within which one constructs the sheaves, and over which one constructs $\mathcal{S}[\mathbb{T}]$? To Grothendieck it would have been classical set theory $\mathbf{Set}$. With the subsequent discovery of elementary toposes, it was found that any elementary topos $\mathcal{S}$ with nno could be used as base for a notion of geometric theory and for constructing generalized spaces (bounded geometric morphisms into $\mathcal{S}$) as classifying toposes. $\mathcal{S}$-indexed categories are used to capture the idea that an object of $\mathcal{S}$ can be used as an indexing set for a colimit diagram (see [7, B1.4]).

That relieves the classical dependency, but unfortunately creates a problem of its own: even if (as in [10]) the working is foundationally robust, one still has to choose a base $\mathcal{S}$ in order to have a mathematical incarnation $\mathcal{S}[\mathbb{T}]$ of the generalized space $[\mathbb{T}]$.

In its conclusions, [10] proposed that $\mathcal{S}$ might be dispensed with if all the working could be reduced to that of arithmetic universes (AUs), with finite colimits and list objects instead of "$\mathcal{S}$-indexed" colimits. By [8], every elementary topos with nno is an AU, and for any geometric morphism f between them, the inverse image part f^* is a (non-strict) AU-functor. Then the infinities of geometric logic, supplied extrinsically by $\mathcal{S}$, would be replaced where possible by intrinsic infinities supplied by the list object construction.

The ultimate ambition would be to develop an entirely "arithmetic" account of generalized topology, using AUs $\mathbf{AU}\langle\mathbb{T}\rangle$, to replace the present geometric account using Grothendieck toposes. How far that can be carried through remains to be tested. The more modest aim of the present paper is to show how arithmetic techniques can give base-independent results in the existing topos theory.

1.2 Outline

Section 2 summarizes the background of AUs, their sketches, and the 2-category $\mathfrak{Con}$ [12]; and of geometric theories and classifying toposes largely as presented in [7, B4.2].

Section 3 discusses the models of AU-sketches in AUs in general, and elementary toposes in particular. A particular issue is whether the models should be *strict* or not. We need both, and the *contexts*, the AU-sketches appearing as objects of $\mathfrak{Con}$, have the special property that every non-strict model has a canonical strict isomorph. We describe two interacting actions on models: one by context maps between theories, and one by non-strict AU-functors between the AUs where the models are found.

Section 4 collects miscellaneous remarks on the 2-fibrational background that allows us to vary the base elementary topos $\mathcal{S}$, and includes (Definition 4.4) a notion of *local representability* that captures, 2-fibrationally, the idea of classifying toposes behaving in an indexed way under pseudopullback along change of base topos. In essence this is the idea of "geometricity" as expressed in [11].

Section 5 examines classifying toposes for contexts. In fact, we deal with a relativized version, with a context extension map $U \colon \mathbb{T}_1 \to \mathbb{T}_0$ (given by $\mathbb{T}_0 \subset \mathbb{T}_1$). If each context represents "the space of its models", then we wish to view U as a bundle: over each model M of $\mathbb{T}_0$, the fibre over it is the "space of models of $\mathbb{T}_1$ that restrict to M". We shall show how these fibres can be represented as classifying toposes.

Now we fibre over pairs $(\mathcal{S}, M)$, where M is a strict model of $\mathbb{T}_0$ in $\mathcal{S}$. We find a geometric (though not arithmetic in general) theory $\mathbb{T}_1/M$ of models of $\mathbb{T}_1$ restricting to M, and it has a classifying topos $\mathcal{S}[\mathbb{T}_1/M] \to \mathcal{S}$ (with its generic model).

Our main result, Theorem 5.12, is that this construction is locally representable, in other words that it is geometric – preserved by pseudopullback along arbitrary geometric morphisms. A corollary is the "geometricity of presentations" result of [11, Section 5].

2. Background

2.1 Sketches for arithmetic universes

We summarize the sketch approach to arithmetic universes as set out in [12]. The sketches are roughly as in [3], with a reflexive graph of nodes and edges for objects and morphisms, a set of "commutativities" to specify commutative triangles, and "universals" (the cones and cocones) for finite limits and finite colimits – specifically: terminals, pullbacks, initials, pushouts. In addition they have universals to specify list objects, thus gaining an nno as $\mathsf{List}\,1$.

In our *sketch extensions* $\mathbb{T} \subset \mathbb{T}'$ such universals may be introduced only for fresh objects, and hence in a definitional way. A *context* is then an extension of the empty sketch $\mathbb{1}$.

In *equivalence extensions* $\mathbb{T} \Subset \mathbb{T}'$, everything fresh that is introduced must have been implicitly present already. This includes composites of composable pairs of edges; commutativities deducible from existing ones (e.g. by unit laws or associativities); universals, fillins for universals and uniqueness of fillins; and inverses for certain edges that must be isomorphisms because of the categorical properties of AUs such as balance, stability and exactness.

Homomorphisms $\mathbb{T} \lessdot \mathbb{T}'$ are structure-preserving homomorphisms for the algebraic theory of sketches. They translate nodes to nodes, edges to edges, commutativities to commutativities and universals to universals. The two kinds of extensions are special cases of this.

Next, we have a notion of *object equalities* between nodes, certain edges that include all identity edges but can also arise as fillins when the same universal construction is applied to equal data. We extend this to object equalities between edges, when their domains have an object equality and so do the codomains, and there are appropriate commutativities to make a commutative square; and then we extend to object equalities between homomorphism of models, using object equalities between corresponding nodes and edges in the image.

Putting these together we get a category $\mathfrak{Con}$ whose objects are contexts. Its morphisms, context *maps*, are the dual of context homomorphisms, but subject to (i) those for equivalence extensions are invertible, and (ii) object equalities become identity morphisms between actually equal objects. Every map $\mathbb{T}_0 \to \mathbb{T}_1$ is an equivalence class of opspans of homomorphisms $\mathbb{T}_0 \Subset \mathbb{T}'_0 \gtrdot \mathbb{T}_1$.

Notice that, for each of the special symbols $\subset$, $\Subset$ and $\lessdot$, the narrow end is at the codomain for the corresponding reduction *map*.

For each context $\mathbb{T}$ there is also a context $\mathbb{T}^{\to}$ for which a model is a pair of models of $\mathbb{T}$, together with a $\mathbb{T}$-homomorphism between them. These enable us to define 2-cells between maps, using maps $\mathbb{T}_0 \to \mathbb{T}_1^{\to}$, and $\mathfrak{Con}$ becomes a 2-category. It has finite PIE-limits (Product, Inserter, Equifier) and pullbacks of extension maps (the duals of the homomorphisms corresponding to extensions).

There is a full and faithful 2-functor from $\mathfrak{Con}$ to the category $\mathbf{AU}_s$ of AUs and strict AU-functors, contravariant on 1-cells, that takes $\mathbb{T} \mapsto \mathbf{AU}\langle\mathbb{T}\rangle$, the AU presented using $\mathbb{T}$ as generators and relations.

A central issue for models of sketches is that of *strictness*. The standard sketch-theoretic notion is non-strict: for a universal, such as a pullback of some given opspan, the pullback cone can be interpreted as any pullback of the opspan. However, we could also seek strict models that use the canonical pullbacks (in categories where they exist). Strictness is essential for the universal algebra that generates $\mathbf{AU}\langle\mathbb{T}\rangle$, but in general it is inconvenient. Significant parts of the present paper are concerned with relating the strict and the non-strict.

Contexts are designed to give us good control over strictness, as summarized by the following proposition.

Proposition 2.1. *Let $U\colon \mathbb{T}_1 \to \mathbb{T}_0$ be an extension map in $\mathfrak{Con}$, that is to say one deriving from an extension $\mathbb{T}_0 \subset \mathbb{T}_1$. Suppose in some AU $\mathcal{A}$ we have a model M_1 of $\mathbb{T}_1$, a strict model M'_0 of $\mathbb{T}_0$, and an isomorphism $\phi_0\colon M'_0 \cong M_1U$ (the restriction of M_1 to $\mathbb{T}_0$).*

$$\begin{array}{ccccc} \mathbb{T}_1 & \quad & M'_1 & \overset{\phi_1}{\underset{\cong}{\dashrightarrow}} & M_1 \\ {\scriptstyle U}\downarrow & & \downarrow & & \downarrow \\ \mathbb{T}_0 & & M'_0 & \overset{\phi_0}{\underset{\cong}{\longrightarrow}} & M_1U \end{array}$$

Then there is a unique model M'_1 of $\mathbb{T}_1$ and isomorphism $\phi_1\colon M'_1 \cong M_1$ such that

1. *M_1' is strict,*

2. *$M_1'U = M_0'$,*

3. *$\phi_1 U = \phi_0$, and*

4. *ϕ_1 is equality on all the primitive nodes for the extension $\mathbb{T}_0 \subset \mathbb{T}_1$.*

The proof can be deduced from the strictness results in [12]. In brief, it is reduced by induction to the case of simple extension steps in $\mathbb{T}_0 \subset \mathbb{T}_1$. Adjoining a primitive node, M_1' and ϕ_1 are determined by (4). Adjoining a primitive edge, M_1' and ϕ_1 are determined by the need to make ϕ_1 an isomorphism. Adjoining a universal, M_1' is determined by (1) and ϕ_1 by (3), as the unique fillin consistent with ϕ_0.

In the case where $\mathbb{T}_0$ is the empty context $\mathbb{1}$, we see the important corollary that for a context $\mathbb{T}$ *every model is uniquely isomorphic to a unique strict model with which it agrees on all primitive nodes.* We call this its *canonical strict isomorph.*

Thus in topos theory, where non-strict AU-functors are liable to transform strict models into non-strict ones, we can regain strictness of models.

Example 2.2. The Proposition does not hold for arbitrary context maps $H\colon \mathbb{T}_1 \to \mathbb{T}_0$. Let $\mathbb{O}, \mathbb{O}^2$ be the contexts that have, respectively, one and two nodes, and nothing else. Consider the diagonal $\Delta\colon \mathbb{O} \to \mathbb{O}^2$ given by the context homomorphism that takes both nodes in $\mathbb{O}^2$ to the node in $\mathbb{O}$. If X is a model of $\mathbb{O}$, then its Δ-reduct $X\Delta = (X, X)$. If we can find $X_1 \cong X \cong X_2$ with $X_1 \neq X_2$, then $(X_1, X_2) \cong X\Delta$ without itself being a Δ-reduct.

2.2 Elephant theories

Here we briefly summarize the account in [7, B4.2] of classifying toposes, over a fixed base elementary topos $\mathcal{S}$.

Central to its treatment is the 2-category $\mathfrak{BTop}/\mathcal{S}$. A 0-cell is a bounded geometric morphism $p\colon \mathcal{E} \to \mathcal{S}$, a *Grothendieck topos over $\mathcal{S}$*. In Definition 4.1 these will appear in the fibre of our $\mathfrak{GTop}$ over $\mathcal{S}$. A 1-cell f is a pair $(\overline{f}, f\Downarrow)$, where $\overline{f}$ is a bounded geometric morphism and $f\Downarrow$ is a specified isomorphism in the triangle over $\mathcal{S}$.

