cahiers de topologie et géométrie différentielle catégoriques

créés par CHARLES EHRESMANN en 1958
dirigés par Andrée CHARLES EHRESMANN

VOLUME LXII-3, 3e trimestre 2021

AMIENS

Cahiers de Topologie et Géométrie Différentielle Catégoriques

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33 rue Saint-Leu, F-80039 Amiens.

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CAHIERS DE TOPOLOGIE ET GEOMETRIE DIFFERENTIELLE CATEGORIQUES

VOLUME LXII-3 (2021)

GLOBULARLY GENERATED DOUBLE CATEGORIES II: THE CANONICAL DOUBLE PROJECTION

Juan Orendain

Résumé. Il s'agit du deuxième volet d'une série d'articles en deux parties portant sur les catégories doubles librement globulairement engendrées. Nous introduisons la construction canonique de la double projection. Celle-ci transporte l'information des catégories doubles librement globulairement engendrées aux catégories doubles définies par le même ensemble de données globulaires et verticales. Nous utilisons cette double projection pour définir des extensions fonctorielles linéaires formelles compatibles de la forme standard de Haagerup et de l'opération de fusion de Connes aux morphismes entre facteurs d'index éventuellement infini. Nous l'utilisons encore pour montrer que la construction de la double catégorie librement globulairement engendrée est adjointe à gauche à l' "horizontalisation décorée". Nous interprétons ainsi les catégories doubles librement globulairement engendrées comme des analogues formellement décorés des catégories doubles de quintettes et comme des générateurs pour l'internalisation.

Abstract. This is the second installment of a two part series of papers studying free globularly generated double categories. We introduce the canonical double projection construction. The canonical double projection translates information from free globularly generated double categories to double categories defined through the same set of globular and vertical data. We use the canonical double projection to define compatible formal linear functorial extensions of the Haagerup standard form and the Connes fusion operation to possibly-infinite index morphisms between factors. We use the canonical double projection to prove that the free globularly generated dou-

ble category construction is left adjoint to decorated horizontalization. We thus interpret free globularly generated double categories as formal decorated analogs of double categories of quintets and as generators for internalizations.
Keywords. Bicategory, double category, 2-group, double groupoid, von Neumann algebra
Mathematics Subject Classification (2010). 18D35, 46M05, 46M20, 46L10

1. Introduction

Globularly generated double categories were introduced by the author in [15] in order to study ways of minimally lifting bicategories into double categories along possible categories of vertical arrows. Free globularly generated double categories were later introduced in [16]. The free globularly generated double category construction minimally associates to every bicategory together with a possible category of vertical arrows, a double category fixing this set of initial data. Free globularly generated double categories are related to free products of groups and monoids, free double categories in the sense of [9] and to the Ehresmann double category of quintets construction [10], they define numerical invariants for both bicategories and double categories, and provide formal linear functorial extensions of operations in the representation theory of von Neumann algebras.

In this paper we study the canonical projection double functor. The canonical double projection transfers information from free globularly generated double categories to other double categories defined through the same set of initial data. In the language of [15, 16] given a decorated bicategory $(\mathcal{B}, \mathcal{B}^*)$, i.e. given a bicategory $\mathcal{B}$ together with a category $\mathcal{B}^*$ having the same set of objects as $\mathcal{B}$, and a globularly generated double category C internalizing $(\mathcal{B}, \mathcal{B}^*)$, i.e. having $\mathcal{B}^*$ as category of objects and $\mathcal{B}$ as horizontal bicategory, the canonical double projection associated to C is a strict double functor

$$\pi^C : Q_{(\mathcal{B},\mathcal{B}^*)} \to C$$

from the free globularly generated double category $Q_{(\mathcal{B},\mathcal{B}^*)}$ associated to $(\mathcal{B}, \mathcal{B}^*)$, to C, such that π^C is surjective on squares and acts as the identity on objects, vertical morphisms, horizontal morphisms and 2-cells of $\mathcal{B}$.

We summarize this by saying that the restriction of the decorated horizontalization pseudofunctor $H^*\pi^C$ of π^C, see [15, Section 2.6], to $(\mathcal{B},\mathcal{B}^*)$, is the identity on $(\mathcal{B},\mathcal{B}^*)$, or equivalently by the equation:

$$H^*\pi^C \restriction_{\mathcal{B}} = id_{\mathcal{B}}$$

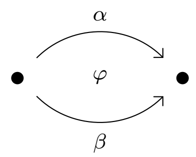
In Theorem 2.1 we prove canonical double projections always exist and that are uniquely determined by the above properties. We interpret the properties defining canonical double projections by considering free globularly generated double categories and canonical double projections as generators and relations presentations of general globularly generated double categories. In Section 3 we exploit this to provide bounds for numerical invariants of double categories, to prove that every lift of a decorated 2-groupoid canonically contains a double groupoid, and to provide compatible formal linear functorial extensions of the Haagerup standard form and the Connes fusion operation extending the corresponding functors provided in [1]. In Section 4 we extend the free globularly generated double category construction to a functor Q : **bCat*** $\to$ **dCat** and in Theorem 5.2 we prove that Q fits into a left adjoint pair (Q,H^*) with the collection of canonical double projections as counit thus making free globularly generated double categories free objects with respect to H^*, see Corollary 5.6. We regard this result as a fibered version of the classic result of [18] and [4] exchanging horizontalization H with decorated horizontalization H^* and the Ehresmann double category of quintets functor **Q** with Q. We provide a more detailed account of the contents and motivation for the main results of the paper.

Internalization

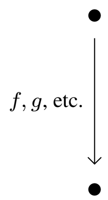
Given a bicategory $\mathcal{B}$ we will say that a category $\mathcal{B}^*$ is a decoration for $\mathcal{B}$ if the collection of 0-cells of $\mathcal{B}$ and the collection of objects of $\mathcal{B}^*$ are equal. In that case we say that the pair $(\mathcal{B}^*,\mathcal{B})$ is a decorated bicategory. Given a double category C the pair (C_0, HC) formed by the category of objects and the horizontal bicategory of C is a decorated bicategory. We will write H^*C for this decorated bicategory. We will call H^*C the decorated horizontalization of C. We are interested in the question of how generic the decorated horizontalization construction is, i.e. we are interested in how and when a given decorated bicategory con be presented as the decorated horizontalization of a double category. We study solutions to the following problem:

Problem 1.1. Let $(\mathcal{B}^*, \mathcal{B})$ be a decorated bicategory. Find double categories C satisfying the equation $H^*C = (\mathcal{B}^*, \mathcal{B})$.

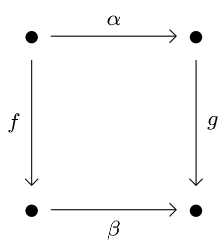
We call any solution C to the equation $H^*C = (\mathcal{B}^*, \mathcal{B})$ an internalization of $(\mathcal{B}^*, \mathcal{B})$. Problem 1.1 admits the following pictorial interpretation: Suppose we are given a collection of globular diagrams of the form:



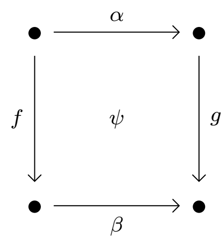
forming a bicategory, together with a collection of vertical arrows of the form:



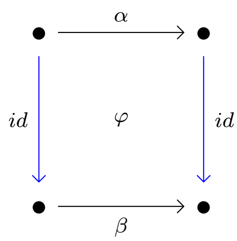
forming a category, satisfying the condition that the collection of vertices of both sets of diagrams coincide. With this data we can form hollow squares of the form:



formed by the edges of the diagrams we are provided with. Problem 1.1 asks about ways to fill these hollow squares *equivariantly* with respect to the globular diagrams in our set of initial conditions. That is, Problem 1.1 asks for the existence of systems of solid squares of the form:



forming a double category such that every square as above admits an interpretation as a globular diagram together with extra structure provided only by our category of vertical arrows, that is such that the only solid squares of the form:



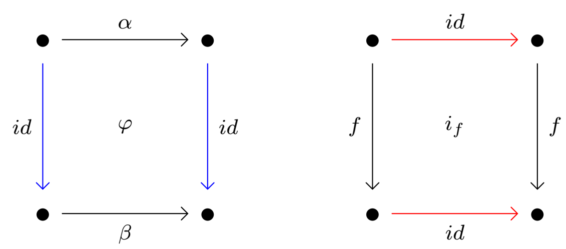
are the globular diagrams provided as set of initial conditions. We regard the decorated horizontalization condition of Problem 1.1 a formalization of the equivariance condition on the above squares.

Constructions of this sort appear in different parts of the theory of double categories. Notably the double category of squares and the double category of commuting squares construction, the Ehresmann double category of quintets construction [10], the double category of adjoint pairs construction [17], and the double categories of spans and cospans constructions all follow the pattern described above. Double categories of squares have categories as globular and vertical sets of initial data, the double category of quintets has a given 2-category and the corresponding category of 1-cells as set of initial data, the double category of adjoints has a given 2-category together with adjoint pairs of 1-cells as set of initial data, and the double category of spans/cospans has the bicategory of spans/cospans of a category with pushouts/pullbacks and the arrows of this category as globular and vertical sets of initial data. In all cases solid squares are carefully chosen so as to encode different aspects of the globular theory.

Our main interest in Problem 1.1 comes from the theory of representations of von Neumann algebras. In [1, 2] a double category of semisimple von Neumann algebras, Hilbert bimodules and finite index bounded equivariant intertwiners was defined. See [3] for applications to conformal field theory and the Stolz-Teichner program. The main goal of this construction is to serve as an intermediate step in the construction of an internal bicategory of coordinate free conformal nets. The main obstruction for the existence of an internal bicategory of general, i.e. not-necessarily-semisimple coordinate free conformal nets, is the existence of a compatible pair of tensor functors extending the Haagerup standard form construction [11] and the Connes fusion operation to not-necessarily-finite index morphisms of semisimple von Neumann algebras. The existence of such tensor functors is equivalent to the existence of a tensor double category of (not-necessarily-semisimple) von Neumann algebras, Hilbert bimodules, and (not-necessarily-finite index) equivariant intertwiners extending the double category defined in [1]. We achieve this in this paper in the case of linear double categories of factors.

Globularly generated double categories

Globularly generated double categories were introduced in [15] as minimal solutions to Problem 1.1. A double category C is globularly generated if C is generated by its collection of globular squares. Pictorially a double category C is globularly generated if every square of C can be written as vertical and horizontal compositions of squares of the form:

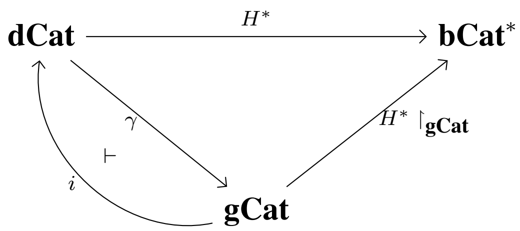


Given a double category C we write γC for the sub-double category of C generated by squares of the above form. We call γ the globularly generated piece of C. γC is globularly generated, satisfies the equation

$$H^*C = H^*\gamma C$$

and is contained in every sub-double category D of C satisfying the equation $H^*C = H^*D$. Moreover, a double category C is globularly generated if and only if C does not contain proper sub-double categories satisfying the above equation. Globularly generated double categories are thus minimal with respect to H^*.

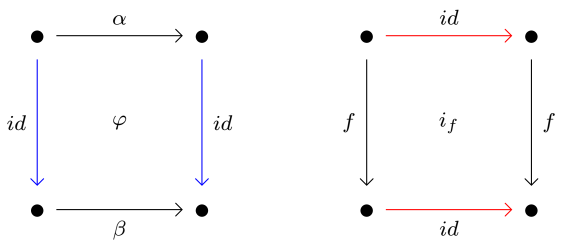
The comments in the previous paragraph admit the following categorical interpretation: Let **dCat**, **gCat** and **bCat*** denote the category of double categories and double functors, the full sub-category of **dCat** generated by globularly generated double categories and the category of decorated bicategories and decorated pseudofunctors respectively. Decorated horizontalization extends to a functor $H^* : \mathbf{dCat} \to \mathbf{bCat}^*$ and the globularly generated piece construction extends to a functor $\gamma : \mathbf{dCat} \to \mathbf{gCat}$. In [15, Proposition 3.6] it is proven that γ is a coreflector of **gCat** in **dCat**. It is easily seen that this implies that γ is a Grothendieck fibration. Moreover, H^* is constant on γ-fibers. We present this through the following diagram:



where i denotes the inclusion of **gCat** in **dCat**. The above diagram breaks Problem 1.1 into the problem of studying bases of γ and then understanding the double categories in each fiber. We follow this strategy and thus study globularly generated double categories, i.e. bases with respect to γ.

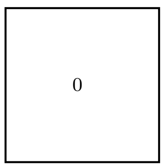
The vertical filtration

Globularly generated double categories admit a helpful combinatorial description provided in the form of a filtration of their categories of squares. Given a globularly generated double category C we write V_C^1 for the category formed by vertical compositions of squares of the form:

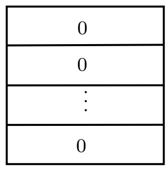


and we denote by H_C^1 the (possibly weak) category formed by horizontal compositions of squares of this form. Assuming we have defined V_C^k and H_C^k through vertical and horizontal compositions respectively, we make V_C^{k+1} to be the category generated by squares in H_C^k and H_C^{k+1} the (possibly weak) category generated by squares in V_C^{k+1}. The category of squares C_1 of C satisfies the equation $C_1 = \varinjlim V_C^k$. We define the length $\ell C \in \mathbb{N} \cup \{\infty\}$ of a double category C as the minimal k such that the equation $\gamma C_1 = V_{\gamma C}^k$ holds. Intuitively the vertical length of a double category C measures the complexity of expressions of squares in C by globular and horizontal identity squares.

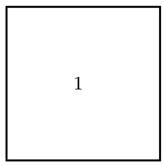
We further explain the vertical filtration construction through the following pictorial representation: We regard the globular and horizontal identity squares of a double category C as the simplest possible squares of C, i.e. we regard these squares as having 'complexity' 0. We thus represent globular and horizontal identity squares diagramatically as squares marked by 0, i.e. as:



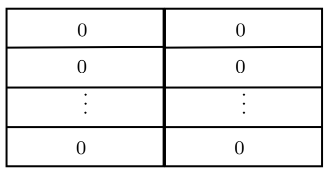
The collection of such squares is what in Section 2 we denote by $\mathbb{G}$. Observe that the collection of 0-marked squares is closed under horizontal composition. Squares in V_C^1 are those squares in C admitting a subdivision as vertical composition of 0-marked squares. Diagrammatically every square in V_C^1 admits a decomposition as:



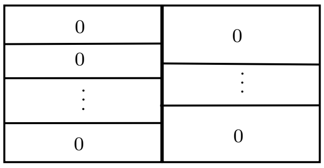
where we draw internal 0-marked squares as rectangles for convenience. If a square as above is not globular or a horizontal identity, i.e. is not 0-marked, we mark it with 1. We represent 1-marked squares pictorially as:



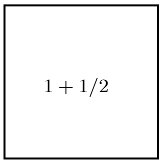
Squares in H_C^1 are thus those squares in C that admit a subdivision as horizontal composition of squares marked with $i \leq 1$. Given two horizontally composable squares φ, ψ in V_C^1 we might be able to find compatible vertical subdivisions of φ and ψ in 0-marked squares, i.e. we might be able to represent the horizontal composition of φ and ψ as:



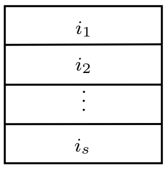
where the internal 0-marked squares of the left and right outer squares match and can be composed horizontally. In that case we can use the exchange identity to re-arrange the above horizontal composition into a vertical subdivision of 0-marked squares. Example [16, Example 4.1] shows that this is not always the case and that there might exist horizontally composable squares φ, ψ such that any two vertical subdivisions into 0-squares look like:



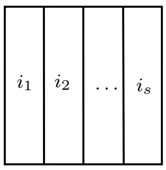
i.e. the internal 0-squares cannot be arranged to match horizontally. Such horizontal compositions are not 1-marked. We represent squares in H_C^1 as above, i.e. squares in $H_C^1 \setminus V_C^1$ as squares marked with 1+1/2, i.e. as:



V_C^2 is thus the category of squares admitting a vertical subdivision into squares marked with $\leq 1 + 1/2$. Inductively, given $k \geq 1$, V_C^k is the category of squares admitting vertical subdivisions as:



where the i_j's are all $\leq k - 1/2$. Squares marked with k are squares in V_C^k not marked with $i < k$. H_C^{k+1} is the (possibly weak) category of squares admitting a horizontal subdivision as:



where the i_j's are all $\leq k$. Squares marked with $k + 1/2$ are those squares in H_C^k such that no subdivision as above can be reduced as a vertical subdivision as i-squares with $i \leq k - 1/2$. In [16] it shown that there exist globularly generated double categories such that squares marked with $k + 1/2$ exist for every $k \geq 0$. The formula $C_1 = \varinjlim V_C^k$ thus means that in a globularly generated double category C every square admits a $\mathbb{N} + 1/2\mathbb{N}$-marking as above. The length of a square φ marked by $x \in \mathbb{N} + 1/2\mathbb{N}$ is $\lceil x \rceil$ and the length ℓC is the maximum of legths of squares in C. The above pictorial representation is only meant to serve as intuition for the vertical filtration construction and we will not use it for the remainder of the paper.

Free globularly generated double categories

The free globularly generated double category construction associates to every decorated bicategory $(\mathcal{B}^*, \mathcal{B})$ a globularly generated double category $Q_{(\mathcal{B}^*,\mathcal{B})}$. The double category $Q_{(\mathcal{B}^*,\mathcal{B})}$ lifts the bicategory structure of $\mathcal{B}$ in the sense that the category of objects $Q_{(\mathcal{B}^*,\mathcal{B})_0}$ of $Q_{(\mathcal{B}^*,\mathcal{B})}$ is equal to $\mathcal{B}^*$, the horizontal morphisms of $Q_{(\mathcal{B}^*,\mathcal{B})}$ are the 1-cells of $\mathcal{B}$ and $\mathcal{B}$ is a sub-bicategory of $HQ_{(\mathcal{B}^*,\mathcal{B})_0}$. The equation

$$H^*Q_{(\mathcal{B}^*,\mathcal{B})} = (\mathcal{B}^*, \mathcal{B})$$

holds only in special cases, e.g. $\mathcal{B}^*$ is reduced or $\mathcal{B}^*$ is the category of factors and unital $*$-morphisms, but the inclusion

$$(\mathcal{B}^*, \mathcal{B}) \subseteq H^*Q_{(\mathcal{B}^*,\mathcal{B})}$$

always holds. Free globularly generated double categories thus not always provide solutions to Problem 1.1. An example where the above inclusion is proper is provided in [16, Example 3.1], where it is proven that in the case in which $\mathcal{B}^*$ is the delooping groupoid $\Omega\mathbb{Z}_2$ of $\mathbb{Z}_2$ and $\mathcal{B}$ is the double delooping 2-group $2\Omega\mathbb{Z}_2$ of $\mathbb{Z}_2$, i.e. when $\Omega\mathbb{Z}_2$ is the groupoid with a single object having $\mathbb{Z}_2$ as group of automorphisms and $2\Omega\mathbb{Z}_2$ is the 2-group having a single object with endomorphism category $\Omega\mathbb{Z}_2$, the horizontal bicategory $HQ_{(2\Omega\mathbb{Z}_2,\Omega\mathbb{Z}_2)}$ associated to the decorated bicategory $(2\Omega\mathbb{Z}_2, \Omega\mathbb{Z}_2)$ is equal to $\Omega(\mathbb{Z}_2 * \mathbb{Z}_2)$. The inclusion $(\mathcal{B}, \mathcal{B}^*) \subseteq H^*Q_{(\mathcal{B},\mathcal{B}^*)}$ is in this case obviously proper. We call decorated bicategories for which their free globularly generated double category provides solutions to Problem 1.1 saturated. Every decorated bicategory $(\mathcal{B}^*, \mathcal{B})$ has a saturated decorated bicategory associated to it with the same free globularly generated double category as $(\mathcal{B}^*, \mathcal{B})$. Free globularly generated double categories are related to free products and free double categories in the sense of [9]. Moreover, free globularly generated double categories provide examples of double categories of arbitrarily large and infinite length and provide formal equivariant functorial extensions of the Haagerup standard form and the Connes fusion operation in the theory of representation of von Neumann algebras.

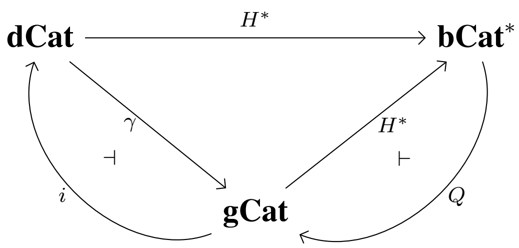
The canonical double projection

The canonical double projection construction relates free globularly generated double categories to general solutions to Problem 1.1. Precisely, given a decorated bicategory $(\mathcal{B}^*, \mathcal{B})$ and a double category C satisfying the equation $H^*C = (\mathcal{B}^*, \mathcal{B})$ the double canonical projection associated to C is a strict double functor $\pi^C : Q_{(\mathcal{B}^*,\mathcal{B})} \to \gamma C$ satisfying the equation:

$$\pi^C \restriction_{(\mathcal{B}^*,\mathcal{B})} = id_{(\mathcal{B}^*,\mathcal{B})}$$

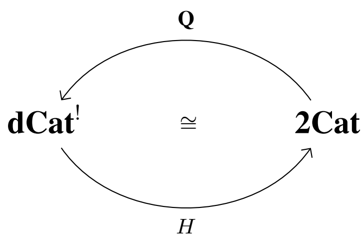
and such that π^C is surjective on squares. Moreover, π^C is unique with respect to this property. We interpret the existence of such double functors as the fact that every globularly generated solution to Problem 1.1 for a decorated bicategory $(\mathcal{B}^*, \mathcal{B})$ can be canonically expressed as a double quotient of $Q_{(\mathcal{B}^*,\mathcal{B})}$. We apply the canonical double projection to length, double groupoids, double deloopings of groups decorated by groups, and to double categories of von Neumann algebras. All applications of the canonical double projections follow the slogan: *Saying something about the free globularly generated double category associated to a decorated bicategory translates to saying something about all its globularly generated internalizations*, the intuition of which clearly follows from the properties defining the canonical double projection.

The canonical double projection construction provides free globularly generated double categories with the structure of universal bases with respect to the fibration γ as follows: We extend the free globularly generated double category construction to a functor $Q : \mathbf{bCat}^* \to \mathbf{gCat}$ using methods analogous to those used in the construction of the canonical double projection. We prove that the set of canonical double projections $\pi^\bullet = \left\{\pi^C : C \in \mathbf{gCat}\right\}$ provides a counit to a left adjunction pair (Q, H^*). We thus obtain a diagram as:



completing the similar diagram above. Further, we prove that the restriction $H^* \restriction_{\mathbf{gCat}}$ is faithful. This provides **gCat** with the structure of a concrete category over **bCat*** and provides Q with the structure of a free contruction with respect to H^*.

We consider the above statement as a generalization of a classic result in nonabelian algebraic topology. In [5] the concept of edge symmetric double category with connection is introduced. In [18] and later in [4] it is proven that the category **dCat**$^!$ of edge symmetric double categories with connection is equivalent to the category **2Cat** of 2-categories, with equivalences provided by the horizontalization functor H and the functor associating to every 2-category B its Ehresmann category of quintets $\mathbf{Q}B$. Pictorially H and **Q** fit into a diagram of the form:



The above diagram can be considered as a statement on fillings of hollow squares. When considering problems of filling squares through data provided by general decorated bicategories and not just by data provided by 2-categories decorated by 1-cells, one wishes to obtain a similar statement. We regard the diagram involving H^* and Q above as a decorated bicategory version of the diagram involving **Q** and H above, fibered by γ.

Notational conventions

We will follow the notational conventions appearing in [15, 16]. We refer the reader to Section 3 of [16] for the details of the notational conventions used in the construction of the free globularly generated double category. We will heavily use the notation and results presented there. In the introduction we have written decorated bicategories in the form $(\mathcal{B}^*, \mathcal{B})$ with $\mathcal{B}^*$ denoting the decoration and $\mathcal{B}$ denoting the underlying bicategory of $(\mathcal{B}^*, \mathcal{B})$ respectively. In what follows we will suppress $\mathcal{B}^*$ from this notation and we will denote $\mathcal{B}$ for a decorated bicategory $(\mathcal{B}^*, \mathcal{B})$.

Contents

In Section 2 we introduce the canonical double projection construction. We prove that the canonical double projection always exists and that it is uniquely determined by the conditions mentioned in the introduction. The construction of the canonical double projection follows a strategy similar to that of the free globularly generated double category construction. In Section 3 we study applications of canonical double projections. We provide upper bounds for lengths of internalizations, we prove that every globularly generated internalization of a decorated 2-groupoid is a double groupoid and we provide compatible formal linear extensions of the Bartels-Douglas-Hénriques Haagerup standard form and Connes fusion functors to the category of factors and possibly-infinite index morphisms. In Section 4 we extend the free globularly generated double category construction to decorated pseudofunctors thus extending the free globularly generated double category construction to a functor. In Section 5 we prove that the pair formed by the free globularly generated double category functor and the decorated horizontalization functor forms a left adjoint pair. Moreover, we prove that the restriction of the decorated horizontalization functor to globularly generated double categories is faithful. We use this to interpret globularly generated double categories as a concrete category over decorated bicategories and the free globularly generated double category construction as a free object.

2. The canonical double projection

In this section we present the canonical double projection construction. Given a decorated bicategory $\mathcal{B}$ the canonical double projection construction associates to every double category C satisfying the equation $H^*C = \mathcal{B}$ a unique strict double functor $\pi^C : Q_{\mathcal{B}} \to \gamma C$ such that π^C acts as the identity on $\mathcal{B}$ and such that π^C is surjective on squares. The following is the main theorem of this section.

Theorem 2.1. *Let $\mathcal{B}$ be a decorated bicategory. Let C be a globularly generated double category such that $H^*C = \mathcal{B}$. In that case there exists a unique strict double functor $\pi^C : Q_{\mathcal{B}} \to C$ such that the equation*

$$H^*\pi^C \upharpoonright_{\mathcal{B}} = id_{\mathcal{B}}$$

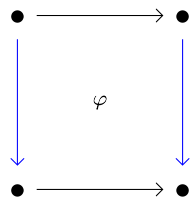
holds, and such that π^C is surjective on squares.

Given a double category C satisfying the conditions above for a decorated bicategory $\mathcal{B}$ we will call the double functor π^C provided in Theorem 2.1 the canonical double projection associated to C. We divide the construction of π^C in several steps. We begin by summarizing the free globularly generated double category construction. We do this in order to set notational conventions used throughout the section and the rest of the paper. The exact details of this construction and the corresponding notational conventions can be found in [16, Section 2].

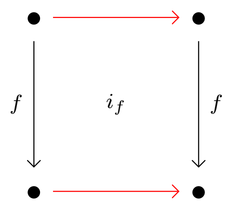
The free globularly generated double category: Quick summary

Given functions $s, t : X \to Y$ between sets X and Y, which we interpret as source and target functions for elements of X, we write $X_{s,t}$ for the set of evaluations of finite compatible words of elements of X with respect to different parentheses patterns. Geometrically $X_{s,t}$ is the set of compatible evaluations, with elements of X, of the vertices of all Stasheff associahedra [19, 20]. The functions s, t extend to functions $\tilde{s}, \tilde{t} : X_{s,t} \to Y$ and concatenation provides a composition $*_{s,t} : X_{s,t} \times_Y X_{s,t} \to X_{s,t}$. Given another pair of functions $s', t' : X' \to Y'$ as above and a pair of functions $\psi : X \to X', \varphi : Y \to Y'$ intertwining s, s' and t, t', evaluation on ψ provides a function $\mu_{\psi,\varphi} : X_{s,t} \to X'_{s'.t'}$ intertwining $\tilde{s}, \tilde{s}'$, $\tilde{t}, \tilde{t}'$ and $*_{s,t}$, $*_{s',t'}$. We apply these conventions to the situation we are interested in as follows.

Let $\mathcal{B}$ be a decorated bicategory. We formally associate to every 2-cell φ in $\mathcal{B}$ a diagram of the form:



and we associate to every morphism f in $\mathcal{B}^*$ a square of the form:



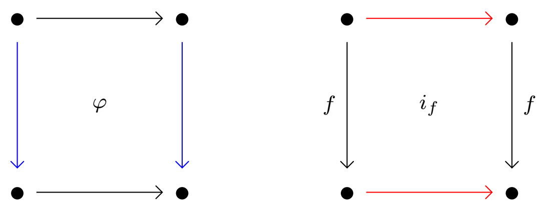
where the blue and red arrows above always denote identity arrows in $\mathcal{B}^*$ and $\mathcal{B}$ respectively. We write $\mathbb{G}$ for the collection of the above diagrams. The free globularly generated double category $Q_{\mathcal{B}}$ is the double category freely generated by $\mathbb{G}$. We explain this in more detail. Going around the edges of the above squares there are obvious vertical domain and codomain functions $d_0, c_0 : \mathbb{G} \to \mathrm{Hom}_{\mathcal{B}_1}$ and obvious horizontal domain and codomain functions $s_0, t_0 : \mathbb{G} \to \mathrm{Hom}_{\mathcal{B}^*}$. We write E_1 for $\mathcal{B}_{1_{s_0,t_0}}$. The functions d_0, c_0 extend to functions on E_1. We write F_1 for the free category generated by E_1 with respect to these extensions. The functions s_0, t_0 extend to functors on F_1. We extend this construction inductively and obtain increasing sequences E_k and F_k equipped with corresponding functions d_k, c_k and functors s_{k+1}, t_{k+1} satisfying certain compatibility conditions, see [16, Lemma 2.5]. We consider limits in **Set** and **Cat** and obtain a category F_∞ together with functions d_∞, c_∞ and functors s_∞, t_∞ extending d_0, c_0 and s_0, t_0 respectively. The category F_∞ does not capture the information contained in $\mathcal{B}^*$.

We thus consider an equivalence relation R_∞ on the set of morphisms E_∞ of F_∞ implementing this information, see [16, Definition 2.10]. We write V_∞ for the quotient F_∞/R_∞. The structure used to define F_∞ descends to V_∞ and provides the pair $(\mathcal{B}^*, V_\infty)$ with the structure of a double category. This is the free globularly generated double category $Q_\mathcal{B}$ associated to $\mathcal{B}$.

The category V_∞ described above comes equipped with a filtration V_k, which we call the free vertical filtration of $Q_\mathcal{B}$ and the set of squares H_∞ of $Q_\mathcal{B}$ comes equipped with a horizontal filtration H_k. We call H_k the free horizontal filtration of $Q_\mathcal{B}$, see [16, Lemma 2.20]. In Section 4 we deal with the free globularly generated double category associated to more than one decorated bicategory. In that case we will write the corresponding decorated bicategory as superscript in the pieces of structure described above. We now proceed to the proof of Theorem 2.1. We first briefly explain our strategy for the proof.

Strategy

The construction in Theorem 2.1 will follow a strategy similar to that employed in the free globularly generated double category construction explained above. Let $\mathcal{B}$ be a decorated bicategory. Let C be a globularly generated double category satisfying the equation $H^*C = \mathcal{B}$. We will begin the construction of π^C by first defining π^C on squares of the form

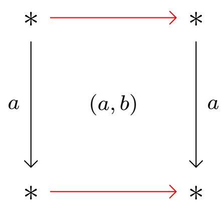


Recall that we denote the set of the above squares by $\mathbb{G}$. We thus first define π^C on $\mathbb{G}$. The equation

$$H^*\pi^C \restriction_\mathcal{B} = id_\mathcal{B}$$

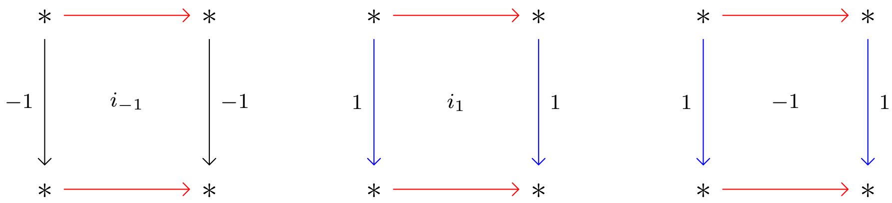
together with the requirement that π^C is a strict double functor, forces π^C to act as the identity in such squares. We extend π^C formally to E_1 and we extend this freely to F_1. We proceed through an induction argument, to extend π^C to E_k, F_k for every positive integer k. We do this carefully so as to make these extensions compatible with the finite terms of the structure data $d_k, c_k, s_k, t_k, *_k$ defined on categories F_k. This is the content of Lemma 2.3. We take limits and define a functor on F_∞. We prove that this functor is well defined with respect to the equivalence relation R_∞ defining $Q_\mathcal{B}$. This is the content of Lemma 2.6 and Lemma 2.7. This will prove that our limit functor descends to a functor from V_∞ to C_1. This will be the morphism functor of the canonical double projection π^C. Finally we take advantage of the vertical filtration on C to prove uniqueness and square surjectivity of π^C.

