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THE AMPLE CLOSURE OF THE CATEGORY OF LOCALLY COMPACT ABELIAN GROUPS

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Dedicated to B. V. M.

Résumé. La catégorie des groupes abéliens localement compactes est une catégorie de Morita, catégorie quasi-abélienne avec une représentation particulière de ses objets au moyen d'une paire intrinsèque de sous-catégories abeliennes de Serre. Pour toute catégorie de Morita, un plongement dans une plus grande catégorie de Morita, la clôture ample, est construite, de sorte que les deux sous-catégories de Serre ne sont pas modifiées. Dans le cas des groupes LCA, ces sous-catégories coïncident respectivement avec la classe des groupes discrets et des groupes compacts. La clôture ample est équivalente à la catégorie de groupes abéliens de Hausdorff totalement bornés, une catégorie tenseur complet et cocomplet, avec une dualité unique prolongeant la dualité de Pontryagin. Cinq différentes caractérisations sont données pour cette catégorie.

Abstract. The category of locally compact abelian groups is shown to be a Morita category, a quasi-abelian category with a particular representation of its objects by means of an intrinsic pair of abelian Serre subcategories. For any Morita category, an embedding into a largest Morita category, the ample closure, is constructed, so that the two Serre subcategories are not changed. In the case of LCA groups, these subcategories coincide with the class of discrete and compact groups, respectively. The ample closure is shown to be equivalent to the category of totally bounded Hausdorff abelian groups, a complete and cocomplete tensor category, with a unique duality extending Pontryagin duality. Five characterizations are given for this category.

Keywords. LCA group, duality, totally bounded abelian group, quasi-abelian category.
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1. Introduction

Extensions of classical Pontryagin duality have been proposed by many authors (see Section 5). First attempts focussed upon closure properties of the category **LCA** of locally compact abelian groups within the category **HAb** of all Hausdorff abelian groups. Kaplan [48] proved that the groups in **HAb** which are Pontryagin-reflexive (=P-reflexive for short) are closed with respect to products, and that inverse limits of sequences of LCA groups are P-reflexive [49]. Freundlich-Smith [35] proved that the additive group of a real Banach space or a reflexive locally convex space is P-reflexive. Further positive results are given in [25]. On the other hand, the category **LCA** is neither complete nor cocomplete [53, 45], and cannot be made into a closed (symmetric monoidal) category [54] (see [57], Theorem 4.3; [12]; [44], Remark 3.15).

In view of this lack of completeness properties, it soon became clear that the compact-open topology has to be modified. Binz [16, 17, 18] suggested to consider groups with an underlying convergence space [34, 19, 30], using the fact that convergence spaces form a cartesian closed category ([19], Satz 5) if the morphism sets are endowed with the continuous convergence structure. The full subcategory of reflexive abelian convergence groups is closed with respect to products and coproducts ([23], Theorem 2.4), but neither complete nor cocomplete [14].

In this paper, we give a self-contained approach to the concept of *Morita category* [64], the rationale behind Morita duality, and apply it to the category of LCA groups as a typical example of a Morita category (Proposition 3.2). Every Morita category is quasi-abelian (Proposition 3.4). One of the widely ignored features of a *quasi-abelian* [66] category $\mathscr{A}$ is the existence of abelian Serre subcategories $\mathscr{A}_\circ$ and $\mathscr{A}^\circ$ ([64], Proposition 4.3), where $\mathscr{A}_\circ$ consists of the objects into which every monomorphism is a kernel. For a Morita category $\mathscr{A}$, these subcategories give rise to a canonical pre-factorization (Proposition 3.3) into classes $\mathscr{E}$ and $\mathscr{M}$ of epimorphisms and monomorphisms, respectively. We call $\mathscr{A}$ *ample* if $(\mathscr{E},\mathscr{M})$ is a factorization system [36], that is, every morphism $f \in \mathscr{A}$ has a factorization $f = me$ with $m \in \mathscr{M}$ and $e \in \mathscr{E}$. Our first main result states that every Morita category $\mathscr{A}$ embeds into a maximal Morita category $\widetilde{\mathscr{A}}$ so that the subcategories $\mathscr{A}_\circ$ and $\mathscr{A}^\circ$ are not altered by the embedding (Theorem 4.2). The category $\widetilde{\mathscr{A}}$ is unique, up to equivalence, and characterized by the property to be an ample Morita category (Corollary 2). We call it the *ample closure* of $\mathscr{A}$.