Any logical description of a theory does implicitly describe the models, but one can also try to use the category of models as a direct semantic description of the theory. Unfortunately this does not work for geometric theories, which may be incomplete – there are not enough models for semantic entailment to agree with the syntactic entailment got from the rules of geometric logic.

The semantic description used to get round this in [7, B4.2] is to describe all the models in *all Grothendieck toposes*. For narrative purposes in the present paper, to make a clear distinction from the logical theories, I shall refer to such an "all model" description as an "elephant theory". Of course that acknowledges their use in [7], but I also want to convey something of the sheer quantity of data encapsulated in one of these theories.

Definition 2.3. *An* elephant theory over $\mathcal{S}$ *is an indexed category* $\mathbb{T}$ *over* $\mathfrak{BTop}/\mathcal{S}$*. Then an object of* $\mathbb{T}(\mathcal{E})$ *is a "model of* $\mathbb{T}$ *in* $\mathcal{E}$*".*

In our applications derived from AU-sketches, the elephant theories will be strict, 2-functors to $\mathfrak{CAT}$.

A particularly important example is the context $\mathbb{O}$, the object classifier, with $\mathbb{O}(\mathcal{E}) = \mathcal{E}$.

Given an elephant theory $\mathbb{T}$ over $\mathcal{S}$, a *geometric construct* on $\mathbb{T}$ is an indexed functor from $\mathbb{T}$ to $\mathbb{O}$.

Definition 2.4. *Let* $\mathbb{T}_0$ *be an elephant theory over* $\mathcal{S}$*. A* geometric extension *of* $\mathbb{T}_0$ *is a theory built, starting from* $\mathbb{T}_0$*, by a finite sequence of the following "simple" steps from* $\mathbb{T}$ *to* $\mathbb{T}'$*.*

- Simple functional extension: *Let* $H_0, H_1 \colon \mathbb{T} \to \mathbb{O}$ *be two geometric constructs. Define the theory* $\mathbb{T}'$ *whose models in* $\mathcal{E}$ *are pairs* (M, u) *where* M *is a model of* $\mathbb{T}$ *in* $\mathcal{E}$ *and* $u \colon MH_0 \to MH_1$ *is a morphism. A morphism from* (M, u) *to* (M', u') *is morphism* $\phi \colon M \to M'$ *such that that following diagram commutes.*

$$\begin{array}{ccc} MH_0 & \xrightarrow{u} & MH_1 \\ {\scriptstyle \phi H_0}\downarrow & & \downarrow{\scriptstyle \phi H_1} \\ M'H_0 & \xrightarrow[u']{} & M'H_1 \end{array} .$$

- Simple geometric quotient: *Let* $\phi\colon H_0 \to H_1$ *be a morphism of geometric constructs on* $\mathbb{T}$. $\mathbb{T}'$ *is the theory whose models in* $\mathcal{E}$ *are those models of* $\mathbb{T}$ *for which* ϕ *is an isomorphism; its morphisms are all* $\mathbb{T}$*-morphisms.*

- Simple extension by primitive object: *We define* $\mathbb{T}'(\mathcal{E}) = \mathbb{T}(\mathcal{E}) \times \mathcal{E}$. *In other words, we may write* $\mathbb{T}' = \mathbb{T} \times \mathbb{O}$.

Then a geometric theory *over* $\mathcal{S}$ *is a geometric extension of* $\mathbb{1}$*, the trivial theory for which every* $\mathbb{1}(\mathcal{E})$ *is the category with one object* $*$ *and its identity morphism.*

Note that [7] does not define the general notion of geometric extension, but simply that of geometric theory as an extension of $\mathbb{O}^n$ (for some finite n) by simple functional extensions and simple geometric quotients. The two are equivalent, because no harm is done if the primitive sorts are all adjoined at the start, and doing this n times to $\mathbb{1}$ gives $\mathbb{O}^n$.

If $\mathbb{T}_1$ is a geometric extension of $\mathbb{T}_0$, then there is a theory morphism from $\mathbb{T}_1$ to $\mathbb{T}_0$ given by model reduction.

For future reference we prove the following result that does not appear to be in [7].

Proposition 2.5. *In the category of elephant theories over* $\mathcal{S}$ *and indexed functors between them, geometric extensions can be pulled back along any morphism.*

Proof. The point is that we have a pullback, not a pseudopullback.

Let $H\colon \mathbb{T}'_0 \to \mathbb{T}_0$ be an indexed functor between elephant theories over $\mathcal{S}$, and let $\mathbb{T}_1$ be a geometric extension of $\mathbb{T}_0$ with indexed functor $U\colon \mathbb{T}_1 \to \mathbb{T}_0$ defined by model reduction. We define the elephant theory $\mathbb{T}'_1$ by argumentwise pullback of categories.

$$\begin{array}{ccc} \mathbb{T}'_1(\mathcal{E}) & \longrightarrow & \mathbb{T}_1(\mathcal{E}) \\ \downarrow & & \downarrow{\scriptstyle U(\mathcal{E})} \\ \mathbb{T}'_0(\mathcal{E}) & \xrightarrow[H(\mathcal{E})]{} & \mathbb{T}_0(\mathcal{E}) \end{array}$$

Thus a model of $\mathbb{T}'_1$ is a pair (M_0, M_1) of models of $\mathbb{T}'_0$ and $\mathbb{T}_1$ for which $M_0 H = M_1 U$.

For reindexing along $f\colon \mathcal{F} \to \mathcal{E}$ (over $\mathcal{S}$), the naive attempt to define $f^*(M_0, M_1)$ as (f^*M_0, f^*M_1) fails because we only have

$$(f^*M_0)H \cong f^*(M_0H) = f^*(M_1U) = (f^*M_1)U.$$

(The last equality can be readily checked for different kinds of simple geometric extension.) The trick then is to define $f^*(M_0, M_1)$ as (f^*M_0, N_1) for some $N_1 \cong f^*M_1$ whose $\mathbb{T}_0$-reduct is $(f^*M_0)H \cong (f^*M_1)U$.

It suffices to check the three kinds of simple geometric extension. For extension by primitive sort, $\mathbb{T}_1 = \mathbb{T}_0 \times \mathbb{O}$, we find that $\mathbb{T}'_1$ as defined by pullback is $\mathbb{T}'_0 \times \mathbb{O}$. For the reindexing question, we have M_1 of the form (M_0H, X) and define $N_1 = ((f^*M_0)H, f^*X)$.

The next case is when $\mathbb{T}_1$ is a simple functional extension of $\mathbb{T}_0$ for two geometric constructs $G_0, G_1\colon \mathbb{T}_0 \to \mathbb{O}$. We find that $\mathbb{T}'_1$, as defined by pullback, is a simple functional extension of $\mathbb{T}'_0$ for HG_0 and HG_1. For the reindexing, we have M_1 of the form $(M_0H, u\colon M_0HG_0 \to M_0HG_1)$. Then we take N_1 to be $((f^*M_0)H, u')$, where u' is so as to make the following diagram commute.

$$\begin{array}{ccccc}
(f^*M_0)HG_0 & \xrightarrow{\cong} & (f^*(M_0H))G_0 & \xrightarrow{\cong} & f^*(M_0HG_0) \\
{\scriptstyle u'}\downarrow & & & & \downarrow{\scriptstyle f^*u} \\
(f^*M_0)HG_1 & \xrightarrow{\cong} & (f^*(M_0H))G_1 & \xrightarrow{\cong} & f^*(M_0HG_1)
\end{array}$$

For the final case, $\mathbb{T}_1$ is an extension of $\mathbb{T}_0$ by simple geometric quotient for a morphism $\phi\colon G_0 \to G_1$ of two geometric constructs on $\mathbb{T}_0$. Now $\mathbb{T}'_1$ is an extension of $\mathbb{T}'_0$ by simple geometric quotient for a morphism $H\phi\colon HG_0 \to HG_1$. □

Definition 2.6. *Let $\mathbb{T}$ be an elephant theory over $\mathcal{S}$. A* classifying topos *for $\mathbb{T}$ is a bounded $\mathcal{S}$-topos $p\colon \mathcal{S}[\mathbb{T}] \to \mathcal{S}$, equipped with a "generic" $\mathbb{T}$-model N_G, such that, for each bounded $\mathcal{S}$-topos $\mathcal{E}$, the functor*

$$\mathfrak{BTop}/\mathcal{S}[\mathcal{E}, \mathcal{S}[\mathbb{T}]] \to \mathbb{T}(\mathcal{E}), \quad f \mapsto f^*N_G,$$

is one half of an equivalence of categories.

In other words, the pseudofunctor $\mathbb{T}\colon \mathfrak{BTop}/\mathcal{S} \to \mathfrak{CAT}$ is representable.

Since all our elementary toposes have nno, [7, Theorem B4.2.9] tells us that *every geometric theory has a classifying topos.*

3. Indexed categories of models

In this section we deal with categories of models of AU-contexts from $\mathfrak{Con}$. For each AU $\mathcal{A}$ and AU-context $\mathbb{T}$ we have a category $\mathcal{A}\text{-}\mathbf{Mod}\text{-}\mathbb{T}$ of models of $\mathbb{T}$ in $\mathcal{A}$, and a full subcategory $\mathcal{A}\text{-}\mathbf{Mod}_s\text{-}\mathbb{T}$ of strict models.

We shall show that $\mathcal{A}\text{-}\mathbf{Mod}_s\text{-}\mathbb{T}$ is acted on strictly (on the right) by $\mathfrak{Con}$, and strictly (on the left) by $\mathbf{AU}$, the category of AUs and *non-strict* AU-functors. This strict left action arises because $\mathbb{T}$, a context, has the strict model corollary of Proposition 2.1: applying a non-strict AU-functor gives us a non-strict model, but we can then replace it by its canonical strict isomorph.[1] The left and right actions commute up to isomorphism, which we express in Theorem 3.6 as a category strictly indexed over the Gray tensor product. However, right action by extension maps commutes *up to equality* with the left actions (Lemma 3.7), and this will be important for us.

Note that the context maps, between contexts $\mathbb{T}$, correspond to *strict* AU-functors between the classifying AUs $\mathbf{AU}\langle\mathbb{T}\rangle$. What we have done, therefore, is in effect to have strict and non-strict AU-functors acting on the right and left respectively, with the Gray tensor action representing the interplay between strict and non-strict.

One might wonder whether we could instead have focused on the non-strict models $\mathcal{A}\text{-}\mathbf{Mod}\text{-}\mathbb{T}$. There is an obvious action on the left by $\mathbf{AU}$, and an action on the right, by model reduction, by the context maps that correspond to context homomorphisms. Those left and right actions commute up to equality. However, the right action does not extend strictly to arbitrary context maps: this is because the maps for context equivalence extensions, which are invertible in $\mathfrak{Con}$, give only equivalences between model categories, not isomorphisms. We prefer to work with the strict action on strict models.

In any case, the non-strict models of a context $\mathbb{T}$ are the strict models of an extension $\mathbb{T}'$. For each node X in $\mathbb{T}$ introduced by a universal, adjoin another copy X' with edges and commutativities to make $X' \cong X$.