We show how the construction of π^C works in a specific example. Let $\mathcal{B}$ be the decorated 2-group $(2\Omega\mathbb{Z}_2, \Omega\mathbb{Z}_2)$ as in [16, Example 3.1]. Consider squares of the form

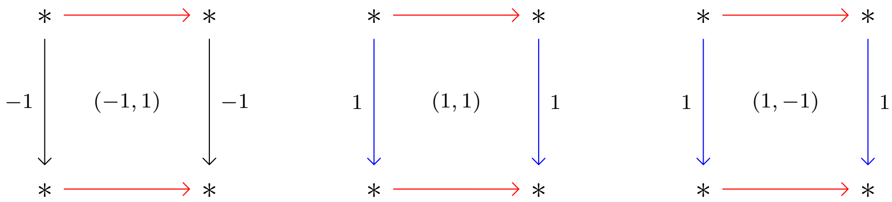


where $a, b \in \mathbb{Z}_2$. The collection of squares as above forms a double groupoid, which we denote by C. The vertical composition of two squares (a, b) and (a', b') in C is the square (aa', bb') and the horizontal composition of two horizontally composable squares (a, b) and (a, b') is (a, bb'). It is easily seen that C is globularly generated, has vertical length 1, and that the groupoid of squares of C is the delooping groupoid ΩV of the Klein 4-group V. Moreover, if we identify the 2-cells in $\mathcal{B}$ with the squares $(1, b)$ with $b \in \mathbb{Z}_2$, then the equation $H^*C = \mathcal{B}$ holds. We briefly describe the procedure to construct π^C in this case.

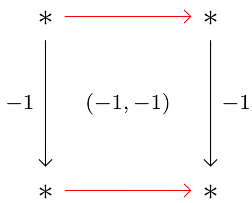
The generating set $\mathbb{G}$ for the free globularly generated double category $Q_\mathcal{B}$ is formed by the formal squares



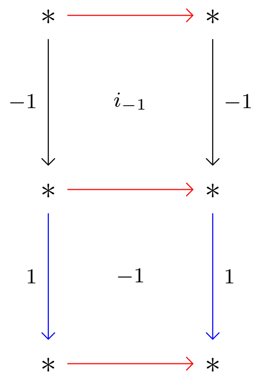
The first step in the construction of π^C associates to the above squares, from left to right, the following squares in C:



The second step of the free globularly generated double category construction for $\mathcal{B}$ considers the free category F_1 on $\mathbb{G}$. In this case F_1 is the delooping category on the free monoid generated by the three squares forming $\mathbb{G}$ above. The second step of the construction of π^C is thus the unique functor from F_1 to C_1 extending the value of π^C on $\mathbb{G}$ described above. We can recover the square



in C as the image, under π^C, of the formal composition



in F_1. By [16, Proposition 5.1] the decorated 2-group $\mathcal{B}$ has free length 1 and thus every square in $Q_{\mathcal{B}}$ can be written as a vertical composition of squares in F_1. It is not difficult to see that, V_∞ which in this case is F^1/R_∞, is equal to the delooping groupoid $\Omega(\mathbb{Z}_2 * \mathbb{Z}_2)$ on the free product $\mathbb{Z}_2 * \mathbb{Z}_2$ that the canonical double projection π^C is the double functor from $Q_{\mathcal{B}}$ to C induced by the projection from $\mathbb{Z}_1 * \mathbb{Z}_2$ to V induced by the square-assignments described above. In the case where a decorated bicategory $\mathcal{B}$ has free length > 1, e.g. [16, Example 4.1] the construction of the canonical double projection π^C follows the above pattern inductively.

Construction

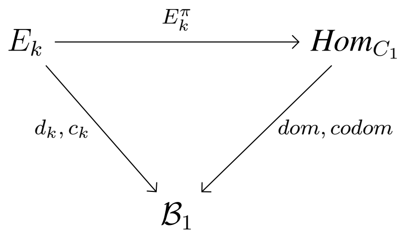
Notation 2.2. Let C be a double category. We denote by q^C the function from $\mathrm{Hom}_{C_{1_{s,t}}}$ to Hom_{C_1} associating to every evaluation Φ of a compatible sequence of squares $\Psi_1, ..., \Psi_k$ in C, the horizontal composition $\Psi_k * \cdots * \Psi_1$ following the parenthesis pattern defining Φ.

Lemma 2.3. *Let $\mathcal{B}$ be a decorated bicategory. Let C be a globularly generated double category satisfying the equation $H^*C = \mathcal{B}$. There exists a pair, formed by a sequence of functions $E_k^\pi : E_k \to Hom_{C_1}$ and a sequence of functors $F_k^\pi : F_k \to C_1$, with $k \geq 1$, such that the following conditions are satisfied:*

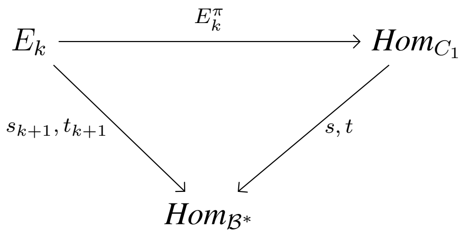
1. *The restriction $E_1^\pi \restriction_{\mathbb{G}}$ is equal to $id_{\mathbb{G}}$.*

2. *For every* $m, k \geq 1$ *such that* $m \leq k$*, the restriction of* E_k^π *to the set of morphisms of* F_m *is equal to the morphism function of* F_m^π*, and the restriction to* E_m *of the morphism function of* F_k^π *is equal to* E_m^π*.*

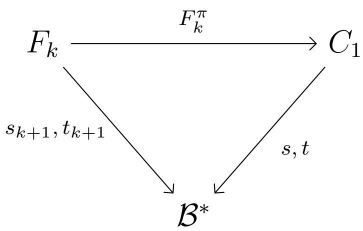
3. *The following two triangles commute for every positive integer* k*:*



4. *The following two triangles commute for every* $k \geq 1$*:*



5. *The following two triangles commute for every* $k \geq 1$*:*



6. *The following square commutes for every* $k \geq 1$*:*

$$\begin{array}{ccc} E_k \times_{Hom_{\mathcal{B}^*}} E_k & \xrightarrow{E_k^\pi \times E_k^\pi} & Hom_{C_1} \times_{Hom_{\mathcal{B}^*}} Hom_{C_1} \\ \downarrow{*_k} & & \downarrow{*} \\ E_k & \xrightarrow[E_k^\pi]{} & Hom_{C_1} \end{array}$$

Moreover, conditions 1-5 above determine the pair of sequences E_k^π *and* F_k^π*.*

Proof. Let $\mathcal{B}$ be a decorated bicategory. Let C be a globularly generated double category such that $H^*C = \mathcal{B}$. We wish to construct a sequence of functions E_k^π from $E_k^{\mathcal{B}}$ to Hom_{C_1} and a sequence of functors F_k^π from $F_k^{\mathcal{B}}$ to C_1 with k running through the collection of positive integers, in such a way that the pair of sequences E_k^π and F_k^π satisfies conditions 1-6 of the lemma.

We proceed inductively on k. We begin with the definition of function E_1^π. Observe first that from the fact that $H^*C = \mathcal{B}$ it follows that the collection of morphisms of $\mathcal{B}^*$ is equal to the collection of vertical morphisms of C. There is thus an obvious identification between the formal horizontal identities of $Q_{\mathcal{B}}$ and the collection of horizontal identities of C. We use this identification and consider the horizontal identities of both $Q_{\mathcal{B}}$ and C as being the same. Observe that that the equation $H^*C = \mathcal{B}$ also implies that the globular squares of C are precisely the 2-cells of $\mathcal{B}$. Thus $\mathbb{G}$ is the set of generators, as a globularly generated double category, of C. We make E_1^π to be the composition $q^C\mu_{id_{\mathbb{G}},id_{\mathcal{B}^*}}$. Thus defined E_1^π is a function from E_1 to Hom_{C_1}. Moreover, from the way it was defined it easily follows that E_1^π satisfies condition 1 and conditions 3-5 in the statement the lemma. We now define the functor F_1^π as follows: Observe first that from the fact that $H^*C = \mathcal{B}$ it follows that the collection of horizontal morphisms of C is equal to $\mathcal{B}_1$. We make the object function of F_1^π to be $id_{\mathcal{B}_1}$. From the fact that E_1^π satisfies condition 3 of the statement of the lemma and from the fact that E_1 freely generates F_1 with respect to d_1, c_1 it follows that there exists a unique extension of E_1^π to a functor from F_1 to C_1. We make F_1^π to be this extension. Thus defined F_1^π trivially satisfies condition 2 of the statement of the lemma with respect to E_1^π. The fact that the functor F_1^π satisfies the condition 5 in the statement of the lemma follows from the fact that the function E_1^π satisfies condition 4 and from the functoriality of s_1 and t_1.

Let $k > 1$. Assume now that for every $m < k$ the function E_m^π from E_m to Hom_{C_1} and the functor F_m^π from F_m to C_1 have been defined, in such a way that the pair of sequences E_m^π and F_m^π with m running through the collection of positive integers strictly less than k satisfies the conditions 1-6 in the statement of the lemma. We now construct a function E_k^π from E_k to Hom_{C_1} and a functor F_k^π from F_k to C_1 such that the pair E_k^π, F_k^π satisfies conditions 1-6 in the statement of the lemma with respect to the pair of sequences E_m^π, F_m^π with m running through the collection of positive integers strictly less than k.

We first define the function E_k^π. Observe first that from the assumption that F_{k-1}^π satisfies condition 5 it follows that the function $\mu_{F_{k-1}^\pi, id_{\mathcal{B}^*}}$ is well defined. We make E_k^π to be composition $q^C \mu_{F_{k-1}^\pi, id_{\mathcal{B}^*}}$. Thus defined E_k^π is a function from E_k to Hom_{C_1}. From the way it was defined it is clear that E_k^π satisfies conditions 4 and 6 of the lemma. From the induction hypothesis it follows that E_k^π satisfies conditions 1 and 2. The function E_k^π satisfies the condition 3 of the lemma by the fact that it satisfies condition 2 and by the functoriality of F_{k-1}^π. We now define the functor F_k^π. By the fact that the function E_k^π satisfies the condition 3 of the lemma it follows that there is a unique extension of E_k^π to a functor from F_k to C_1. We make F_k^π to be this functor. Thus defined F_k^π satisfies the condition 2 of the lemma. This follows from the way F_k^π was constructed and from the fact that condition 2 is already satisfied by the function E_k^π. From the fact that E_k^π satisfies condition 4 it follows that the functor F_k^π satisfies the condition 5 of the lemma. We have thus constructed, recursively, a pair of sequences E_k^π, F_k^π satisfying the conditions in the statement of the lemma. This concludes the proof. □

Observation 2.4. Let $\mathcal{B}$ be a decorated bicategory. Condition 2 of Lemma 2.3 implies that for every pair $m, k \geq 1$ such that $m \leq k$ the following two equations hold:

$$E_k^\pi \upharpoonright_{E_m} = E_m^\pi \text{ and } F_k^\pi \upharpoonright_{F_m} = F_m^\pi$$

Notation 2.5. Let $\mathcal{B}$ be a decorated bicategory. Let C be a globularly generated double category satisfying the equation $H^*C = \mathcal{B}$. In that case we write E_∞^π for the limit $\varinjlim E_k^\pi$ in **Set** and we write F_∞^π for the limit $\varinjlim F_k^{\pi^C}$ **Cat**. Thus defined E_∞^π is a function from E_∞ to the set of squares of C and

F_∞^π is a functor from F_∞ to the category of squares of C. The function E_∞^π is the morphism function of F_∞^π.

The following lemma follows directly from Lemma 2.3 and Observation 2.4.

Lemma 2.6. *Let $\mathcal{B}$ be a decorated bicategory. Let C be a globularly generated double category such that $H^*C = \mathcal{B}$. In that case E_∞^π and F_∞^π satisfy the following conditions:*

1. *The equations $E_\infty^\pi \restriction_{E_k} = E_k^\pi$ and $F_k^\pi \restriction_{F_k} = F_k^\pi$ hold for every $k \geq 1$.*
2. *The following two triangles commute:*

$$\begin{array}{ccc} F_\infty & \xrightarrow{F_\infty^\pi} & C_1 \\ & {\scriptstyle s_\infty, t_\infty} \searrow \quad \swarrow {\scriptstyle s,t} & \\ & \mathcal{B}^* & \end{array}$$

3. *The following square commutes:*

$$\begin{array}{ccc} E_\infty \times_{Hom_{\mathcal{B}^*}} E_\infty & \xrightarrow{E_\infty^\pi \times E_\infty^\pi} & Hom_{C_1} \times_{Hom_{\mathcal{B}^*}} Hom_{C_1} \\ \downarrow {\scriptstyle *_\infty} & & \downarrow {\scriptstyle *} \\ E_\infty & \xrightarrow[E_\infty^\pi]{} & Hom_{C_1} \end{array}$$

Lemma 2.7. *Let $\mathcal{B}$ be a decorated bicategory. Let C be a globularly generated double category such that $H^*C = \mathcal{B}$. In that case the functor F_∞^π is well defined with respect to the equivalence relation R_∞.*

Proof. Let $\mathcal{B}$ be a decorated bicategory. Let C be a globularly generated double category such that $H^*C = \mathcal{B}$. We wish to prove that F^π_∞ is well defined with respect to the equivalence relation R_∞. The fact that F^π_∞ is well defined with respect to relation 1 in the definition of R_∞ follows from the functoriality of F^π_∞ together with the fact that F^π_∞ satisfies conditions 5 and 6 of Lemma 2.3.

We now prove that F^π_∞ is well defined with respect to relation 2 in the definition of R_∞. Let first Φ and Ψ be globular squares of $\mathcal{B}$ such that the pair Φ, Ψ is compatible with respect to d_∞ and c_∞. In that case the image $F^\pi_\infty \Psi \bullet_\infty \Phi$ of the vertical composition $\Psi \bullet_\infty \Phi$ of under F^π_∞ is equal to the image $F^\pi_1 \Psi \bullet_\infty \Psi$ of $\Psi \bullet_\infty \Phi$ under F^π_1, which is, by functoriality of F^π_1 equal to the composition $F^\pi_1 \Psi \bullet F^\pi_1 \Phi$ in C of $F^\pi_1 \Phi$ and $F^\pi_1 \Psi$, which is equal, by the definition of F^π_1 to $\Psi \bullet \Phi$. Now, $F^\pi_\infty \Psi \bullet \Phi$ is equal to $E^\pi_1 \Psi \bullet \Phi$, which is equal, by the way E^π_1 was defined, to $\Psi \bullet \Phi$. The functor F^π_∞ is thus well defined with respect to relation 2 in the definition of R_∞ when restricted to the 2-cells of $\mathcal{B}$. Let now α and β be morphisms of $\mathcal{B}$ such that the pair α, β is composable. In that case $F^\pi_\infty i_\beta \bullet_\infty i_\alpha$ is equal to $F^\pi_1 i_\beta \bullet_\infty i_\alpha$, which is equal to $F^\pi_1 i_\beta \bullet F^\pi_1 i_\beta$. This is equal, again by the definition of F^π_1, to $i_\beta \bullet i_\alpha$. Now, $F^\pi_\infty i_{\beta\alpha}$ is equal to $E^\pi_1 i_{\beta\alpha}$ which is, by the way E^π_1 was defined, equal to $i_{\beta\alpha}$, that is, $F^\pi_\infty i_{\beta\alpha}$ is equal to $i_\beta \bullet i_\alpha$. We conclude that F^π_∞ is well defined with respect to relation 2 in the definition of R_∞ when restricted to formal horizontal identities and thus F^π_∞ is well defined with respect to relation 2 in the definition of R_∞.

We now prove that F^π_∞ is well defined with respect to relation 3 in the definition of R_∞. Let Φ and Ψ be globular squares in $\mathcal{B}$ such that the pair Φ, Ψ is compatible with respect to s_∞ and t_∞. In that case $F^\pi_\infty \Psi *_\infty \Phi$ is equal to $E^\pi_1 \Psi *_1 \Phi$. This is equal, by the fact that E^π_1 satisfies condition 6 of Lemma 2.3, to $E^\pi_1 \Psi * E^\pi_1 \Phi$, which, by the definition of E^π_1 is equal to $\Psi * \Phi$. Now, $F^\pi_\infty \Psi * \Psi$ is equal to $E^\pi_1 \Phi * \Psi$. This is equal, again by the way E^π_1 was defined, to $\Psi * \Phi$. We conclude that F^π_∞ is well defined with respect to relation 3 in the definition of R_∞.

Finally, the fact that F^π_∞ is well defined with respect to relations 4 and 5 in the definition of relation R_∞ follows from conditions 3 and 5 of Lemma 2.3 and from the fact that $id_{\mathcal{B}}$ carries left and right identity transformations to left and right identity transformations and associators to associators. This concludes the proof of the lemma.

□

Notation 2.8. Let $\mathcal{B}$ be a decorated bicategory. Let C be a globularly generated double category such that $H^*C = \mathcal{B}$. In that case will write V_∞^π for the functor from V_∞ to C_1 induced by F_∞^π and R_∞. We write H_∞^π for the morphism function of V_∞^π.

The proof of the following lemma follows directly from Lemma 2.6 by taking limits.

Lemma 2.9. *Let $\mathcal{B}$ be a decorated bicategory. Let C be a globularly generated double category such that $H^*C = \mathcal{B}$. In that case V_∞^π satisfies the following conditions:*

1. *The following two triangles commute:*

$$\begin{array}{ccc} V_\infty & \xrightarrow{V_\infty^\pi} & C_1 \\ & {\scriptstyle s_\infty, t_\infty} \searrow \quad \swarrow {\scriptstyle s,t} & \\ & \mathcal{B}^* & \end{array}$$

2. *The following square commutes for every $k \geq 1$:*

$$\begin{array}{ccc} V_\infty \times_{\mathcal{B}^*} V_\infty & \xrightarrow{V_\infty^\pi \times V_\infty^\pi} & C_1 \times_{\mathcal{B}^*} C_1 \\ {\scriptstyle *_\infty} \downarrow & & \downarrow {\scriptstyle *} \\ V_\infty & \xrightarrow[V_\infty^\pi]{} & C_1 \end{array}$$

Existence

We now prove the existence part of Theorem 2.1.

Proof: Let $\mathcal{B}$ be a decorated bicategory. Let C be a globularly generated double category such that $H^*C = \mathcal{B}$. We wish to construct a double functor $\pi^C : Q_{\mathcal{B}} \to C$ such that $H^*\pi = d_{\mathcal{B}}$.

We make π^C to be equal to the pair $(id_{\mathcal{B}^*}, V_\infty^\pi)$. The pair π^C is a double functor from $Q_{\mathcal{B}}$ to C by Lemma 2.9 and by the fact that it clearly intertwines the horizontal identity functor i_∞ in $Q_{\mathcal{B}}$ and the horizontal identity functor i in C. The fact that $H^*\pi \restriction_{\mathbb{G}}$ is equal to $id_{\mathcal{B}}$ follows directly from the way V_∞^π was defined. This concludes the proof. ■

Definition 2.10. *Let $\mathcal{B}$ be a decorated bicategory. Let C be a globularly generated double category such that $H^*C = \mathcal{B}$. We call the double functor π^C defined in the above the canonical double projection associated to C.*

When necessary we will write $V_\infty^{\pi^C}$ for the morphism functor V_∞^π of the canonical double projection associated to a globularly generated double category C. We will use the same convention for $F_k^\pi, H_k^\pi, V_k^\pi$ and H_k^π

Surjectivity

We now prove the surjectivity on squares part of Theorem 2.1. We begin with the following lemma.

Lemma 2.11. *Let $\mathcal{B}$ be a decorated bicategory. Let C be a globularly generated double category such that $H^*C = \mathcal{B}$. Let k be a positive integer. The image of $H_\infty^{\pi^C} \restriction_{H_k}$ is equal to H_C^k and the image category of $V_\infty^\pi \restriction_{V_k}$ is equal to V_C^k.*

Proof. Let $\mathcal{B}$ be a decorated bicategory. Let C be a globularly generated double category such that $H^*C = \mathcal{B}$. Let k be a positive integer. We wish to prove that the image of $H_\infty^\pi \restriction_{H_k}$ is equal to H_C^k and that the image category of $V_\infty \restriction_{V_k}$ is equal to V_C^k of vertical filtration associated to C. We proceed by induction on k.

We prove first that $H_\infty^\pi H_1$ is equal to H_C^1. From the obvious fact that H_C^1 is contained in H_1, and from the fact that π is a double functor, it follows that $H_\infty H_C^1$ is contained in H_C^1. Now, H_∞^π acts as the identity function when restricted to 2-cells and horizontal identities of $\mathcal{B}$. It follows, from this, from the fact that H^π_∞ satisfies condition 2 of lemma 2.6, and from the way H^1_C is defined, that $H_\infty H^1_C$. We conclude that πH_1 is equal to H^1_C. We now prove that the image category of V_1 under V^π_∞ is equal to V^1_C. From the previous argument, and from the fact that V^π_∞ satisfies condition 2 of lemma 2.6 it follows that $V^\pi_\infty H_1$ is equal to H^1_C. This, together with the fact that V^π_∞ is a functor, implies that the image category of V^π_1 under V^π_∞ is precisely V^1_C.

Let now k be a positive integer such that $k > 1$. Suppose that for every $m < k$, $H^\pi_\infty H_m$ is equal to H^m_C and that the image category of V_m, under V^π_∞, is equal to V^m_C. We now prove that $H^\pi_\infty H_k$ is H^k_C. From the fact that H_k is obviously contained in H_k and from the fact that π is a double functor it follows that $H^\pi_\infty H_k$ is contained in H^k_C. Now, H^π_∞ satisfies condition 1 of Lemma 2.6, the induction hypothesis implies that $H^\pi_\infty \mathrm{Hom}_{V_{k-1}}$ is precisely $\mathrm{Hom}_{V^{k-1}_C}$. It follows, from this, from the fact that H^π_∞ satisfies condition 3 of lemma 2.6 and from the fact that every square in H^k_C is the horizontal composition of a composable sequence of squares in $\mathrm{Hom}_{V^{k-1}_C}$ that $H^\pi_\infty H_k$ contains H^k_C. We thus conclude that $H^\pi_\infty H_k$ is equal to H^k_C. Finally, we prove that the image category, under V^π_∞, of V_k is precisely V^k_C. From the the previous argument, from Observation 2.4 and from the fact that V^π_∞ satisfies condition 1 of Lemma 2.6 it follows that the image of H_k under V^π_∞ is equal to H^k_C. This, together with functoriality of V^π_k implies that the image category of the restriction to V_k, of V^π_∞, is equal to V^k_C. This concludes the proof. □

We now prove the surjectivity part of Theorem 2.1.

Proof: Let $\mathcal{B}$ be a decorated bicategory. Let C be a globularly generated double category such that $H^*C = \mathcal{B}$. We wish to prove that V^π_∞ is full.

Let k be a positive integer. The restriction, to V_k of V_∞ defines, by Lemma 2.11, a functor from V_k to V^k_C. We denote this functor by $\tilde{V}^\pi_k$. The fact that V^π_∞ satisfies condition 1 of Lemma 2.6 implies that for every pair of integers m, k such that $m \leq k$, the functor $\tilde{V}^\pi_m$ is equal to the restriction, to $\tilde{V}_m$, of $\tilde{V}^\pi_k$. The sequence $\tilde{V}^\pi_k$ is thus a directed system in **Cat**. The functor V^π_∞ is equal to its limit $\varinjlim \tilde{V}^\pi_k$ in **Cat**. This, together with the fact, following Lemma 2.11, that $\tilde{V}^\pi_k$ is full for every positive integer k completes the proof of the proposition. ■

Uniqueness

We begin the proof of uniqueness part of Theorem 2.1 by extending the notation used in the above proof.

Notation 2.12. Let $\mathcal{B}$ be a decorated bicategory. Let C be a globularly generated double category. Let $T : Q_{\mathcal{B}} \to C$ be a double functor. Let k be a positive integer. We write $\tilde{H}_k^T$ for $H_k^T \upharpoonright_{\tilde{H}_k}$. Thus defined $\tilde{H}_k^T$ is a function from H_k to H_C^k. Moreover, we write $\tilde{V}_k^T$ for $V_k^T \upharpoonright_{V_k}$. Thus defined $\tilde{V}_k^T$ is a functor from V_k to the k-th vertical category V_C^k of C.

Lemma 2.13. *Let $\mathcal{B}$ be a decorated bicategory. Let C be a globularly generated double category. Let $T, L : Q_{\mathcal{B}} \to C$ be double functors. If $\tilde{H}_1^T$ and $\tilde{H}_1^L$ are equal, then for every $k \geq 1$, $\tilde{H}_k^T$ and $\tilde{H}_k^L$ are equal and $\tilde{V}_k^T$ and $\tilde{V}_k^L$ are equal.*

Proof. Let $\mathcal{B}$ be a decorated bicategory. Let C be a globularly generated double category. Let $T, L : Q_{\mathcal{B}} \to C$ be double functors. Let $k > 1$. Suppose that $\tilde{H}_1^T = \tilde{H}_1^L$. We wish to prove the equations $\tilde{H}_k^T = \tilde{H}_k^L$ and $\tilde{V}_k^T = \tilde{V}_k^L$.

We proceed by induction on k. We first prove that $\tilde{V}_1^T = \tilde{V}_1^L$. Observe first that the restriction of the morphism function of $\tilde{V}_1^T$ to H_1 is equal to $\tilde{H}_1^T$ and that the restriction of the morphism function of $\tilde{V}_1^L$ to H_1 is equal to $\tilde{H}_1^L$. From this and from the assumption of the lemma it follows that the restrictions of the morphism functions of $\tilde{V}_1^T$ and $\tilde{V}_1^L$ to H_1 are equal. We conclude, from this, from the fact that H_1 generates V_1, and from the functoriality of $\tilde{V}_1^T$ and $\tilde{V}_1^L$, that $\tilde{V}_1^T$ and $\tilde{V}_1^L$ are equal.

Let now $k > 1$. Suppose that for every $m < k$ the equations $\tilde{H}_m^T = \tilde{H}_m^L$ and $\tilde{V}_m^T = \tilde{V}_m^L$ hold. We now prove that the equation $\tilde{H}_k^T = \tilde{H}_k^L$ holds. Observe first that the restriction of $\tilde{H}_k^T$ to $\mathrm{Hom}_{V_{k-1}^{\mathcal{B}}}$ is equal to the morphism function of $\tilde{V}_{k-1}^T$ and that the restriction of $\tilde{H}_k^L$ to $\mathrm{Hom}_{V_{k-1}^{\mathcal{B}}}$ is equal to the morphism function of $\tilde{V}_{k-1}^L$. From this, from induction hypothesis, and from the fact that both T and L are double functors that the equation $\tilde{H}_k^T = \tilde{H}_k^L$ holds. We now prove the equation $\tilde{V}_k^T = \tilde{V}_k^L$ holds. Observe again that the restriction of the morphism function of $\tilde{V}_k^T$ to H_k is equal to $\tilde{H}_k^T$ and that the restriction of the morphism function of $\tilde{V}_k^L$ to H_k is equal to $\tilde{H}_k^L$. From this, from the previous argument, from the fact that $H_k^{\mathcal{B}}$ generates the category

V_k, and from the functoriality of $\tilde{V}_k^T$ and $\tilde{V}_k^L$ it follows that the equation $\tilde{V}_k^T = \tilde{V}_k^L$ holds. This concludes the proof. □

Given a double functor $T : Q_{\mathcal{B}} \to C$ from the free globularly generated double category associated to a decorated bicategory $\mathcal{B}$ to a globularly generated double category C, it is a straightforward observation that the morphism functor T_1 of T is equal to $\varinjlim \tilde{V}_k^T$ in **Cat**. This, together with Lemma 2.13 implies the following proposition. We interpret this by saying that a double functor with domain a free globularily generated double category is completely determined by its value on globular squares.

Proposition 2.14. *Let $\mathcal{B}$ be a decorated bicategory. Let C be a globularly generated double category. Let $T, L : Q_{\mathcal{B}} \to C$ be double functors. If $\tilde{H}_1^T$ and $\tilde{H}_1^L$ are equal then T_1 and L_1 are equal.*

The uniqueness part of Theorem 2.1 follows directly from the above proposition. We interpreted the surjectivity part of Theorem 2.1 by saying that every globularly generated internalization of a decorated bicategory $\mathcal{B}$ could be interpreted as a quotient of the free globularly generated double category $Q_{\mathcal{B}}$ associated to $\mathcal{B}$ via the canonical projection double functor. We interpret the uniqueness part of Theorem 2.1 by saying that in this case the choice of canonical projections as projection is canonical.

Linear canonical double projection

Let k be a field. Let $\mathcal{B}$ be a k-linear decorated bicategory. In that case the free globularly generated double category construction can be modified to produce a k-linear free globularly generated double category $Q_{\mathcal{B}}^k$ associated to $\mathcal{B}$, see the final comments of [16, Section 2]. Given k-linear decorated bicategories $\mathcal{B}, \mathcal{B}'$ we will say that a decorated pseudofunctor $G : \mathcal{B} \to \mathcal{B}'$ is linear if G is linear on 2-cells and vertical arrows of $\mathcal{B}$. It is easily seen that the canonical double projection π^C associated to a linear globularly generated double category C satisfying the equation $H^*C = \mathcal{B}$ is a linear pseudofunctor. We will make use of this fact in the next section.

3. Applications

In this section we make use of the canonical double projection to obtain information about solutions to Problem 1.1. We study applications of Theorem 2.1 to length, double groupoids, single 1- and 2-cell decorated bicategories and double categories of von Neumann algebras.

Length

Recall that the length of a globularly generated double category C, ℓC, is the minimal $k \in \mathbb{N} \cup \{\infty\}$ for which $V_C^k = C_1$. In the non-globularly generated case we define the length of a double category C as $\ell\gamma C$. The length of a double category C is meant to serve as a measure of complexity on the interplay between horizontal and vertical compositions of globular and horizontal squares of C. Equivalently ℓC serves as a measure of complexity on presentations of globularly generated squares of C. Double categories of arbitrarily large and infinite lengths were constructed in [16]. Using the free globularly generated double category construction we translate the definition of length to decorated bicategories. Given a decorated bicategory $\mathcal{B}$ we define the length $\ell\mathcal{B}$ of $\mathcal{B}$ as $\ell Q_{\mathcal{B}}$. We prove the following proposition.

Proposition 3.1. *Let $\mathcal{B}$ be a decorated bicategory. Let C be a double category. If $H^*C = \mathcal{B}$ then the following inequality holds:*

$$\ell C \leq \ell\mathcal{B}$$

Proof. Let $\mathcal{B}$ be a decorated bicategory. Let C be a double category such that $H^*C = \mathcal{B}$. We wish to prove that $\ell C \leq \ell\mathcal{B}$.

From the equations $H^*\gamma C = H^*C$ and $\ell\gamma C = \ell C$ we may assume that C is globularly generated. Let k be a positive integer. Suppose $\ell\mathcal{B} = k$. We wish to prove that $\ell C \leq k$. To prove this it is enough to prove that H_C^{k+1} is closed under vertical compositions. Let φ, ψ be vertically compatible squares in H_C^{k+1}. We wish to prove that $\varphi \bullet \psi \in H_C^{k+1}$. By the fact that π^C is surjective on squares the function $H_{k+1}^\pi : H_{Q_{\mathcal{B}}}^{k+1} \to H_C^{k+1}$ is epic. Let $\varphi', \psi' \in H_{Q_{\mathcal{B}}}^{k+1}$ such that

$$H_{k+1}^\pi \varphi' = \varphi \text{ and } H_{k+1}^\pi \psi' = \psi$$

By the fact that π intertwines vertical domain and codomains of $Q_{\mathcal{B}}$ and C it follows that the squares φ', ψ' are vertically compatible. From the fact that $\ell\mathcal{B} = \ell Q_{\mathcal{B}} = k$ it follows that $\varphi' \bullet_\infty \psi' \in H^{k+1}$ and thus the square:

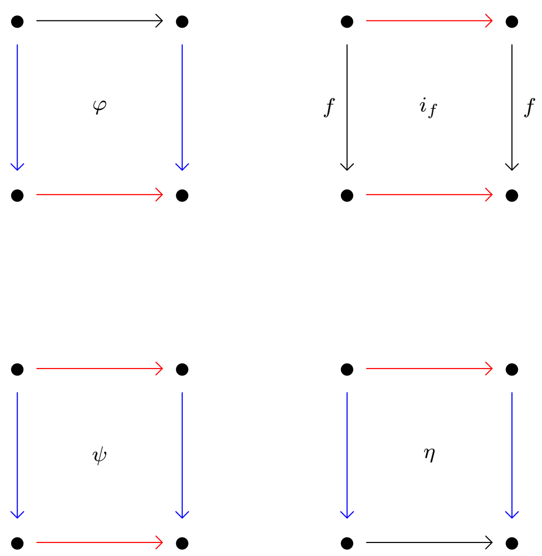
$$H^\pi_{k+1}(\varphi' \bullet_\infty \psi') = \varphi \bullet \psi$$

is a square in H^{k+1}_C. We conclude that $\ell C \leq k$. The case in which $\ell\mathcal{B} = \infty$ is trivial. This concludes the proof of the proposition. □

An important case of Proposition 3.1 is when the length of the decorated bicategory $\mathcal{B}$ is assumed to be 1. This is contained in the following immediate corollary.