For the Morita category $\mathscr{A} = \mathbf{LCA}$ of LCA groups, $\mathscr{A}_\circ$ consists of the discrete groups, while $\mathscr{A}^\circ$ consists of the compact groups. (Incidentally, this shows that these subcategories are categorical invariants.) Removing the deficiencies of **LCA**, the ample closure is a complete and cocomplete closed category with a unique duality, extending Pontryagin duality (Theorem 5.1). So there are internal hom-objects with an adjoint tensor product. The dual of an object is obtained via the circle group R/Z, as in **LCA**. In particular, this gives a very simple proof of Pontryagin duality for LCA groups and its uniqueness [56, 63]. Indeed, any duality takes $\mathscr{A}_\circ$ to $\mathscr{A}^\circ$. Since $\mathscr{A}_\circ \approx \mathbf{Mod}(\mathbb{Z})$ has no non-identical self-equivalence, uniqueness of the duality follows by the structure of a Morita category.

In Section 2, we provide five different realizations of the ample closure of **LCA**. The first one is inspired by the *dual systems* [33] in functional analysis, which can be adapted to produce a large class of quasi-abelian categories ([64], Section 2, Example 4). Barr used them for the construction of $*$-autonomous categories [11, 12, 13]. The ample closure of **LCA** is equivalent to the category chu in [13], a reduced version of the so-called Chu-construction [12]. It is also equivalent to the category **TBA** of totally bounded Hausdorff abelian groups with homomorphism groups endowed with the weak topology. Totally bounded abelian groups have been studied intensively in connection with Prontryagin duality (e. g., [67, 29, 24, 41, 42, 26, 3, 38]). Theorem 5.1 now shows that the full embedding $\mathbf{LCA} \hookrightarrow \mathbf{TBA}$ hinges entirely on the above mentioned invariant Serre subcategories $\mathscr{A}_\circ$ and $\mathscr{A}^\circ$, and that **TBA** can be identified as the ample closure of **LCA**. Further incarnations of **TBA** are given in Propositions 2.1-2.2 and a subsequent remark.

For any Morita category $\mathscr{A}$, there is also a smallest full subcategory which leaves the Serre subcategories $\mathscr{A}_\circ$ and $\mathscr{A}^\circ$ unaltered. It is again a Morita category. For $\mathscr{A} = \mathbf{LCA}$ this category consists of the LCA groups which admit an open compact subgroup (Proposition 3.6). Equivalently, these LCA groups don't have R as a direct summand.

Since compactly generated Hausdorff spaces (also called Hausdorff k-*spaces* [50]) form a complete and cocomplete cartesian closed category [54], it was natural to study topological groups with an underlying k-space [58]. Glicksberg's theorem [40] implies that LCA groups are of that type. It states that the topology of an LCA group is determined by its compact subgroups, hence by the associated totally bounded group. A misconception of the relationship between k-spaces and duality led to an inadequate characterization of P-reflexive groups [71], and the belief that Glicksberg's theorem would hold for all these groups [70]. The latter was corrected by Remus and Trigos-Arrieta [62] who were thus led to study the category **PKAb** of P-reflexive groups satisfying Glicksberg's property.

The category of all P-reflexive Hausdorff abelian groups, correctly characterized by Hernández [41], has no better closure properties than **LCA**, which shows the inadequacy of the compact-open topology beyond LCA groups. We prove that the category **PKAb** of Remus and Trigos-Arrieta [62] admits a full embedding into the ample closure of **LCA** which cannot be extended to a bigger category of P-reflexive groups (Theorem 5.2).

2. Totally bounded abelian groups

A Hausdorff topological abelian group A is said to be *totally bounded* [73, 28] if for any neighbourhood U of 0 there is a finite set $F \subset A$ with $A = \bigcup_{a\in F} a + U$. Equivalently, A is totally bounded if and only if its completion is compact. With continuous homomorphisms as morphisms, the category **TBA** of totally bounded Hausdorff abelian groups is a full subcategory of the category **HAb** of Hausdorff topological abelian groups. For any $A \in$ **HAb** we write A_d for the underlying discrete abelian group.

There are several ways to describe the objects of **TBA**. Firstly, let $\mathsf{R} \in$ **HAb** denote the additive group of reals, and Z the subgroup of integers. So $\mathsf{T} := \mathsf{R}/\mathsf{Z}$ is a compact abelian group. We define a *dual system* of abelian groups to be a biadditive map

$$\beta\colon A \times B \to \mathsf{T}_d \tag{1}$$

of abelian groups A, B which is *non-degenerate* in the sense that the associated homomorphisms $\beta_\ell\colon A \to \mathrm{Hom}(B, \mathsf{T}_d)$ and $\beta_r\colon B \to \mathrm{Hom}(A, \mathsf{T}_d)$ are injective. For dual systems $\beta\colon A \times B \to \mathsf{T}_d$ and $\beta'\colon A' \times B' \to \mathsf{T}_d$, a *morphism* $\beta \to \beta'$ is given by a pair of homomorphisms $f\colon A \to A'$ and $g\colon B' \to B$ such that $\beta'(f(a), b') = \beta(a, g(b'))$ holds for $a \in A$ and $b' \in B'$, that is, the commutative diagram