Definition 3.1. *Let $\mathcal{A}$ be an AU and $\mathbb{T}$ a context. Then $\mathcal{A}\text{-}\mathbf{Mod}_s\text{-}\mathbb{T}$ is the category of strict models of $\mathbb{T}$ in $\mathcal{A}$.*

[1] In fact, the definitions of extension and context in [12] were made in anticipation of these results.

Lemma 3.2. *For each arithmetic universe $\mathcal{A}$, we can define a 2-functor*

$$\mathcal{A}\text{-}\mathbf{Mod}_s\text{-}\bullet \colon \mathfrak{Con} \to \mathfrak{CAT}$$

for which $\mathcal{A}\text{-}\mathbf{Mod}_s\text{-}\bullet(\mathbb{T}) = \mathcal{A}\text{-}\mathbf{Mod}_s\text{-}\mathbb{T}$.

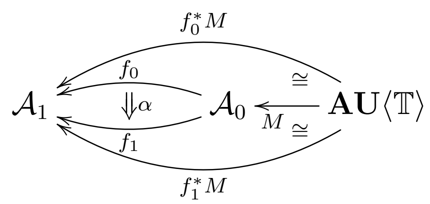
Proof. Since those models are in bijection with strict AU-functors from $\mathbf{AU}\langle\mathbb{T}\rangle$ to $\mathcal{A}$, and we have a (full and faithful) 2-functor from $\mathfrak{Con}$ to $\mathbf{AU}_s^{op}$, this extends to a 2-functor $\mathcal{A}\text{-}\mathbf{Mod}_s\text{-}\bullet$ as desired. □

If M is a strict model in $\mathcal{A}\text{-}\mathbf{Mod}_s\text{-}\mathbb{T}_0$ and $H \colon \mathbb{T}_0 \to \mathbb{T}_1$ is a context map, then we write MH for $\mathcal{A}\text{-}\mathbf{Mod}_s\text{-}H(M)$. If H is the dual of a context homomorphism then MH is got by model reduction. If H is the inverse of the dual for an equivalence extension $\mathbb{T}_0 \Subset \mathbb{T}_1$, then MH is got by interpreting all the adjoined ingredients of $\mathbb{T}_1$ in the unique strict way.

Now we fix $\mathbb{T}$ and let $\mathcal{A}$ vary.

Definition 3.3. *Let $f \colon \mathcal{A}_0 \to \mathcal{A}_1$ be an AU-functor, $\mathbb{T}$ a context and M a model in $\mathcal{A}_0\text{-}\mathbf{Mod}_s\text{-}\mathbb{T}$. Then we define $f^*M = f\text{-}\mathbf{Mod}_s\text{-}\mathbb{T}(M)$ as follows. We first define $f \cdot M$ as the non-strict model got by applying f to M. Then f^*M is (using Proposition 2.1) the canonical strict isomorph of $f \cdot M$.*

We extend this to 2-cells $\alpha \colon f_0 \to f_1$ by treating them as AU-functors from $\mathcal{A}_0$ to the comma AU $\mathcal{A}_1 \downarrow \mathcal{A}_1$. $\alpha^*M \colon f_0^*M \to f_1^*M$ is then calculated by pasting the following diagram.



Proposition 3.4. *For each context $\mathbb{T}$ we have a 2-functor*

$$\bullet\text{-}\mathbf{Mod}_s\text{-}\mathbb{T} \colon \mathbf{AU} \to \mathfrak{CAT}$$

for which $\bullet\text{-}\mathbf{Mod}_s\text{-}\mathbb{T}(\mathcal{A}) = \mathcal{A}\text{-}\mathbf{Mod}_s\text{-}\mathbb{T}$ *and* $\bullet\text{-}\mathbf{Mod}_s\text{-}\mathbb{T}(f)(M) = f^*(M)$.

Proof. The main point is that it is strictly functorial on 1-cells f. Suppose we have AU-functors

$$\mathcal{A}_2 \xleftarrow{f_1} \mathcal{A}_1 \xleftarrow{f_0} \mathcal{A}_0 .$$

Then $f_1^* f_0^* M$ and $(f_0 f_1)^* M$ are both the canonical strict isomorph of $f_1 \cdot f_0 \cdot M$.

After this, the rest follows by pasting diagrams. $\square$

The equation $f_1^* f_0^* M = (f_0 f_1)^* M$ will seem notationally perverse for morphisms in $\mathbf{AU}$, composed diagrammatically, but it looks more natural for geometric morphisms, where the AU-functor for f is f^*.

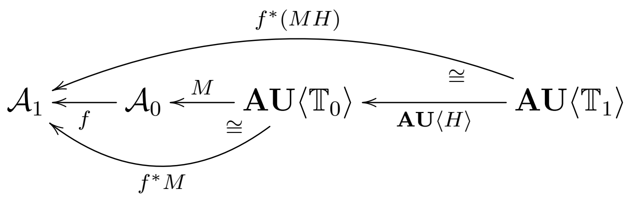
Definition 3.5. *Suppose we have 1-cells $f\colon \mathcal{A}_0 \to \mathcal{A}_1$ in $\mathbf{AU}$ and $H\colon \mathbb{T}_0 \to \mathbb{T}_1$ in $\mathfrak{Con}$. Then we define a natural isomorphism $\Sigma_{f,H}$ as follows.*

$$\begin{array}{ccc} \mathcal{A}_0\text{-}\mathbf{Mod}_s\text{-}\mathbb{T}_0 & \xrightarrow{\mathcal{A}_0\text{-}\mathbf{Mod}_s\text{-}H} & \mathcal{A}_0\text{-}\mathbf{Mod}_s\text{-}\mathbb{T}_1 \\ {\scriptstyle f\text{-}\mathbf{Mod}_s\text{-}\mathbb{T}_0}\downarrow & \Sigma_{f,H}\Downarrow & \downarrow{\scriptstyle f\text{-}\mathbf{Mod}_s\text{-}\mathbb{T}_1} \\ \mathcal{A}_1\text{-}\mathbf{Mod}_s\text{-}\mathbb{T}_0 & \xrightarrow[\mathcal{A}_1\text{-}\mathbf{Mod}_s\text{-}H]{} & \mathcal{A}_1\text{-}\mathbf{Mod}_s\text{-}\mathbb{T}_1 \end{array} \tag{1}$$

For each M in $\mathcal{A}_0\text{-}\mathbf{Mod}_s\text{-}\mathbb{T}_0$, we define the isomorphism

$$\Sigma_{f,H}(M)\colon f^*(MH) \cong (f^*M)H$$

by pasting the following diagram.



Naturality is clear.

Theorem 3.6. *The two actions on $\bullet$-$\mathbf{Mod}_s$-$\bullet$ by $\mathbf{AU}$ and $\mathfrak{Con}$, together with the pseudo-naturality isomorphisms $\Sigma_{f,H}$, make up a "cubical functor" from $\mathbf{AU} \times \mathfrak{Con}$ to $\mathfrak{CAT}$ in the sense of [5], and hence a 2-functor from the Gray tensor product $\mathbf{AU} \otimes \mathfrak{Con}$ to $\mathfrak{CAT}$.*

Proof. There are three conditions to be checked. The first two are that the squares (1) paste together correctly, either horizontally or vertically, for composition of 1-cells in either $\mathfrak{Con}$ or $\mathbf{AU}$. The third is that it pastes correctly with 2-cells in $\mathfrak{Con}$ and $\mathbf{AU}$. All are clear by pasting the appropriate isomorphisms from the definition of f^*. □

Lemma 3.7.

1. *If U is an extension map (for $\mathbb{T}_0 \subset \mathbb{T}_1$) then $(f^*M)U = f^*(MU)$ for every f and M, and $\Sigma_{f,U}(M)$ is the identity morphism.*
2. *If U is an equivalence extension map ($\mathbb{T}_0 \Subset \mathbb{T}_1$), then $(f^*M)U^{-1} = f^*(MU^{-1})$, and $\Sigma_{f,U^{-1}}(M)$ is the identity morphism.*

Proof. (1) $f^*(MU)$ is the canonical strict isomorph of $f \cdot (MU)$.

On the other hand $(f^*M)U \cong (f \cdot M)U = f \cdot (MU)$ and they are equal on all the primitive nodes of $\mathbb{T}_0$ because they are also primitive in the extension $\mathbb{T}_1$.

(2) Apply part (1) to MU^{-1}. □

Example 3.8. Equality in Lemma 3.7 can fail for a map $H\colon \mathbb{T}_1 \to \mathbb{T}_0$ involving a context homomorphism that maps primitive nodes to non-primitives. Consider the context $\mathbb{T}$ with a single node T, declared terminal, and $H\colon \mathbb{T} \to \mathbb{O}$ given by the sketch homomorphism that takes the single node X in $\mathbb{O}$ to T.

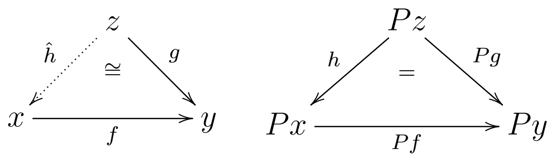
If M is the unique strict model of $\mathbb{T}$ in $\mathcal{A}$, then MH simply picks out the canonical terminal object, and $(f^*M)H$ does the same in $\mathcal{A}'$. $f^*(MH)$ picks out the image under f of the canonical terminal in $\mathcal{A}$.

Finally, we can translate these results to elementary toposes. For each AU-context $\mathbb{T}$ we have a 2-category $\bullet$-$\mathbf{Mod}_s$-$\mathbb{T}$, strictly indexed over $\mathfrak{eTop}$, and it restricts to $\mathfrak{BTop}/\mathcal{S}$, with the geometric morphisms $p\colon \mathcal{E} \to \mathcal{S}$ playing no role in the reindexing. Thus it gives a strict elephant theory over $\mathcal{S}$ for $\mathbb{T}$. Also, each context map $H\colon \mathbb{T}_0 \to \mathbb{T}_1$ gives a corresponding indexed functor from $\mathbb{T}_0$ to $\mathbb{T}_1$ as elephant theories.

4. Remarks on 2-fibrations

In the 2-functor $\bullet\text{-}\mathbf{Mod}_s\text{-}\mathbb{T}\colon \mathbf{AU} \to \mathfrak{CAT}$ we have already seen a category strictly indexed over the 2-category $\mathbf{AU}^{op}$. As we proceed, however, we shall encounter non-strict indexations, with pseudofunctors, and for these we shall prefer a fibrational approach. Thus we avoid confronting coherence conditions for indexed 2-categories.

For the appropriate notion of 2-fibration we shall follow Buckley's account [4], which in turn was based on earlier work of Hermida [6] and Baković [1]. Definitions are given for fibrations both between 2-categories and between bicategories. Note that, although we deal only with 2-categories, and 2-functors between them, we shall still need to use the bicategorical notion of fibration once we go beyond strictly indexed categories. The essential difference, for a 2-functor $P\colon \mathcal{E} \to \mathcal{B}$, is that the properties characterizing a cartesian 1-cell $f\colon x \to y$ in $\mathcal{E}$ are weaker. Given $g\colon z \to y$ and $h\colon Pz \to Px$ with $h(Pf) = Pg$, we can lift h to $\hat{h}\colon z \to x$ but the corresponding triangle in $\mathcal{E}$ commutes only up to isomorphism.