Corollary 3.2. *Let $\mathcal{B}$ be a decorated bicategory. Suppose $\ell\mathcal{B} = 1$. If C is a double category such that $H^*C = \mathcal{B}$ then $\ell C = 1$.*

Corollary 3.2 says that if we assume that $\ell\mathcal{B} = 1$ then we have a good control on expressions of all squares of any globularly generated double category satisfying $H^*C = \mathcal{B}$. More precisely, every square φ of a globularly generated double category C satisfying the equation $H^*C = \mathcal{B}$ admits a decomposition as a vertical composition of squares of the following four forms:

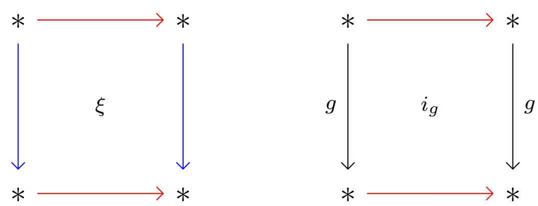


and the horizontal composition of any such squares in C can be re-arranged so as to be written in the above form. The following are examples of decorated bicategories of length 1.

1. **Groups decorated by groups:** In [16, Proposition 5.1] the following equation is proven:

$$\ell(\Omega G, 2\Omega A) = 1$$

for every pair of groups G, A with A abelian, where recall that ΩG and $2\Omega G$ are the delooping groupoid and the double delooping 2-group of G respectively, i.e. ΩG is the groupoid with a single object $*$ such that $Aut_{\Omega G}(*) = G$ and $2\Omega A$ is the 2-group with a single object, which we also denote by $*$, such that $End_{2\Omega A}(*)$ is equal to ΩA. Every square φ in any globularly generated double category C satisfying the equation $H^*C = (\Omega G, 2\Omega A)$ can thus be written as a vertical composition of squares of the form:

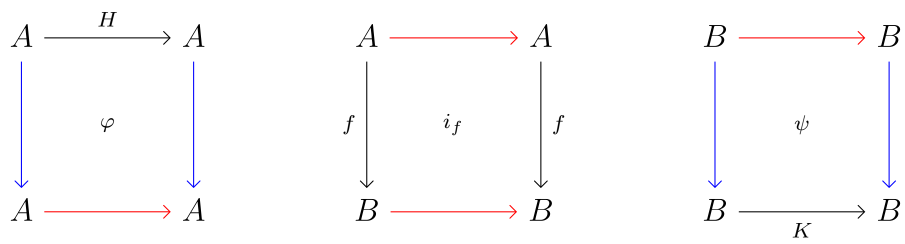


where $*$ denotes the only object in ΩG, ξ is an element of the monoid of squares of the only horizontal morphism i_* of C and where g is any element of G. In Corollary 3.5 we obtain more information about double categories of this form.

2. **von Neumann algebras:** In [16, Proposition 6.1] the following equation was proven:

$$\ell Q^{\mathbb{C}}_{W^*_{fact}} = 1$$

where W^*_{fact} denotes the bicategory of factors, Hilbert bimodules and intertwiners, decorated by the category of possibly infinite index unital *-morphisms. Every square in any linear globularly generated double category C satisfying the equation $H^*C = W^*_{fact}$ can thus be written as a multiple of a vertical composition of squares of the form:



where A, B are factors, H is a left-right A-bimodule, φ is a bounded intertwiner from H to $L^2(A)$, $f : A \to B$ is a possibly infinite index unital *-morphism, K is a left-right B-bimodule, and ψ is a bounded intertwiner from $L^2(B)$ to K. In Proposition 3.6 we obtain more information of double categories of this form.

2-groupoids and double groupoids

Double groupoids and 2-groupoids categorify crossed modules and are thus used to model homotopy 2-types [6, 12]. Relations between double groupoids and 2-groupoids have been studied in [5] in the case of edge-symmetric double groupoids with special connection. We apply the results obtained in Section 2 to study relations between decorated 2-groupoids and general double groupoids. We say that a decorated bicategory $\mathcal{B}$ is a decorated 2-groupoid if $\mathcal{B}$ is a 2-groupoid and $\mathcal{B}^*$ is a groupoid. Decorated bigroupoids are defined analogously. Given a 3-filtered topological space (X, A, C), the pair $(\Pi_1(A; C), W(X; A, C))$, where $W(X; A, C)$ is Moerdijk-Svensson's Whitehead homotopy 2-groupoid associated to (X, A, C) [14] and $\Pi_1(A; C)$ is the fundamental groupoid of A relative to C, is a decorated 2-groupoid. The Brown-Higgins fundamental double groupoid $\rho(X; A, C)$ [7] satisfies the equation

$$H^*\rho(X; A, C) = (\Pi_1(A, C), W(X; A, C))$$

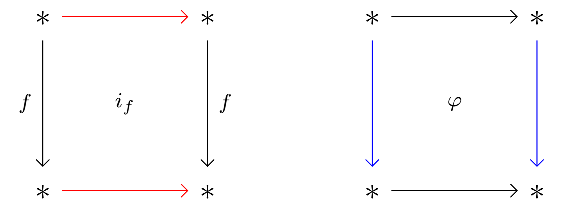
Decorated 2-groupoids of the form $(\Pi_1(A; C), W(X; A, C))$ thus always admit solutions to Problem 1.1 and these solutions can always be chosen to be double groupoids. A similar statement holds for homotopy 2-groupoids $G_2(X)$ associated to Hausdorff topological spaces X by Hardie, Kamps and Kieboom in [13] decorated by the full fundamental groupoid $\Pi_1(X)$, with internalization provided by the Brown-Hardie-Kamps-Porter homotopy double groupoid $\rho_2^{\square}(X)$ defined in [8].

Invertibility is perhaps the most essential condition on structures involved in the homotopy hypothesis. In our context it is thus an important question whether every decorated 2-groupoid can always be internalized by a double groupoid. The Brown-Spencer theorem [5] applies in the context of special double groupoids with special connections [6], and thus every 2-groupoid $\mathcal{B}$, decorated by its groupoid of horizontal arrows is internalized by a double groupoid, its Ehresmann double category of quintets. We treat the general case of 2-groupoids decorated by groupoids which are not-necessarily groupoids of horizontal arrows. We prove that given a general decorated 2-groupoid (more generally a decorated bigroupoid) $\mathcal{B}$, if there exists a double category C (not-necessarily a double groupoid) such that $H^*C = \mathcal{B}$ then γC is a double groupoid. We begin with the following lemma.

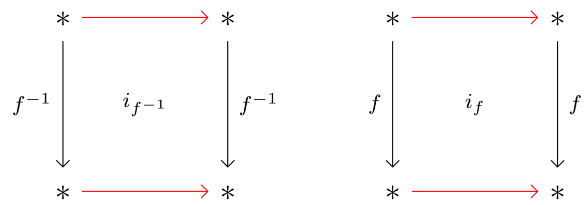
Proposition 3.3. *Let $\mathcal{B}$ be a decorated bicategory. If $\mathcal{B}$ is a decorated 2-groupoid then $Q_{\mathcal{B}}$ is a double groupoid.*

Proof. Let $\mathcal{B}$ be a decorated 2-groupoid. We wish to prove that $Q_{\mathcal{B}}$ is a double groupoid.

We prove by induction on k that every square φ in V_k is vertically and horizontally invertible. By the condition that $\mathcal{B}$ is a decorated 2-groupoid all squares of $Q_{\mathcal{B}}$ of the form:



are vertically and horizontally invertible, with the vertical and horizontal inverse of a square on the left-hand side above given by



respectively. Given any globular or horizontal identity square φ in $Q_{\mathcal{B}}$ we will write $v\varphi^{-1}$ and $h\varphi^{-1}$ for its its vertical and its horizontal inverse in $Q_{\mathcal{B}}$ respectively. Suppose φ is a general square in V_1. Write φ as a vertical composition of the form $\varphi = \varphi_k \bullet \ldots \varphi_1$ where the $\varphi_i' s$ are squares of C as above. In that case the vertical inverse of φ is given by the composition $v\varphi_1^{-1} \bullet \cdots \bullet v\varphi_k^{-1}$ and the horizontal inverse of φ is given by the vertical composition $h\varphi_k^{-1} \bullet \cdots \bullet h\varphi_1^{-1}$.

Let n be a positive integer such that $n > 1$. Suppose that for every $m < n$ every square in V_m is both vertically and horizontally invertible. We prove that every square in V_n is vertically and horizontally invertible. Let φ first be a square in H_n. Write φ as a horizontal composition $\varphi = \varphi_k * \cdots * \varphi_1$ with

φ_i in V_{n-1}. By the induction hypothesis the squares φ_i are all vertically and horizontally invertible. We again write $v\varphi_i^{-1}$ and $h\varphi_i^{-1}$ for the horizontal and the vertical inverse of φ_i respectively. The vertical inverse $v\varphi^{-1}$ of φ is given by the horizontal composition $v\varphi_k^{-1} * \cdots * v\varphi_1^{-1}$ and the horizontal inverse $h\varphi^{-1}$ of φ is given by the horizontal composition $h\varphi_1^{-1} * \cdots * h\varphi_k^{-1}$. Thus every square in H_n admits both a horizontal and a vertical inverse. Using this and the same argument used in the previous paragraph every square in V_n is vertically and horizontally invertible. This concludes the proof. □

Corollary 3.4. *Let C be a double category. If H^*C is a decorated 2-groupoid then γC is a double groupoid.*

Proof. Let C be a double category. Suppose that H^*C is a decorated 2-groupoid. We wish to prove that γC is a double groupoid.

It is enough to prove that every square in γC is both vertically and horizontally invertible. This follows directly from Proposition 3.3 and Theorem 2.1. This concludes the proof of the corollary. □

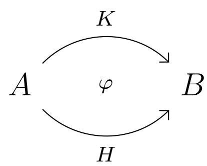
Observe that Proposition 3.3 and Corollary 3.4 still hold if we assume that $\mathcal{B}$ is a decorated bigroupoid. The following corollary follows directly from Proposition 3.1, Proposition 3.3, and [16, Corollary 5.2] by considering decorated 2-groupoids of the form $(\Omega G, 2\Omega A)$ whith G, A groups and A abelian.

Corollary 3.5. *Let G, A be groups. Suppose A is abelian. Let C be a globularly generated double category. If $H^*C = (\Omega G, 2\Omega A)$ then the category of squares C_1 is of the form ΩH for a group H such that H is a quotient of $G * A$.*

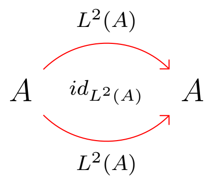
von Neumann algebras

We study linear double categories of von Neumann algebras and their Hilbert bimodules. In [1] a tensor double category of semisimple von Neumann algebras, Hilbert bimodules, equivariant intertwiners and finite index morphisms was constructed in order to express the fact that the Haagerup standard form and the Connes fusion operation admit compatible extensions to tensor functors. In [16] it was proven that the bicategory of factors, Hilbert bimodules, and intertwiners, decorated by possibly infinite index morphisms is saturated and thus its linear free globularly generated double category is an internalization, providing formal linear functorial extensions of both the Haagerup standard form and the Connes fusion operations. We investigate the relation of these two constructions through the canonical double projection and we use this to construct a linear extension of the double category of factors and finite morphisms accommodating possibly infinite index morphisms.

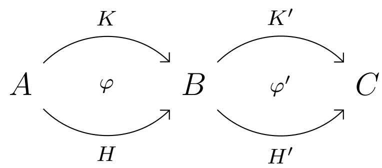
We write $\mathbf{Mod}^{fact}$ for the linear bicategory whose 2-cells are of the form:



where A, B are factors, H, K are left-right Hilbert A-B-bimodules and where φ is an intertwiner operator from H to K. Horizontal identity 2-cells in $\mathbf{Mod}^{fact}$ are given by 2-cells of the form:



where A is a factor and $L^2(A)$ is the Haagerup standard form of A, see [11]. Given horizontally compatible 2-cells in $\mathbf{Mod}^{fact}$ of the form:



their horizontal composition is provided by the 2-cell:

$$A \overset{K \boxtimes_B K'}{\underset{H \boxtimes_B H'}{\overset{\varphi \boxtimes_B \varphi'}{\rightrightarrows}}} C$$

where $H \boxtimes_B H'$ and $K \boxtimes_B K'$ denote the Connes fusion of H, H' and K, K' and where $\varphi \boxtimes_B \varphi'$ denotes the Connes fusion of φ and φ'. We write $\mathbf{vN}^{fact}$ for the category of factors and unital *-morphisms $f : A \to B$ with $[f(A), B]$ possibly infinite. We write $\mathbf{vN}^{fin}$ for the subcategory of $\mathbf{vN}^{fact}$ generated by *-morphisms $f : A \to B$ such that $[f(A), B] < \infty$. The pairs $(\mathbf{vN}^{fact}, \mathbf{Mod}^{fact})$ and $(\mathbf{vN}^{fin}, \mathbf{Mod}^{fin})$ are linear decorated bicategories. We write W^*_{fact} and W^*_{fin} for these decorated bicategories. In [1] an internalization of W^*_{fin} is constructed through functorial extensions, to $\mathbf{vN}^{fin}$ of the Haagerup standard form construction and the Connes fusion operation construction. We write BDH for this double category. In [16] the author proves that W^*_{fact} and thus W^*_{fin} are saturated, i.e. $H^*Q_{W^*_{fact}} = W^*_{fact}$ and $H^*Q_{W^*_{fin}} = W^*_{fin}$. The exact relation between BDH and $Q_{W^*_{fact}}$ is provided by the canonical projection. We have the following consequence of 2.1.

Proposition 3.6. *γBDH is a double quotient of $Q_{W^*_{fact}}$ through π^{BDH}.*

The category of squares BDH_1 of BDH is the category whose objects and morphisms are Hilbert bimodules over factors and finite index equivariant intertwiners, i.e. the morphisms of BDH are the squares of the form:

$$\begin{array}{ccc} A & \xrightarrow{H} & B \\ {\scriptstyle f}\downarrow & (f, \varphi, g) & \downarrow{\scriptstyle g} \\ A' & \xrightarrow[H']{} & B' \end{array}$$

where A, A', B, B' are factors, H is a left-right A, B-Hilbert bimodule, H' is a left-right A', B'-Hilbert bimodule, $f : A \to A'$ and $g : B \to B'$ are unital *-morphisms satisfying the inequalities

$$[f(A), A'] < \infty \text{ and } [g(B), B'] < \infty$$

and φ is a bounded operator from H to K satisfying the equation

$$\varphi(a\xi b) = f(a)\varphi(\xi)g(b)$$

for every $\xi \in H$ and $a \in A, b \in B$. In [15] the category of squares of γBDH_1 of γBDH was computed as the category of 2-subcyclic equivariant intertwiners. The fact that BDH is a tensor double category means that there exists a tensor functor

$$L^2 : \mathbf{vN}^{fin} \to BDH_1$$

associating to every factor A the Haagerup standar form $L^2(A)$ of A, and a tensor functor

$$\boxtimes_\bullet : BDH_1 \times_{\mathbf{vN}^{fin}} BDH_1 \to BDH_1$$

associating to every compatible pair of squares ${}_AH_{B,B}K_A$ its Connes fusion ${}_AH \boxtimes_B K_C$. The fact that these functors are compatible is expressed by the fact that BDH is a tensor double category. The fact that these are operations on Hilbert bimodules and finite equivariant intertwiners is expressed by the equation

$$H^*BDH = W^*_{fin}$$

This equation is minimized by γBDH and the image category of the L^2-functor above is in γBDH_1. We are thus interested in extending the functors

$$L^2 : \mathbf{vN}^{fin} \to \gamma BDH_1$$

and

$$\boxtimes_\bullet : \gamma BDH_1 \times_{\mathbf{vN}^{fin}} \gamma BDH_1 \to \gamma BDH_1$$

to compatible functors on $\mathbf{vN}^{fact}$. The following proposition does this.

Proposition 3.7. *There exists a linear double category $B\tilde{D}H$ such that $H^*B\tilde{D}H = W^*_{fact}$ and such that γBDH is a sub-double category of $B\tilde{D}H$ satisfying the following condition: Given vertically or horizontally compatible squares φ, ψ in $B\tilde{D}H$ if the vertical or horizontal composition of φ and ψ is in BDH so are φ and ψ.*

Proof. We wish to prove that there exists a linear double category $B\tilde{D}H$ satisfying the equation $H^*B\tilde{D}H = W^*_{fact}$ and having γBDH as sub-double category in such a way that given every pair of squares φ, ψ in $B\tilde{D}H$ such that either the vertical or the horizontal composition of φ and ψ is a square in BDH then both φ and ψ are in BDH.

Write R for the equivalence relation on the collection of squares of $Q_{W^*_{fact}}$ defined as: $\varphi R \psi$ if $\pi^{\gamma BDH}\varphi = \pi^{\gamma BDH}\psi$. Thus defined R is compatible with the vertical and horizontal structure data of $Q_{W^*_{fact}}$ and γBDH and thus $Q_{W^*_{fact}}/R$ is a globularly generated double category. We make $B\tilde{D}H$ to be this double category. From the equation $H^*Q_{W^*_{fact}} = W^*_{fact}$ the equation $H^*B\tilde{D}H = W^*_{fact}$ follows. $Q_{W^*_{fin}}$ is a sub-double category of $Q_{W^*_{fact}}$. Moreover, it is easily seen that the equtation

$$Q_{W^*_{fin}} = \pi^{\gamma BDH-1}(\gamma BDH)$$

holds. We thus obtain an isomorphism of double categories

$$\pi^{\gamma BDH} \restriction_{Q_{W^*_{fin}}} \cong \gamma BDH$$

We make use of the above isomorphism to identify γBDH with a sub-double category of $B\tilde{D}H$. The fact that pairs of squares φ, ψ in γBDH satisfy the required condition inside $B\tilde{D}H$ follows by an easy induction argument on $min\{\ell\varphi, \ell\psi\}$ using the fact that given morphisms $f : A \to B$ and $g : B \to C$ in $\mathbf{vN}^{fact}$ such that $[gf(A), C] < \infty$ then $[f(A), B], [g(B), C] < \infty$. This concludes the proof. □

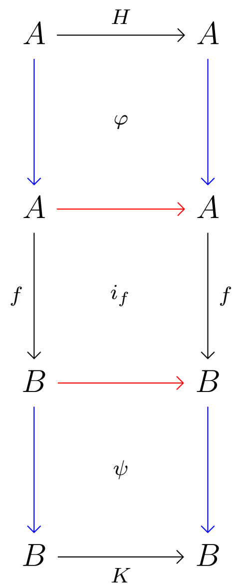
The horizontal identity functor and the horizontal composition functor of $B\tilde{D}H$ are functors:

$$L^2 : \mathbf{vN}^{fact} \to B\tilde{D}H_1$$

and

$$\boxtimes_{\bullet} : B\tilde{D}H_1 \times_{\mathbf{vN}^{fact}} B\tilde{D}H_1 \to B\tilde{D}H_1$$

compatible in the sense that $B\tilde{D}H$ is a double category and such that they restrict to the corresponding functors on γBDH. By [16, Proposition 6.1] and Theorem 2.1 the space of morphisms of $B\tilde{D}H_1$ is the complex vector space spanned by formal vertical compositions of the form:



where A, B are factors, f is a unital *-morphism of possibly infinite index, H is a left-right A Hilbert bimodule, K is a left-right B Hilbert bimodule, φ, ψ are bounded intertwiners from H to $L^2(A)$ and from $L^2(B)$ to K respectively, and $L^2(f)$ is a formal object in $B\tilde{D}H_1$. Whenever f satisfies the inequality:

$$[f(A), B] < \infty$$

the formal symbol $L^2(f)$ is the image of f under the L^2 functor of [1], the three term composition above is the corresponding composition in BDH. Moreover, this three term formal composition is a square in γBDH if and only if $[f(A), B] < \infty$.

In the construction presented in Proposition 3.6 we have not addressed the fact that we wish for $B\tilde{D}H$ to inherit, from γBDH the structure of a symmetric tensor double category. We will address this issue elsewhere.

Representability

In Proposition 3.6 we have obtained a linear double category $B\tilde{D}H$ satisfying the equation

$$H^*B\tilde{D}H = W^*_{fact}$$

and having γBDH as sub-double category. This provides compatible linear functors of the Haagerup standard form and the Connes fusion operation on linear categories of Hilbert bimodules and formal equivariant bounded intertwiners. We would like to obtain such functors, not on formal equivariant intertwiners, but on the category of Hilbert bimodules and actual equivariant intertwiners. We are not able to do this at the moment but Theorem 2.1 provides a possible solution to this. Assuming such functorial extensions exist, compatibility would provide a linear double category C satisfying the equation

$$H^*C = W^*_{fact}$$

having BDH as a sub-double category and such that the category of squares C_1 is a linear sub-category of the category $\mathbf{Mod}^{fact}$ of Hilbert bimodules and equivariant intertwiners. Such category would be in the γ-fiber of a globularly generated double category, γC, satisfying the equation

$$H^*\gamma C = W^*_{fact}$$

having γBDH as a sub-double category and such that γC_1 is a linear subcategory of $\mathbf{Mod}^{fact}$. In that case the morphism functor $\pi_1^{\gamma C}$ of $\pi^{\gamma C}$ will be a linear functor from $Q^{\mathbb{C}}_{W^*_{fact1}}$ to $\mathbf{Mod}^{fact}$ satisfying invariance conditions with respect to the double category structures of $Q^{\mathbb{C}}_{W^*_{fact}}$ and BDH. This suggests we should study the structure of the 2-category

$$Fun(Q^{\mathbb{C}}_{W^*_{fact1}}, \mathbf{Mod}^{fact})$$

under a possible set of initial conditions. We will analyze this point of view elsewhere, but we would like to obtain a categorical version of the above comments. In order to do this we need a way to understand functors between free globularly generated double categories. In the next section we study free double functors.

4. Free double functors

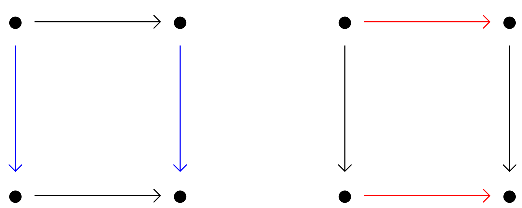
In this section we introduce free double functors between free globularly generated double categories. We will use the free double functor construction to extend the free globularly generated double category construction to a functor. We use this construction to prove the results of Section 5. The methods employed in the construction of free double functors mimic the construction of the canonical double projection of Section 2.

Strategy

Given a pseudofunctor $G : \mathcal{B} \to \mathcal{B}'$ between decorated bicategories $\mathcal{B}, \mathcal{B}'$ the free double functor Q_G associated to G will be a double functor from $Q_\mathcal{B}$ to $Q_{\mathcal{B}'}$ satisfying the equation:

$$H^* Q_G \restriction_\mathcal{B} = G$$

The strategy for the construction of Q_G will be analogous to that of the construction of the canonical double projection of Section 2. We first define Q_G on formal squares of the form:



The requirements in the definition of Q_G force Q_G to be uniquely defined by G on the above squares. We freely extend this to a functor $F_1^G : F_1^\mathcal{B} \to F_1^{\mathcal{B}'}$.

We extend this to a functor from $F_k^G : F_k^{\mathcal{B}} \to F_k^{\mathcal{B}'}$ inductively for all k and we take the corresponding limit $F_\infty^G : F_\infty^{\mathcal{B}} \to F_\infty^{\mathcal{B}'}$. We prove that F_∞^G is compatible with both the structure data and the equivalence relations R_∞ defining $Q_{\mathcal{B}}$ and $Q_{\mathcal{B}'}$ and that thus descends to the morphism functor of a double functor Q_G from $Q_{\mathcal{B}}$ to $Q_{\mathcal{B}'}$. The coherence data for Q_G will be inherited from that of G. Most of the technical results used in the construction of the free globularly generated double functor are analogous to arguments used in Section 2. The precise statements are useful. We will thus record statements for these results but we will usually omit proofs.

Construction

Let $\mathcal{B}, \mathcal{B}'$ be decorated bicategories. Let $G : \mathcal{B} \to \mathcal{B}'$ be a decorated pseudofunctor. We begin the construction of Q_G with the following lemma. Its proof is analogous to that of Proposition 2.3 and we will omit it.

Lemma 4.1. *There exists a pair, formed by a sequence of functions $E_k^G : E_k^{\mathcal{B}} \to E_k^{\mathcal{B}'}$, and a sequence of functors $F_k^G : F_k^{\mathcal{B}} \to F_k^{\mathcal{B}'}$, with k running over all positive integers, such that the following conditions are satisfied:*

1. *The equations $E_1^G \varphi = G\varphi$ and $E_1^G i_\alpha = i_\alpha$ hold for every 2-cell φ in $\mathcal{B}$ and for every morphism α in $\mathcal{B}^*$.*
2. *For every pair of positive integers m, k such that $m \leq k$, the restriction of E_k^G to the collection of morphisms of $F_m^{\mathcal{B}}$ is equal to the morphism function of F_m^G, and the restriction of the morphism function of F_k^G to $E_m^{\mathcal{B}}$ is equal to E_m^G.*
3. *The following two squares commute for every positive integer k:*

$$\begin{array}{ccc} E_k^{\mathcal{B}} & \xrightarrow{E_k^G} & E_k^{\mathcal{B}'} \\ {\scriptstyle d_k^{\mathcal{B}}, c_k^{\mathcal{B}}} \downarrow & & \downarrow {\scriptstyle d_k^{\mathcal{B}'}, c_k^{\mathcal{B}'}} \\ \mathcal{B}_1 & \xrightarrow[G]{} & \mathcal{B}_1' \end{array}$$

4. *The following two squares commute for every positive integer k:*

$$\begin{array}{ccc} E_k^{\mathcal{B}} & \xrightarrow{E_k^G} & E_k^{\mathcal{B}'} \\ {\scriptstyle s_k^{\mathcal{B}},\, t_k^{\mathcal{B}}}\downarrow & & \downarrow{\scriptstyle s_k^{\mathcal{B}'},\, t_k^{\mathcal{B}'}} \\ Hom_{\mathcal{B}^*} & \xrightarrow[G^*]{} & Hom_{\mathcal{B}'^*} \end{array}$$

5. *The following two squares commute for every positive integer k:*

$$\begin{array}{ccc} F_k^{\mathcal{B}} & \xrightarrow{F_k^G} & F_k^{\mathcal{B}'} \\ {\scriptstyle s_{k+1}^{\mathcal{B}},\, t_{k+1}^{\mathcal{B}}}\downarrow & & \downarrow{\scriptstyle s_{k+1}^{\mathcal{B}'},\, t_{k+1}^{\mathcal{B}'}} \\ \mathcal{B}^* & \xrightarrow[G^*]{} & \mathcal{B}'^* \end{array}$$

6. *The following square commutes for every positive integer k*

$$\begin{array}{ccc} E_k^{\mathcal{B}} \times_{Hom_{\mathcal{B}^*}} E_k^{\mathcal{B}} & \xrightarrow{E_k^G \times_G E_k^G} & E_k^{\mathcal{B}'} \times_{Hom_{\mathcal{B}'^*}} E_k^{\mathcal{B}'} \\ {\scriptstyle *_k^{\mathcal{B}}}\downarrow & & \downarrow{\scriptstyle *_k^{\mathcal{B}'}} \\ E_k^{\mathcal{B}} & \xrightarrow[E_k^G]{} & E_k^{\mathcal{B}'} \end{array}$$

Moreover, conditions 1-5 above determine the pair of sequences E_k^G and F_k^G.

Observation 4.2. Let m, k be positive integers such that $m \leq k$. Condition 2 of Proposition 4.1 implies that the equations hold:

$$E_k^G \restriction_{E_k^{\mathcal{B}}} = E_m^G \text{ and } F_k^G \restriction_{F_m^{\mathcal{B}}} = F_m^G$$

Notation 4.3. We will write E_∞^G for the limit $\varinjlim E_k^G$ in **Set** of the sequence E_k^G. Thus defined E_∞^G is a function from $E_\infty^{\mathcal{B}}$ to $E_\infty^{\mathcal{B}'}$. Further, we will write F_∞^G for the limit $\varinjlim F_k^G$ in **Cat** of the sequence of functors F_k^G. Thus defined, F_∞^G is a functor from $F_\infty^{\mathcal{B}}$ to $F_\infty^{\mathcal{B}'}$.

The following observation follows directly from Lemma 4.1 and Observation 4.2.

Observation 4.4. The pair E_∞^G, F_∞^G satisfies the following conditions:

1. E_∞^G is equal to the morphism function of F_∞^G.
2. Let k be a positive integer. The following equations hold:

$$E_\infty^G \restriction_{E_k^{\mathcal{B}}} = E_k^G \text{ and } F_\infty^G \restriction_{F_k^{\mathcal{B}}} = F_k^G$$

3. The following squares commute:

$$\begin{array}{ccc} F_\infty^{\mathcal{B}} & \xrightarrow{F_\infty^G} & F_\infty^{\mathcal{B}'} \\ {\scriptstyle s_\infty^{\mathcal{B}}, t_\infty^{\mathcal{B}}} \downarrow & & \downarrow {\scriptstyle s_\infty^{\mathcal{B}'}, t_\infty^{\mathcal{B}'}} \\ \mathcal{B}^* & \xrightarrow[G^*]{} & \mathcal{B}'^* \end{array}$$

4. The following square commutes:

$$\begin{array}{ccc} E_\infty^{\mathcal{B}} \times_{\mathrm{Hom}_{\mathcal{B}^*}} E_\infty^{\mathcal{B}} & \xrightarrow{E_\infty^G \times_G E_\infty^G} & E_\infty^{\mathcal{B}'} \times_{\mathrm{Hom}_{\mathcal{B}'^*}} E_\infty^{\mathcal{B}'} \\ {\scriptstyle *_\infty^{\mathcal{B}}} \downarrow & & \downarrow {\scriptstyle *_\infty^{\mathcal{B}'}} \\ E_\infty^{\mathcal{B}} & \xrightarrow[E_\infty^G]{} & E_\infty^{\mathcal{B}'} \end{array}$$

It is easily seen, from the above observation, that the functor F_∞^G is compatible with the equivalence relations $R_\infty^{\mathcal{B}}$ and $R_\infty^{\mathcal{B}'}$. We will write V_∞^G for the functor from $V_\infty^{\mathcal{B}}$ to $V_\infty^{\mathcal{B}'}$ induced by F_∞^G and the equivalence relations $R_\infty^{\mathcal{B}}$ and $R_\infty^{\mathcal{B}'}$. We will write H_∞^G for the morphism function of V_∞^G. Thus defined H_∞^G is function from $H_\infty^{\mathcal{B}}$ to $H_\infty^{\mathcal{B}'}$ induced by the function E_∞^G and the equivalence relations $R_\infty^{\mathcal{B}}$ and $R_\infty^{\mathcal{B}'}$. The following proposition follows directly from Observation 4.4.

Proposition 4.5. *V_∞^G satisfies the following conditions:*

1. *The following squares commute:*

$$\begin{array}{ccc} V_\infty^{\mathcal{B}} & \xrightarrow{V_\infty^G} & V_\infty^{\mathcal{B}'} \\ {\scriptstyle s_\infty^{\mathcal{B}},\, t_\infty^{\mathcal{B}}}\downarrow & & \downarrow{\scriptstyle s_\infty^{\mathcal{B}'},\, t_\infty^{\mathcal{B}'}} \\ \mathcal{B}^* & \xrightarrow[G^*]{} & \mathcal{B}'^* \end{array}$$

2. *The following square commutes*

$$\begin{array}{ccc} V_\infty^{\mathcal{B}} \times_{\mathcal{B}^*} V_\infty^{\mathcal{B}} & \xrightarrow{V_\infty^G \times_G H_\infty^G} & V_\infty^{\mathcal{B}'} \times_{\mathcal{B}'^*} V_\infty^{\mathcal{B}'} \\ {\scriptstyle *_\infty^{\mathcal{B}}}\downarrow & & \downarrow{\scriptstyle *_\infty^{\mathcal{B}'}} \\ V_\infty^{\mathcal{B}} & \xrightarrow[V_\infty^G]{} & V_\infty^{\mathcal{B}'} \end{array}$$

Notation 4.6. Let $\mathcal{B}, \mathcal{B}'$ be decorated bicategories. Let $G : \mathcal{B} \to \mathcal{B}'$. We write Q_G for the pair (G^*, V_∞^G).