To summarize Buckley's definitions, –

- A 1-cell f in $\mathcal{E}$ is cartesian if it lifts 1-cells up to isomorphism, and lifts 2-cells coherently with the lifted isos. The uniqueness of lifted 2-cells implies that lifted 1-cells are unique up to a coherent isomorphism.
- A 2-cell $\alpha\colon f \Rightarrow g\colon x \to y$ in $\mathcal{E}$ is cartesian if it is cartesian as a 1-cell for the functor $P_{xy}\colon \mathcal{E}(x,y) \to \mathcal{B}(Px,Py)$.
- P is a fibration if for every $f\colon b \to Pe$ in $\mathcal{B}$, there is a cartesian $h\colon a \to e$ with $Ph = f$; each P_{xy} is a fibration of categories; and the cartesian 2-cells are closed under whiskering on both sides.

4.1 The fibred 2-category of Grothendieck toposes

By "Grothendieck topos", we mean a *bounded* geometric morphism from some elementary topos $\mathcal{E}$ to some, understood, base elementary topos $\mathcal{S}$.[2] The 2-category of Grothendieck toposes over $\mathcal{S}$ is studied in [7, B4] as $\mathfrak{BTop}/\mathcal{S}$.

A notable property of $\mathfrak{BTop}/\mathcal{S}$ is that any geometric theory $\mathbb{T}$ (geometric, that is, with respect to $\mathcal{S}$) has a classifying topos $\mathcal{S}[\mathbb{T}]$ that behaves in many respect as "the space of models of $\mathbb{T}$"; indeed, the whole of $\mathfrak{BTop}/\mathcal{S}$ may then be viewed as the 2-category of generalized spaces relative to $\mathcal{S}$: 0-cells are spaces, 1-cells are (continuous) maps, and 2-cells are generalized specializations (morphisms, not order).

Our interest in using arithmetic universes is to deal with theories $\mathbb{T}$ that depend on the base $\mathcal{S}$ only to the extent that nnos are required to exist. Our aim here will be to prove results about Grothendieck toposes that are fibred over choice of base.

From the point of view of indexed categories, the key result [7, B3.3.6] is that bounded geometric morphisms can be pseudo-pulled-back along arbitrary geometric morphisms.[3] Thus for any geometric morphism $f\colon \mathcal{S}_0 \to \mathcal{S}_1$ we get a reindexing $f^*\colon \mathfrak{BTop}/\mathcal{S}_1 \to \mathfrak{BTop}/\mathcal{S}_0$. This does not extend to arbitrary natural transformations $\alpha\colon f \to g$ unless the Grothendieck toposes are restricted to fibrations or opfibrations over $\mathcal{S}$, so instead we restrict the αs at the base level to be isomorphisms.

We write $\mathfrak{eTop}_{\cong}$ for the 2-category of elementary toposes (with nno), geometric morphisms and natural isomorphisms.

We now express $\mathcal{S} \mapsto \mathfrak{BTop}/\mathcal{S}$ as a fibred 2-category $\mathfrak{GTop}$ of Grothendieck toposes.

Definition 4.1. *The data for the 2-category $\mathfrak{GTop}$ is defined as follows.*
A 0-cell is a bounded geometric morphism $p\colon \mathcal{E} \to \mathcal{S}$.

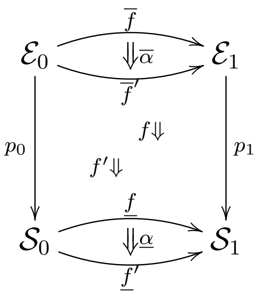
[2] As always for us, our elementary toposes are assumed to have nnos.
[3] Beware that, in 2-categorical contexts, [7] consistently omits "pseudo-" – see B1.1.

A 1-cell $f = (\overline{f}, f\Downarrow, \underline{f})$ *from* $\mathcal{E}_0 \xrightarrow{p_0} \mathcal{S}_0$ *to* $\mathcal{E}_1 \xrightarrow{p_1} \mathcal{S}_1$ *is a square*

$$\begin{array}{ccc} \mathcal{E}_0 & \xrightarrow{\overline{f}} & \mathcal{E}_1 \\ {\scriptstyle p_0}\downarrow & f\Downarrow & \downarrow{\scriptstyle p_1} \\ \mathcal{S}_0 & \xrightarrow[\underline{f}]{} & \mathcal{S}_1 \end{array}$$

in which $f\Downarrow\colon \overline{f}p_1 \to p_0\underline{f}$ *is an isomorphism.*

Given two such 1-cells, f *and* f' *from* p_0 *to* p_1*, a 2-cell* $\alpha\colon f \to f'$ *is a pair of natural transformations* $\overline{\alpha}\colon \overline{f} \to \overline{f}'$ *and* $\underline{\alpha}\colon \underline{f} \to \underline{f}'$



such that the obvious diagram of 2-cells commutes. Moreover, as mentioned earlier, we require $\underline{\alpha}$ *to be an isomorphism.*

It is clear that $\mathfrak{GTop}$ is a 2-category

Proposition 4.2. *There is a 2-functor* $\mathfrak{GTop}^{co} \to \mathfrak{eTop}^{co}_{\cong}$ *that forgets all but the downstairs part. Although it is strict, we consider it as a homomorphsm of* bicategories *for the purposes of [4, 3.1].*

1. *A 1-cell is cartesian iff it is a pseudopullback square in* $\mathfrak{eTop}$.

2. *A 2-cell* α *is cartesian iff* $\overline{\alpha}$ *is an isomorphism.*

3. *The 2-functor is a fibration of bicategories.*

Proof. (1): This is essentially the same as the proof of the result for 1-categories, that for the codomain fibration $\mathsf{cod}\colon \mathcal{C}^{\to} \to \mathcal{C}$, a morphism for $\mathcal{C}^{\to}$ is cartesian iff it is a pullback square in $\mathcal{C}$. The conditions for pseudopullbacks and cartesian 1-cells both bring in the 2-cells in the same way. For the "$\Rightarrow$" direction, note that an arbitrary elementary topos $\mathcal{E}$ can be treated as a 0-cell in $\mathfrak{GTop}$ using the identity geometric morphism.

(2): If $\overline{\alpha}$ is an isomorphism then so is the 2-cell α, and it is then clearly cartesian. For the converse, suppose $\alpha\colon f \to g$ is a cartesian 2-cell. (Note that because we are going to dualize, α is really cocartesian in $\mathfrak{GTop}$.) Downstairs, $\underline{\alpha}$ is invertible and so by lifting $\underline{\alpha}^{-1}$ we get $\alpha'\colon g \to f$, with $\alpha\alpha' = \mathsf{Id}_f$. By considering Id_g and $\alpha'\alpha$ as lifts of $\mathsf{Id}_{\underline{g}}$ we see that they are equal.

(3) Cartesian lifting of 1-cells arises because, in $\mathfrak{eTop}$, pseudopullbacks of bounded geometric morphisms along arbitrary geometric morphisms always exist [7, B3.3.6].

Cartesian lifting of 2-cells is easy – in fact we can ensure that the upstairs part of the lifted 2-cell is an identity. □

Of course, $\mathfrak{eTop}_{\cong}^{co} \cong \mathfrak{eTop}_{\cong}$, so we could equally well consider $\mathfrak{GTop}^{co}$ as fibred over $\mathfrak{eTop}_{\cong}$.

4.2 Representability

In Definition 2.6, "classifying topos" is *defined* in terms of representability of an indexed category, a pseudofunctor $\mathbb{T}\colon (\mathfrak{BTop}/\mathcal{S})^{op} \to \mathfrak{CAT}$. We now look at how this appears in terms of fibrations.

To work abstractly, suppose $\mathcal{C}$ is a 2-category, and $F\colon \mathcal{C}^{coop} \to \mathfrak{CAT}$ a pseudofunctor. We shall describe the Grothendieck construction for it. In our applications, for elephant theories deriving from AU-contexts, F will be strict and the Grothendieck construction is described in [4, 2.2] as a fibration of 2-categories. For the present section, however, we shall not assume strictness: thus we retain the connection with general elephant theories. Because of this we need to use [4, 3.3.3], which describes the Grothendieck construction as a fibration of bicategories. Nonetheless, our situation is somewhat simpler than Buckley's. We have not allowed $\mathcal{C}$ to be a bicategory, and we have taken each $F(X)$ to be a category, not a bicategory. Because of this, our fibred bicategory $\mathcal{E}$ is actually a 2-category, though not fibred as such. It has –

0-cells are pairs (x, x_-) of objects of $\mathcal{C}$ and Fx.

1-cells are pairs $(f, f_-)\colon (x, x_-) \to (y, y_-)$ where

$$f\colon x \to y \text{ and } f_-\colon x_- \to Ff(y_-).$$

2-cells $(f, f_-) \to (g, g_-)\colon (x, x_-) \to (y, y_-)$ are 2-cells $\alpha\colon f \to g$ such that the following diagram commutes.

$$\begin{array}{ccc} x_- & \xrightarrow{f_-} & Ff(y_-) \\ & {\scriptstyle g_-}\searrow \quad = \quad \nearrow{\scriptstyle F\alpha_{y_-}} & \\ & Fg(y_-) & \end{array}$$

Then the 1-cell (f, f_-) is cartesian iff f_- is an isomorphism. Every 2-cell α is cartesian.

In the following proposition we characterize representability of the pseudofunctor F in a purely fibrational way, independent of F as choice of cleavage.

Proposition 4.3. *Let* $F\colon \mathcal{C}^{coop} \to \mathfrak{CAT}$ *be a pseudofunctor as above, and let* $P\colon \mathcal{E} \to \mathcal{C}$ *be its Grothendieck construction. Then* F *is representable iff there is an object* (x, x_-) *in* $\mathcal{E}$ *(a* representing object*) with the following properties.*

1. *For each* (y, y_-) *in* $\mathcal{E}$*, there is a cartesian 1-cell* $(f, f_-)\colon (y, y_-) \to (x, x_-)$.

2. *Each cartesian 1-cell* $(f, f_-)\colon (y, y_-) \to (x, x_-)$ *is terminal in the category* $\mathcal{E}((y, y_-), (x, x_-))$.

Proof. By definition, F is represented by (x, x_-) iff for every y the functor $K_y\colon \mathcal{C}(y, x)^{op} \to Fy$, given by $f \mapsto Ff(x_-)$, is an equivalence.

Condition (1) says that each K_y is essentially surjective. It remains to show that, for each y, K_y is full and faithful iff condition (2) holds.

Suppose K_y is full and faithful and, for a given y_-, we have

$$(f, f_-), (g, g_-)\colon (y, y_-) \to (x, x_-)$$

with (f, f_-) cartesian, i.e. f_- an isomorphism. Then there is a unique $\alpha\colon g \to f$ such that $F\alpha_{x_-} = f_-^{-1}; g_-$, in other words a unique 2-cell from (g, g_-) to (f, f_-).

Conversely, suppose condition (2) holds for a given y, and suppose we have $f, g\colon y \to x$ and $g_-\colon Ff(x_-) \to Fg(x_-)$. We then have two 1-cells

$$(f, \mathsf{Id}), (g, g_-)\colon (y, Ff(x_-)) \to (x, x_-).$$

Since (f, Id) is cartesian we get a unique 2-cell $\alpha\colon (g, g_-) \to (f, \mathsf{Id})$, in other words, a unique $\alpha\colon g \to f$ such that $K_y(\alpha) = g_-$. $\square$

By the usual means, one can show that if x is a representing object for P, then for any object x' in $\mathcal{E}$ we have that x' is a representing object iff it is equivalent to x.

We now extend the above discussion to a situation where $\mathcal{C}$ too is fibred: we have fibrations

$$\mathcal{E} \xrightarrow{P} \mathcal{C} \xrightarrow{Q} \mathcal{B}\ .$$

In our applications, P will again be got from a pseudofunctor (in fact a 2-functor) $\mathcal{C}^{coop} \to \mathfrak{CAT}$, but Q will be more general. The paradigm example for Q is $\mathfrak{GTop}^{co}$ fibred over $\mathfrak{eTop}^{co}_{\cong}$.

We also assume (as in the paradigm) that all 2-cells in $\mathcal{B}$ are isomorphisms.

Note that $f\colon x \to y$ in $\mathcal{E}$ is cartesian for $P;Q$ iff it is cartesian for P and Pf is cartesian for Q. For the "$\Leftarrow$" direction, we just lift in two stages. For "$\Rightarrow$", consider cartesian lifts $\hat{f}\colon \hat{x} \to Py$ of $Q(Pf)$, and then $\hat{\hat{f}}\colon \hat{\hat{x}} \to y$ of $\hat{f}$. We get an equivalence $x \simeq \hat{\hat{x}}$ and deduce the result from that.

Now each object w of $\mathcal{B}$ has a fibre over it, a fibration $P_w\colon \mathcal{E}_w \to \mathcal{C}_w$: it comprises the 0-cells of $\mathcal{C}$ and $\mathcal{E}$ that map to w, and the 1- and 2-cells that map to identities at w. We are now interested in the situation where each P_w is representable, and in how the representing objects transform under 1-cells in $\mathcal{B}$.

Since we are assuming P arises from a pseudofunctor, it is easy to see that a 1-cell or 2-cell in $\mathcal{E}_w$ is cartesian for P_w iff it is cartesian for P.

Definition 4.4. *P is* locally representable *(over Q) iff*

1. *Each fibre P_w is representable.*
2. (Geometricity) *Suppose P_w is represented by x_w, $f\colon w' \to w$ in $\mathcal{B}$, and $h\colon y \to x_w$ is $P;Q$-cartesian over f. Then y is a representing object for $P_{w'}$.*

We call condition (2) "geometricity" in line with [11], because it concerns a property that is preserved by pseudopullback in $\mathfrak{eTop}$. Note that it suffices to verify it for *some* x_w and *some* h. This is because representing objects are equivalent, and so too are cartesian liftings.

As defined, local representability focuses on the fibres P_w. We can express the property in a way that says more about the interaction with change of base.

Proposition 4.5. *P is locally representable over Q iff, for each object w of $\mathcal{B}$, we have an object x_w of $\mathcal{E}$ over it that satisfies the following conditions.*

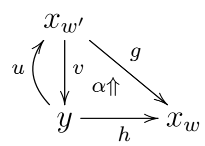
1. *For every object y of $\mathcal{E}$, and 1-cell $f\colon Q(Py) \to w$ in $\mathcal{B}$, there is some $\hat{f}\colon y \to x_w$ over f that is cartesian with respect to P.*

2. *Suppose $h_0, h_1\colon y \to x_w$ in $\mathcal{E}$, with h_1 being P-cartesian.*

   If $\alpha\colon Q(Ph_0) \to Q(Ph_1)$, then there is a unique $\hat{\alpha}\colon h_0 \to h_1$ over α.

Proof. $\Leftarrow$: Clearly any x_w satisfying the conditions must be a representing object for P_w. It remains to show that the representing objects transform correctly under base 1-cells $f\colon w' \to w$.

Suppose x_w and $x_{w'}$ satisfy the conditions. By the conditions for x_w we have P-cartesian $g\colon x_{w'} \to x_w$ over f. Suppose also that $h\colon y \to x_w$ is $P;Q$-cartesian over f. By the conditions on $x_{w'}$ we get P-cartesian $u\colon y \to x_{w'}$ over $\mathsf{Id}_{w'}$, and by cartesianness of h we get $v\colon x_{w'} \to y$ over $\mathsf{Id}_{w'}$ with an isomorphism $\alpha\colon vh \to g$ over Id_f.



Since both g and h are P-cartesian, so is v. It follows by the conditions on $x_{w'}$ that there is a unique isomorphism $vu \cong \mathsf{Id}_{x_{w'}}$ in $P_{w'}$. Also, by the $P;Q$-cartesian property of h, there is a unique isomorphism $uv \cong \mathsf{Id}_y$ in $P_{w'}$. Hence y is equivalent to $x_{w'}$, and so represents $P_{w'}$ as required.

$\Rightarrow$: Let x_w be a representing object for P_w. We show it has the two properties stated.

Suppose y is an object in $\mathcal{E}$, and $f\colon w' = Q(Py) \to w$ a 1-cell in $\mathcal{B}$. Let $g\colon x_{w'} \to x_w$ be $P;Q$-cartesian over f, so that $x_{w'}$ is a representing object for $P_{w'}$. Then there is a P-cartesian 1-cell $u\colon y \to x_{w'}$ in $P_{w'}$, and $ug\colon y \to x_w$ is P-cartesian (because u and g are) over f.

Now suppose $h_0, h_1\colon y \to x_w$ are two 1-cells, with h_1 cartesian for P, and with $f_i = Q(Ph_i)\colon w' \to w$, and $\alpha\colon f_0 \to f_1$. Recall our assumption that all 2-cells in $\mathcal{B}$ are isomorphisms. Let $g_i\colon z_i \to x_w$ be a $P;Q$-cartesian lifting of f_i, with $u_i\colon y \to z_i$ in $P_{w'}$ and $\beta_i\colon u_i g_i \cong h_i$ over f_i. By [4, 3.1.15], there is an equivalence $k\colon z_0 \simeq z_1$ with isomorphism $kg_1 \cong g_0$ over α, and the pair is unique up to unique isomorphism between ks in $P_{w'}$. Thus 2-cells $h_0 \to h_1$ over α are in bijection with 2-cells $u_0 k g_1 \to u_1 g_1$ over f_1, and hence (because g_1 is $P;Q$-cartesian) with 2-cells $u_0 k \to u_1$ in $P_{w'}$. Since z_1 is a representing object for $P_{w'}$, and u_1 is P-cartesian (because h_1 and g_1 are), and hence cartesian in $P_{w'}$, we get a unique 2-cell $u_0 k \to u_1$ in $P_{w'}$. $\square$

5. Context extensions as bundles

In this Section we gather together the previous remarks to get results on classifying toposes in a form that is fibred over a category of bases.

This is most easily understood in the simple case of a single context $\mathbb{T}$. For each Grothendieck topos $p\colon \mathcal{E} \to \mathcal{S}$ we have a category $\mathcal{E}$-$\mathbf{Mod}_s$-$\mathbb{T}$ of strict models of $\mathbb{T}$ in $\mathcal{E}$. This extends to a 2-functor from $\mathfrak{GTop}^{op} = (\mathfrak{GTop}^{co})^{coop}$ to $\mathfrak{CAT}$, and its Grothendieck construction can be written as $P\colon (\mathfrak{GTop}\text{-}\mathbb{T})^{co} \to \mathfrak{GTop}^{co}$.

In constructing that fibration we ignored the parts $\xrightarrow{p} \mathcal{S}$, but when we bring in $\mathcal{S}$ we find that the classifying topos $\mathcal{S}[\mathbb{T}]$ provides a representing object for $P_{\mathcal{S}}$.

The main novelty here is that those representing objects transform according to Definition 4.4: that (Theorem 5.11) the pseudopullback along any $\underline{f}\colon \mathcal{S}_0 \to \mathcal{S}_1$ preserves classifiers. Our proof is non-trivial, and shows that the steps constructing the classifier are preserved under pseudopullback.

As mentioned in Section 1.2, we shall prove local representability more generally, dealing not just with a single context $\mathbb{T}$, but in the relativized situation for an extension $\mathbb{T}_0 \subset \mathbb{T}_1$.

Why extensions, and not arbitrary $H\colon \mathbb{T}_1 \to \mathbb{T}_0$? The main reason is the repeated use of Proposition 2.1, sometimes via Lemma 3.7.

5.1 Models for a context extension

Definition 5.1. *Let* $\mathbb{T}_0 \subset \mathbb{T}_1$ *be an extension of contexts, with corresponding extension map* $U\colon \mathbb{T}_1 \to \mathbb{T}_0$*, and let* $p\colon \mathcal{E} \to \mathcal{S}$ *be a bounded geometric morphism. A* strict model of U in p *is a pair* (M, N) *where* M *is a strict model of* $\mathbb{T}_0$ *in* $\mathcal{S}$*,* N *a strict model of* $\mathbb{T}_1$ *in* $\mathcal{E}$*, and* $NU = p^*M$*.*

A morphism from one such strict model, (M, N)*, to another,* (M', N')*, is a pair* $\phi = (\phi_-, \phi^-)$ *where* $\phi_-\colon M \to M'$ *and* $\phi^-\colon N \to N'$ *are homomorphisms and* $\phi_- U = p^*\phi^-$*.*

For given U we thus get, for each p, a category p-$\mathbf{Mod}_s$-U. It is strictly indexed over $\mathfrak{GTop}$ in the following way.

First suppose $f\colon p_0 \to p_1$ is a 1-cell in $\mathfrak{GTop}$, as in Definition 4.1. If (M, N) is a strict model in p_1, then we define a strict model $f^*(M, N) = (\underline{f}^*M, f^*N)$

$$\begin{array}{ccc} f^*N & \xleftarrow{\;\cong\;} & \overline{f}^*N \\ \downarrow & & \downarrow \\ p_0^*\underline{f}^*M & \xleftarrow[(f\Downarrow)^*M]{} & \overline{f}^*p_1^*M \end{array}$$

where the upstairs isomorphism is the canonical one obtained from Proposition 2.1. The action extends to morphisms between strict models of U, and we obtain a functor f-$\mathbf{Mod}_s$-$U\colon p_1$-$\mathbf{Mod}_s$-$U \to p_0$-$\mathbf{Mod}_s$-U.