The following is the main theorem of this section.

Theorem 4.7. *Let $\mathcal{B}, \mathcal{B}'$ be decorated bicategories. Let $G : \mathcal{B} \to \mathcal{B}'$ be a decorated pseudofunctor. In that case Q_G is the unique double functor from $Q_\mathcal{B}$ to $Q_{\mathcal{B}'}$ satisfying the equation:*

$$H^* Q_G \restriction_\mathcal{B} = G$$

Proof. Let $\mathcal{B}$ and $\mathcal{B}'$ be decorated bicategories. Let $G : \mathcal{B} \to \mathcal{B}'$ be a decorated pseudofunctor. We wish to prove that in that case the pair Q_G is a double functor from $Q_\mathcal{B}$ to $Q_{\mathcal{B}'}$ satisfying the equation

$$H^* Q_G \restriction \mathcal{B} = C$$

A direct computation proves that the pair Q_G intertwines the horizontal identity functors $i_\infty^\mathcal{B}$ and $i_\infty^{\mathcal{B}'}$ of $\mathcal{B}$. This, together with a direct application of Proposition 4.5 implies that the pair Q_G is a double functor, with the coherence data of G as coherence data. The object function of V_∞^G is equal to the restriction of G to $\mathcal{B}_1$ and the restriction of V_∞^G to 2-cells of $\mathcal{B}$ is equal to the 2-cell function of G. This together with the fact that the object functor Q_G is G^* implies that the restriction to $\mathcal{B}$ of $H^* Q_G$ is equal to G. This concludes the proof. □

Definition 4.8. *We call Q_G above the free double functor associated to G.*

Observation 4.9. In the more general case in which G is a lax/oplax decorated functor, the double functor Q_G is also lax/oplax respectively.

Functoriality

We now prove that the pair formed by the function associating $Q_\mathcal{B}$ to every decorated bicategory $\mathcal{B}$ and Q_G to every decorated pseudofunctor G is a functor from **bCat*** to **gCat**. We begin with the following lemma.

Lemma 4.10. *Let $\mathcal{B}, \mathcal{B}', \mathcal{B}''$ be decorated bicategories. Let $G : \mathcal{B} \to \mathcal{B}'$ and $G' : \mathcal{B}' \to \mathcal{B}''$ be decorated pseudofunctors. The equations:*

$$H_k^{G'G} = H_k^{G'} H_k^G \text{ and } V_k^{G'G} = V_k^{G'} V_k^G$$

and

$$H_k^{id_{\mathcal{B}}} = id_{H_k^{\mathcal{B}}} \text{ and } V_k^{id_{\mathcal{B}}} = id_{V_k^{\mathcal{B}}}$$

hold for every positive integer k.

Proof. Let $\mathcal{B}, \mathcal{B}', \mathcal{B}''$ be decorated bicategories. Let $G : \mathcal{B} \to \mathcal{B}'$ and $G' : \mathcal{B}' \to \mathcal{B}''$ be decorated pseudofunctors. Let k be a positive integer. We wish to prove that $H_k^{G'G} = H_k^{G'} H_k^G$, that $V_k^{G'G} = V_k^{G'} V_k^G$, that $H_k^{id_{\mathcal{B}}} = id_{H_k^{\mathcal{B}}}$ and that $V_k^{id_{\mathcal{B}}} = id_{H_k^{\mathcal{B}}}$. We prove the first two of these equations. The proof of the remaining equations will be analogous.

We proceed by induction on k. We first prove the equation $H_1^{G'G} = H_1^{G'} H_1^G$. Let φ be a square in $\mathbb{G}^{\mathcal{B}}$. Suppose first that φ is a 2-cell of $\mathcal{B}$. In that case $H_1^{G'G}\varphi = G'G\varphi$, which is equal to $G'\varphi G\varphi = H_1^{G'} H_1^G$. The equation clearly holds for horizontal identity squares in $Q_{\mathcal{B}}$. This proves the equation $H_1^{G'G} = H_1^{G'} H_1^G$ is true when restricted to $\mathbb{G}^{\mathcal{B}}$. This and the fact that $H_1^G, H_1^{G'}$ and $H_1^{G'G}$ satisfy condition 2 of Proposition 4.1 proves that the equality extends to $H_1^{\mathcal{B}}$. Now $V_1^{G'G}$ and $V_1^{G'} V_1^G$ are equal to $H_1^{G'G}$ when restricted to $H_1^{\mathcal{B}}$. This and the fact that $H_1^{\mathcal{B}}$ generates $V_1^{\mathcal{B}}$ proves the equation $V_1^{G'G} = V_1^{G'} V_1^G$.

Let now k be a positive integer such that $k > 1$. Suppose that for every $m < k$ the equations $H_m^{G'G} = H_m^{G'} H_m^G$ and $V_m^{G'G} = V_m^{G'} V_m^G$ hold. We now prove that the equations $H_k^{G'G} = H_k^{G'} H_k^G$ and $V_k^{G'G} = V_k^{G'} V_k^G$ hold. We first prove $H_k^{G'G} = H_k^{G'} H_k^G$. Both function $H_k^{G'G}$ and $H_k^{G'} H_k^G$ are equal to the morphism function of $V_{k-1}^{G'G}$ when restricted to the morphisms of $V_{k-1}^{\mathcal{B}}$. This, together with the way $H_k^{G'G}$ is defined, and the fact that $V_{k-1}^{G'G}$ satisfies condition 2 of Proposition 4.1 implies that $H_k^{G'G} = H_k^{G'} H_k^G$ holds. Now, both $V_k^{G'G}$ and $V_k^{G'} V_k^G$ are equal to $H_k^{G'G}$ on the set of generators $H_k^{\mathcal{B}}$ of $V_k^{\mathcal{B}}$. This, together with the fact that $H_k^{G'G}$ satisfies condition 1 of Porposition 4.1 implies the equation $V_k^{G'G} = V_k^{G'} V_k^G$ of V_k^G and $V_k^{G'}$. The proof of the remaining two equations is analogous. This concludes the proof. □

Corollary 4.11. *The pair formed by the function associating $Q_{\mathcal{B}}$ to every decorated bicategory $\mathcal{B}$ and Q_G to every decorated pseudofunctor G is a functor from **bCat*** to **gCat**.*

Notation 4.12. We will write Q for the functor defined in corollary 4.11. We will call Q the free globularly generated double category functor.

Let k be a positive integer. If we write $H_k^\bullet$ for the pair formed by the function associating $H_k^{\mathcal{B}}$ to every decorated bicategory $\mathcal{B}$ and H_k^G to every decorated pseudofunctor G then $H_k^\bullet$ is a functor from **bCat*** to **Set** by Lemma 4.10. Similarly the pair $V_k^\bullet$ formed by the function associating $V_k^{\mathcal{B}}$ to every $\mathcal{B}$ and the functor V_k^G to every decorated pseudofunctor G is a functor from **bCat*** to **Cat**. Further, if we write $H_\infty^\bullet$ for the pair formed by the function associating $H_\infty^{\mathcal{B}}$ to every $\mathcal{B}$ and H_∞^G to every decorated bifunctor G then $H_\infty^\bullet$ is a functor from **bCat*** to **Set** and if we write $V_\infty^\bullet$ for the pair formed by the function associating $V_\infty^{\mathcal{B}}$ to every decorated bicategory $\mathcal{B}$ and the function associating V_∞^G to every G then $V_\infty^\bullet$ is a functor from **bCat*** to **Cat**. Thus defined $H_k^\bullet, H_\infty^\bullet$ relate by the equation $H_\infty^\bullet = \varinjlim V_k^\bullet$ and $V_k^\bullet, V_\infty^\bullet$ are related by the equation $V_\infty^\bullet = \varinjlim V_k^\bullet$.

5. Freeness

In this section we prove that the free globularly generated double category functor Q defined in Section 4 is left adjoint to the restriction $H^* \restriction_{\mathbf{gCat}}$ of H^* to **gCat**, i.e. we prove the relation:

$$Q \dashv H^* \restriction_{\mathbf{gCat}}$$

Further, we prove that $H^* \restriction_{\mathbf{gCat}}$ is faithful thus making **gCat** into a concrete category over **bCat*** and Q into a free functor on **gCat**. We interpret the results of this section by saying that the free globularly generated double category provides universal bases for γ-fibers with respect to H^*.

Adjoint relation

We define a counit-unit pair for the adjuntion $Q \dashv H^* \restriction_{\mathbf{gCat}}$. We will write $\pi^\bullet$ for the collection of canonical double projections π^C with C running over the objects of **gCat**. We prove the following proposition.

Proposition 5.1. $\pi^\bullet$ *is a natural transformation.*

Proof. We wish to prove that $\pi^\bullet$ is a natural transformation from H^*Q to identity $id_{\mathbf{gCat}}$. That is, we wish to prove that for every double functor

$T : C \to C'$ from a globularly generated double category C to a globularly generated double category C' the following square commutes:

$$\begin{array}{ccc} Q_{H^*C} & \xrightarrow{Q_{H^*T}} & Q_{H^*C'} \\ {\scriptstyle \pi^C}\downarrow & & \downarrow{\scriptstyle \pi^{C'}} \\ C & \xrightarrow[T]{} & C' \end{array}$$

Let C, C' be globularly generated double categories. Let $T : C \to C'$ be a double functor. We first prove that for each positive integer k the following two squares commute:

$$\begin{array}{ccc} H_k^{H^*C} & \xrightarrow{H_k^{H^*T}} & H_k^{H^*C'} \\ {\scriptstyle H_k^{\pi^C}}\downarrow & & \downarrow{\scriptstyle H_k^{\pi^{C'}}} \\ \mathrm{Hom}_{C_1} & \xrightarrow[T]{} & \mathrm{Hom}_{C'_1} \end{array} \qquad \begin{array}{ccc} V_k^{H^*C} & \xrightarrow{V_k^{H^*T}} & V_k^{H^*C'} \\ {\scriptstyle V_k^{\pi^C}}\downarrow & & \downarrow{\scriptstyle V_k^{\pi^{C'}}} \\ C_1 & \xrightarrow[T]{} & C'_1 \end{array}$$

We do this by induction on k. We fist prove that square on the left hand side commutes in the case $k = 1$. Let φ be a square in $\mathbb{G}^{\mathcal{B}}$. Suppose first that φ is a 2-cell in $\mathcal{B}$. In that case $H_1^{H^*T}\varphi = H^*T\varphi$, which is equal to $T\varphi$. Now, $H_1^{\pi^C}\varphi = \varphi$ and thus the lower left corner of the left hand side square above is also equal to $T\varphi$. The square thus commutes in the values $k = 1$ and φ globular. An equally easy evaluation proves that the square also commutes for the values $k = 1$ and $\varphi = i_f$ for any morphism f in $\mathcal{B}^*$. We conclude that diagram on the left had side above commutes when restricted to collection $\mathbb{G}^{\mathcal{B}}$ in the case in which $k = 1$. This together with the fact that $H_1^{H^*T}$ satisfies condition 6 of Proposition 4.1 and the fact that T is a double functor, implies that square commutes $H_1^{\mathcal{B}}$. Now, the square on the right hand side above restricts to the square on the left when restricted to the set of generators $E_1^{\mathcal{B}}$ of $V_1^{\mathcal{B}}$. This, together with the fact that all edges involved are functors implies that the square on the right commutes in the value $k = 1$.

Let now k be a positive integer such that $k > 1$. Suppose that for every positive $m < k$ the squares above commute. We now prove that the squares above commute for k. The square on the left hand side commutes when restricted to the collection of morphisms of $V_{k-1}^{H^*C}$ by the induction hypothesis. This, together with the fact that the upper edge of the square satisfies condition 4 of Proposition 4.1 and its left and right edges satisfy condition 5 of Proposition 4.1 implies that the full square commutes. Now, the square on the right above is equal to the square on the left when restricted to the set $E_k^{\mathcal{B}}$ of generators of $V_k^{\mathcal{B}}$. This, together with the fact that all the edges of the square are functors, implies that the full square commutes on k. The result follows from this by taking limits.

□

Let $\mathcal{B}$ be a decorated bicategory. In that case $\mathcal{B}$ is a sub-decorated bicategory of $H^*Q_{\mathcal{B}}$. We denote by $j^{\mathcal{B}}$ the inclusion of $\mathcal{B}$ in $H^*Q_{\mathcal{B}}$. We write j for the collection of decorated pseudofunctors $j^{\mathcal{B}}$ with $\mathcal{B}$ running through the objects of **bCat***. As defined above j is clearly a natural transformation from $id_{\mathbf{bCat}^*}$ to H^*Q.

Theorem 5.2. *Q and $H^* \restriction_{\mathbf{gCat}}$ satisfy the relation:*

$$Q \dashv H^* \restriction_{\mathbf{gCat}}$$

with the pair $(\pi^\bullet, j)$ as counit-unit pair.

Proof. We wish to prove that pair formed by Q and $H^* \restriction_{\mathbf{gCat}}$ forms a left adjoint pair with $\pi^\bullet$ and j as counit and unit respectively.

It has already been established that $\pi^\bullet$ is a natural transformation from QH^* to $id_{\mathbf{gCat}}$, and that j is a natural transformation from identity endofunctor $id_{\mathbf{bCat}^*}$ of **bCat*** to H^*Q. We thus only need to prove that the pair $(\pi^\bullet, j)$ satisfies the triangle equations for a counit-unit pair. We begin by proving that the following triangle commutes:

$$\begin{array}{ccc} H^* & \xrightarrow{jH^*} & H^*QH^* \\ & \searrow^{id_{H^*}} & \downarrow{\scriptstyle H^*\pi^\bullet} \\ & & H^* \end{array}$$

Let C be a globularly generated double category. The decoration and the collection of 1-cells of both H^*C and $H^*Q_{H^*C}$ are equal to C_0 and to the collection of horizontal morphisms of C respectively. Moreover, the restriction of j_{H^*C} to both C_0 and the collection of horizontal morphisms of C is the identity. The restriction of j_{H^*C} to the collection of 2-cells of H^*C is the inclusion of the collection of globular squares of C to the collection of globular squares of Q_C. Now, again the restriction to both the decoration and the collection of horizontal morphisms of $H^*\pi^C$ is equal to the identity in both cases. The restriction of $H^*\pi^C$ to the collection of 2-cells of $H^*Q_{H^*C}$ is equal to the restriction, to the collection of globular squares of C, of π^C, which in turn is equal to the identity. It follows that $H^*\pi^C j_{H^*C}$ is equal to id_{H^*C}. We conclude that triangle above commutes.

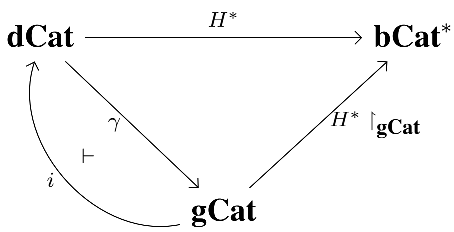
We now prove that the following triangle is commutative:

$$\begin{array}{ccc} Q & \xrightarrow{Qj} & QH^*Q \\ & \searrow^{id_{Q}} & \downarrow{\scriptstyle \pi^\bullet Q} \\ & & Q \end{array}$$

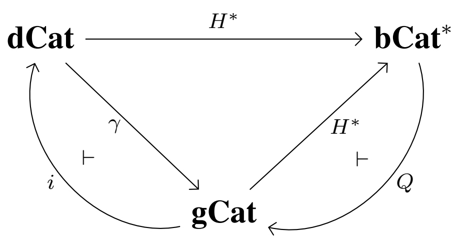
Let $\mathcal{B}$ be a decorated bicategory. In this case the restriction to both C_0 and to the collection of horizontal morphisms of C, of $j_\mathcal{B}$ in $H^*Q_\mathcal{B}$ is equal to the identity. The restriction, to both C_0 and to the collection of horizontal morphisms of C, now of $\pi_{Q_\mathcal{B}}$, is again equal to the identity. We now prove that the restriction, to $\mathcal{B}$, of $H^*\pi^{Q_\mathcal{B}}Q_{j_\mathcal{B}}$ of composition $\pi^{Q_\mathcal{B}}Q_{j_\mathcal{B}}$ defines a decorated endopseudofunctor of $\mathcal{B}$. It has already been established that the restriction, to both the decoration and the collection of horizontal morphisms of $\mathcal{B}$, of both $Q_{j_\mathcal{B}}$ and $\pi^{Q_\mathcal{B}}$ and thus of $\pi^{Q_\mathcal{B}}Q_{j_\mathcal{B}}$ is equal to the identity. Now, let φ be a 2-cell in $\mathcal{B}$. In that case $Q_{j_{\mathcal{B}}}\varphi$ is equal to $j_{\mathcal{B}}\varphi$, which is equal to $\mathcal{B}$. Now, $\pi^{Q_{\mathcal{B}}}\varphi$ is again equal to φ. We conclude that the restriction to $\mathcal{B}$ of $H^*\pi^{Q_{\mathcal{B}}}Q_{j_{\mathcal{B}}}$ defines a decorated endopseudofunctor of $\mathcal{B}$. Moreover, this decorated endopseudofunctor of $\mathcal{B}$ is the identity endopseudofunctor of $\mathcal{B}$. It follows, from this, and from Proposition 5.1 that the composition $\pi^{Q_{\mathcal{B}}}Q_{j_{\mathcal{B}}}$ is equal to the identity endopseudofunctor of $Q_{\mathcal{B}}$. We conclude that triangle above commutes. This concludes the proof.

□

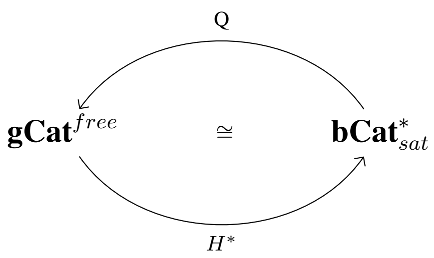
As explained in the introduction we interpret Theorem 5.2 as a generalization of [4, Theorem 5.3] and as a way to complete the diagram



to a diagram of the form:



Moreover, if we write $\mathbf{bCat}^*_{sat}$ for the full subcategory of $\mathbf{bCat}^*$ generated by saturated bicategories and we write $\mathbf{gCat}^{free}$ for the full subcategory of **gCat** generated by the image of the object function of Q, then Theorem 5.2 and [16, Corollary 3.4] say that H^* and Q establish an equivalence between $\mathbf{bCat}^*_{sat}$ y $\mathbf{gCat}^{free}$. That is, we obtain the diagram:



Faithfulness of decorated horizontalization

We now prove that the decorated horizontalization functor H^* is faithful when restricted to the category **gCat** of globularly generated double categories thus allowing an interpretation of **gCat** as a concrete category over **bCat*** and of Q as a free construction. We begin with the following proposition.

Lemma 5.3. *Let C, C' be globularly generated double categories. Let G, G' : $C \to C'$ be double functors. Suppose that the equation $H^*G = H^*G'$ holds. In that case the equations*

$$H^k_G = H^k_{G'} \text{ and } V^k_G = V^k_{G'}$$

hold for every $k \geq 1$.

Proof. Let C, C' be globularly generated double categories. Let G, G' : $C \to C'$ be double functors. Suppose that the equation $H^*G = H^*G'$ holds. Let k be a positive integer. We wish to prove the equations $H^G_k = H^{G'}_k$ and $V^G_k = V^{G'}_k$ hold.

We proceed by induction on k. We first prove the equation $H^G_1 = H^{G'}_1$. Let φ be a globular square in C. In that case $H^G_1 = G\varphi$. This is equal, given that φ is a globular square, to $H^*G\varphi$, which by the assumption of the lemma, is equal to $H^*G'\varphi$, and this is equal to $H^{G'}_1\varphi$. This, together with the fact that both G and G' are double functors and thus preserve horizontal identities implies that the functions H^G_1 and $H^{G'}_1$ are equal. We now prove the equation $V^G_1 = V^{G'}_1$. Observe first that the restriction of the morphism function of V^G_1 to H^C_1 is equal to H^G_1 and that the restriction of the morphism function of $V^{G'}_1$ to H^C_1 is equal to $H^{G'}_1$. This, the previous argument, the fact that H_1^C generates V_1^C, and the functoriality of V_1^G and $V_1^{G'}$, implies that the equation $V_1^G = V_1^{G'}$ holds.

Let now k be a positive integer such that $k > 1$. Suppose that for every $n < k$ the equations $H_n^G = H_n^{G'}$ and $V_n^G = V_n^{G'}$ hold. We prove that under these assumptions the equation $H_k^G = H_k^{G'}$ holds. Observe first that the restriction of H_k^G to $\mathrm{Hom}_{V_{k-1}^C}$ is equal to the morphism function of V_{k-1}^G and that the restriction of $H_k^{G'}$ to $\mathrm{Hom}_{V_{k-1}^C}$ is equal to the morphism function of $V_{k-1}^{G'}$. This, together with the induction hypothesis, and the fact that G and G' intertwine the source and target functors and the horizontal composition funtor of C and C' implies that $H_k^G = H_k^{G'}$. We now prove that V_k^G and $V_k^{G'}$ are equal. Observe that the restriction of the morphism function of V_k^G to H_k^C is equal to H_k^G and that the restriction of the morphism function of $V_k^{G'}$ to H_k^C is equal to the function $H_k^{G'}$. This, together with the previous argument, the fact that H_k^C generates V_k^C, and the functoriality of both V_k^G and $V_k^{G'}$ proves that the functors V_k^G and $V_k^{G'}$ are equal. This concludes the proof. □

Corollary 5.4. *Let C, C' be double categories. Suppose C is globularly generated. Let $G, G' : C \to C'$ be double functors. If the equation $H^*G = H^*G'$ holds then the equation $G = G'$ also holds.*

Proof. Let C, C' be double categories. Let $G, G' : C \to C'$ be double functors. Suppose that C is globularly generated and that the equation $H^*G = H^*G'$ holds. We wish to prove that the equation $G = G'$ holds.

The globular pieces γG and $\gamma G'$ of G and G' are both double functors from C to $\gamma C'$. We have the equalities $H^*\gamma G = H^*G$ and $H^*\gamma G' = H^*G'$. It follows, from the assumption of the corollary that γG and $\gamma G'$ satisfy the assumptions of Proposition 5.3 and thus the equation $V_k^{\gamma G} = V_k^{\gamma G'}$ are equal for everu k. Both γG and $\gamma G'$ admit decompositions as limits $\varinjlim V_k^{\gamma G}$ and $\varinjlim V_k^{\gamma G'}$. It follows, from this, that $\gamma G = \gamma G'$. Finally by the assumption that C is globularly generated γG and $\gamma G'$ are equal to the codomain restrictions, from C' to $\gamma C'$, of G and G' respectively and thus γG and $\gamma G'$ are equal if and only if G and G' are equal. This concludes the proof. □

Proposition 5.5. *$H^* \restriction_{\mathbf{gCat}}$ is faithful.*

Proposition 5.5 allows us to interpret **gCat** as a concrete category over **bCat*** through $H^* \restriction_{\mathbf{gCat}}$. From this and from Theorem 5.2 we have the following corollary.

Corollary 5.6. *Q is a free functor with respect to $H^* \restriction_{\boldsymbol{gCat}}$.*

We interpret Corollary 5.6 by saying that the free globularly generated double category construction provides universal bases of fibers of the globularly generated piece fibration γ and thus provides generators for globularly generated solutions to Problem 1.1.

Acknowledgements: The author would like to thank the anonymous referee, whose comments and suggestions have greatly improved the paper. The author would also like to thank Robert Paré for his support, encouragement and interest in this project.

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CAHIERS DE TOPOLOGIE ET
GEOMETRIE DIFFERENTIELLE
CATEGORIQUES

VOLUME LXII-3 (2021)

THE AMPLE CLOSURE OF THE CATEGORY OF LOCALLY COMPACT ABELIAN GROUPS

Wolfgang RUMP

Dedicated to B. V. M.

Résumé. La catégorie des groupes abéliens localement compactes est une catégorie de Morita, catégorie quasi-abélienne avec une représentation particulière de ses objets au moyen d'une paire intrinsèque de sous-catégories abeliennes de Serre. Pour toute catégorie de Morita, un plongement dans une plus grande catégorie de Morita, la clôture ample, est construite, de sorte que les deux sous-catégories de Serre ne sont pas modifiées. Dans le cas des groupes LCA, ces sous-catégories coïncident respectivement avec la classe des groupes discrets et des groupes compacts. La clôture ample est équivalente à la catégorie de groupes abéliens de Hausdorff totalement bornés, une catégorie tenseur complet et cocomplet, avec une dualité unique prolongeant la dualité de Pontryagin. Cinq différentes caractérisations sont données pour cette catégorie.

Abstract. The category of locally compact abelian groups is shown to be a Morita category, a quasi-abelian category with a particular representation of its objects by means of an intrinsic pair of abelian Serre subcategories. For any Morita category, an embedding into a largest Morita category, the ample closure, is constructed, so that the two Serre subcategories are not changed. In the case of LCA groups, these subcategories coincide with the class of discrete and compact groups, respectively. The ample closure is shown to be equivalent to the category of totally bounded Hausdorff abelian groups, a complete and cocomplete tensor category, with a unique duality extending Pontryagin duality. Five characterizations are given for this category.

Keywords. LCA group, duality, totally bounded abelian group, quasi-abelian category.
Mathematics Subject Classification (2010). 22B05, 18E05, 43A95, 43A40.

1. Introduction

Extensions of classical Pontryagin duality have been proposed by many authors (see Section 5). First attempts focussed upon closure properties of the category **LCA** of locally compact abelian groups within the category **HAb** of all Hausdorff abelian groups. Kaplan [48] proved that the groups in **HAb** which are Pontryagin-reflexive (=P-reflexive for short) are closed with respect to products, and that inverse limits of sequences of LCA groups are P-reflexive [49]. Freundlich-Smith [35] proved that the additive group of a real Banach space or a reflexive locally convex space is P-reflexive. Further positive results are given in [25]. On the other hand, the category **LCA** is neither complete nor cocomplete [53, 45], and cannot be made into a closed (symmetric monoidal) category [54] (see [57], Theorem 4.3; [12]; [44], Remark 3.15).

In view of this lack of completeness properties, it soon became clear that the compact-open topology has to be modified. Binz [16, 17, 18] suggested to consider groups with an underlying convergence space [34, 19, 30], using the fact that convergence spaces form a cartesian closed category ([19], Satz 5) if the morphism sets are endowed with the continuous convergence structure. The full subcategory of reflexive abelian convergence groups is closed with respect to products and coproducts ([23], Theorem 2.4), but neither complete nor cocomplete [14].

In this paper, we give a self-contained approach to the concept of *Morita category* [64], the rationale behind Morita duality, and apply it to the category of LCA groups as a typical example of a Morita category (Proposition 3.2). Every Morita category is quasi-abelian (Proposition 3.4). One of the widely ignored features of a *quasi-abelian* [66] category $\mathscr{A}$ is the existence of abelian Serre subcategories $\mathscr{A}_\circ$ and $\mathscr{A}^\circ$ ([64], Proposition 4.3), where $\mathscr{A}_\circ$ consists of the objects into which every monomorphism is a kernel. For a Morita category $\mathscr{A}$, these subcategories give rise to a canonical pre-factorization (Proposition 3.3) into classes $\mathscr{E}$ and $\mathscr{M}$ of epimorphisms and monomorphisms, respectively. We call $\mathscr{A}$ *ample* if $(\mathscr{E}, \mathscr{M})$ is a factorization system [36], that is, every morphism $f \in \mathscr{A}$ has a factorization $f = me$ with $m \in \mathscr{M}$ and $e \in \mathscr{E}$. Our first main result states that every Morita category $\mathscr{A}$ embeds into a maximal Morita category $\widetilde{\mathscr{A}}$ so that the subcategories $\mathscr{A}_\circ$ and $\mathscr{A}^\circ$ are not altered by the embedding (Theorem 4.2). The category $\widetilde{\mathscr{A}}$ is unique, up to equivalence, and characterized by the property to be an ample Morita category (Corollary 2). We call it the *ample closure* of $\mathscr{A}$.

For the Morita category $\mathscr{A} = \mathbf{LCA}$ of LCA groups, $\mathscr{A}_\circ$ consists of the discrete groups, while $\mathscr{A}^\circ$ consists of the compact groups. (Incidentally, this shows that these subcategories are categorical invariants.) Removing the deficiencies of **LCA**, the ample closure is a complete and cocomplete closed category with a unique duality, extending Pontryagin duality (Theorem 5.1). So there are internal hom-objects with an adjoint tensor product. The dual of an object is obtained via the circle group R/Z, as in **LCA**. In particular, this gives a very simple proof of Pontryagin duality for LCA groups and its uniqueness [56, 63]. Indeed, any duality takes $\mathscr{A}_\circ$ to $\mathscr{A}^\circ$. Since $\mathscr{A}_\circ \approx \mathbf{Mod}(\mathbb{Z})$ has no non-identical self-equivalence, uniqueness of the duality follows by the structure of a Morita category.

In Section 2, we provide five different realizations of the ample closure of **LCA**. The first one is inspired by the *dual systems* [33] in functional analysis, which can be adapted to produce a large class of quasi-abelian categories ([64], Section 2, Example 4). Barr used them for the construction of $*$-autonomous categories [11, 12, 13]. The ample closure of **LCA** is equivalent to the category chu in [13], a reduced version of the so-called Chu-construction [12]. It is also equivalent to the category **TBA** of totally bounded Hausdorff abelian groups with homomorphism groups endowed with the weak topology. Totally bounded abelian groups have been studied intensively in connection with Prontryagin duality (e. g., [67, 29, 24, 41, 42, 26, 3, 38]). Theorem 5.1 now shows that the full embedding $\mathbf{LCA} \hookrightarrow \mathbf{TBA}$ hinges entirely on the above mentioned invariant Serre subcategories $\mathscr{A}_\circ$ and $\mathscr{A}^\circ$, and that **TBA** can be identified as the ample closure of **LCA**. Further incarnations of **TBA** are given in Propositions 2.1-2.2 and a subsequent remark.

For any Morita category $\mathscr{A}$, there is also a smallest full subcategory which leaves the Serre subcategories $\mathscr{A}_\circ$ and $\mathscr{A}^\circ$ unaltered. It is again a Morita category. For $\mathscr{A} = \mathbf{LCA}$ this category consists of the LCA groups which admit an open compact subgroup (Proposition 3.6). Equivalently, these LCA groups don't have R as a direct summand.

Since compactly generated Hausdorff spaces (also called Hausdorff k-*spaces* [50]) form a complete and cocomplete cartesian closed category [54], it was natural to study topological groups with an underlying k-space [58]. Glicksberg's theorem [40] implies that LCA groups are of that type. It states that the topology of an LCA group is determined by its compact subgroups, hence by the associated totally bounded group. A misconception of the relationship between k-spaces and duality led to an inadequate characterization of P-reflexive groups [71], and the belief that Glicksberg's theorem would hold for all these groups [70]. The latter was corrected by Remus and Trigos-Arrieta [62] who were thus led to study the category **PKAb** of P-reflexive groups satisfying Glicksberg's property.

The category of all P-reflexive Hausdorff abelian groups, correctly characterized by Hernández [41], has no better closure properties than **LCA**, which shows the inadequacy of the compact-open topology beyond LCA groups. We prove that the category **PKAb** of Remus and Trigos-Arrieta [62] admits a full embedding into the ample closure of **LCA** which cannot be extended to a bigger category of P-reflexive groups (Theorem 5.2).

2. Totally bounded abelian groups

A Hausdorff topological abelian group A is said to be *totally bounded* [73, 28] if for any neighbourhood U of 0 there is a finite set $F \subset A$ with $A = \bigcup_{a \in F} a + U$. Equivalently, A is totally bounded if and only if its completion is compact. With continuous homomorphisms as morphisms, the category **TBA** of totally bounded Hausdorff abelian groups is a full subcategory of the category **HAb** of Hausdorff topological abelian groups. For any $A \in$ **HAb** we write A_d for the underlying discrete abelian group.