If $\alpha\colon f \to f'$ is a 2-cell in $\mathfrak{GTop}$, then it gives a natural transformation from f-$\mathbf{Mod}_s$-U to f'-$\mathbf{Mod}_s$-U. We obtain a strict 2-functor from $\mathfrak{GTop}^{op}$ to $\mathfrak{CAT}$. Its Grothendieck construction is a fibration $(\mathbf{Mod}_s\text{-}U)^{co} \to \mathfrak{GTop}^{co}$.

Definition 5.2. *The data for the 2-category* $\mathbf{Mod}_s$-U *is defined as follows. In each case, a 0-, 1- or 2-cell is the corresponding item for* $\mathfrak{GTop}$*, equipped with extra structure in the form of models of* U*.*

A 0-cell is a bounded geometric morphism $p\colon \mathcal{E} \to \mathcal{S}$*, equipped with a strict model* (M, N) *of* U*.*

A 1-cell from (p_0, M_0, N_0) *to* (p_1, M_1, N_1) *is a 1-cell* $f\colon p_0 \to p_1$ *from* $\mathfrak{GTop}$*, equipped with a homomorphism* $(f_-, f^-)\colon (M_0, N_0) \to f^*(M_1, N_1)$*. (Note that the letter* f *is highly decorated: we have* $\underline{f}$*,* $f\Downarrow$*,* $\overline{f}$*,* f_- *and* f^-*.)*

Given 1-cells (f, f_-, f^-) *and* (f', f'_-, f'^-)*, with the same domain and codomain, a 2-cell from one to the other is a 2-cell* $\alpha\colon f \to f'$ *in* $\mathfrak{GTop}$ *such* *that*

$$(f_-, f^-)(\alpha^*(M_1, N_1)) = (f'_-, f'^-).$$

It is clear that $\mathbf{Mod}_s$-U is a 2-category, with a functor $F'\colon \mathbf{Mod}_s\text{-}U \to \mathfrak{GTop}$ that forgets the model, and by construction F'^{co} is a split fibration. Note that –

1. A 1-cell (f, f_-, f^-) is cartesian iff f_- and f^- are isomorphisms.

2. Every 2-cell α is (co-)cartesian.

Note the special case of a trivial extension $\mathbb{T}_0 = \mathbb{T}_0$. A model of this in p is simply a model M of $\mathbb{T}_0$ in $\mathcal{S}$, since the corresponding model in $\mathcal{E}$ has to be p^*M. In this case we write $\mathbf{Mod}_s$-$(\mathbb{T}_0 \subset \mathbb{T}_0)$.

We have an obvious forgetful functor from $\mathbf{Mod}_s$-U to $\mathbf{Mod}_s$-$(\mathbb{T}_0 \subset \mathbb{T}_0)$, which (or its co-dual) is almost, but not quite, a fibration. The problem is that $\mathbb{T}_0$-homomorphisms $\phi_-\colon M \to M'$ do not lift to functors for the categories of U-models over them. To rectify this, we restrict to isomorphisms downstairs.

Definition 5.3. *$\mathfrak{GTop}$-U is the sub-2-category of $\mathbf{Mod}_s$-U with all the 0-cells, but with only the 1-cells (f, f_-, f^-) for which f_- is an isomorphism. It is full on 2-cells.*

Proposition 5.4. *We write $P^{co}\colon \mathfrak{GTop}\text{-}U \to \mathfrak{GTop}\text{-}(\mathbb{T}_0 \subset \mathbb{T}_0)$ for the forgetful functor. Then $P\colon (\mathfrak{GTop}\text{-}U)^{co} \to (\mathfrak{GTop}\text{-}(\mathbb{T}_0 \subset \mathbb{T}_0))^{co}$ is a split fibration. A 1-cell (f, f_-, f^-) is cartesian iff its f^- is an isomorphism. Every 2-cell is cartesian.*

Proof. It is the Grothendieck construction for the evident 2-functor from $(\mathfrak{GTop}\text{-}(\mathbb{T}_0 \subset \mathbb{T}_0))^{op}$ to $\mathfrak{CAT}$. □

We now fibre over pairs $(\mathcal{S}, M)$.

Definition 5.5. *The 2-category $\mathfrak{eTop}_{\cong}$-$\mathbb{T}$ has structure as follows. A* 0-cell *is a pair $(\mathcal{S}, M)$ where $\mathcal{S}$ is an elementary topos and M a model of $\mathbb{T}$ in $\mathcal{S}$. A* 1-cell *from $(\mathcal{S}_0, M_0)$ to $(\mathcal{S}_1, M_1)$ is a pair $(\underline{f}, f_-)$ where $\underline{f}\colon \mathcal{S}_0 \to \mathcal{S}_1$ is a geometric morphism and $f_-\colon M_0 \to \underline{f}^*M_1$ is an isomorphism. A* 2-cell *from $(\underline{f}, f_-)$ to $(\underline{g}, g_-)$ is a natural isomorphism $\underline{\alpha}\colon \underline{f} \to \underline{g}$ such that $f_-; \underline{\alpha}^*M_1 = g_-$.*

The 2-category $\mathfrak{GTop}$-$(\mathbb{T} \subset \mathbb{T})$ *is made from* $\mathfrak{GTop}$ *by adding components* M *and* f_-*, and the condition on* $\underline{\alpha}$*, in the same way as* $\mathfrak{eTop}_{\cong}$-$\mathbb{T}$ *is made from* $\mathfrak{eTop}_{\cong}$*.*

Proposition 5.6. *Let* $Q^{co}\colon \mathfrak{GTop}\text{-}(\mathbb{T} \subset \mathbb{T}) \to \mathfrak{eTop}_{\cong}\text{-}\mathbb{T}$ *be the evident forgetful functor. Then* $Q = (Q^{co})^{co}$ *is a fibration of bicategories.*

Proof. Much as in Proposition 4.2. □

We now get a diagram of 2-functors as follows, where the Ps and Qs are fibrations. The left hand tower is for the relativized situation $\mathbb{T}_0 \subset \mathbb{T}_1$, while the right hand tower is the special case $\mathbb{T}_0 = \mathbb{1}$.

$$\begin{array}{ccc}
(\mathfrak{GTop}\text{-}U)^{co} & \searrow & \\
\downarrow P & & (\mathfrak{GTop}\text{-}\mathbb{T}_1)^{co} \\
(\mathfrak{GTop}\text{-}(\mathbb{T}_0 \subset \mathbb{T}_0))^{co} & \searrow & \downarrow P \\
\downarrow Q & & \mathfrak{GTop}^{co} \\
(\mathfrak{eTop}_{\cong}\text{-}\mathbb{T}_0)^{co} & \searrow & \downarrow Q \\
 & & \mathfrak{eTop}_{\cong}^{co}
\end{array} \tag{2}$$

5.2 Context extensions fibred over models

Our aim now is to show that, in diagram (2), each P is locally representable over its Q. (Note that the right hand one is a special case of the left hand, for when $\mathbb{T}_0 = \mathbb{1}$.) The existence of the representing objects (as classifying toposes) is straightforward; what seems more novel is their preservation by pseudopullback.

Proposition 5.7. *Let* $\mathbb{T}_0 \subset \mathbb{T}_1$ *be a context extension. Then, over any elementary topos* $\mathcal{S}$*, it is also a geometric extension of elephant theories.*

Proof. It suffices to check the different kinds of simple context extension. Note that any node X in $\mathbb{T}_0$ gives a context homomorphism $\mathbb{O} \lessdot \mathbb{T}_0$, so a map $\mathbb{T}_0 \to \mathbb{O}$, and hence a geometric construct on $\mathbb{T}_0$. Likewise, any edge or composite of edges gives a map $\mathbb{T}_0 \to \mathbb{O}^{\to}$, and hence a morphism between geometric constructs.

An extension by primitive node is a geometric extension by primitive sort.

A simple functional extension of contexts (adjoining a primitive edge) is also a simple functional extension of geometric theories.

An extension by a universal is essentially no geometric extension at all, as the categories of (strict) models are isomorphic.

An extension by commutativities is a simple geometric quotient, as imposing an equality between morphisms is equivalent to requiring the equalizer to be an isomorphism. □

Proposition 5.8. *Let $\mathbb{T}_0$ be a context, and M a strict model of $\mathbb{T}_0$ in an elementary topos $\mathcal{S}$. Then there is an elephant morphism $M\colon \mathbb{1} \to \mathbb{T}_0$ that, on bounded $\mathcal{S}$-topos $(\mathcal{E}, p)$, takes $*$ to p^*M.*

Proof. Note that, although the elephant theories for both $\mathbb{1}$ and $\mathbb{T}_0$ are strictly indexed, M is not a strict morphism. Consider a morphism of $\mathcal{S}$-toposes

$$\begin{array}{ccc} \mathcal{F} & \xrightarrow{\overline{f}} & \mathcal{E} \\ & {\scriptstyle q}\searrow \; f\Downarrow \; \swarrow{\scriptstyle p} & \\ & \mathcal{S} & \end{array}, \qquad \begin{array}{ccc} \mathbb{1}(\mathcal{F}) & = & \mathbb{1}(\mathcal{E}) \\ {\scriptstyle M(\mathcal{F})}\downarrow & \cong & \downarrow{\scriptstyle M(\mathcal{E})} \\ \mathbb{T}_0(\mathcal{F}) & \xleftarrow[\mathbb{T}_0(\overline{f})]{} & \mathbb{T}_0(\mathcal{E}) \end{array}$$

On the right is a pseudo-naturality square, subject to the isomorphism

$$(f\Downarrow)^*M\colon \overline{f}^*p^*M \cong q^*M.$$

□

Of course, $M\colon \mathbb{1} \to \mathbb{T}_0$ is not a map of contexts in general.

Definition 5.9. *Let $\mathbb{T}_0 \subset \mathbb{T}_1$ be a context extension and M a strict model of $\mathbb{T}_0$ in an elementary topos $\mathcal{S}$. By Proposition 2.5 we can pull back the geometric extension for $\mathbb{T}_0 \subset \mathbb{T}_1$ along $M\colon \mathbb{1} \to \mathbb{T}_0$, getting a geometric theory $\mathbb{T}_1/M$ over $\mathcal{S}$. Its models in $(\mathcal{E}, p)$ are the strict models of $\mathbb{T}_1$ whose $\mathbb{T}_0$-reducts are equal to p^*M. It has a classifying topos $\mathcal{S}[\mathbb{T}_1/M]$.*

Proposition 5.10. *Let $\mathbb{T}_0 \subset \mathbb{T}_1 \subset \mathbb{T}_2$ be a sequence of context extensions, with extension maps* $\mathbb{T}_2 \xrightarrow{U'} \mathbb{T}_1 \xrightarrow{U} \mathbb{T}_0$.

Let M be a strict model of $\mathbb{T}_0$ in an elementary topos $\mathcal{S}$, and consider the classifying toposes $p\colon \mathcal{S}' = \mathcal{S}[\mathbb{T}_1/M] \to \mathcal{S}$ with generic model N_G, and $p'\colon \mathcal{S}'' = \mathcal{S}'[\mathbb{T}_2/N_G] \to \mathcal{S}'$ with generic model N'_G.

Then $(\mathcal{S}'', p'p)$ serves as classifier for $\mathbb{T}_2/M$, with generic model N'_G.