There are several ways to describe the objects of **TBA**. Firstly, let $\mathsf{R} \in$ **HAb** denote the additive group of reals, and Z the subgroup of integers. So $\mathsf{T} := \mathsf{R}/\mathsf{Z}$ is a compact abelian group. We define a *dual system* of abelian groups to be a biadditive map

$$\beta \colon A \times B \to \mathsf{T}_d \tag{1}$$

of abelian groups A, B which is *non-degenerate* in the sense that the associated homomorphisms $\beta_\ell \colon A \to \mathrm{Hom}(B, \mathsf{T}_d)$ and $\beta_r \colon B \to \mathrm{Hom}(A, \mathsf{T}_d)$ are injective. For dual systems $\beta \colon A \times B \to \mathsf{T}_d$ and $\beta' \colon A' \times B' \to \mathsf{T}_d$, a *morphism* $\beta \to \beta'$ is given by a pair of homomorphisms $f \colon A \to A'$ and $g \colon B' \to B$ such that $\beta'(f(a), b') = \beta(a, g(b'))$ holds for $a \in A$ and $b' \in B'$, that is, the commutative diagram

$$\begin{array}{ccc} A \times B' & \xrightarrow{1 \times g} & A \times B \\ \downarrow{\scriptstyle f \times 1} & & \downarrow{\scriptstyle \beta} \\ A' \times B' & \xrightarrow{\beta'} & \mathsf{T}_d \end{array} \tag{2}$$

commutes. Note that dual systems of topological vector spaces, introduced by Dieudonné [33], play an important role in functional analysis (see [65], Chapter IV). To see that dual systems form a category (denoted as $\mathbf{DS}(\mathbf{Ab})$), we replace (1) by its adjoint map $\beta_\ell \colon A \to \mathrm{Hom}(B, \mathsf{T}_d)$. Then (2) takes the form of a commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{\beta_\ell} & \mathrm{Hom}(B, \mathsf{T}_d) \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle \mathrm{Hom}(g, \mathsf{T}_d)} \\ A' & \xrightarrow{\beta'_\ell} & \mathrm{Hom}(B', \mathsf{T}_d). \end{array} \tag{3}$$

By Pontryagin duality, the functor $\mathrm{Hom}(-, \mathsf{T})$ gives an equivalence between the category **Ab** of abelian groups and the full subcategory **CA** of compact abelian groups in **HAb**. Let **DCA** be the subcategory of $\mathbf{Ab} \times \mathbf{CA}$ given by the objects (A, C) where A is a dense subgroup of C. Then the above discussion yields

Proposition 2.1. *The categories* **TBA** *(totally bounded ab. groups),* $\mathbf{DS}(\mathbf{Ab})$ *(dual systems), and* **DCA** *(dense subgroups of compact abelian groups) are equivalent.*

Proof. Consider a dual system (1) and the corresponding adjoint map β_ℓ. The non-degeneracy of β says that $\beta_\ell \colon A \to \mathrm{Hom}(B, \mathsf{T}_d)$ and $\beta_r \colon B \to$

$\mathrm{Hom}(A, \mathsf{T}_d)$ are injective. For β_r this means that the map which carries a homomorphism $g\colon \mathsf{Z} \to B$ to the composed homomorphism

$$A \xrightarrow{\ \beta_\ell\ } \mathrm{Hom}(B, \mathsf{T}_d) \xrightarrow{\mathrm{Hom}(g,\mathsf{T}_d)} \mathrm{Hom}(\mathsf{Z}, \mathsf{T}_d)$$

is injective. Since $\mathrm{Hom}(\mathsf{Z}, \mathsf{T}) = \mathsf{T}$ is a cogenerator of **CA**, the condition states that the embedding $\beta_\ell\colon A \to \mathrm{Hom}(B, \mathsf{T}_d)$ is dense, where $C := \mathrm{Hom}(B, \mathsf{T}_d)$ is viewed as a compact abelian group with Pontryagin dual B. So the morphisms (3) coincide with the corresponding morphisms in **DCA**, which proves that $\mathbf{DS}(\mathbf{Ab}) \approx \mathbf{DCA}$.

Now let $A \hookrightarrow C$ be an object of **DCA**, that is, A is a dense subgroup of the compact abelian group C. If we endow A with the induced topology from C, then A becomes a totally bounded abelian group. Conversely, each object $A \in$ **TBA** gives rise to a dense embedding $A \hookrightarrow C$ into its compact completion C. Since every morphism $A \to B$ of totally bounded abelian groups extends uniquely to the completions, this gives the equivalence **TBA** $\approx$ **DCA**. □

There is a fourth description of the category **TBA**. Let **ChA** be the category of abelian groups A together with a distinguished set $X \subset \mathrm{Hom}(A, \mathsf{T}_d)$ of characters which *separate points* in A, that is, the canonical map $A \to \mathsf{T}_d^X$ is injective. Of course, X separates points if and only if the subgroup of $\mathrm{Hom}(A, \mathsf{T}_d)$ generated by X separates points. So we can assume without loss of generality that X is a subgroup X_A of $\mathrm{Hom}(A, \mathsf{T}_d)$. By [28], Theorem 1.9 (cf. the argument in the above proof), point separation thus means that X_A is dense in $\mathrm{Hom}(A, \mathsf{T}_d)$. Morphisms in **ChA** are group homomorphisms $f\colon A \to B$ such that every character $\chi \in X_A$ factors through f. In what follows, we write $\mathit{Hom}(A, B)$ for the group of continuous homomorphisms between topological abelian groups.

Proposition 2.2. *The categories of Proposition 2.1 are equivalent to* **ChA**.

Proof. Let $i\colon A \hookrightarrow C$ be an object of **DCA**. Define X_A to be the image of

$$\mathit{Hom}(i, \mathsf{T})\colon \mathit{Hom}(C, \mathsf{T}) \to \mathit{Hom}(A, \mathsf{T}).$$

Thus X_A makes A into an object of **ChA**. Conversely, every object $A \in$ **ChA** gives rise to an embedding $A \hookrightarrow \mathrm{Hom}(X_A, \mathsf{T}_d)$ which maps $a \in A$ to $\chi \mapsto \chi(a)$. Thus $C := \mathrm{Hom}(X_A, \mathsf{T}_d)$ is a compact abelian group with character group $Hom(C, \mathsf{T}) \cong X_A$, and there is no non-zero character of C which annihilates A. Hence $A \subset C$ is dense. Thus **DCA** $\approx$ **ChA**. □

Remark. Alfsen and Fenstad [1] established an equivalence between totally bounded uniform structures and proximity spaces. Accordingly, there is a fifth description of **TBA** as a category of abelian groups with a compatible proximity structure. We leave it to the reader to carry this out.

Next we show that the full subcategory **LCA** of locally compact abelian groups (LCA groups for short) in **HAb** admits a full embedding into **TBA**. For any $A \in$ **LCA** there is a dense embedding $A \hookrightarrow \mathfrak{b}A$ into the Bohr compactification $\mathfrak{b}A := \mathrm{Hom}(Hom(A, \mathsf{T})_d, \mathsf{T})$ of A. By A^+ we denote the group A with the induced topology of $\mathfrak{b}A$. The following result is due to Trigos-Arrieta [67]. For convenience, we give a short proof.

Proposition 2.3. *The functor $A \mapsto A^+$ gives a full embedding* **LCA** $\hookrightarrow$ **TBA**.

Proof. For $A \in$ **LCA**, the group A^+ is totally bounded. So $A \mapsto A^+$ gives a faithful functor **LCA** $\hookrightarrow$ **TBA**. To show that it is full, let $f\colon A^+ \to B^+$ be a morphism in **TBA** with $A, B \in$ **LCA**. Then f extends uniquely to a morphism $f'\colon \mathfrak{b}A \to \mathfrak{b}B$ in **CA**. Let V be a neighbourhood of 0 in B. For any compact 0-neighbourhood K in A, the set $f(K) = f'(K)$ is compact in $\mathfrak{b}B$. By Glicksberg's theorem [40], $f(K)$ is compact in B. Hence $V \cap f(K)$ is a 0-neighbourhood in $f(K)$. Since $f'|_K$ is continuous, $f^{-1}(V) \cap K = f^{-1}(V \cap f(K)) \cap K$ is a 0-neighbourhood in A. Thus $A \mapsto A^+$ is full. □

3. The Morita category of LCA groups

Recall that an additive category is said to be *preabelian* [60] if it has kernels and cokernels. For a preabelian category $\mathscr{A}$, we call a sequence

$$A_0 \xrightarrow{a} A_1 \xrightarrow{b} A_2 \tag{4}$$

of morphisms *short exact* if $a = \ker b$ and $b = \operatorname{cok} a$. As usual, we depict kernels by tailed arrows $A_0 \rightarrowtail A_1$ and cokernels by two-head arrows $A_1 \twoheadrightarrow A_2$. A full subcategory $\mathscr{S}$ of $\mathscr{A}$ is said to be a *Serre subcategory* if for every short exact sequence (4), the middle term A_1 belongs to $\mathscr{S}$ if and only if the end terms A_0, A_2 belong to $\mathscr{S}$. If cokernels are stable under pullback and kernels are stable under pushout, $\mathscr{A}$ is said to be *quasi-abelian* [66]. By [64], Proposition 1 and Corollary 1, this implies that the short exact sequences form an exact structure in the sense of Quillen [61]. In [64], Section 8, we have shown that the category **LCA** is quasi-abelian.

For a preabelian category $\mathscr{A}$, let $\mathscr{A}_\circ$ denote the full subcategory of objects $A \in \mathscr{A}$ such that every monomorphism $A' \to A$ is a kernel. Similarly, we define $\mathscr{A}^\circ$ to be the full subcategory of objects $A \in \mathscr{A}$ such that every epimorphism $A \to A'$ is a cokernel. We call an object $P \in \mathscr{A}$ *projective* if for every short exact sequence (4), any morphism $P \to A_2$ factors through b. Similarly, $I \in \mathscr{A}$ is *injective* if each morphism $A_0 \to I$ factors through a. Let $\mathsf{S}_\circ\mathscr{A}$ denote the full subcategory of objects $P \in \mathscr{A}_\circ$ which are projective in $\mathscr{A}$, and let $\mathsf{S}^\circ\mathscr{A}$ be the full subcategory of objects $I \in \mathscr{A}^\circ$ which are injective in $\mathscr{A}$. We say that $f\colon A \to B$ is a $\circ$*-epimorphism* if every morphism $P \to B$ with $P \in \mathsf{S}_\circ\mathscr{A}$ factors through f. Similarly, f is $\circ$*-monic* if every morphism $A \to I$ with $I \in \mathsf{S}^\circ\mathscr{A}$ factors through f.

For example, $\mathbf{LCA}_\circ \approx \mathbf{Ab}$ and $\mathbf{LCA}^\circ \approx \mathbf{CA}$. Indeed, any object $A \in \mathbf{LCA}$ determines a sequence of morphisms

$$\mathsf{Z}^{(H)} \xrightarrow{\ p\ } A \xrightarrow{\ i\ } \mathsf{T}^{H'} \tag{5}$$

with $H := \mathrm{Hom}(\mathsf{Z}, A)$ and $H' := \mathit{Hom}(A, \mathsf{T})$. The coimage $\mathrm{cok}(\ker p)$ of p is A_d, while the image $\ker(\mathrm{cok}\, i)$ of i is the Bohr compactification $\mathfrak{b}A = \overline{i(A)}$. Hence $A \in \mathbf{LCA}_\circ$ if and only if $A = A_d$ and $A \in \mathbf{LCA}^\circ$ if and only if A is a compact group. The objects in $\mathsf{S}_\circ\mathbf{LCA}$ are the free abelian groups $\mathsf{Z}^{(\kappa)}$, while $\mathsf{S}^\circ\mathbf{LCA}$ consists of the cofree compact groups T^κ. Besides these projectives and injectives in **LCA** there are only the vector groups R^n which are projective and injective. The following concept was introduced in [64]. Here we give a different formulation without using PI-varieties.

Definition 3.1. We define a *Morita category* to be a preabelian category with the following properties:

(a) Each object $A \in \mathscr{A}$ admits a $\circ$-epimorphism $P \to A$ and a $\circ$-monomorphism $A \to I$ with $P \in \mathsf{S}_\circ\mathscr{A}$ and $I \in \mathsf{S}^\circ\mathscr{A}$.

(b) Any $\circ$-epic $\circ$-monomorphism is invertible.

Proposition 3.2. *The category* **LCA** *is a Morita category.*

Proof. Condition (a) follows immediately by (5). Condition (b) follows by a theorem of Kaplansky and Glicksberg ([40], Corollary 2.4). □

A morphism $e\colon A \to A'$ in an arbitrary category is said to be *left orthogonal* to $m\colon B \to B'$ (and m is said to be *right orthogonal* to e) if for each commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \downarrow{\scriptstyle e} & & \downarrow{\scriptstyle m} \\ A' & \xrightarrow{f'} & B' \end{array} \tag{6}$$

there exists a morphism $h\colon A' \to B$ with $he = f$ and $mh = f'$. The crucial condition (b) of Definition 3.1 implies:

Proposition 3.3. *For a Morita category $\mathscr{A}$, the $\circ$-epimorphisms are left orthogonal to the $\circ$-monomorphisms.*

Proof. Let (6) be a commutative diagram such that e is $\circ$-epic and m is $\circ$-monic. The pushout

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \downarrow{\scriptstyle e} & & \downarrow{\scriptstyle p} \\ A' & \xrightarrow{g} & C \end{array}$$

gives rise to a morphism $h\colon C \to B'$ with $hg = f'$ and $hp = m$. Since $(g\ p)\colon A' \oplus B \to C$ is a cokernel, any morphism $c\colon P \to C$ with $P \in \mathsf{S}_\circ\mathscr{A}$ factors through $(g\ p)$. So there are morphisms $u\colon P \to A'$ and $v\colon P \to B$ with $c = gu + pv$. Since u factors through e, this implies that c factors through p. Thus p is $\circ$-epic. Since m factors through p, we infer that p is also a $\circ$-monomorphism. Thus p is invertible, so that $p^{-1}g$ gives the desired diagonal in (6). □

Another consequence of Definition 3.1(b) is the following

Proposition 3.4. *Every Morita category $\mathscr{A}$ is quasi-abelian. In particular, $\mathscr{A}_\circ$ and $\mathscr{A}^\circ$ are abelian Serre subcategories of $\mathscr{A}$.*

Proof. Let (6) be a pullback with a cokernel f'. With $k := \ker f$, this implies that $ek = \ker f'$ and $f' = \operatorname{cok} ek$. Let $c\colon A \twoheadrightarrow C$ be the cokernel of k. Then $f = rc$ for some $r\colon C \to B$. Since f' is $\circ$-epic, it follows that r is

$\circ$-epic. On the other hand, let $i\colon C \to I$ be a morphism with $I \in \mathsf{S}^\circ\mathscr{A}$. Since $\binom{e}{f}$ is a kernel, ic factors through $\binom{e}{f}$. So there are morphisms $u\colon A' \to I$ and $v\colon B \to I$ with $ic = ue + vf$. Hence $uek = 0$, which yields a morphism $u'\colon B' \to I$ with $u = u'f'$. Thus $ic = u'f'e + vf = u'mf + vf$, which yields $i = (u'm + v)r$. So r is $\circ$-monic and $\circ$-epic, hence invertible. By symmetry, this implies that $\mathscr{A}$ is quasi-abelian. The assertions on $\mathscr{A}_\circ$ and $\mathscr{A}^\circ$ hold by [64], Proposition 8. □

Corollary 1. *Let $\mathscr{A}$ be a Morita category. Every $\circ$-epimorphism in $\mathscr{A}$ is epic, and every $\circ$-monomorphism is monic.*

Proof. By [64], Corollary 1 of Proposition 1, every morphism $f\colon A \to B$ in $\mathscr{A}$ has a factorization $f = ie$ into an epimorphism e and a kernel i. Assume that f is $\circ$-epic. Then i is $\circ$-epic and $\circ$-monic, hence invertible. So f is epic. By symmetry, this proves the corollary. □

If an abelian category $\mathscr{A}$ has enough projectives, the projective objects form a full subcategory $\mathscr{P}$ with $\mathscr{A} \approx \mathbf{mod}(\mathscr{P})$, see [64], Section 3 ([4], III.1) for the definition of $\mathbf{mod}(\mathscr{P})$. If $\mathscr{P}$ is skeletally small, $\mathbf{mod}(\mathscr{P})$ can be identified with the category of finitely presented additive functors $\mathscr{P}^{\mathrm{op}} \to \mathbf{Ab}$. Similarly, if $\mathscr{A}$ has enough injectives, making up a full subcategory $\mathscr{I}$, we have $\mathscr{A} \approx \mathbf{com}(\mathscr{I}) := \mathbf{mod}(\mathscr{I}^{\mathrm{op}})^{\mathrm{op}}$.

Corollary 2. *Let $\mathscr{A}$ be a Morita category. Then $\mathscr{A}_\circ \approx \mathbf{mod}(\mathsf{S}_\circ\mathscr{A})$ and $\mathscr{A}^\circ \approx \mathbf{com}(\mathsf{S}^\circ\mathscr{A})$.*

Proof. Since $\mathscr{A}_\circ$ is an abelian Serre subcategory of $\mathscr{A}$, every short exact sequence (4) in $\mathscr{A}_\circ$ is short exact in $\mathscr{A}$. Hence $\mathsf{S}_\circ\mathscr{A}$ consists of projective objects in $\mathscr{A}_\circ$. By Corollary 1, the objects $P \in \mathsf{S}_\circ\mathscr{A}$ provide enough projective objects in $\mathscr{A}_\circ$. By symmetry, this proves the claim. □

Definition 3.5. Let $\mathscr{C}$ be a full subcategory of a Morita category $\mathscr{A}$. We say that $\mathscr{C} \hookrightarrow \mathscr{A}$ is a $\circ$-*embedding* and that $\mathscr{C}$ is a $\circ$-*subcategory* of $\mathscr{A}$ if $\mathscr{C}$ is closed with respect to kernels and cokernels such that $\mathscr{C}_\circ = \mathscr{A}_\circ$ and $\mathscr{C}^\circ = \mathscr{A}^\circ$.

Thus any $\circ$-subcategory $\mathscr{C}$ of a Morita category $\mathscr{A}$ is a Morita category. If $i\colon C \rightarrowtail A$ is a kernel of a morphism $f\colon A \to B$ in $\mathscr{A}$ with $A \in \mathscr{C}$, there is a monomorphism $j\colon B \to I$ with $I \in \mathsf{S}^\circ\mathscr{A}$. Hence $i = \ker jf$, and thus $C \in \mathscr{C}$. By [64], Proposition 1, it follows that any Morita category $\mathscr{A}$ admits a smallest $\circ$-subcategory $\underset{\sim}{\mathscr{A}}$, consisting of the subquotients of objects $P \oplus I$ with $P \in \mathsf{S}_\circ\mathscr{A}$ and $I \in \mathsf{S}^\circ\mathscr{A}$ (cf. [64], Proposition 21). By a *subquotient* of $C \in \mathscr{A}$ we mean an object S which admits a kernel $S \rightarrowtail A$ and a cokernel $C \twoheadrightarrow A$ (equivalently, a cokernel $B \twoheadrightarrow S$ and a kernel $B \rightarrowtail C$). So the objects A of $\underset{\sim}{\mathscr{A}}$ are related to pairs $P \in \mathsf{S}_\circ\mathscr{A}$ and $I \in \mathsf{S}^\circ\mathscr{A}$ as follows:

$$\begin{array}{ccccc} & & P \oplus I & & \\ & \nearrow & & \searrow & \\ B & & & & C \\ & \searrow & & \nearrow & \\ & & A & & \end{array}$$

In the category of LCA groups, this subcategory is of special importance. It consists of the groups for which the projective-injective object R does not occur as a direct summand:

Proposition 3.6. *The smallest* $\circ$*-subcategory of* **LCA** *consists of the locally compact abelian groups which admit an open compact subgroup.*

Proof. By [43], Theorem 24.30, every LCA group is of the form $\mathsf{R}^n \oplus A$ such that A admits an open compact subgroup. As R is projective and injective, it can be removed from **LCA** to get a $\circ$-subcategory $\mathscr{A}$ which consists of the objects A with an open compact subgroup C. In other words, A admits a short exact sequence

$$C \overset{i}{\rightarrowtail} A \overset{p}{\twoheadrightarrow} D$$

with a discrete abelian group D. So there exists a kernel $j\colon C \rightarrowtail \mathsf{T}^\kappa$ for some cardinal κ. The pushout of i along j gives a split short exact sequence $\mathsf{T}^\kappa \rightarrowtail \mathsf{T}^\kappa \oplus D \twoheadrightarrow D$ and a kernel $A \rightarrowtail \mathsf{T}^\kappa \oplus D$. Thus A is a subquotient of an object of the form $\mathsf{T}^\kappa \oplus \mathsf{Z}^{(\lambda)}$. □

4. The ample closure

In [64], Morita categories are introduced as a special class of PI-categories, which implies that every Morita category $\mathscr{A}$ admits a largest Morita category with $\mathscr{A}$ as a $\circ$-subcategory. In this section, we give a direct and more symmetric construction of the *ample closure* of a Morita category, and determine it for the category of LCA groups.

Definition 4.1. A Morita category $\mathscr{A}$ is said to be *ample* [64] if every morphism $f \in \mathscr{A}$ admits a factorization $f = me$ with a $\circ$-epimorphism e and a $\circ$-monomorphism m.

Theorem 4.2. *Every Morita category $\mathscr{A}$ admits a $\circ$-embedding into an ample Morita category $\widetilde{\mathscr{A}}$ which is unique up to equivalence.*

Proof. Let $\mathscr{A}$ be a Morita category, and let $\mathscr{C}$ be the category of monic epimorphisms $r\colon D \to C$ in $\mathscr{A}$ with $D \in \mathscr{A}_\circ$ and $C \in \mathscr{A}^\circ$. Morphisms in $\mathscr{C}$ are commutative squares

$$\begin{array}{ccc} D & \xrightarrow{\ f\ } & D' \\ \downarrow{\scriptstyle r} & & \downarrow{\scriptstyle r'} \\ C & \xrightarrow{\ g\ } & C'. \end{array} \tag{7}$$

The kernel $r_0\colon D_0 \to C_0$ of such a morphism is obtained by taking the kernel $k\colon D_0 \rightarrowtail D$ of f in $\mathscr{A}_\circ$. Since $\mathscr{A}$ is quasi-abelian, the monomorphism rk admits a factorization $rk = \ell r_0$ with a kernel ℓ and a monic epimorphism $r_0\colon D_0 \to C_0$. By Proposition 3.4, the object C_0 belongs to $\mathscr{A}^\circ$. It is easily verified that the so constructed $r_0 \to r$ is a kernel of the morphism (7) in $\mathscr{C}$. Thus, by symmetry, $\mathscr{C}$ is preabelian.

To any $A \in \mathscr{A}$, there is a $\circ$-epimorphism $p\colon P \to A$ with $P \in \mathsf{S}_\circ\mathscr{A}$ and a $\circ$-monomorphism $i\colon A \to I$ with $I \in \mathsf{S}^\circ\mathscr{A}$. Since $\mathscr{A}$ is quasi-abelian, there are factorizations $p = r_A q$ and $i = jr^A$ with a cokernel q, a kernel j, and monic epimorphisms $r_A\colon A_\circ \to A$ and $r^A\colon A \to A^\circ$. By Proposition 3.4, $A_\circ \in \mathscr{A}_\circ$ and $A^\circ \in \mathscr{A}^\circ$. For any morphism $f\colon A \to B$ in $\mathscr{A}$, there is a morphism $f'\colon P \to B_\circ$ with $fp = r_B f'$ because r_B is $\circ$-epic. Since $r_B f'$ annihilates the kernel of q, it follows that f' annihilates the kernel of q. So f' factors through q, which shows that fr_A factors through r_B. By symmetry, this gives a faithful additive functor $\mathscr{A} \to \mathscr{C}$ which maps A to $r^A r_A$. Up to isomorphism, $r_A\colon A_\circ \to A$ is uniquely determined as a $\circ$-epic monomorphism with $A_\circ \in \mathscr{A}_\circ$. To show that the functor $\mathscr{A} \to \mathscr{C}$ is full, let a morphism $r^A r_A \to r^B r_B$ be given by a commutative diagram

$$\begin{array}{ccccc} A_\circ & \xrightarrow{r_A} & A & \xrightarrow{r^A} & A^\circ \\ \downarrow f & & & & \downarrow g \\ B_\circ & \xrightarrow{r_B} & B & \xrightarrow{r^B} & B^\circ \end{array}$$

By Proposition 3.3, r_A is left orthogonal to r^B. Hence there is a unique morphism $h\colon A \to B$ with $hr_A = r_B f$ and $r^B h = gr^A$. So we have a full embedding $\mathscr{A} \hookrightarrow \mathscr{C}$.

In particular, each object $r\colon D \to C$ in $\mathscr{C}$ gives rise to a commutative diagram

$$\begin{array}{ccccc} D & = & D & \longrightarrow & C_\circ \\ \downarrow r^D & & \downarrow r & & \downarrow r_C \\ D^\circ & \longrightarrow & C & = & C \end{array} \tag{8}$$

where r^D and r_C can be viewed as objects of the full subcategories $\mathscr{A}_\circ$ and $\mathscr{A}^\circ$, respectively. Furthermore, the morphism $r^D \to r$ is monic and epic, which shows that $\mathscr{C}_\circ$ is equivalent to a full subcategory of $\mathscr{A}_\circ$. Conversely, let (7) be a monomorphism in $\mathscr{C}$ with $r' = r^{D'}$. Then f is monic in $\mathscr{A}$, hence a kernel. Therefore, $gr = r'f$ is $\circ$-monic, which shows that r is isomorphic to an object in $\mathscr{A}_\circ$. On the other hand, there is a factorization $gr = js$ with a kernel j and a monic epimorphism s. Thus $s\colon D \to C''$ is again $\circ$-monic, and $C'' \in \mathscr{A}^\circ$. Hence g is a kernel. Now let $D' \twoheadrightarrow D''$ be the cokernel of f. Then it follows immediately that r is the kernel of the induced morphism $r' \to r^{D''}$. Thus $\mathscr{C}_\circ \approx \mathscr{A}_\circ$. In particular, this shows that $\mathscr{A} \subset \mathscr{C}$ is closed with respect to kernels. By symmetry, $\mathscr{C}^\circ \approx \mathscr{A}^\circ$, and $\mathscr{A} \subset \mathscr{C}$ is closed with respect to cokernels.

So the diagram (8) implies that $\mathscr{C}$ satisfies condition (a) of Definition 3.1. (It is easily checked that $\mathsf{S}_\circ\mathscr{A}$ consists of the projective objects in $\mathscr{C}$.) To verify (b), consider a morphism (7) in $\mathscr{C}$ which is $\circ$-monic and $\circ$-epic. Choose a kernel $i\colon C \rightarrowtail I$ with $I \in \mathsf{S}^\circ\mathscr{A}$. Then there exists a morphism $j\colon D \to I_\circ$ with $r_I j = ir$. Hence i factors through g. By [64], Proposition 2, this implies that g is a kernel. By duality, f is a cokernel. Since r annihilates the kernel of f, there exists a morphism $h\colon D' \to C$ with $hf = r$ and

$gh = r'$. Therefore, f is a monic cokernel, hence invertible. Similarly, g is invertible, which proves that $\mathscr{C}$ is a Morita category such that $\mathscr{A}$ is equivalent to a $\circ$-subcategory.

To show that $\mathscr{C}$ is ample, let (7) be any morphism $\varphi \in \mathscr{C}$. Taking the image of f and g, we obtain a factorization $\varphi = \mu\varepsilon$ where ε is given by a commutative diagram (7) with cokernels f, g, and μ is given by such a commutative diagram with kernels f, g. Thus, by symmetry, it is enough to verify that a morphism (7) is $\circ$-epic whenever f and g are cokernels. Assuming this, let $f' : P \to D'$ and $g' : P^\circ \to C'$ be morphisms with $P \in \mathsf{S}_\circ\mathscr{A}$ and $r'f' = g'r^P$. Then $f' = fh$ for some $h : P \to D$. Since $C \in \mathscr{A}^\circ$, we find a morphism $h' : P^\circ \to C$ with $rh = h'r^P$. Thus $g'r^P = r'f' = r'fh = gh'r^P$, which yields $g' = gh'$. So the morphism (7) is $\circ$-epic.

We set $\widetilde{\mathscr{A}} := \mathscr{C}$. To prove the uniqueness statement, let $\mathscr{A} \hookrightarrow \mathscr{B}$ be a $\circ$-embedding into an ample Morita category $\mathscr{B}$. Then the above constructed ample category $\mathscr{C}$ is the same for both $\mathscr{A}$ and $\mathscr{B}$. Since $\mathscr{B}$ is ample, any object $r : D \to C$ in $\mathscr{C}$ satisfies $r = me$ with a $\circ$-epimorphism e and a $\circ$-monomorphism m. Hence $\mathscr{B} \approx \mathscr{C}$. □

The ample category $\widetilde{\mathscr{A}}$ will be called the *ample closure* of $\mathscr{A}$. By the preceding proof, we have

Corollary 1. *Let $\mathscr{A}$ be a Morita category. For any object $A \in \mathscr{A}$ there are monic epimorphisms $r_A : A_\circ \to A$ and $r^A : A \to A^\circ$ with $A_\circ \in \mathscr{A}_\circ$ and $A^\circ \in \mathscr{A}^\circ$ such that r_A is $\circ$-epic and r^A is $\circ$-monic. Up to isomorphism, the morphisms r_A and r^A are unique.*

Recall that a *factorization system* [36, 37] in a category is given by a pair $(\mathscr{E}, \mathscr{M})$ of morphism classes, containing the isomorphisms and closed under composition, such that the morphisms $e \in \mathscr{E}$ are left orthogonal to the morphisms $m \in \mathscr{M}$ and each $f \in \mathscr{A}$ admits a factorization $f = me$ with $m \in \mathscr{M}$ and $e \in \mathscr{E}$.

Corollary 2. *For a Morita category $\mathscr{A}$, the following are equivalent:*

(a) *Every $\circ$-embedding into a Morita category is an equivalence.*

(b) *The $\circ$-epimorphisms and $\circ$-monomorphisms form a factorization system.*

(c) *$\mathscr{A}$ is ample.*

Proof. (a) ⇔ (c) follows by Theorem 4.2. Proposition 3.3 yields the equivalence (b) ⇔ (c). □

Corollary 3. *The category* **TBA** *of totally bounded abelian groups is the ample closure of the category* **LCA** *of locally compact abelian groups.*

Proof. By Proposition 2.1, **TBA** is equivalent to the category **DCA**. By the proof of Theorem 4.2, $\mathbf{DCA} \approx \widetilde{\mathbf{LCA}}$. □

As is well known, the category **LCA** is neither complete nor cocomplete [53, 45].

Proposition 4.3. *The ample category* **TBA** *of totally bounded abelian groups is complete and cocomplete.*

Proof. Using Proposition 2.1, we consider the equivalent category **DCA**. Since **DCA** is preabelian, it is enough to show that **DCA** has products and coproducts. Thus let (r_i) be a family of objects in **DCA**. So the $r_i\colon D_i \to C_i$ are dense embeddings of discrete groups D_i into compact groups C_i. Thus $r := \prod r_i$ is a dense embedding $r\colon \prod D_i \to \prod C_i$, and it is easily verified that r is a product of the r_i in **DCA**. Since **DCA** is equivalent to **DS**(**Ab**) by Proposition 2.1, the category **DCA** has a natural duality which maps an object $r\colon D \to C$ to $\mathit{Hom}(r, \mathsf{T})\colon \mathit{Hom}(C, \mathsf{T}) \to \mathit{Hom}(D, \mathsf{T})$. Thus **DCA** is cocomplete. □

5. Duality

By a *duality* of a category $\mathscr{C}$ we mean an involution $\mathscr{C}^{\mathrm{op}} \to \mathscr{C}$. The Pontryagin-van Kampen theorem ([59, 68]; [43], Theorem 24.8) states that $A \mapsto \mathit{Hom}(A, \mathsf{T})$ is a duality of **LCA** if the continuous dual of an LCA group is endowed with the compact-open topology.

Extensions of Pontryagin duality have been investigated by many authors, e. g., [48, 49, 53, 58, 69, 70, 71, 18, 22, 31, 32, 72, 52, 8, 62, 55, 5, 41, 39, 3]. Kaplan [48] proved that products of LCA groups are *Pontryagin-reflexive* (*P-reflexive* for short) in the sense that the natural map to the bidual is a topological isomorphism. He raised the problem to determine the class of all P-reflexive topological abelian groups. Freundlich-Smith [35] proved that the additive group of a real Banach space or a reflexive locally convex space is P-reflexive. In 1976, Venkataraman [71] proposed a solution to Kaplan's problem which contains a wrong statement of [69], similar to the incorrect characterization of P-reflexive locally convex spaces in [52] (see [5], Section 8). Correct solutions are given in [41] and [39], respectively. Deviating from the compact-open topology, Binz [15, 16, 17, 18] and Butzmann [21, 22, 23] studied duality within a class of abelian convergence groups [30, 51] and convergence vector spaces.

Note that abelian convergence groups form a *closed category* [54], that is, a symmetric monoidal category $\mathscr{A}$ with internal hom-objects $\mathit{Hom}(A, B)$. An object D of $\mathscr{A}$ is said to be *dualizing* if the natural morphism $A \to \mathit{Hom}(\mathit{Hom}(A, D), D)$ is invertible for each object A. A closed category with a dualizing object D is said to be $*$-*autonomous* [10].

Morita [56] and Roeder [63] proved that Pontryagin duality is essentially unique as a duality of the category **LCA**. More generally, we have the following

Theorem 5.1. *The ample closure of* **LCA** *is a complete and cocomplete* $*$-*autonomous category with an essentially unique duality which extends the Pontryagin duality of* **LCA**.

Proof. Any duality $X \mapsto \widehat{X}$ of $\mathscr{A} := \widetilde{\mathbf{LCA}}$ maps $\mathscr{A}_\circ = \mathbf{Ab}$ to $\mathscr{A}^\circ = \mathbf{CA}$. Up to isomorphism, **Ab** contains a unique indecomposable projective object Z. Hence $\widehat{\mathsf{Z}} \cong \mathsf{T}$. So the duality restricts to the Pontryagin duality on **Ab**. Identifying $\widetilde{\mathbf{LCA}}$ with **DCA**, the objects of this category are given as monic epimorphisms $r\colon D \to C$ with $D \in \mathbf{Ab}$ and $C \in \mathbf{CA}$. Now the Pontryagin-dual of $r_C\colon C_d \to C$ is the Bohr compactification $r^{\widehat{C}}\colon \widehat{C} \to \mathfrak{b}\widehat{C}$. Since $\circ$-epimorphisms correspond to $\circ$-monomorphisms under the duality, this shows that the dual of the morphism $r\colon D \xrightarrow{r} C_d \xrightarrow{r_C} C$ in **DCA** coincides with the Pontryagin-dual of this map in **LCA**. So the dual of the object $r \in \mathbf{DCA}$ is given by the Pontryagin-dual $\widehat{r}\colon \widehat{C} \to \widehat{D}$ of the morphism $r \in \mathbf{LCA}$. Thus $r \mapsto \widehat{r}$ is the unique duality of $\mathbf{DCA} \approx \widetilde{\mathbf{LCA}}$.

Now let $A \in \mathbf{LCA}$ be given. By dualizing the sequence $A_d \to A \to \mathfrak{b}A$, we obtain a sequence $\mathit{Hom}(\mathfrak{b}A, \mathsf{T}) \to \mathit{Hom}(A, \mathsf{T}) \to \mathrm{Hom}(A_d, \mathsf{T})$ with $\mathit{Hom}(A, \mathsf{T})_d = \mathit{Hom}(\mathfrak{b}A, \mathsf{T})$. Furthermore, Pontryagin duality implies that $\mathit{Hom}(A, \mathsf{T}) \to \mathrm{Hom}(A_d, \mathsf{T})$ is the Bohr compactification of $\mathit{Hom}(A, \mathsf{T})$. Thus $\mathit{Hom}(A, \mathsf{T})$ coincides with the dual of A in **DCA**.

To show that $\mathbf{TBA} \approx \widetilde{\mathbf{LCA}}$ is a closed category, we endow the group $\mathit{Hom}(A, B)$ of continuous homomorphisms for $A, B \in \mathbf{TBA}$ with the weak topology, induced by the embedding $\mathit{Hom}(A,B) \hookrightarrow B^A$, and write $\mathit{Hom}^\sigma(A,B)$ for this group. Thus $\mathit{Hom}^\sigma(A,B) \in$ **TBA**. For $B := \mathsf{T}$, this gives the weak dual $A^\sigma := \mathit{Hom}^\sigma(A,\mathsf{T})$, in accordance with the duality in **DCA**. Every biadditive map $\beta\colon A \times B \to C$ with LCA groups A, B, C corresponds to a group homomorphism $\beta_\ell\colon A \to \mathrm{Hom}(B,C)$, which maps A into $\mathit{Hom}^\sigma(B,C)$ if and only if the partial map $\beta(a,-)$ is continuous for all $a \in A$. The embedding $\mathit{Hom}^\sigma(B,C) \hookrightarrow C^B$ shows that β_ℓ is continuous if and only if the partial maps $\beta(-,b)$ are continuous for all $b \in B$. Together with the embeddings

$$\mathit{Hom}^\sigma(A,\mathit{Hom}^\sigma(B,C)) \hookrightarrow \mathit{Hom}^\sigma(B,C)^A \hookrightarrow (C^B)^A \cong C^{A\times B},$$

this gives a topological isomorphism

$$\mathit{Hom}^\sigma(A,\mathit{Hom}^\sigma(B,C)) \cong \mathit{Hom}^\sigma(B,\mathit{Hom}^\sigma(A,C)).$$

In particular,

$$\mathit{Hom}^\sigma(A^\sigma,B^\sigma) \cong \mathit{Hom}^\sigma(B,A^{\sigma\sigma}) \cong \mathit{Hom}^\sigma(B,A).$$

Hence

$$\begin{aligned}\mathit{Hom}^\sigma(A,\mathit{Hom}^\sigma(B,C)) &\cong \mathit{Hom}^\sigma(A,\mathit{Hom}^\sigma(C^\sigma,B^\sigma))\\ &\cong \mathit{Hom}^\sigma(C^\sigma,\mathit{Hom}^\sigma(A,B^\sigma))\\ &\cong \mathit{Hom}^\sigma(\mathit{Hom}^\sigma(A,B^\sigma)^\sigma,C),\end{aligned}$$

and thus $A \otimes B := \mathit{Hom}^\sigma(A,B^\sigma)^\sigma \cong \mathit{Hom}^\sigma(B,A^\sigma)^\sigma$ yields the desired isomorphism

$$\mathit{Hom}^\sigma(A\otimes B,C) \cong \mathit{Hom}^\sigma(A,\mathit{Hom}^\sigma(B,C)).$$

Note that the associativity of $\otimes$ follows by this formula. With the dualizing object T, the category **TBA** is $*$-autonomous. By Proposition 4.3, it is complete and cocomplete. □

Remarks. 1. Theorem 5.1 suggests an alternative proof of classical Pontryagin duality. Since LCA groups are topological k-spaces, and $\circ$-epimorphisms are transformed into $\circ$-monomorphisms under duality, one only has to verify that subsets K of $\mathit{Hom}(A,\mathsf{T})$ which are compact in $\mathrm{Hom}(A_d,\mathsf{T})$ are compact in the compact-open topology of $Hom(A, \mathsf{T})$. Using the Arzelà-Ascoli theorem for LCA groups ([7], Theorem 4), it suffices to show that K is evenly continuous [50]. Using [58], Lemma 2.1, this is easily verified.

2. Note that the unique duality of **TBA** is not the Pontryagin duality in **TBA** (see [3], Example 2.6). For example, let B be the image of $\mathsf{Z} \hookrightarrow \mathfrak{b}\mathsf{Z}$, with the induced topology. By Glicksberg's theorem [40], the compact subsets of B are finite. So the Pontryagin duals of B and Z are topologically isomorphic. Since Z is not totally bounded, this shows that B is not P-reflexive. By [24], Theorem 2, the image of a dense embedding $\mathsf{Z} \hookrightarrow \mathsf{T}$ with the induced topology is not P-reflexive either.

3. It is natural to ask how far Theorem 5.1 can be extended to the category of (not necessarily abelian) locally compact groups. A first step would be to describe the category of compact groups, which is semi-abelian in the sense of [47], within the category of all locally compact groups, in analogy to the subcategory $\mathscr{A}^\circ$ of a quasi-abelian category. Such a study will probably require concepts beyond those discussed in [20].

Let **KAb** be the category of topological abelian groups which *respect compactness* [62], that is, weakly compact subsets are compact. This category contains the nuclear groups [9] and more generally, the locally quasi-convex Schwartz groups ([6], Theorem 4.4). Glicksberg's theorem [40] states that every LCA group belongs to **KAb**. For a while, it was believed ([70], Theorem 1.1) that **KAb** also contains the category **PAb** of P-reflexive Hausdorff abelian groups. This was disproved by Remus and Trigos-Arrieta [62] who provided a class of locally convex spaces as counterexamples, including the additive group of the separable Hilbert space $\ell^2(\mathbb{R})$. More precisely, it is known [35] that the additive group of a reflexive locally convex space X is P-reflexive. By [62], Theorem 1.4, such a group belongs to **KAb** if and only if X is a *Montel space* [65]. This leads to the category **PKAb** $:=$ **PAb** $\cap$ **KAb** which was introduced by Trigos-Arrieta ([67], 1.8) and studied in [62], Section 2. By [62], Corollary 2.2, the full subcategory **PKAb** of **PAb** is closed with respect to products, which shows that **PKAb** strictly contains **LCA**.

Theorem 5.2. *The category* **PKAb** *of P-reflexive Hausdorff abelian groups respecting compactness admits a duality preserving full embedding into the ample closure of* **LCA**.

Proof. Let A be a Hausdorff abelian group in **PAb**. By [2], Theorem 1, A has a Bohr compactification $r^A\colon A \to \mathfrak{b}A$ with $\mathfrak{b}A \cong \mathrm{Hom}(\mathit{Hom}(A,\mathsf{T})_d,\mathsf{T})$ [46, 27], so that every character $A \to \mathsf{T}$ factors uniquely through r^A. Since A has enough characters, r^A is a dense embedding. Dually, there is a continuous bijection $r_A\colon A_d \to A$. We associate the object $r^A r_A \in$ **DCA** to A. This gives a faithful additive functor $F\colon$ **PAb** $\to$ **DCA**.

We show first that F respects duality. Indeed, $\mathit{Hom}(A,\mathsf{T})_d = \mathit{Hom}(\mathfrak{b}A,\mathsf{T})$. With $X := \mathit{Hom}(A,\mathsf{T})$, this shows that $r_X\colon X_d \to X$ is given by

$$\mathit{Hom}(r^A,\mathsf{T})\colon \mathit{Hom}(\mathfrak{b}A,\mathsf{T}) \to \mathit{Hom}(A,\mathsf{T}).$$

Hence $r^X\colon X \to \mathfrak{b}X$ is given by $\mathit{Hom}(r_A,\mathsf{T})\colon \mathit{Hom}(A,\mathsf{T}) \to \mathrm{Hom}(A_d,\mathsf{T})$. So the duality of **DCA** restricts to the duality of **PAb**.

To show that the restriction $F_k\colon$ **PKAb** $\to$ **DCA** of F is full, let $FA \to FB$ be a morphism in **DCA** with $A, B \in$ **PKAb**. This gives a commutative diagram

$$\begin{array}{ccccc} A_d & \xrightarrow{r_A} & A & \xrightarrow{r^A} & \mathfrak{b}A \\ \downarrow{\scriptstyle f} & & & & \downarrow{\scriptstyle g} \\ B_d & \xrightarrow{r_B} & B & \xrightarrow{r^B} & \mathfrak{b}B. \end{array} \tag{9}$$

We show that the unique map $h\colon A \to B$ with $hr_A = r_B f$ is continuous. Let K be a compact subset of A. Then $r^B h(K) = gr^A(K)$ is compact. Hence $h(K)$ is compact in B. Thus h maps compact sets to compact sets. Dualizing the diagram (9) leads to a map $\widehat{g}\colon \mathit{Hom}(\mathfrak{b}B,\mathsf{T}) \to \mathit{Hom}(\mathfrak{b}A,\mathsf{T})$ which induces a map $\widehat{h}\colon \mathit{Hom}(B,\mathsf{T}) \to \mathit{Hom}(A,\mathsf{T})$ with $\widehat{h}(\chi) = \chi h$. For a 0-neighbourhood W in T, consider the 0-neighbourhood ($K \subset A$ compact)

$$U(K,W) := \{\chi \in \mathit{Hom}(A,\mathsf{T}) \mid \chi(K) \subset W\}$$

in $\mathit{Hom}(A,\mathsf{T})$. Then

$$\chi \in \widehat{h}^{-1}(U(K,W)) \Leftrightarrow \chi h(K) \subset W \Leftrightarrow \chi \in U(h(K),W),$$

which yields $\widehat{h}^{-1}(U(K,W)) = U(h(K),W)$. Thus $\widehat{h}$ is continuous. By duality, this implies that h is continuous. So the functor F_k is a full embedding. □

Remark. The first part of the proof yields a duality preserving faithful functor **PAb** → **DCA**. Without the counterexamples to [70], Theorem 1.1, this functor would be a full embedding. The failure of this, analysed in [62], led to the category **PKAb** which is more well-behaved with respect to P-duality.

Conversely, the proof of Theorem 5.2 shows that **PKAb** is the largest full subcategory of **PAb** which allows a full embedding into the ample closure of **LCA**. Namely, if $A \in$ **PAb** does not respect compactness, the totally bounded image A^+ of r^A gives a commutative diagram

$$\begin{array}{ccccc} A^+_d & \xrightarrow{r_{A^+}} & A^+ & \xrightarrow{r^{A^+}} & \mathfrak{b}A^+ \\ \| & & \downarrow{\scriptstyle h} & & \| \\ A_d & \xrightarrow{r_A} & A & \xrightarrow{r^A} & \mathfrak{b}A \end{array}$$

with a non-continuous bijection h. So the compact-open topology becomes inadequate beyond **PKAb**. In topological terms, Theorem 5.2 implies that a weakly continuous homomorphism between topological groups in **PKAb** is continuous. For LCA groups this was proved by Trigos-Arrieta ([67], Theorem 1.2).

Acknowledgement. The author thanks an anonymous referee for thoughtful remarks which helped to improve the presentation of the paper.

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CAHIERS DE TOPOLOGIE ET GEOMETRIE DIFFERENTIELLE CATEGORIQUES

VOLUME LXII-3 (2021)

EXAMPLES AND NON-EXAMPLES OF INTEGRAL CATEGORIES AND THE ADMISSIBLE INTERSECTION PROPERTY

Souheila HASSOUN, Amit SHAH and Sven-Ake WEGNER

Dedicated to Prof. Wolfgang Rump on the occasion of his 70th birthday

Résumé. Les catégories intégrales forment une sous-classe des catégories pré-abéliennes, ces dernières ayant été initialement étudiées par Rump en 2001. Dans la première partie de ce travail, on détermine si certaines catégories d'espaces vectoriels topologiques et bornologiques sont intégrales. Ensuite, on prouve que la classe des catégories intégrales n'est pas contenue dans la classe des catégories quasi-abéliennes, et qu'il existe des catégories semi-abéliennes ni intégrales, ni quasi-abéliennes. Enfin, on trouve une nouvelle caractérisation des catégories quasi-abéliennes, en utilisant la propriété des sommes admissibles, récemment considérée par Brüstle, Hassoun et Tattar. On note qu'une classe de catégories additives non-abéliennes, jouant un rôle très important en analyse fonctionnelle, satisfait à cette propriété.

Abstract. Integral categories form a sub-class of pre-abelian categories whose systematic study was initiated by Rump in 2001. In the first part of this article we determine whether several categories of topological and bornological vector spaces are integral. Moreover, we establish that the class of integral categories is not contained in the class of quasi-abelian categories, and that there exist semi-abelian categories that are neither integral nor quasi-abelian. In the last part of the article we show that a category is quasi-abelian if and only if it has admissible intersections, in the sense considered recently by Brüstle, Hassoun and Tattar. This exhibits that a rich class of non-abelian categories having this property arises naturally in functional analysis.

Keywords. Integral category, quasi-abelian category, projective object, quasi-projective object, topological vector space, bornological vector space, exact category, admissible intersections.

Mathematics Subject Classification (2020). 18E05, 18E10, 46A08, 46A13, 46A17, 46M10, 46M15.

Acknowledgements. The authors would like to thank T. Brüstle and A. Tattar for sharing their results whilst [4] was still in progress. This exchange of ideas was the trigger for the current paper. The authors would also like to thank the referee for their very helpful comments and suggestions. Furthermore, the first author is supported by the 'thésards étoiles' scholarship of the ISM, Bishop's University and Université de Sherbrooke. The second author is grateful for financial support from the EPSRC grant EP/P016014/1 'Higher Dimensional Homological Algebra' and from the London Mathematical Society through an Early Career Fellowship with support from Heilbronn Institute for Mathematical Research. The third author would like to express his gratitude to the Institut des Hautes Études Scientifique for their hospitality during a stay in the summer 2019, where part of this research began, and to Teesside University, where he carried out a majority of the work on this project.

1. Introduction

Since the 1960s there has been much research on additive, non-abelian categories. This has led to the development of a spectrum of classes of categories ranging from pre-abelian to abelian. Our goal in this note is to explain the following diagram:

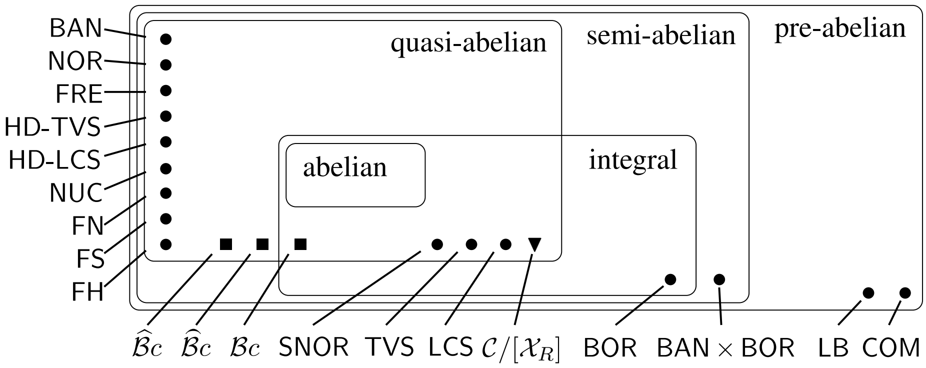
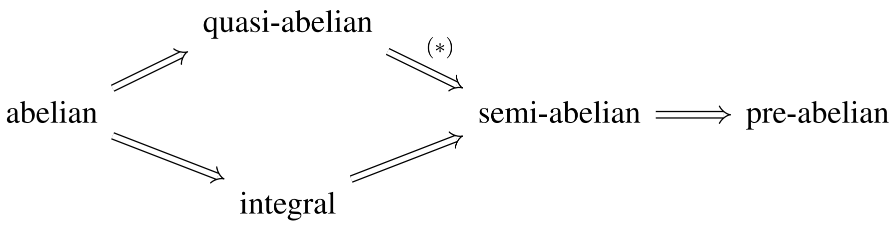


Figure 1: A graphic summary of the categories studied in this article.

We focus on the concrete examples from functional analysis and representation theory that appear therein. We refer the reader to §§2–4 for precise definitions.

A *pre-abelian* category is an additive category in which every morphism has a kernel and a cokernel. Within the class of pre-abelian categories, one defines the following notions. *Semi-abelian* categories are the ones in which the canonical morphism between the coimage and image is always both monic and epic, but not necessarily an isomorphism as one would expect in an abelian category. *Quasi-abelian* categories are the ones in which kernels are stable under pushout and cokernels are stable under pullback. On the other hand, *integral* categories are the ones in which monomorphisms are stable under pushout and epimorphisms are stable under pullback. One can check that the implications



are relatively straightforward; see e.g. Rump [32]. A conjecture of Raĭkov, which has been solved in the negative, was that the converse of $(*)$ holds; see Remark 3.7.

Despite recent progress on integral categories, which appears to be predominantly in algebra (see Remark 2.7), the question if there exist integral categories that are not quasi-abelian seems to date to be open. We answer this positively; see Corollary 3.6. With an idea communicated to the authors by J. Wengenroth, we also prove that the class of semi-abelian categories is not merely the union of the classes of integral and quasi-abelian categories; see Theorem 3.8. Furthermore, we systematically investigate integrality for many examples found in the functional analyst's category theory toolbox; see Theorems 3.1–3.5, 4.1 and 4.2. As a consequence of these results, we derive that most of the categories in Figure 1 have neither enough projectives nor enough injectives; see Theorem 5.3.

Non-abelian categories appear in abundance in functional analysis and have applications for instance in the theory of partial differential equations; see Wengenroth [43], and Frerick and Sieg [9], and the references therein. Indeed, as can be seen from Figure 1, most of the categories we study here are quasi-abelian but not abelian. However, this is still enough intrinsic structure to conduct homological algebra as Schneiders [39] did. He also observed that on each quasi-abelian category the class of all kernel-cokernel pairs forms an *exact structure* in the sense of Quillen [30] (see also Yoneda's 'quasi-abelian $\mathscr{S}$-categories' [45]). In contrast to the internal structure of a category, like pre-, semi- and quasi-abelian, an exact structure is extrinsic.

In studying lengths of objects in exact categories, Brüstle, Hassoun, Langford and Roy [3, Exam. 6.9] showed that an analogue of the classic Jordan-Hölder property can fail for an arbitrary exact category; see also Enomoto [8]. Motivated partly by this, Brüstle, Hassoun and Tattar [4] have recently considered additive categories with a mix of intrinsic and extrinsic structures. More specifically, they consider pre-abelian categories equipped with an exact structure that has ‘admissible intersections’; see §6. Building on their groundwork, we show in Theorem 6.1 that this property is satisfied if and only if the category is quasi-abelian, thereby giving a new characterisation for quasi-abelian categories.

2. A reminder on pre-abelian categories

We recall some definitions of additive categories more general than abelian ones. For more details we refer the reader to [32]. Recall that an additive category is called *pre-abelian* if every morphism has a kernel and a cokernel.

For the remainder of this section, let $\mathcal{A}$ be a pre-abelian category.

Definition 2.1. [32, p. 167] If each morphism $f\colon A \to B$ in $\mathcal{A}$ can be expressed as $f = i \circ p$ for some monomorphism i and some cokernel p, then $\mathcal{A}$ is said to be *left semi-abelian*. Dually, if each morphism f can be written as $f = i \circ p$ for some kernel i and some epimorphism p, then $\mathcal{A}$ is said to be *right semi-abelian*. If $\mathcal{A}$ is both left and right semi-abelian, then it is called *semi-abelian*.

Definition 2.2. Let $\mathcal{X}$ be a class of morphisms in $\mathcal{A}$. We say that $\mathcal{X}$ is *stable under pullback* if, in any pullback square

$$\begin{array}{ccc} A & \xrightarrow{a} & B \\ {\scriptstyle b}\downarrow & \text{PB} & \downarrow{\scriptstyle c} \\ C & \xrightarrow[d]{} & D \end{array}$$

a is in $\mathcal{X}$ whenever d is in $\mathcal{X}$. Being *stable under pushout* is defined dually.

Definition 2.3. [32, p. 168] If cokernels in $\mathcal{A}$ are stable under pullback, then $\mathcal{A}$ is called *left quasi-abelian*. Dually, if kernels in $\mathcal{A}$ are stable under pushout, then $\mathcal{A}$ is called *right quasi-abelian*. If $\mathcal{A}$ is both left and right quasi-abelian, then it is called *quasi-abelian*.

We note here that different authors, in particular Palamodov [25, 26] and Raĭkov [31], have used the above notions in different senses. For example, we follow Palamodov in the use of 'semi-abelian'; see also Kopylov and Wegner [18] for different characterisations. In the non-additive setting, this same name is used to describe a category that is pointed Barr-exact protomodular, admitting binary coproducts; see Janelidze, Márki and Tholen [16]. We refer to the introductions of [38], [18], and [44] for historic references.

Definition 2.4. [32, p. 168] If epimorphisms in $\mathcal{A}$ are stable under pullback, then $\mathcal{A}$ is called *left integral*. Dually, if monomorphisms in $\mathcal{A}$ are stable under pushout, then $\mathcal{A}$ is called *right integral*. If $\mathcal{A}$ is both left and right integral, then it is called *integral*.

Certain relationships between the categories defined above can then be established.

Proposition 2.5. *[32, p. 169, Cor. 1]*

(i) *If $\mathcal{A}$ is a left (respectively, right) quasi-abelian category, then it is left (respectively, right) semi-abelian.*

(ii) *If $\mathcal{A}$ is a left (respectively, right) integral category, then it is left (respectively, right) semi-abelian.*

It follows then that the classes of quasi-abelian and integral categories are both contained in the class of semi-abelian categories.

Proposition 2.6. [32, Prop. 3 and p. 173, Cor.] *Suppose $\mathcal{A}$ is semi-abelian.*

(i) *The category $\mathcal{A}$ is left quasi-abelian if and only if it is right quasi-abelian.*

(ii) *The category $\mathcal{A}$ is left integral if and only if it is right integral.*

We conclude this section with the following remark on integral categories.

Remark 2.7. Although Rump introduced the name 'integral' for a category, such categories were known to Bănică and Popescu [1]. By extending results of [1], Rump proved that a pre-abelian category is integral if and only if it admits a faithful embedding into an abelian category which preserves kernel-cokernel pairs; see [32, Prop. 7]. In particular, he observed that the class of simultaneously monic and epic morphisms in an integral category admits a *calculus of fractions* in the sense of Gabriel and Zisman [10], and hence the (canonical) localisation of the category at this class is an abelian category. This is one reason why integral categories have gained popularity among representation theorists:

(i) Rump [33, 34, 35, 36] himself showed, among other things, that the torsion-free class of a hereditary torsion theory in an abelian category is integral.

(ii) Buan and Marsh [5] showed that, for a certain triangulated category $\mathcal{C}$ and a rigid object $R \in \mathcal{C}$, the quotient category $\mathcal{C}/[\mathcal{X}_R]$, where $\mathcal{X}_R = \operatorname{Ker}\operatorname{Hom}_{\mathcal{C}}(R, -)$, is integral. From this they proved that the canonical localisation of $\mathcal{C}/[\mathcal{X}_R]$ is a module category over $(\operatorname{End}_{\mathcal{C}} R)^{\mathrm{op}}$.

(iii) By introducing *hearts* of twin cotorsion pairs on triangulated categories, Nakaoka [22] generalised this construction of $\mathcal{C}/[\mathcal{X}_R]$. He showed that the heart is always semi-abelian and gave a sufficient condition for it to be integral. A condition for the heart to be quasi-abelian was given by Shah [40]. Furthermore, analogous concepts have been studied by Liu [19] for exact categories, and by Liu and Nakaoka [20], and Hassoun and Shah [13] for *extriangulated* categories (in the sense of Nakaoka and Palu [23]).

3. Categories of topological vector spaces

In this section we look at categories of *topological vector spaces*. The objects of such a category are pairs (X, τ), where X is a vector space and τ is a topology on X that makes the vector space operations continuous. The morphisms are continuous linear maps. For unexplained notation from functional analysis we refer the reader to Meise and Vogt [21].

Our first result extends Rump's observation [32, §2.2] that the topological abelian groups form an integral category.

Theorem 3.1. *Let $k \in \{\mathbb{R}, \mathbb{C}\}$ be fixed. The categories*

(i) SNOR *of semi-normed spaces;*

(ii) LCS *of (Hausdorff and non-Hausdorff) locally convex spaces; and*

(iii) TVS *of (Hausdorff and non-Hausdorff) topological vector spaces*

over k, each furnished with linear and continuous maps as morphisms, are quasi-abelian and integral.

Proof. It is well-known that all three categories are quasi-abelian; see e.g. [39, Prop. 3.2.4], Prosmans [28, Prop. 2.1.11] and [9, Exam. 4.14]. In SNOR, LCS and TVS the kernel of a morphism $f\colon X \to Y$ is the inclusion $f^{-1}(0) \to X$, where $f^{-1}(0)$ is furnished with the induced topology. Denote by $\operatorname{ran} f$ the range of f. Then the cokernel of f is the quotient $Y \to Y/\operatorname{ran} f$ with the quotient topology; see e.g. [39, Lem. 3.2.3], [28, Prop. 2.1.8] and [9, Exam. 2.14]. Thus, f is monic if and only if f is injective, and f is epic if and only if f is surjective. Since pushouts and pullbacks compute algebraically precisely as in $\mathsf{Mod}\, k$, the two conditions in Definition 2.4 hold. □

Our second result exhibits a collection of quasi-abelian categories that are not integral. This is due to the Hausdorff property that we require below. Although all categories in Theorem 3.2 are full subcategories of TVS, their cokernels and thus pushouts compute algebraically differently than in $\mathsf{Mod}\, k$. Theorem 3.2 extends Rump's results [32, §2.2] on Hausdorff topological abelian groups.

Theorem 3.2. *Let $k \in \{\mathbb{R}, \mathbb{C}\}$ be fixed. The categories*

(i) BAN *of Banach spaces;*

(ii) NOR *of normed spaces;*

(iii) FRE *of Fréchet spaces;*

(iv) HD-LCS *of Hausdorff locally convex spaces;*

(v) HD-TVS *of Hausdorff topological vector spaces;*

(vi) NUC *nuclear spaces;*

(vii) FN *of nuclear Fréchet spaces;*

(viii) FS *of Fréchet-Schwartz spaces; and*

(ix) FH *of Fréchet-Hilbert spaces*

over k, each furnished with linear and continuous maps as morphisms, are quasi-abelian but not (left or right) integral.

Proof. Again, it is well-known that these categories are quasi-abelian; see e.g. Prosmans [27, Prop. 3.1.7], [39, Prop. 3.2.17], [28, Prop. 4.4.5] and [28, Prop. 3.1.8] for direct proofs for the first four. The most efficient approach, however, is to establish explicitly that HD-TVS is quasi-abelian, which can be achieved with a slight modification of the proofs just cited. In doing so, one observes that given a morphism $f\colon X \to Y$ in HD-TVS, its kernel is the inclusion $f^{-1}(0) \to X$, and its cokernel is the quotient $Y \to Y/\overline{\operatorname{ran} f}$. These spaces are endowed with the subspace and the quotient topology, respectively. Since the defining properties for the other categories, like Banach, normed, Fréchet, etc., are inherited by closed subspaces and quotients by closed subspaces[1], these categories reflect the kernels and cokernels of HD-TVS. From this it follows that all these categories are also quasi-abelian by, for example, [9, Prop. 4.20].

By Propositions 2.5 and 2.6, left integrality is equivalent to right integrality for all categories in our list; thus, below we show that they all are not right integral. For this we bear in mind that in all nine categories, a morphism is monic if and only if it is injective. This follows from our observations above about kernels in these categories.

(i)–(v): Consider the Banach spaces

$$\mathrm{c}_0 = \left\{ x = (x_j)_{j\in\mathbb{N}} \in k^{\mathbb{N}} \;\middle|\; \lim_{j\to\infty} x_j = 0 \right\}$$

and

$$\ell^1 = \left\{ x \in k^{\mathbb{N}} \;\middle|\; \|x\|_1 = \sum_{j=1}^{\infty} |x_j| < \infty \right\}$$

of null sequences and of absolutely summable sequences, respectively. Here, c_0 is endowed with the supremum norm given by $\|x\|_\infty = \sup_{j\in\mathbb{N}} |x_j|$ and ℓ^1 is endowed with the 1-norm $\|\cdot\|_1$ indicated above. The field k is endowed with the absolute value as a norm. We denote by $i\colon \ell^1 \to \mathrm{c}_0$ the inclusion

[1]For the not-so-explicitly-studied categories in (vi)–(ix), this can be found in [21, Prop. 28.6, Prop. 24.18 and Rmk. 29.15].

and by $\Sigma\colon \ell^1 \to k$ the map that sends a sequence to its sum. Now we put $P = (\mathrm{c}_0 \oplus k)/\overline{\operatorname{ran}\left[\begin{smallmatrix} i \\ -\Sigma \end{smallmatrix}\right]}$, where $\mathrm{c}_0 \oplus k$ carries the product topology, the closure is taken in $\mathrm{c}_0 \oplus k$, and P is furnished with the quotient topology. We denote by $p\colon \mathrm{c}_0 \oplus k \to P$ the quotient map, and by $i_1\colon \mathrm{c}_0 \to \mathrm{c}_0 \oplus k$ and $i_2\colon k \to \mathrm{c}_0 \oplus k$ the inclusion maps. We claim that in all five categories the diagram

$$\begin{array}{ccc} \ell^1 & \xrightarrow{\ i\ } & \mathrm{c}_0 \\ {\scriptstyle\Sigma}\big\downarrow & & \big\downarrow{\scriptstyle p\circ i_1} \\ k & \xrightarrow[p\circ i_2]{} & P \end{array}$$

is a pushout square, and that i is a monomorphism but $p \circ i_2$ is not.