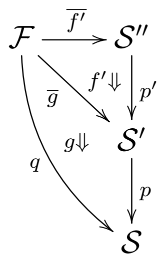
Proof. Note that, using Lemma 3.7, $N'_G U'U = p'^* N_G U = (p'p)^* M$.

For the "essential surjectivity" part, suppose N is a model of $\mathbb{T}_2/M$ in $(\mathcal{F}, q)$. Then NU' is a model of $\mathbb{T}_1/M$, so we get $g = (\overline{g}, g\Downarrow)\colon (\mathcal{F}, q) \to (\mathcal{S}', p)$ with $NU' \cong g^* N_G$ as models of $\mathbb{T}_1/M$; also $g^* N_G \cong \overline{g}^* N_G$ as models of $\mathbb{T}_1$. Now using Proposition 2.1 we can find a strict model $N' \cong N$ of $\mathbb{T}_2$ with $N'U' = \overline{g}^* N_G$, so N' is a model of $\mathbb{T}_2/N_G$ in $(\mathcal{F}, \overline{g})$. Hence there is a morphism

$$f' = (\overline{f'}, f'\Downarrow)\colon (\mathcal{F}, \overline{g}) \to (\mathcal{S}'', p')$$

such that $f'^* N'_G \cong N'$. Now define

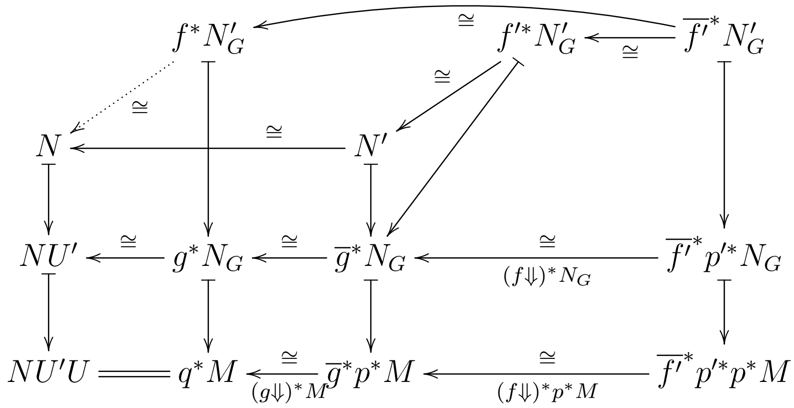
$$f = (\overline{f'}, ((f'\Downarrow) \cdot p; g\Downarrow))\colon (\mathcal{F}, q) \to (\mathcal{S}'', p'p)$$



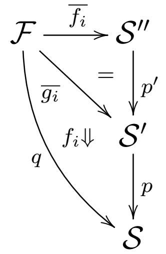
.

As models of $\mathbb{T}_2$, we have $f^* N'_G \cong \overline{f'}^* N'_G \cong f'^* N'_G \cong N' \cong N$; and we see from the following diagram that this composite isomorphism restricts under

$U'U$ to the identity – it is an isomorphism for $\mathbb{T}_2/M$.

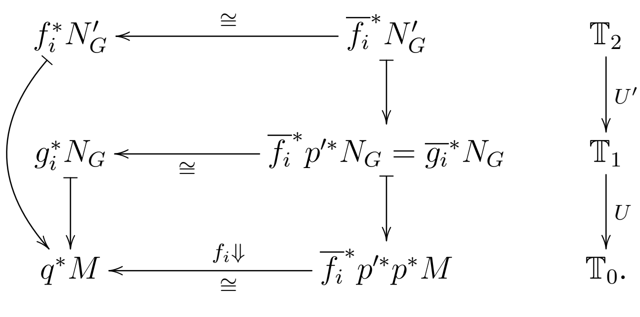


Now suppose we have two morphisms $f_i = (\overline{f_i}, f_i\Downarrow) \colon (\mathcal{F}, q) \to (\mathcal{S}'', p'p)$ $(i = 0, 1)$. Let us write $\overline{g_i} = \overline{f_i}p'$ and

$g_i = (\overline{g_i}, f_i\Downarrow) \colon (\mathcal{F}, q) \to (\mathcal{S}', p)$  .

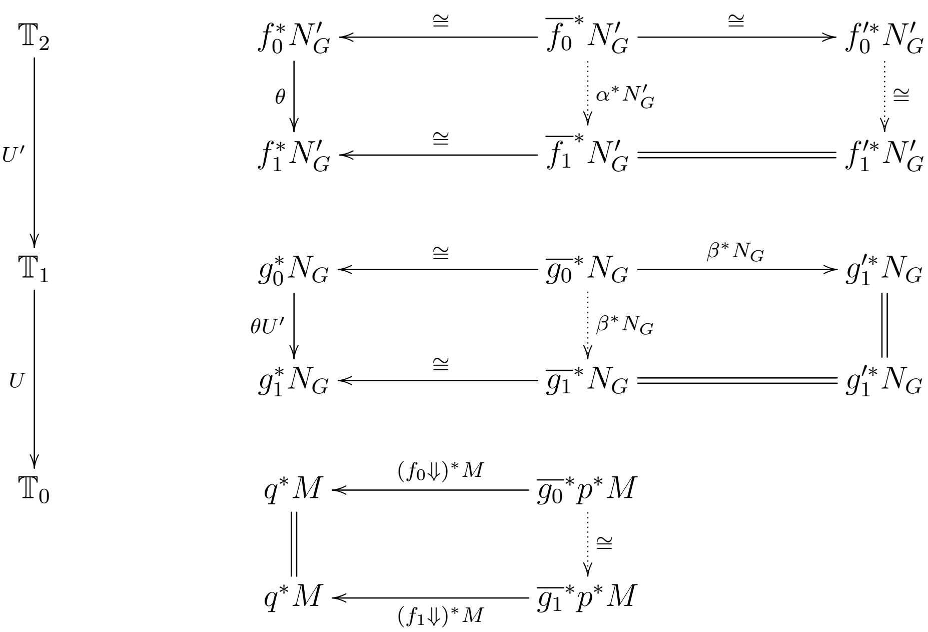
This makes $\mathcal{F}$ two separate toposes $(\mathcal{F}, \overline{g_i})$ over $\mathcal{S}'$.

Suppose also we have a $\mathbb{T}_2/M$-morphism $\theta \colon f_0^* N'_G \to f_1^* N'_G$. Our aim is to show that there is a unique 2-cell $\alpha \colon f_0 \to f_1$ such that $\alpha^* N'_G = \theta$. Consider the diagram



We find that $(f_i^* N'_G)U' = g_i^* N_G$, as it has the correct properties according to Proposition 2.1. Hence we have $\theta U' \colon g_0^* N_G \to g_1^* N_G$, and there is a unique $\beta \colon g_0 \to g_1$ such that $\theta U' = \beta^* N_G$. (This is modulo the appropriate isomorphisms, for β is actually a natural transformation from $\overline{g_0}$ to $\overline{g_1}$.)

Let us first deal with the case where θ is an isomorphism, and β likewise. We thus have two morphisms $f'_i \colon (\mathcal{F}, \overline{g_1}) \to (\mathcal{S}'', p')$, given by $f'_0 = (\overline{f_0}, \beta)$ and $f'_1 = (\overline{f_1}, \mathsf{Id})$. In the diagrams below, three levels are for $\mathbb{T}_2$, $\mathbb{T}_1$ and $\mathbb{T}_0$, successively reduced by U' and U. The horizontal isomorphisms '$\cong$' come from Proposition 2.1, and the vertical ones are defined to make their outer squares commute. We then find a unique $\alpha \colon f'_0 \to f'_1$ (which is the same as saying $\alpha \cdot p' = \beta$) such that $\alpha^* N'_G$ is the isomorphism $f_0'^* N'_G \cong f_1'^* N'_G$ at top right in the diagram. Then $\alpha \colon f_0 \to f_1$ and is unique such that $\theta = \alpha^* N'_G$.



We now generalize to arbitrary morphisms θ. Let $(\mathcal{G}, q')$ be the cocomma object in $\mathfrak{BTop}/\mathcal{S}$ of the identity on $(\mathcal{F}, q)$ against itself, with cocomma injections $h_i \colon (\mathcal{F}, q) \to (\mathcal{G}, q')$ and $\eta \colon h_0 \to h_1$. By [7, B3.4.7], $\mathcal{G}$ as a category is just the comma category $\mathcal{F} \downarrow \mathcal{F}$. It follows from [9] that there is a bijection between, on the one hand, morphisms $\theta \colon N_0 \to N_1$ between strict $\mathbb{T}_2/M$-models in $\mathcal{F}$, and, on the other, strict $\mathbb{T}_2/M$-models in $\mathcal{G}$. Applying the essential surjectivity property (already proved) for $\mathcal{S}''$, in relation to $\mathcal{G}$, we see that for every such θ there is a morphism $f' \colon (\mathcal{G}, q') \to (\mathcal{S}'', p'p)$, hence a pair of morphisms $f'_i \colon (\mathcal{F}, q) \to (\mathcal{S}'', p'p)$, and a 2-cell α' between them, with a commuting diagram

$$\begin{array}{ccc} N_0 & \xrightarrow{\cong} & f_0'^* N'_G \\ {\scriptstyle\theta}\downarrow & & \downarrow{\scriptstyle\alpha'^* N'_G} \\ N_1 & \xrightarrow{\cong} & f_1'^* N'_G \end{array}$$

We return to the case of interest, where $N_i = f_i^* N'_G$. By the restricted case, with θ an isomorphism, we find 2-cell isomorphisms $\beta_i \colon f_i \to f'_i$ that, applied to N'_G, give the horizontal isomorphisms above. Then, taking $\alpha = \beta_0; \alpha'; \beta_1^{-1}$, we get $\theta = \alpha^* N'_G$. This proves fullness.

Finally we must prove faithfulness. Suppose we have f_0 and f_1 as before, and 2-cells $\alpha, \alpha' \colon f_0 \to f_1$ with $\alpha^* N'_G = \alpha'^* N'_G$. We deduce that $\alpha \cdot p' = \alpha' \cdot p'$ because $\mathcal{S}'$ is a classifier. Hence we have two geometric morphisms $g = (\overline{f_0}, \alpha, \overline{f_1})$ and $g' = (\overline{f_0}, \alpha', \overline{f_1})$ from $\mathcal{G}$ to $\mathcal{S}''$, with $gp' = g'p'$. We have $g^* N'_G = g'^* N'_G$, so from the properties of $\mathcal{S}''$ as classifying topos we get a unique 2-cell $\beta \colon g \to g'$ such that $\beta^* N'_G$ is the identity. This gives two 2-cells $\beta_i \colon \overline{f_i} \to \overline{f_i}$, making a commutative square with α and α', with $\beta_i^* N'_G$ the identity on $\overline{f_i}^* N'_G$. We deduce that each β_i is an identity, and it follows that $\alpha = \alpha'$. □

Theorem 5.11. *Let $\mathbb{T}_0 \subset \mathbb{T}_1$ be a context extension and M a strict model of $\mathbb{T}_0$ in an elementary topos $\mathcal{S}_1$. Let the following diagram be a cartesian 1-cell f in $\mathfrak{GTop}$ over $\mathfrak{eTop}_{\cong}$, hence a pseudopullback in $\mathfrak{eTop}$.*

$$\begin{array}{ccc} \mathcal{E}_0 & \xrightarrow{\quad \overline{f} \quad} & \mathcal{E}_1 = \mathcal{S}_1[\mathbb{T}_1/M] \\ {\scriptstyle p_0}\downarrow & f\Downarrow & \downarrow{\scriptstyle p_1} \\ \mathcal{S}_0 & \xrightarrow[\quad \underline{f} \quad]{} & \mathcal{S}_1 \end{array}$$

Then $p_0 \colon \mathcal{E}_0 \to \mathcal{S}_0$ serves as a classifying topos $\mathcal{S}_0[\mathbb{T}_1/\underline{f}^* M]$.