Since the pushout of i along Σ is the cokernel of $\left[\begin{smallmatrix} i \\ -\Sigma \end{smallmatrix}\right]\colon \ell^1 \to \mathrm{c}_0 \oplus k$, our initial remarks establish the first claim and imply that for the second claim it is enough to show that $p \circ i_2$ is not injective. In order to achieve this we will establish that $(p \circ i_2)(1) = 0$ in P. Applying the definition of $p \circ i_2$, we see that we need to show that

$$\left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right] \in \overline{\left\{ \left[\begin{smallmatrix} i \\ -\Sigma \end{smallmatrix}\right](x) \;\middle|\; x \in \ell^1 \right\}}$$

holds. For this we define a sequence $(x^n)_{n\in\mathbb{N}}$ in ℓ^1 as follows. For positive integers n and j we put $x^n_j = -1/n$ whenever $1 \leqslant j \leqslant n$, and $x^n_j = 0$ otherwise. Since for each n only finitely many entries of

$$x^n = (-1/n, -1/n, \dots, -1/n, 0 \dots)$$

are non-zero, we get $(x^n)_{n\in\mathbb{N}} \subseteq \ell^1$. In view of $\|x^n\|_\infty = 1/n$ and $i(x^n) = x^n$ we see that $(i(x^n))_{n\in\mathbb{N}}$ converges to 0 in c_0. On the other hand, we have

$$\big|1 - (-\Sigma(x^n))\big| = \big|1 + \sum_{j=1}^{n} -1/n\big| = 0$$

for every n. Whence, $(-\Sigma(x^n))_{n\in\mathbb{N}}$ converges to 1 in k and $\left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right] \in \overline{\operatorname{ran}\left[\begin{smallmatrix} i \\ -\Sigma \end{smallmatrix}\right]}$, as desired.

(vi)–(ix): Since nuclear Fréchet spaces are Fréchet-Hilbert and Fréchet-Schwartz by [21, Lem. 28.1 and Cor. 28.5], we construct a pushout diagram like in the first part of the proof but with all spaces being nuclear Fréchet. As a locally convex space is simultaneously Banach and nuclear if and only if it is of finite dimension, we need nuclear replacements for c_0 and ℓ^1. First, consider the space $k^{\mathbb{N}}$ of all sequences, which carries the topology of pointwise convergence given by $|x|_m = \sup_{1 \leqslant j \leqslant m} |x_j|$. Secondly, let

$$s = \left\{ x \in k^{\mathbb{N}} \;\middle|\; \forall\, m \in \mathbb{N}\colon \|x\|_m = \sum_{j=1}^{\infty} j^m |x_j| < \infty \right\}$$

be the space of rapidly decreasing sequences, which is endowed with the topology generated by the semi-norms $(\|\cdot\|_m)_{m \in \mathbb{N}}$. Both spaces are nuclear[2]. Since $|x|_m \leqslant \|x\|_m$ and $|\Sigma(x)| \leqslant \|x\|_m$ hold for every $m \in \mathbb{N}$ and every $x \in s$, the inclusion $i'\colon s \to k^{\mathbb{N}}$ and the summation $\Sigma'\colon s \to k$ are both well-defined and continuous. The space $P' = (k^{\mathbb{N}} \oplus k)/\overline{\operatorname{ran}\left[\begin{smallmatrix} i' \\ -\Sigma' \end{smallmatrix}\right]}$ and the quotient map $p'\colon k^{\mathbb{N}} \oplus k \to P'$ are defined analogously to the first part.

Our observations at the beginning of this proof imply that the following holds in all four categories. Firstly, the pushout of i' along Σ' is given by the cokernel of $\left[\begin{smallmatrix} i' \\ -\Sigma' \end{smallmatrix}\right]$, and thus precisely by P'. Secondly, i' is monic. Therefore the diagram

$$\begin{array}{ccc} s & \xrightarrow{\;i'\;} & k^{\mathbb{N}} \\ {\scriptstyle \Sigma'}\big\downarrow & & \big\downarrow{\scriptstyle p' \circ i'_1} \\ k & \xrightarrow[\;p' \circ i'_2\;]{} & P' \end{array}$$

is a pushout. By employing the same sequence $(x^n)_{n \in \mathbb{N}}$ as in the first part, we can see that $p' \circ i'_2$ is not monic. □

We now consider two examples of categories that are neither semi-abelian nor integral. Both are full subcategories of HD-LCS; the first reflects the kernels and the second the cokernels of HD-LCS. However, in the first one cokernels compute differently than in HD-LCS, and in the second the kernels do.

Theorem 3.3. *Let $k \in \{\mathbb{R}, \mathbb{C}\}$ be fixed. The category* COM *of complete Hausdorff locally convex spaces over k, furnished with linear and continuous maps as morphisms, is right quasi-abelian but neither left semi-abelian nor right integral.*

[2] This follows from [21, Prop. 28.16], because both $s = \lambda^1((j^m)_{j,m})$ and $k^{\mathbb{N}} = \lambda^\infty((\mathbb{1}_{\{1,\ldots,m\}}(j))_{j,m})$ are from the class of Köthe echelon spaces.

Proof. The category COM reflects kernels of HD-LCS, and thus a morphism $f\colon X \to Y$ in COM is monic if and only if it is injective. To compute the cokernel

$$Y \to \widehat{Y/\overline{\operatorname{ran} f}}$$

of f in COM, one must take a completion. From this it was derived in [18, Exam. 4.2] that COM is not left semi-abelian. If, on the other hand, we go through the first example in the proof of Theorem 3.2, but in the category COM, we see that P is already complete since we are dealing with Banach spaces. The diagram constructed in the proof of Theorem 3.2 is thus also a pushout in COM, and hence COM is not right integral.

It remains to see that COM is right quasi-abelian. We remark that by [28, Prop. 4.1.10 and Cor. 2.1.9], a morphism in COM is a kernel if and only if it is injective and open onto its range. Notice that this implies automatically that $\operatorname{ran} f$ is closed; see [28, Rmk. 4.1.11(i)]. Let $f\colon X \to Y$ be a kernel and $g\colon X \to Z$ be an arbitrary morphism. We put now $Q = (Y \oplus Z)/\overline{\operatorname{ran}\left[\begin{smallmatrix} g \\ -f \end{smallmatrix}\right]}$. Notice that the pushout of f along g taken in COM factors through the pushout taken in HD-LCS. Thus, there is a diagram

$$\begin{array}{ccc}
X & \xrightarrow{\;f\;} & Y \\
{\scriptstyle g}\big\downarrow & & \big\downarrow \\
Z & \xrightarrow[q\circ i_2]{} & Q \\
\big\| & & \big\downarrow{\scriptstyle i} \\
Z & \xrightarrow[i\circ q\circ i_2]{} & \widehat{Q}
\end{array}$$

in which the outer rectangle is the pushout in COM and the upper square is the pushout in HD-LCS. Here, $i_2\colon Z \to Y \oplus Z$ is the inclusion, $q\colon Y \oplus Z \to Q$ is the quotient map and $i\colon Q \to \widehat{Q}$ is the inclusion of Q into its completion. Since HD-LCS is quasi-abelian, $q \circ i_2$ is a kernel in HD-LCS, and thus injective and open onto its range; see [28, Cor. 3.1.5]. Since i is an isomorphism onto its range, we see that $i \circ q \circ i_2$ is injective and open onto its range, too. Thus, it is a kernel in COM, and we are done. □

Theorem 3.4. *Let $k \in \{\mathbb{R}, \mathbb{C}\}$ be fixed. The category* LB *of countable Hausdorff locally convex inductive limits of Banach spaces over k, furnished with* *linear and continuous maps as morphisms, is left quasi-abelian but neither right semi-abelian nor left integral.*

Proof. The category LB reflects cokernels of HD-LCS, and thus a morphism $f\colon X \to Y$ is epic in LB if and only if it has dense range. Forming the kernel

$$f^{-1}(0)^\flat \to X$$

of f requires endowing $f^{-1}(0)$ with a possibly strictly finer topology; see Wegner [42, Proof of Prop. 14]). The proof in [42] shows that LB is left semi-abelian but not semi-abelian—and therefore necessarily not right semi-abelian.

Furthermore, LB is left quasi-abelian, as noted without proof already in [18, p. 540]. Indeed, first observe that since LB reflects cokernels of HD-LCS, and since every cokernel is the cokernel of its own kernel, all cokernels of LB are surjective. Conversely, if $f\colon X \to Y$ is a surjective morphism in LB, then it satisfies the universal property of a cokernel. Assume now that f is a cokernel and let $g\colon Z \to Y$ be an arbitrary morphism in LB. Then the pullback of f along g is

$$\begin{array}{ccc} [g\ \text{-}f]^{-1}(0)^\flat & \xrightarrow{i_1} & Z \\ {\scriptstyle i_2}\downarrow & \text{PB} & \downarrow{\scriptstyle g} \\ X & \xrightarrow[f]{} & Y \end{array}$$

which is algebraically the pullback taken in $\mathsf{Mod}\,k$. Thus, we see that i_1 is surjective and hence a cokernel in LB by the argument just above.

Finally, we use that BAN is a subcategory of LB, in order to show that the latter is not left integral. Since BAN is not left integral by Theorem 3.2, we can find a pullback diagram in BAN such that the bottom morphism is epic but the top one is not. Since for a Banach space X and a closed subspace $U \subseteq X$, the topology of $U^\flat$ coincides with the topology induced by X, this diagram is also a pullback in LB. From this we see that LB is not left integral. □

In view of Proposition 2.5, we note that it follows from Theorem 3.3 that COM cannot be left quasi-abelian or left integral. Similarly, using Theorem 3.4, LB cannot be right quasi-abelian or right integral.

So far we have witnessed that there exist examples of quasi-abelian categories that are not integral. Next we give an example of an integral category that is not quasi-abelian. This establishes that the class of integral categories is not contained in the class of quasi-abelian ones. To the knowledge of the authors this seemed to be previously unknown. Notice that the cokernels appearing below have no closure in the denominator, since we deal here again with a category whose objects are in general not Hausdorff.

Theorem 3.5. *Let $k \in \{\mathbb{R}, \mathbb{C}\}$ be fixed. The category* BOR *of bornological (Hausdorff and non-Hausdorff) locally convex spaces over k, furnished with linear and continuous maps as morphisms, is integral but neither left nor right quasi-abelian.*

Proof. The category BOR reflects cokernels in LCS. Analogously to LB, the kernel of a morphism $f\colon X \to Y$ is the inclusion $f^{-1}(0)^{\text{BOR}} \to X$, where the 'associated bornological topology' of $f^{-1}(0)^{\text{BOR}}$ can be strictly finer than the topology induced by X; see Sieg and Wegner [41, Exam. 4.1]. We thus get that f is monic if and only if f is injective, and that f is epic if and only if f is surjective. Consequently, pushouts and pullbacks compute algebraically precisely as in $\mathsf{Mod}\,k$. Similarly to Theorem 3.1 we conclude that BOR is integral.

A counterexample constructed by Bonet and Dierolf in [2] (see [41, Exam. 4.1]) shows that BOR is not left quasi-abelian. However, by Proposition 2.6, BOR cannot be right quasi-abelian either as BOR is semi-abelian by Proposition 2.5. Note that it was known already that BOR is semi-abelian but not quasi-abelian, cf. Remark 3.7. □

Corollary 3.6. *The class of integral categories is not contained in the class of quasi-abelian categories.*

We recall the connection between Raĭkov's conjecture and the category BOR.

Remark 3.7. Recall from §1 that *Raĭkov's conjecture* states that a category is semi-abelian if and only if it is quasi-abelian. It was posed around 1970 and answered negatively some 30 years later. Disproving it brought together aspects from algebra and analysis. The category BOR is one of the first two counterexamples given in the literature that falsify it. The other of these is due to Rump [37, Exam. 1] and is a category of the form A-proj, where A is a tilted algebra of Dynkin-type $\mathbb{E}_6$. We refer to [38] for historical details on Raĭkov's conjecture, and to [44] for an extended survey on why the conjecture must naturally fail from the analytic point of view.

We conclude this section on a related note. All the examples of semi-abelian categories we have studied so far (and even those we will see in §4) are either integral or quasi-abelian. Therefore, it is natural to ask if there exists a semi-abelian category which is neither integral nor quasi-abelian. The authors would like to kindly thank J. Wengenroth for proposing a method to obtain a positive answer to this question. Indeed, he suggested that the product category $\mathcal{A}\times\mathcal{B}$ of a non-integral category $\mathcal{A}$ and a non-quasi-abelian category $\mathcal{B}$ would give such an example. This would ensure $\mathcal{A}\times\mathcal{B}$ is neither integral nor quasi-abelian. On the other hand, choosing $\mathcal{A}$ and $\mathcal{B}$ so that they are semi-abelian in their own right ensures $\mathcal{A}\times\mathcal{B}$ is also semi-abelian.

Theorem 3.8. *There exist semi-abelian categories that are neither integral nor quasi-abelian. In particular, the product category* $\mathsf{BAN}\times\mathsf{BOR}$ *is an example of such a category.*

Proof. Let $\mathcal{A}$ denote a semi-abelian category that is not integral (e.g. BAN) and let $\mathcal{B}$ denote a semi-abelian category that is not quasi-abelian (e.g. BOR). Consider the *product* category $\mathcal{A}\times\mathcal{B}$. The objects of $\mathcal{A}\times\mathcal{B}$ are pairs (A,B), where $A\in\operatorname{obj}(\mathcal{A})$ and $B\in\operatorname{obj}(\mathcal{B})$, and morphisms in $\mathcal{A}\times\mathcal{B}$ are pairs (f,g), where f is a morphism in $\mathcal{A}$ and g is a morphism in $\mathcal{B}$.

It is straightforward to check that $\mathcal{A}\times\mathcal{B}$ is additive and pre-abelian. In particular, (co)kernels in $\mathcal{A}\times\mathcal{B}$ are constructed component-wise; for example, the kernel of (f,g) is

$$(\ker f,\ker g)\colon (\operatorname{Ker} f,\operatorname{Ker} g)\to(A,B)$$

for $f\in\operatorname{Hom}_{\mathcal{A}}(A,A')$ and $g\in\operatorname{Hom}_{\mathcal{B}}(B,B')$.

As observed in [32, pp. 167–168], a pre-abelian category is semi-abelian if and only if the *parallel* morphism $h^{\sim}\colon\operatorname{Coim} h\to\operatorname{Im} h$ (that is, the canonical morphism from the coimage to the image) of a morphism h is both monic and epic. It is easy to show that (f,g) is monic (respectively, epic) if and only if f,g are monic (respectively, epic) in their respective categories. Thus, since $\mathcal{A}, \mathcal{B}$ are both semi-abelian, the parallel morphism $(f,g)^{\sim} = (f^{\sim}, g^{\sim})$ of (f,g) is both monic and epic, and hence $\mathcal{A} \times \mathcal{B}$ is semi-abelian.

Since $\mathcal{A}$ is not integral, but is semi-abelian, it cannot be left or right integral by Proposition 2.6. Therefore, as $\mathcal{A}$ is not left integral, there is a pullback square

$$\begin{array}{ccc} P & \xrightarrow{f_1'} & A_2 \\ {\scriptstyle f_2'}\downarrow & \text{PB} & \downarrow{\scriptstyle f_2} \\ A_1 & \underset{f_1}{\twoheadrightarrow} & A \end{array}$$

in $\mathcal{A}$, where f_1 is an epimorphism but f_1' is not. Since kernels in $\mathcal{A} \times \mathcal{B}$ are constructed component-wise, it follows that pullbacks are also determined by their components. Hence, we have the pullback square

$$\begin{array}{ccc} (P, 0_{\mathcal{B}}) & \xrightarrow{(f_1',0)} & (A_2, 0_{\mathcal{B}}) \\ {\scriptstyle (f_2',0)}\downarrow & \text{PB} & \downarrow{\scriptstyle (f_2,0)} \\ (A_1, 0_{\mathcal{B}}) & \xrightarrow[(f_1,0)]{} & (A, 0_{\mathcal{B}}) \end{array}$$

in $\mathcal{A} \times \mathcal{B}$, where $0_{\mathcal{B}}$ is the zero object in $\mathcal{B}$. Moreover, as $f_1 \in \operatorname{Hom}_{\mathcal{A}}(A_1, A)$ and $0 \in \operatorname{Hom}_{\mathcal{B}}(0_{\mathcal{B}}, 0_{\mathcal{B}})$ are both epic, we see that $(f_1, 0)$ is epic; and $(f_1', 0)$ cannot be epic since f_1' is not. Consequently, $\mathcal{A} \times \mathcal{B}$ is not left integral and hence not integral.

Similarly, one can show that $\mathcal{A} \times \mathcal{B}$ is not quasi-abelian, and this concludes the proof. $\square$

4. Categories of bornological vector spaces

Below we consider categories of bornological vector spaces, in the sense introduced by Buchwalter [6] and Hogbe-Nlend [14, 15]. We follow the notation of Prosmans and Schneiders [29], and consider categories whose objects are pairs $(X, \mathcal{B}_X)$ where X is a k-vector space and $\mathcal{B}_X$ is a convex bornology. Their morphisms are the so-called *bounded* linear maps $f\colon X \to Y$, i.e. linear maps for which $f(B) \in \mathcal{B}_Y$ holds whenever $B \in \mathcal{B}_X$. See [29, §1] for more details. Notice that the term 'bornological' in this section has a different meaning than in Theorem 3.5. Here the bornology is an additional structure on a vector space, whereas in §3 being bornological is a property that a locally convex space either enjoys or not.

We start again with identifying a category that is both quasi-abelian and integral.

Theorem 4.1. *Let* $k \in \{\mathbb{R}, \mathbb{C}\}$. *The category* $\mathcal{B}c$ *of (separated and non-separated) bornological vector spaces over* k*, furnished with bounded linear maps as morphisms, is quasi-abelian and integral.*

Proof. By [29, Prop. 1.8] the category is quasi-abelian. By [29, Prop. 1.5], for a morphism $f\colon X \to Y$ the kernel is the inclusion map $f^{-1}(0) \to X$ and the cokernel is the quotient map $Y \to Y/\operatorname{ran} f$. Here, $f^{-1}(0)$ is endowed with the induced bornology and $Y/\operatorname{ran} f$ with the quotient bornology; see [29, Def. 1.4]. One can now proceed as in the proof of Theorem 3.1. □

The other two categories that are usually studied in the context of bornologies are both quasi-abelian, but neither of them is integral.

Theorem 4.2. *Let* $k \in \{\mathbb{R}, \mathbb{C}\}$. *The categories*

(i) $\widehat{\mathcal{B}}c$ *of separated bornological vector spaces; and*

(ii) $\widehat{\mathcal{B}}c$ *of complete bornological vector spaces,*

over k*, furnished with bounded linear maps as morphisms, are quasi-abelian but neither left nor right integral.*

Proof. By [29, Prop. 4.10 and Prop. 5.6] both categories are quasi-abelian. If $f\colon X \to Y$ is a morphism in either one of the two categories, then its cokernel is given by the quotient map $Y \to Y/\overline{\operatorname{ran} f}$; see [29, Prop. 4.6 and Prop. 5.6]. The *closure* $\overline{\operatorname{ran} f}$ is given as the intersection of all closed subspaces U of Y containing $\operatorname{ran} f$. A subspace U is *closed* if limits of sequences in U that converge in X belong to U; see [29, Def. 4.3]. Finally, convergence is defined as follows: $(x_n)_{n\in\mathbb{N}} \subseteq X$ *converges to* $x \in X$ if there exists an absolutely convex set $B \in \mathcal{B}_X$, such that $(x_n)_{n\in\mathbb{N}}$ converges to x in the normed space

$$X_B = (\operatorname{span} B, \|\cdot\|_B) \;\text{ where }\; \|x\|_B = \inf\{\, \lambda > 0 \mid x \in \lambda B \,\};$$

see [29, Def. 4.1].

Assume now that $X = (X, \|\cdot\|)$ is a Banach space that we furnish with the bornology $\mathcal{B}$ of norm-bounded sets; see, for example, [15, p. 21]. Then $(X, \mathcal{B})$ is an object of both $\widehat{\mathcal{B}c}$ and $\widehat{\mathcal{B}c}$. Moreover, a sequence $(x_n)_{n\in\mathbb{N}} \subseteq X$ converges in norm to $x \in X$ if and only if $(x_n)_{n\in\mathbb{N}}$ converges to x with respect to $\mathcal{B}$. Indeed, if $(x_n)_{n\in\mathbb{N}}$ converges to x in norm, then we choose B to be the unit ball of X and in view of $(X, \|\cdot\|) = (X_B, \|\cdot\|_B)$ we get convergence in bornology. Conversely, if $(x_n)_{n\in\mathbb{N}}$ converges to x in some X_B for $B \in \mathcal{B}$ absolutely convex, we conclude that $(x_n)_{n\in\mathbb{N}}$ converges to x in norm from the fact that the inclusion $(X_B, \|\cdot\|_B) \to (X, \|\cdot\|)$ is continuous; see [21, p. 282].

Now consider the maps $i\colon \ell^1 \to \mathrm{c}_0$ and $\Sigma\colon \ell^1 \to k$ from the proof of Theorem 3.2 in $\widehat{\mathcal{B}c}$. This is possible by the above and since continuous linear maps between Banach spaces send bounded sets to bounded sets. In view of the first part of this proof the pushout of i along Σ in $\widehat{\mathcal{B}c}$ is given by

$$\begin{array}{ccc} \ell^1 & \xrightarrow{\;i\;} & \mathrm{c}_0 \\ {\scriptstyle\Sigma}\downarrow & & \downarrow{\scriptstyle q\circ i_1} \\ k & \xrightarrow[q\circ i_2]{} & P \end{array}$$

where $P = (\mathrm{c}_0 \oplus k)/\overline{\operatorname{ran}\left[\begin{smallmatrix} i \\ -\Sigma \end{smallmatrix}\right]}$ coincides, as a vector space, with the space P from the proof of Theorem 3.2. Thus, in the above diagram, i is injective and $q \circ i_2 = 0$. By [29, Prop. 4.6], in $\widehat{\mathcal{B}c}$ the kernel of a morphism is the preimage of zero endowed with the induced bornology. Thus, i is monic but its pushout is not.

To complete the proof it is enough to observe that the preceding paragraph can be repeated verbatim for $\widehat{\mathcal{B}c}$. □

5. Projectives and injectives

Projective objects in an arbitrary category generalise the notion of projective modules arising in algebra. As such, they have become important objects of study in homological algebra. However, suitable notions of projectivity have also been studied in the categories we have seen so far. We focus on projectivity and leave the dual notions related to injectivity to the reader. Let $\mathcal{A}$ be a locally small category. An object $P \in \mathcal{A}$ is called *projective* if, for every epimorphism $f\colon X \to Y$ the induced map $\mathrm{Hom}_{\mathcal{A}}(P, f)\colon \mathrm{Hom}_{\mathcal{A}}(P, X) \to \mathrm{Hom}_{\mathcal{A}}(P, Y)$ is surjective. We say $\mathcal{A}$ has *enough projectives* if for each $A \in \mathcal{A}$ there is an epimorphism $P \to A$ with P projective.

In addition to the above, the following concept has been introduced by Osborne [24], in order to address the fact that in non-abelian categories the classes of epimorphisms and cokernels do not coincide.

Definition 5.1. [24, Def. 7.52] Let $\mathcal{A}$ be a pre-abelian category. An object $P \in \mathcal{A}$ is called *quasi-projective* if, for every cokernel $f\colon X \to Y$, the map $\mathrm{Hom}_{\mathcal{A}}(P, f)$ is surjective. We say $\mathcal{A}$ has *enough quasi-projectives* if for each $A \in \mathcal{A}$ there is a cokernel $P \to A$ with P quasi-projective.

We now use a connection between the notions of §2 and the ones just introduced, in order to derive some interesting consequences of our main results. For Proposition 5.2(i) notice that in [32] the phrase 'has strictly enough projectives' is equivalent to the phrase 'has enough quasi-projectives' that we use here.

Proposition 5.2. *Suppose $\mathcal{A}$ is a pre-abelian category.*

(i) *[32, Prop. 11] If $\mathcal{A}$ has enough quasi-projectives (respectively, quasi-injectives), then $\mathcal{A}$ is left (respectively, right) quasi-abelian.*

(ii) *[5, Prop. 3.9] If $\mathcal{A}$ has enough projectives (respectively, injectives), then $\mathcal{A}$ is left (respectively, right) integral.*

Suppose $\mathcal{A}$ is a pre-abelian category. Although being projective implies being quasi-projective, having enough projectives does not necessarily imply having enough quasi-projectives for $\mathcal{A}$; see [24, pp. 242–243]. In particular, this means that the conclusion of Proposition 5.2(i) cannot be included in the conclusion of Proposition 5.2(ii).

Theorem 5.3. *The following statements hold.*

(i) *The categories* BAN, NOR, FRE, HD-LCS, HD-TVS, NUC, FN, FS, FH, *$\widehat{\mathcal{B}c}$ and $\widehat{\mathcal{B}}c$ have neither enough projectives nor enough injectives.*

(ii) *The category* BOR *has neither enough quasi-projectives nor enough quasi-injectives.*

(iii) *The category* COM *has neither enough quasi-projectives, nor enough projectives, nor enough injectives.*

(iv) *The category* LB *has neither enough quasi-injectives, nor enough projectives, nor enough injectives.*

Proof. (i): Let

$$\mathcal{A} \in \{\, \mathsf{BAN}, \mathsf{NOR}, \mathsf{FRE}, \mathsf{HD\text{-}LCS}, \mathsf{HD\text{-}TVS}, \mathsf{NUC}, \mathsf{FN}, \mathsf{FS}, \mathsf{FH} \,\}.$$

Then $\mathcal{A}$ is quasi-abelian by Theorem 3.2, and so semi-abelian by Proposition 2.5. Thus, left integrality is equivalent to right integrality for $\mathcal{A}$ by Proposition 2.6. But $\mathcal{A}$ is not integral by Theorem 3.2, and so cannot have either enough projectives or enough injectives by Proposition 5.2. For $\mathcal{A} \in \{\overset{\frown}{\mathcal{B}c}, \widehat{\mathcal{B}c}\}$ one can argue analogously by employing corresponding results from §4.

(ii): Similar to (i), using Theorem 3.5.

(iii): The category COM is neither left quasi-abelian, nor left integral, not right integral by Theorem 3.3. Thus, by Proposition 5.2, COM can have neither enough quasi-projectives, nor enough projectives, nor enough injectives.

(iv): Similar to (iii), using Theorem 3.4. □

We mention that for some of the quasi-abelian categories we have seen so far, it has been previously established whether or not they have enough quasi-projectives or quasi-injectives. Indeed, BAN, FRE and LCS have enough quasi-injectives; see [43, Thm. 2.2.1]. Moreover, $\mathcal{B}c$, $\overset{\frown}{\mathcal{B}c}$ and $\widehat{\mathcal{B}c}$ have enough quasi-projectives; see [29, Prop. 2.13, Prop. 4.11 and Prop. 5.8]. Finally, LCS does not have enough quasi-projectives; see Geĭler [11].

We remark also that in the references just cited, the term ‘projective’ is used to mean what we call quasi-projective. Furthermore, in a quasi-abelian category, an object is quasi-projective if and only if it is ‘projective’ in the sense of Bühler [7, Def. 11.1].

We refer the reader to Figure 1 for a graphic summary of all the examples that we have studied in this article.

6. The admissible intersection property

Let $\mathcal{A}$ be a pre-abelian category. We say that $\mathcal{A}$ has *admissible intersections* if there exists an exact structure $\mathcal{E}$ on $\mathcal{A}$ such that for any admissible monomorphisms $c\colon B \rightarrowtail D$ and $d\colon C \rightarrowtail D$, in the pullback diagram

$$\begin{array}{ccc} A & \xrightarrow{\;a\;} & B \\ {\scriptstyle b}\big\downarrow & \text{PB} & \big\downarrow{\scriptstyle c} \\ C & \underset{d}{\rightarrowtail} & D \end{array}$$

in $\mathcal{A}$, the morphisms a and b are also admissible monomorphisms. This property was introduced by Hassoun and Roy in [12] and has been recently considered by Brüstle, Hassoun and Tattar in [4, §4], where they showed that if $\mathcal{A}$ has admissible intersections, then $\mathcal{A}$ is quasi-abelian. We prove here that the converse also holds, and hence together a new characterisation of quasi-abelian categories is established. For the convenience of the reader, and with the kind permission of the authors of [4], we also include their part of the proof below.

Theorem 6.1. (Brüstle, Hassoun, Shah, Tattar, Wegner) *A pre-abelian category $\mathcal{A}$ is quasi-abelian if and only if it has admissible intersections.*

Proof. ($\Longrightarrow$) Let $\mathcal{A}$ be a quasi-abelian category. Endowing it with the class $\mathcal{E}$ of all kernel-cokernel pairs in $\mathcal{A}$ yields an exact category $(\mathcal{A}, \mathcal{E})$ as $\mathcal{A}$ is quasi-abelian; see [39, Rmk. 1.1.11]. The class of admissible monomorphisms in $(\mathcal{A}, \mathcal{E})$ is thus precisely the class of kernels in $\mathcal{A}$. Let $c\colon B \rightarrowtail D$ and $d\colon C \rightarrowtail D$ be arbitrary admissible monomorphisms in $(\mathcal{A}, \mathcal{E})$, i.e. c, d are kernels. Then in the pullback diagram

$$\begin{array}{ccc} A & \xrightarrow{\;a\;} & B \\ {\scriptstyle b}\big\downarrow & \text{PB} & \big\downarrow{\scriptstyle c} \\ C & \underset{d}{\rightarrowtail} & D \end{array}$$

the morphisms a and b are also kernels in $\mathcal{A}$ by the dual of Kelly [17, Prop. 5.2]. That is, a, b are admissible monomorphisms, and we see that $\mathcal{A}$ has admissible intersections.

($\Longleftarrow$) Conversely, suppose $\mathcal{A}$ has admissible intersections and let $\mathcal{E}$ be an exact structure on $\mathcal{A}$ witnessing this. We claim that $\mathcal{E}$ coincides with the class of all kernel-cokernel pairs in $\mathcal{A}$. Assume for contradiction that $A \xrightarrow{f} B \xrightarrow{g} C$ is a kernel-cokernel pair not belonging to $\mathcal{E}$. Then the morphisms $\left[\begin{smallmatrix}1\\g\end{smallmatrix}\right]\colon B \to B \oplus C$ and $\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]\colon B \to B \oplus C$ are both sections, and thus admissible monomorphisms. The pullback of these two morphisms is given by

$$\begin{array}{ccc} A & \xrightarrow{\quad f \quad} & B \\ {\scriptstyle f}\downarrow & \text{PB} & \downarrow{\scriptstyle \left[\begin{smallmatrix}1\\g\end{smallmatrix}\right]} \\ B & \underset{\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]}{\rightarrowtail} & B \oplus C \end{array}$$

Thus, we conclude that f is an admissible monomorphism since $\mathcal{A}$ has admissible intersections. Contradiction. Hence, $\mathcal{E}$ must contain all kernel-cokernel pairs, and so every (co)kernel is admissible. Finally, using the axioms for an exact category (see e.g. [7, Def. 2.1]), we see that in $\mathcal{A}$ kernels are stable under pushout and cokernels are stable under pullback, i.e. $\mathcal{A}$ is quasi-abelian. □

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CAHIERS DE TOPOLOGIE ET
GEOMETRIE DIFFERENTIELLE
CATEGORIQUES

VOLUME LXII-3 (2021)

REGULAR AND EFFECTIVE REGULAR CATEGORIES OF LOCALES

Panagis KARAZERIS and Konstantinos TSAMIS

Résumé. Nous examinons les analogues pour les catégories de locales de deux résultats bien connus sur la régularité et l'effectivité de certaines catégories d'espaces. Nous montrons que la catégorie des locales réguliers compacts est régulière effective (= Barr-exacte). Nous montrons également que la catégorie des locales de Hausdorff compactment engendrés est régulière, à condition qu'elle soit coréflexive dans la catégorie des locales de Hausdorff. Nous ne faisons pas appel à l'existence de points (ce qui rendrait les deux résultats triviaux) mais nous traitons le sujet à l'aide de méthodes valables dans la logique interne d'un topos. En chemin vers le résultat sur les locales com-pactement engendrés, nous arrivons à une généralisation d'un résultat de B. Day et R. Street, dérivant la régularité pour une catégorie cocomplète contenant une sous-catégorie dense régulière, fermée par limites et colimites finies, et satisfaisant une certaine condition de compatibilité des pullbacks avec des colimites appropriées

Abstract. We examine the analogues for the respective categories of locales of two well-known results about regularity and effectiveness of some categories of spaces. We show that the category of compact regular locales is effective regular (=Barr-exact). We also show that the category of compactly generated Hausdorff locales is regular, provided that it is coreflective within Hausdorff locales. We do not appeal to the existence of points (which would render the two results trivial) but rely on the treatment of the subject by methods that are valid in the internal logic of a topos. On the course to the result about compactly generated locales we arrive at a generalization of a result of B. Day and R. Street, deriving regularity for a cocomplete category containing a dense regular subcategory closed under finite limits and colimits and satisfying a certain compatibility condition of pullbacks with appropriate colimits.

Keywords. regular category, effective category, compactly generated locale, compact Hausdorff locale.