If N_G is a generic model for $\mathbb{T}_1/M$, then $f^* N_G$ serves as generic model for $\mathbb{T}_1/\underline{f}^* M$.

Proof. First, pseudopullback squares are preserved under composition with equivalences over $\mathcal{S}_0$ and $\mathcal{S}_1$, so it suffices to show that there is *some* pseudopullback square whose vertical maps are classifiers as stated.

By Proposition 5.10 we can reduce to the case where the extension is simple.

For extension by primitive node, we have the task of constructing an object classifier, and this is a special case of classifying torsors (internal flat presheaves) over an internal category $\mathcal{C}$, here the category of finite sets: objects are natural numbers, morphisms defined in the appropriate way.

For extension by commutativity, we have already remarked that this is equivalent to inverting a morphism.

For a simple functional extension, adjoining a morphism from X to Y, we can decompose the classification problem into two steps of the above kinds. First, we adjoin a subjobject of $X \times Y$ for the graph of the morphism, and this is equivalent to adjoining a torsor (ideal) for the poset $\mathcal{F}(X \times Y)$, the Kuratowski finite powerobject (free semilattice). Next we impose some axioms for single-valuedness and totality, and this is equivalent to making some morphism invertible.

It follows that we reduce to two cases over $\mathbb{T}_0$: adjoining a torsor for an internal category $\mathcal{C}$, and forcing the invertibility of some morphism. (Although these are not simple extensions of contexts, we can still work with them as single steps.) We show that our classifiers $\mathcal{S}_1[\mathbb{T}_1/M]$ can be found in a way that is preserved under pseudopullback. The argument parallels that of [7, B3.3.6].

In one case, $\mathbb{T}_1$ adjoins a torsor (flat presheaf) for an internal category $\mathcal{C}$ in $\mathcal{S}_1$. Here we can take the classifier to be $[\mathcal{C}, \mathcal{S}_1]$ by Diaconescu's Theorem, and this can be pulled back along any $\underline{f}\colon \mathcal{S}_0 \to \mathcal{S}_1$ to $[\underline{f}^*\mathcal{C}, \mathcal{S}_0]$ over $\mathcal{S}_0$. (See [7, B3.2.7, B3.2.14].)

For any geometric theory $\mathbb{T}$, the models of $\mathbb{T}$ in $[\mathcal{C}, \mathcal{S}_1]$ are the internal $\mathcal{C}$-indexed families of models of $\mathbb{T}$ in $\mathcal{S}_1$, and in the particular case of $\mathcal{C}$-torsors the generic model is the Yoneda embedding $\mathcal{Y}$, with the representable torsor $\mathcal{Y}(j)$ for each object j of $\mathcal{C}$. To express this more concretely as a $(p_1^*\mathcal{C})$-indexed family of $\mathcal{C}$-torsors in $[\mathcal{C}, \mathcal{S}_1]$, use the morphism

$$\mathcal{C}_1 \xrightarrow{\langle d,c \rangle} \mathcal{C}_0 \times \mathcal{C}_0 \xrightarrow{\pi_2} \mathcal{C}_0$$

It becomes the object part of an internal presheaf over $p_1^*\mathcal{C}$, and is the generic torsor. Its construction is geometric (arithmetic, even) and so is preserved by f^*.

In the other case, $\mathbb{T}_1$ imposes invertibility for a morphism $u\colon X \to Y$ in $\mathcal{S}_1$. Here $p_1\colon \mathcal{E}_1 \to \mathcal{S}_1$ is an inclusion, and by [7, A4.3.11] it can be taken to be the topos of sheaves for the smallest local operator for which $\mathrm{im}(u) \rightarrowtail Y$ and $X \rightarrowtail \mathrm{kp}(u)$, the kernel pair, are both dense. Inverting both of these monomorphisms will make u invertible. By [7, A4.5.14(e)] its pseudopullback along $\underline{f}$ is also an inclusion, in fact for the smallest local operator that makes $\underline{f}^*u$ an isomorphism. The generic model is p_1^*M, so $f^*p_1^*M \cong p_0^*\underline{f}^*M$ is a generic model in $\mathcal{E}_0$. □

Putting together these results, we now obtain our main *Local Representability Theorem* –

Theorem 5.12. *In diagram* (2)*, the left hand fibration P is locally representable (Definition 4.4) over its Q.*

Proof. Given $(\mathcal{S}, M)$, then, as noted in Definition 5.9, the classifying topos $\mathcal{S}[\mathbb{T}_1/M]$ exists, and, by Proposition 4.3, this gives condition (1) of Definition 4.4. The geometricity condition (2) is Theorem 5.11. □

As we have already mentioned, by taking $\mathbb{T}_0 = \mathbb{1}$ we get that the right hand P in diagram (2) is also locally representable. This tells us that the assignment $\mathcal{S} \mapsto \mathcal{S}[\mathbb{T}_1]$ is preserved under change of base $\mathcal{S}$.

After the main theorem, Proposition 4.5 now provides us with ways to use the classifying toposes $\mathcal{S}[\mathbb{T}_1/M]$ in places beyond $\mathfrak{BTop}/\mathcal{S}$. In particular, –

Corollary 5.13. *Let $\mathbb{T}_0 \subset \mathbb{T}_1$ be a context extension and M a strict model of $\mathbb{T}_0$ in an elementary topos $\mathcal{S}$. Then $\mathcal{E} = \mathcal{S}[\mathbb{T}_1/M]$ has the classifying topos property for arbitrary $\underline{q}\colon \mathcal{F} \to \mathcal{S}$, not necessarily bounded.*

Proof. Apply Proposition 4.5 to models of U in $\mathsf{Id}\colon \mathcal{F} \to \mathcal{F}$ for which the $\mathbb{T}_0$ part is $\underline{q}^*M$. □

Example 5.14. Let $\mathbb{T}_0$ be the context whose models are "GRD-systems" as in [11]. It has three nodes G, R, D, together with (amongst other ingredients) a further node $\mathcal{F}G$ constrained to be the Kuratowski finite powerset of G.

(For instance, it can be constructed as a quotient of the list object $\mathsf{List}\, G$.) Finally, it has edges

$$\begin{array}{ccc} & & D \\ & \overset{\rho}{\swarrow} & \downarrow \pi \\ \mathcal{F}G & \underset{\lambda}{\longleftarrow} & R \end{array}$$

This can be used to present a frame, with generators $g \in G$ subject to relations (for $r \in R$)

$$\bigwedge \lambda(r) \leq \bigvee \{ \bigwedge \rho(d) \mid \pi(d) = r \}.$$

The points of the corresponding locale, the subsets $F \subseteq G$ respecting the relations, are models of a context $\mathbb{T}_1$ that extends $\mathbb{T}_0$. It has a node for F, with an edge $F \to G$ constrained to be monic, nodes for $X = \{r \in R \mid \lambda(r) \subseteq F\}$ and $Y = \{r \in R \mid (\exists d)(\pi(d) = r \wedge \rho(d) \subseteq F)\}$ (which can be constructed in the AU-sketches) and an edge $X \subseteq Y$.

Then the local representability Theorem 5.12 implies [11, Corollary 5.4], the geometricity of presentations.

6. Conclusion

What our main result has done is to elaborate the idea that a map $U\colon \mathbb{T}_1 \to \mathbb{T}_0$, a $\mathbb{T}_0$-valued map on $\mathbb{T}_1$, may also be a *bundle:* that is to say, a *space*-valued map on the *co*domain $\mathbb{T}_0$, transforming points to the corresponding fibres.

This interpretation is often tacit in a morphism in a category, and is particularly important in type theory. We have made it concrete in the particular case of a morphism U in $\mathfrak{Con}$ that arises from a context extension.

Note that U certainly is a "$\mathbb{T}_0$-valued map on $\mathbb{T}_1$", if we think of the points of a context as its strict models. This is shown in Section 3 and does not need toposes – the models can be taken in any AU.

To get U as a bundle, we interpret "space" as Grothendieck topos and look for the classifying toposes for the fibres. However, the base toposes are now allowed to vary, and in Theorem 5.11 we showed the geometricity property that when you change the base, and the corresponding base point of $\mathbb{T}_0$, the classifier (representing the fibre) transforms by pseudopullback.

This result, which I have not been able to find in the literature, relies on a difference between the "arithmetic" theories of $\mathfrak{Con}$ and the geometric theories that are classified. An arithmetic theory depends only on the existence of an nno, whereas a geometric theory depends on the choice of some base topos $\mathcal{S}$.

To avoid the intricacies of coherence for the choices made in indexed 2-categories, we have adopted a fibrational approach to classifiers. As part of that, the definition of classifier as representing object for an indexed 2-category has been reformulated in terms of fibrations. Then their indexed behaviour was formulated (our "local representability", Definition 4.4) in terms of towers of two 2-fibrations. It may be that the formulation in Proposition 4.5 has broader usefulness. I sense that the classifying objects x_w may be trying to fulfill some notion of "cartesian 0-cell", though I have not been able to make that idea any more precise.

The results here are a piece in the broad programme of using AU techniques to prove base-independent, geometric results for toposes in those situations that do not need the full power of $\mathcal{S}$-indexed colimits for some $\mathcal{S}$. One already mentioned is the "geometricity of presentations", Example 5.14.

On the other hand, the results also provide clues to how one might seek a self-standing arithmetic logic of spaces, developing [9]. They suggest that the extension maps might be the correct analogues of bounded geometric morphisms.

A final comment regards the word "topos" itself, which Grothendieck chose to suggest "those things of which topology is the study". The very word "topos" should strongly carry the idea of generalized space, but with the advent of the elementary topos this inherent meaning has become obscure. It is not so much the elementary toposes themselves that are the generalized spaces, as the bounded geometric morphisms between them, and those are what are called "Grothendieck toposes" in the present paper. One might dream that the true toposes, the generalized spaces, the subjects of topology, are arithmetic universes, and [9, 12] were written with that in mind. All the same, there are significant gaps between that and Grothendieck's vision, which was partly of "those categories with the structure needed to do sheaf cohomology". Much as one might hope that, suitably formulated, the basic results of algebraic topology are foundationally robust enough to work with AUs, we have no idea at present as to how to do that.

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