Mathematics Subject Classification (2010). 18B25, 18B35, 06D22

1. Introduction

While regular and, even more so, effective regular categories occur more frequently in the realm of algebra there are two well-known cases of categories of spaces that have these features. The category of compact Hausdorff spaces is effective regular and the category of compactly generated (weakly) Hausdorff spaces is regular [1]. It is a rather natural question to ask whether the corresponding categories of locales maintain these features.

For the case of compact Hausdorff locales we know from [11] that they form a regular category. We show here that it is also effective. The extra step, effectiveness of equivalence relations, almost exists implicitly in the work of Vermeulen [13] on proper maps of locales, in particular his result that proper closed equivalence relations on compact locales are effective.

The situation concerning compactly generated Hausdorff locales is much more complicated. First of all we adopt the definition of compactly generated locales introduced in [5], which constitutes the main, if not the only, study of such locales: a Hausdorff locale is compactly generated if it is isomorphic to the colimit of the (directed, extremal monomorphic) diagram of its compact sublocales (via the canonical comparison as a co-cone for that diagram). The major question that is left open in that work is whether such locales form a coreflective subcategory of that of Hausdorff locales. This would be the case if, for every Hausdorff locale, the colimit in question was Hausdorff (as for example would be the case if the canonical comparison described above was monomorphic), in which case the comparison map would be the co-unit of the adjunction. The question of coreflectivity is important for the way products (and hence also pullbacks) are calculated in that category, namely whether they are calculated by applying coreflection to the localic product. This in turn affects our strategy for approaching the question of regularity of compactly generated Hausdorff locales. For that we adapt the argument due to [4] for deriving regularity of the inductive completion of a category from the regularity of the given category. The argument is familiar in the theory of locally presentable categories but its essential ingredients do not require local presentability. We present a generalization of the relevant result in [4] showing regularity for a a cocomplete category containing a dense regular subcategory closed under finite limits and colimits and satisfying a certain compatibility condition of pullbacks with appropriate colimits. One key step is the "uniformity lemma", namely that if objects (like the compactly generated locales) are built up from building blocks (like their compact sublocales), then every finite diagram of such objects can be expressed as a colimit of diagrams of the same type with the vertices among the building blocks (and all objects of the given diagram to be presented as colimits of building blocks over the same indexing category). This uses only the density of the building blocks and their closure under finite colimits in the broader category. Another property used in their proof, in connection with the existence of regular epi - mono factorizations and the stability of regular epis under pullback, is the commutation of pullbacks with a particular type of colimits (directed extremal monomorphic ones, in our case). We show that it is sufficient that the canonical comparison from the colimit of pullbacks to the pullback of the colimits is epimorphic. This is where the nature of the products plays a role. If we assume coreflectivity we arrive at that epimorphicity result and subsequently at the regularity of compactly generated Hausdorff locales.

Our terminology is, we believe, standard. A map of locales $f\colon X \to Y$ is determined by a map $f^*\colon OY \to OX$ between the respective frames that preserves finite infima and all suprema. Hence it has a right adjoint $f_* \vdash f^*$. The map is a surjection if f^* reflects order. It is proper if f_* preserves directed suprema and, for all $U \in OX$, $V \in OY$, $f_*(U \vee f^*V) = f_*U \vee V$. Under the equivalence of the category of locales over X with that of locales internal in sheaves on X, proper maps in the former correspond to compact locales in the latter [8]. A locale X is Hausdorff if its diagonal $X \to X \times X$ is closed. The category of Hausdorff locales is a reflective subcategory of the category of locales. The reflection associates with a given locale the one that is determined by the largest Hausdorff subframe of the underlying frame of the given locale ([3], Theorem 1.2.2).

Acknowledgements The second named author benefited from the co-financed by Greece and the European Union (European Social Fund- ESF) Operational Programme "Human Resources Development, Education and Lifelong Learning" in the context of the project "Strengthening Human Resources Research Potential via Doctorate Research" (MIS-5000432). Both authors are indebted to Vassilis Aravantinos - Sotiropoulos and Christina Vasilakopoulou for useful discussions. We are particularly thankful to the referee for pointing out an important mistake in an early version of this work, which resulted to a substantial revision of our initial approach, and for many other suggestions that improved the presentation of this work.

2. Compactly generated Hausdorff locales

Let us recall from [5] that a Hausdorff locale X (i.e one whose diagonal is closed) is called compactly generated if the canonical comparison map

$$\varepsilon_X \colon \mathrm{colim} C_i \to X,$$

where the colimit is taken over all the compact sublocales of X (hence is a directed diagram of inclusions), is an isomorphism. For an arbitrary Hausdorff locale X the above map is not known to be a monomorphism in the category of locales. In case it is, or at least the resulting colimit is Hausdorff, Escardo shows that it constitutes the counit of an adjunction, rendering the category CGHLoc of compactly generated Hausdorff locales a coreflective subcategory of Hausdorff locales. Let us refer to the assumption that CGHLoc is coreflective in Hausdorff locales as the *coreflectivity hypothesis*. We have

Proposition 2.1. *Under the coreflectivity hypothesis, if $(t_{ij} \colon X_i \to X_j)$ is the directed diagram of inclusions of the compact sublocales of a compactly generated Hausdorff one and Y is any other such locale, then the canonical map*

$$\mathrm{colim}_i (X_i \times Y) \to (\mathrm{colim}_i X_i) \times Y$$

in CGHLoc *is a split epimorphism.*

Proof. One has to be aware of the fact that even under the coreflectivity hypothesis a directed colimit of inclusions of Hausdorff locales that is calculated in the category of locales need not be Hausdorff, while a product of compactly generated locales, calculated in the category of locales, need not be compactly generated. The colimit $\mathrm{colim}_i X_i$, calculated in the category of locales, is by assumption compactly generated Hausdorff hence it is a colimit in the sense of CGHLoc. Its product with the Hausdorff locale Y remains Hausdorff. The products $X_i \times Y$, again calculated in the category of locales, have a factor which is compact Hausdorff. Compact Hausdorff locales are locally compact, hence exponentiable. Taking product with such a locale preserves colimits and we conclude that the products $X_i \times Y$ are compactly generated Hausdorff themselves (so they are products in CGHLoc). On the other hand the colimit that occurs as the domain of the canonical morphism, does not coincide with the colimit taken in the category of locales since the latter need not be a Hausdorff locale. It is though an epimorphic image of the latter as explained at the end of the previous section.

The product in the right hand side has to be the one in CGHLoc which means that we have (under the coreflectivity hypothesis) to apply the coreflection functor to the ordinary localic product. So it is isomorphic to the colimit $\mathrm{colim}_k C_k$ of the system of all compact sublocales of the localic product $(\mathrm{colim}_i X_i) \times Y$. The inclusion of each C_k into the localic product followed by the projection to $\mathrm{colim}_i X_i$ factors through a compact sublocale of that colimit, in particular $C_k \to X_{i(k)} \rightarrowtail \mathrm{colim}_i X_i$ and similarly for the projection to the other factor, $C_k \to D_k \rightarrowtail Y$. Hence the compact sublocales $X_{i(k)} \times D_k$ are final among the C_k so their colimit is isomorphic to $(\mathrm{colim}_i X_i) \times Y$ in CGHLoc. On the other hand the maps $X_{i(k)} \times D_k \to X_{i(k)} \times Y \to \mathrm{colim}_i (X_i \times Y)$ induce a map $\mathrm{colim}_k (X_{i(k)} \times D_k) \to \mathrm{colim}_i (X_i \times Y)$ which fits in a commutative diagram (where we denote emphatically the colimit and product in CGHLoc where necessary).

$$\begin{array}{ccccc}
\mathrm{colim}_k (X_{i(k)} \times D_k) & \xrightarrow{\cong} & \mathrm{colim}_k C_k & \xrightarrow{\cong} & (\mathrm{colim}_i X_i) \times^{CGH} Y \\
\downarrow & & & \nearrow & \\
\mathrm{colim}_i (X_i \times Y) & \twoheadrightarrow & \mathrm{colim}_i^{GCH} (X_i \times Y) & &
\end{array}$$

where the horizontal composite is an isomorphism hence the canonical

$$\mathrm{colim}_i (X_i \times Y) \to (\mathrm{colim}_i X_i) \times Y$$

is a split epimorphism.

□

Proposition 2.2. *Under the coreflectivity hypothesis, if Z is an object in* CGHLoc, $(t_{ij}: f_i \to f_j)$ *is a directed diagram of inclusions between maps* $(f_i: X_i \to Z)$ *with compact Hausdorff domain over it and* $g: Y \to Z$ *another map in the same category, then the canonical map*

$$\mathrm{colim}_i(X_i \times_Z Y) \to (\mathrm{colim}_i X_i) \times_Z Y$$

is a split epimorphism.

Proof. We want to apply the previous result relativized over a base Z, that is to exploit the previous result as a statement about products in $\mathrm{CGHLoc}(\mathrm{Shv}Z)$. In order to do that, we need to make sure that the data of this Proposition, in particular the map $f: \mathrm{colim}_i X_i \to Z$, give data of the previous Proposition when relativized over Z, more specifically that $f: \mathrm{colim}_i X_i \to Z$ corresponds to a compactly generated locale in $\mathrm{Shv}(Z)$. Since this map is the colimit in Loc/Z of the $f_i: X_i \to Z$, we want to show that each such is a proper map. Indeed, for each for each i the composite $X_i \to \mathrm{colim}_i X_i \to Z$ has a factorization $X_i \to K_i \rightarrowtail Z$, where K_i is a compact sublocale of Z, which is proper: The map $X_i \to K_i$ is proper being a map between compact Hausdorff locales ([11] 3.6.1), while $K_i \rightarrowtail Z$ is proper being a closed inclusion (the image K_i is closed as a compact sublocale of a Hausdorff one). Moreover, as a locale in $\mathrm{Shv}Z$, $X \to Z$ is Hausdorff when X is because the diagonal in $\mathrm{Shv}Z$, $X \to X \times_Z X$, is closed when $X \to X \times X$ is. Obviously then the colimit of these composites $\mathrm{colim}_i X_i \to Z$ corresponds to a compactly generated locale in $\mathrm{Loc}(\mathrm{Shv}Z)$ and so does the map $Y \to Z$ by the same argument, hence we can apply the previous Proposition. Referring to the locales in $\mathrm{Shv}Z$ by the names of their corresponding maps to Z, we notice that since the f_i are compact, the localic products $f_i \times g$ in $\mathrm{Shv}Z$ are also products in $\mathrm{CGHLoc}(\mathrm{Shv}Z)$. As explained earlier, their colimit in $\mathrm{CGHLoc}(\mathrm{Shv}Z)$ will be an epimorphic image of the colimit in $\mathrm{Loc}(\mathrm{Shv}Z)$. Seen as map in Loc/Z, the domain of the latter will be just $\mathrm{colim}_i(X_i \times_Z Y)$, where the colimit is calculated in the category of locales. Both the domain of $\mathrm{colim}_i(f_i \times g)$ in $\mathrm{CGHLoc}(\mathrm{Shv}Z)$, as well as $\mathrm{colim}_i(X_i \times_Z Y)$ in CGHLoc arise as epimorphic images of that object. The former maps epimorphically to the latter. This is so because the frame corresponding to it is a subframe of the one corresponding to the latter, both being subframes of $O(\mathrm{colim}_i(X_i \times_Z Y))$. This in turn is due to the fact that a subframe of a frame that happens to be under OZ with the property that it has a closed diagonal in Loc will also be the (underlying frame of a) domain of a map whose diagonal is closed in $\mathrm{Loc}(\mathrm{Sh}Z)$.

Now by the previous proposition there exists a split epimorphism

$$\varepsilon\colon \mathrm{colim}_i^{CGH(ShZ)}(f_i \times g) \twoheadrightarrow \mathrm{colim}_i f_i \times^{CGH(ShZ)} g$$

in $\mathrm{CGHLoc}(\mathrm{Shv}Z)$, where the latter product is meant in the sense of this category. The domain of the map $\mathrm{colim}_i f_i \times g$ as a product in $\mathrm{CGHLoc}(\mathrm{Shv}Z)$ need not coincide with the product $\mathrm{colim}_i X_i \times_Z Y$ in CGHaus (there may exist sublocales $S \to Z$ of the locale $\mathrm{colim}_i X_i \times_Z Y \to Z$ which are proper as maps to Z but need not have a compact domain). But since $\mathrm{colim}_i X_i \times_Z Y$ is a cone for the discrete diagram formed by $\mathrm{colim}_i f_i$ and g over Z, there is a factorization through the (domain of the) product

$$\beta\colon \mathrm{colim}_i X_i \times_Z Y \to \partial_0(\mathrm{colim}_i f_i \times g)$$

over Z. It is easily seen that this is a monomorphism. We have the following commutative diagram

$$\begin{array}{ccccc}
 & & \partial_0(\mathrm{colim}_i^{CGH(ShZ)}(f_i \times g)) & \overset{\varepsilon}{\underset{\mu}{\rightleftarrows}} & \partial_0(\mathrm{colim}_i f_i \times^{CGH(ShZ)} g) \\
 & \nearrow & \downarrow \alpha & & \uparrow \beta \\
\mathrm{colim}_i(X_i \times_Z Y) & & & & \\
 & \searrow & & & \\
 & & \mathrm{colim}_i^{CGH}(X_i \times_Z Y) & \xrightarrow{\gamma} & \mathrm{colim}_i(X_i \times_Z^{GCH} Y)
\end{array}$$

Finally we get that, if μ is a splitting for ε then $\alpha \cdot \mu \cdot \beta$ is a splitting for the comparison $\gamma\colon \mathrm{colim}_i(X_i \times_Z Y) \to (\mathrm{colim}_i X_i) \times_Z Y$. □

3. Regularity of the category of compactly generated Hausdorff locales

We begin by generalizing a lemma due to B. Day and R. Street that is well-known for the case of locally presentable categories [4] . Its statement has only to do with density assumptions (of the presentable objects in the original case) and the closure of the dense subcategory under certain colimits. We include the proof for the sake of completeness of exposition.

Lemma 3.1. *Let $\mathcal{K}$ be a cocomplete category containing a dense subcategory $\mathcal{C}$ which is closed in $\mathcal{K}$ under finite colimits. Then for any small category with finite hom-sets $\mathcal{D}$ and diagram $D \in [\mathcal{D}, \mathcal{K}]$ we have that*

$$D \cong \mathrm{colim}\ ([\mathcal{D},\mathcal{C}] \downarrow D \to\ [\mathcal{D},\mathcal{C}] \to\ [\mathcal{D},\mathcal{K}])$$

Proof. We show that, for all $d \in \mathcal{D}$, the evaluation at d of the canonical morphism from the colimit to D is an isomorphism in $\mathcal{K}$. Colimits in $[\mathcal{D},\mathcal{K}]$ are given object-wise so, denoting $i\colon \mathcal{C} \to \mathcal{K}$ the inclusion and

$$\partial_0\colon [\mathcal{D},\mathcal{C}] \downarrow D \to\ [\mathcal{D},\mathcal{C}]$$

the domain functor

$$\begin{aligned} \mathrm{colim}\ ([\mathcal{D},\mathcal{C}] \downarrow D \to\ [\mathcal{D},\mathcal{C}] \to\ [\mathcal{D},\mathcal{K}])(d) \quad &\cong \\ ev_d(\ \mathrm{colim}\ ([\mathcal{D}, i] \cdot \partial_0 :\ [\mathcal{D},\mathcal{C}] \downarrow D \to [\mathcal{D},\mathcal{C}] \to [\mathcal{D},\mathcal{K}])\ ) \quad &\cong \\ \mathrm{colim}(\ ev_d \cdot [\mathcal{D}, i] \cdot \partial_0 :\ [\mathcal{D},\mathcal{C}] \downarrow D \to\ [\mathcal{D},\mathcal{C}] \to\ [\mathcal{D},\mathcal{K}] \to \mathcal{K}) \quad &(1) \end{aligned}$$

On the other hand the density of $\mathcal{C}$ in $\mathcal{K}$ means that for all d

$$Dd \cong \mathrm{colim}(\mathcal{C} \downarrow Dd \to \mathcal{C} \to \mathcal{K}),$$

while inspection gives that the composite

$$ev_d \cdot [\mathcal{D}, i] \cdot \partial_0 :\ [\mathcal{D},\mathcal{C}] \downarrow D \to\ [\mathcal{D},\mathcal{C}] \to\ [\mathcal{D},\mathcal{K}] \to \mathcal{K} \quad (2)$$

is naturally isomorphic to the composite

$$i \cdot \partial_0 \cdot (ev_d \downarrow D) :\ [\mathcal{D},\mathcal{C}] \downarrow D \to \mathcal{C} \downarrow Dd \to \mathcal{C} \to \mathcal{K} \quad (3)$$

Moreover the functor $ev_d \downarrow D\colon [\mathcal{D},\mathcal{C}] \downarrow D \to \mathcal{C} \downarrow Dd$ is final, essentially because $ev_d\colon [\mathcal{D},\mathcal{C}] \to \mathcal{C}$ has a left adjoint given by $C \mapsto \coprod_{\mathcal{D}(d,-)} C$ (whose existence is granted by the fact that $\mathcal{D}$ has finite hom-sets and $\mathcal{C}$ is closed under finite colimits in $\mathcal{K}$.) Hence combining the isomorphisms $(1),\ (2),\ (3)$ with the latter finality result we get the desired isomorphism. □

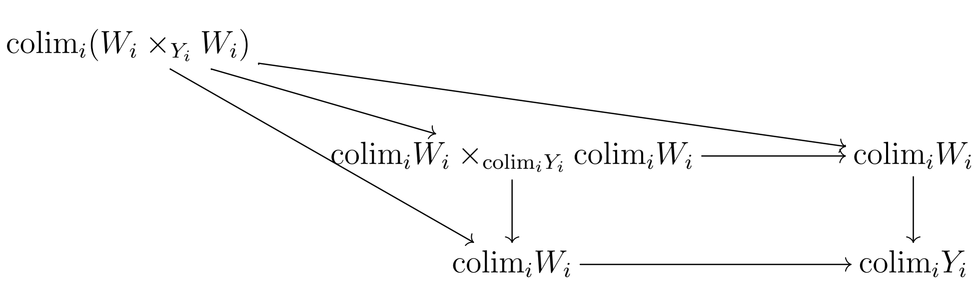
The following proposition generalizes the main result of [4]. It relies on the possibility of conveniently writing a pullback diagram where the one leg is regular epi as colimit of diagrams with vertices in the dense subcategory. This is a key step for arriving at the characterization of regular locally finitely presentable categories in [2] (cf. Lemma 12 and Theorem 14 there). We elaborate on it, showing that the previous lemma, using only density assumptions, suffices.

Proposition 3.2. *Let $\mathcal{K}$ be a cocomplete and finitely complete category, such that it contains a dense regular subcategory $\mathcal{C}$ which is closed in $\mathcal{K}$ under finite limits and finite colimits. Assume that the objects of $\mathcal{K}$ are expressed (by density of $\mathcal{C}$) as colimits of objects from $\mathcal{C}$ of such kind that the canonical comparison map to the pullback of such colimits from the colimit of the pullbacks of its components in $\mathcal{K}$ is a regular epimorphism. Then $\mathcal{K}$ is also regular.*

Proof. First we show the existence of regular epi- mono factorizations in $\mathcal{K}$. We apply the above lemma for $\mathcal{D}$ the category $\bullet \to \bullet$ so that we express every morphism $X \to Y$ in $\mathcal{K}$ as a colimit of morphisms $X_i \to Y_i$ between objects in the full subcategory $\mathcal{C}$. Using the regularity of $\mathcal{C}$ we take the regular epi - mono factorization $X_i \to W_i \to Y_i$ of every such morphism. Taking colimit of the appropriate kind we get a factorization

$$X \cong \mathrm{colim}_i X_i \to \mathrm{colim}_i W_i \to \mathrm{colim}_i Y_i \cong Y,$$

where the first morphism is obviously regular epi. We claim that the second one is mono because its kernel-pair consists of equal legs: Considering the pullbacks $W_i \times_{Y_i} W_i$ we have that the two legs to W_i are equal since $W_i \to Y_i$ is mono. Taking colimit over i we get that the two outer morphisms $\mathrm{colim}_i(W_i \times_{Y_i} W_i) \to \mathrm{colim}_i W_i$ in

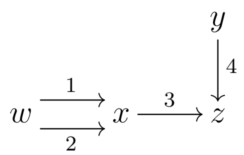


are equal. Hence the two composites

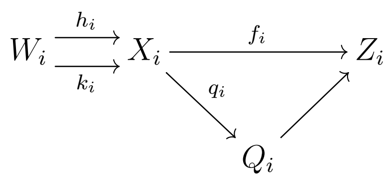
$$\mathrm{colim}_i(W_i \times_{Y_i} W_i) \to \mathrm{colim}_i W_i \times_{\mathrm{colim}_i Y_i} \mathrm{colim}_i W_i \to \mathrm{colim}_i W_i$$

are equal. Since $\mathrm{colim}_i(W_i \times_{Y_i} W_i) \to \mathrm{colim}_i W_i \times_{\mathrm{colim}_i Y_i} \mathrm{colim}_i W_i$ is an epimorphism we conclude that the two legs of the pullback are equal.

Next we show stability of regular epis under pullbacks. Given a regular epi $f: X \to Z$ which occurs as the coequalizer of the pair of morphisms $h,\ k: W \to X$ and any morphsim $g: Y \to Z$ in $\mathcal{K}$ we are applying the lemma to the category $\mathcal{D}$ given as



while $D: \mathcal{D} \to \mathcal{K}$ is defined by $Dw = W$, $Dx = X$, $Dz = Z$, $D4 = g$, $D1 = h$, $D2 = k$, $D3 = f$. It follows from the lemma that we can write $X \cong \mathrm{colim}_i X_i$, $Y \cong \mathrm{colim}_i Y_i$, $W \cong \mathrm{colim}_i W_i$, $h = \mathrm{colim}_i h_i$, $k = \mathrm{colim}_i k_i$, $f = \mathrm{colim}_i f_i$ and $g = \mathrm{colim}_i g_i$ over the same indexing category, with X_i, Y_i, W_i in $\mathcal{C}$. Then the coequalizers $q_i: X_i \to Q_i$ of the pairs (h_i, k_i) will have their codomains in $\mathcal{C}$ and they will factor as in the diagram



Their pullbacks $X_i \times_{Z_i} Y_i \twoheadrightarrow Q_i \times_{Z_i} Y_i$ along the respective $Q_i \times_{Z_i} Y_i \to Q_i$ will be regular epimorphisms, hence the same will be $\mathrm{colim}_i(X_i \times_{Z_i} Y_i) \twoheadrightarrow \mathrm{colim}_i(Q_i \times_{Z_i} Y_i)$. Using the commutation of colimits with coequalizers, the colimit $Q = \mathrm{colim}_i Q_i$ of these coequalizers will be

$$\mathrm{colim}_i Q_i \cong\ \mathrm{coeq}(h, k: W \to X) \cong Z$$

Hence we have a commutative diagram as below, with the lower rectangles being pullbacks.

$$\begin{array}{ccccc}
\mathrm{colim}_i(X_i \times_{Z_i} Y_i) & \twoheadrightarrow & \mathrm{colim}_i(Q_i \times_{Z_i} Y_i) & & \\
\downarrow\!\!\downarrow & & \downarrow\!\!\downarrow & & \\
X \times_Z Y & \longrightarrow & Q \times_Z Y & \xrightarrow{\cong} & \mathrm{colim}_i Y_i \\
\downarrow & & \downarrow & & \downarrow \\
\mathrm{colim}_i X_i & \longrightarrow & \mathrm{colim}_i Q_i & \xrightarrow{\cong} & \mathrm{colim}_i Z_i
\end{array}$$

The morphism $X \times_Z Y \to Q \times_Z Y$ is such that when composed with an epimorphism gives a regular epimorphism. Hence it is itself a regular epimorphism and so is its composition with the isomorphism $Q \times_Z Y \to Y$. This proves the stability of regular epimorphisms under pullback. □

Our intention is to apply the above to the category of compactly generated Hausdorff locales. We have seen that in that category, under the coreflectivity hypothesis, the colimit of the pullbacks of the compact sublocales of a compactly generated one along any morphism, maps epimorphically to the pullback of the locale along that morphism. We need epimorphicity of the comparison between the colimit of pullbacks of compact sublocales to the pullback of the colimits in order to use the above. To that end recall that when $\mathcal{I} \to \mathrm{Loc}$ is a directed system of inclusions of locales one has that the morphisms $X_i \to \mathrm{colim}_i X_i$ are also inclusions. More precisely, stated as a result about sup-lattices, following result appears in the proof of [9] Proposition I.2 and gives the corresponding result for frames.

Lemma 3.3. *Let $(t_{ij}\colon A_i \to A_j)$ be an inverse directed diagram in the category of sup-lattices, such that all the transition maps t_{ij} are surjective. Then the projections*

$$p_i\colon \lim_i A_i \to A_i$$

are also surjective.

Lemma 3.4. *Assume that a category $\mathcal{K}$ has the property that for a monomorphic directed diagram $X\colon \mathcal{I} \to \mathcal{K}$ over Z and a morphism $Y \to Z$, the canonical map*

$$\mathrm{colim}_i(X_i \times_Z Y) \to (\mathrm{colim}_i X_i) \times_Z Y$$

is a regular epimorphism. Then for a monomorphic directed system $X_i \to Z_i \leftarrow Y_i$ indexed by $\mathcal{I}$ the canonical comparison

$$\mathrm{colim}_i(X_i \times_{Z_i} Y_i) \to \mathrm{colim}_i X_i \times_{\mathrm{colim}_i Z_i} \mathrm{colim}_i Y_i$$

is a regular epimorphism.

Proof. Consider the pullback of the diagram

$$\begin{array}{ccc} & & \mathrm{colim}_i Y_i \\ & & \downarrow \\ \mathrm{colim}_i X_i & \longrightarrow & \mathrm{colim}_i Z_i \end{array}$$

Then

$$\mathrm{colim}_i \mathrm{colim}_{i'}(X_i \times_{\mathrm{colim}_i Z_i} Y_{i'}) \;\cong\; \mathrm{colim}_i(X_i \times_{\mathrm{colim}_i Z_i} Y_i),$$

by directedness of $\mathcal{I}$ and we have regular epimorphisms

$$\mathrm{colim}_i \mathrm{colim}_{i'}(X_i \times_{\mathrm{colim}_i Z_i} Y_{i'}) \to \mathrm{colim}_i \; (X_i \times_{\mathrm{colim}_i Z_i} \mathrm{colim}_i Y_i)$$

$$\to \mathrm{colim}_i X_i \times_{\mathrm{colim}_i Z_i} \mathrm{colim}_i Y_i$$

by our assumption. Finally, since each $Z_i \to \mathrm{colim}_i Z_i$ is monomorphism

$$X_i \times_{\mathrm{colim}_i Z_i} Y_i \cong X_i \times_{Z_i} Y_i$$

as the following diagram of pullbacks indicates

$$\begin{array}{ccccc} X_i \times_{\mathrm{colim}_i Z_i} Y_i & \longrightarrow & Y_i & \longrightarrow & Y_i \\ \downarrow & & \downarrow & & \downarrow \\ X_i & \longrightarrow & Z_i & \xrightarrow{id} & Z_i \\ \downarrow & & \downarrow{\scriptstyle id} & & \downarrow \\ X_i & \longrightarrow & Z_i & \longrightarrow & \mathrm{colim}_i Z_i \end{array}$$

□

In view of the above Proposition and the previous lemma we get

Theorem 3.5. *Under the coreflectivity hypothesis, namely that it is coreflective in the category of Hausdorff locales, the category of compactly generated Hausdorff locales is regular.*

Proof. Apply Proposition 3.2 for $\mathcal{K}$ the category of compactly generated Hausdorff locales, $\mathcal{C}$ the category of compact Hausdorff locales, which is regular by [11] 3.6.3. Recall that it is also closed under finite colimits inside the former. A finite coproduct of compact Hausdorff locales is obviously compact Hausdorff, while in the coequalizer $q: X \to Q$ in CGHLoc of a pair of maps to a compact locale X, Q is the directed colimit of its compact sublocales. But $q[X]$ is one of them and it equals Q. □

4. Effectivity of the category of compact Hausdorff locales

Recall that a locale X is regular if every element of its frame of opens is the supremum of all the elements of the frame that are well inside it. An element of a frame U is well inside V, written $U \leqslant V$, if there exists a W such that $U \wedge W = 0$ and $W \vee V = X$. Recall also that a locale is compact Hausdorff iff it is compact regular (a result due to [12], see also [11], Theorem 3.4.2, for a different proof). A surjective map of locales is one where the inverse image of the corresponding map between the respective frames reflects order.

Proposition 4.1. *The image of a compact locale by a surjection is compact.*

Proof. Let $q: X \to Q$ be a surjection of locales, $q^*: OQ \to OX$ its inverse image and assume that $Q = \bigvee U_i$, where the union is directed. Then

$$X = q^*Q = q^*(\bigvee U_i) = \bigvee q^*U_i$$

hence there is an i such that $X = q^*U_i$. It follows that

$$Q = q_*q^*U_i = U_i,$$

where the last equation follows by the fact that q^* reflects order. □

The following is Proposition 2 in [6]:

Proposition 4.2. *The image of a regular locale by a proper surjection is regular.*

Proof. For a proper surjection $q\colon X \to Q$ with X is regular we have that for every $V \in OQ$

$$q^*V = \bigvee\{U \in OX \mid U \preccurlyeq q^*V\}$$

from which we get

$$V = q_*q^*V = q_*(\bigvee\{U \in OX \mid U \preccurlyeq q^*V\}) = \bigvee\{q_*U \in OX \mid U \preccurlyeq q^*V\}$$

since the involved supremum is directed hence preserved by q_* ([7], III 1.1). Now $U \preccurlyeq q^*V$ implies $q_*U \preccurlyeq V$ because if $W \in OX$ is a witness for the first relation, i.e we have

$$U \wedge W = 0 \qquad \text{and} \qquad q^*V \vee W = X$$

then

$$q_*U \wedge q_*W = q_*0 = 0$$

(the latter because $Z \leq q_*0$ iff $q^*Z \leq 0 = q^*0$ and q^* reflects $\leq$) and also

$$Q = q_*X = q_*(q^*V \vee W) = V \vee q_*W$$

by properness of q. We conclude that q_*W is a witness for $q_*U \preccurlyeq V$, hence

$$V = \bigvee\{q_*U \in OQ \mid U \preccurlyeq q^*V\} \leq \bigvee\{q_*U \in OQ \mid q_*U \preccurlyeq V\}.$$

□

Theorem 4.3. *The category of compact Hausdorff locales is effective regular (= Barr-exact).*

Proof. First of all the category CHausLoc of compact Hausdorff locales is regular by [11] 3.6.3. Equivalence relations in this category are proper and closed, as every map between compact Hausdorff locales is proper. We know from [13] 5.17 that closed, proper equivalence relations on compact locales are effective, so they are the kernel pairs of their coequalizers in the category of locales. But the coequalizer of a proper equivalence relation is proper by [13] 5.5. Hence the coequalizer in the category of locales of a (proper as it will be) equivalence relation between compact regular locales is compact regular, by the above two propositions. Since limits in the category in question are constructed as in the category of locales, we conclude that every equivalence relation in CHausLoc, being proper, is the kernel pair of its coequalizer in the category of locales, which lives in CHausLoc. □

Theorem 4.4. *The category of compact Hausdorff locales is a pretopos.*

Proof. We know from [9], Proposition IV.4.1, that coproducts in the category of locales are universal. They are also disjoint since the pullback $X \times_{X+Y} Y$ of the injections of two locales into their coproduct is given by the frame $OX \bigotimes_{OX \times OY} OY$ that occurs as the tensor product in preframes of the corresponding frames over their product. Writing p_1, p_2 for the projections of the product, an element $a \otimes b$ in the latter is then

$$a \otimes b = (1 \wedge p_1(a, b)) \otimes b = 1 \otimes (p_2(a, b) \wedge b) = 1 \otimes b$$

$$= 1 \otimes (p_2(1, b) \wedge 1) = (1 \wedge p_1(1, b)) \otimes 1 = 1 \otimes 1$$

so the pullback is trivial.

Finite coproducts of compact Hausdorff locales are obviously compact Hausdorff themselves and, moreover, the category $\mathrm{CHausLoc}$ is closed in the category of locales under finite limits (in particular under pullbacks) [11], Lemma 3.6.3. Hence $\mathrm{CHausLoc}$ inherits from the category of locales the universality and disjointness of finite coproducts. □

Remark: The category of compact Hausdorff spaces admits a characterization as the unique, up to equivalence, non-trivial, well-pointed, filtral pretopos with set-indexed copowers of its terminal object [10]. As the referee suggested, it would be interesting to know if the category of compact Hausdorff locales admits a similar "pointless" characterization.

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Commission paritaire n° 58964
ISSN 1245-530X (IMPRIME)
1SSN 2681-2363 (EN LIGNE)

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