$$\begin{array}{ccc} A \times B' & \xrightarrow{1 \times g} & A \times B \\ \Big\downarrow{\scriptstyle f \times 1} & & \Big\downarrow{\scriptstyle \beta} \\ A' \times B' & \xrightarrow{\beta'} & \mathsf{T}_d \end{array} \tag{2}$$

commutes. Note that dual systems of topological vector spaces, introduced by Dieudonné [33], play an important role in functional analysis (see [65], Chapter IV). To see that dual systems form a category (denoted as $\mathbf{DS}(\mathbf{Ab})$), we replace (1) by its adjoint map $\beta_\ell\colon A \to \mathrm{Hom}(B, \mathsf{T}_d)$. Then (2) takes the form of a commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{\beta_\ell} & \mathrm{Hom}(B, \mathsf{T}_d) \\ \Big\downarrow{\scriptstyle f} & & \Big\downarrow{\scriptstyle \mathrm{Hom}(g, \mathsf{T}_d)} \\ A' & \xrightarrow{\beta'_\ell} & \mathrm{Hom}(B', \mathsf{T}_d). \end{array} \tag{3}$$

By Pontryagin duality, the functor $\mathrm{Hom}(-, \mathsf{T})$ gives an equivalence between the category **Ab** of abelian groups and the full subcategory **CA** of compact abelian groups in **HAb**. Let **DCA** be the subcategory of **Ab** $\times$ **CA** given by the objects (A, C) where A is a dense subgroup of C. Then the above discussion yields

Proposition 2.1. *The categories* **TBA** *(totally bounded ab. groups),* $\mathbf{DS}(\mathbf{Ab})$ *(dual systems), and* **DCA** *(dense subgroups of compact abelian groups) are equivalent.*

Proof. Consider a dual system (1) and the corresponding adjoint map β_ℓ. The non-degeneracy of β says that $\beta_\ell\colon A \to \mathrm{Hom}(B, \mathsf{T}_d)$ and $\beta_r\colon B \to$

$\mathrm{Hom}(A, \mathsf{T}_d)$ are injective. For β_r this means that the map which carries a homomorphism $g\colon \mathsf{Z} \to B$ to the composed homomorphism

$$A \xrightarrow{\beta_\ell} \mathrm{Hom}(B, \mathsf{T}_d) \xrightarrow{\mathrm{Hom}(g, \mathsf{T}_d)} \mathrm{Hom}(\mathsf{Z}, \mathsf{T}_d)$$

is injective. Since $\mathrm{Hom}(\mathsf{Z}, \mathsf{T}) = \mathsf{T}$ is a cogenerator of **CA**, the condition states that the embedding $\beta_\ell\colon A \to \mathrm{Hom}(B, \mathsf{T}_d)$ is dense, where $C := \mathrm{Hom}(B, \mathsf{T}_d)$ is viewed as a compact abelian group with Pontryagin dual B. So the morphisms (3) coincide with the corresponding morphisms in **DCA**, which proves that $\mathbf{DS}(\mathbf{Ab}) \approx \mathbf{DCA}$.

Now let $A \hookrightarrow C$ be an object of **DCA**, that is, A is a dense subgroup of the compact abelian group C. If we endow A with the induced topology from C, then A becomes a totally bounded abelian group. Conversely, each object $A \in \mathbf{TBA}$ gives rise to a dense embedding $A \hookrightarrow C$ into its compact completion C. Since every morphism $A \to B$ of totally bounded abelian groups extends uniquely to the completions, this gives the equivalence $\mathbf{TBA} \approx \mathbf{DCA}$. □

There is a fourth description of the category **TBA**. Let **ChA** be the category of abelian groups A together with a distinguished set $X \subset \mathrm{Hom}(A, \mathsf{T}_d)$ of characters which *separate points* in A, that is, the canonical map $A \to \mathsf{T}_d^X$ is injective. Of course, X separates points if and only if the subgroup of $\mathrm{Hom}(A, \mathsf{T}_d)$ generated by X separates points. So we can assume without loss of generality that X is a subgroup X_A of $\mathrm{Hom}(A, \mathsf{T}_d)$. By [28], Theorem 1.9 (cf. the argument in the above proof), point separation thus means that X_A is dense in $\mathrm{Hom}(A, \mathsf{T}_d)$. Morphisms in **ChA** are group homomorphisms $f\colon A \to B$ such that every character $\chi \in X_A$ factors through f. In what follows, we write $\mathit{Hom}(A, B)$ for the group of continuous homomorphisms between topological abelian groups.

Proposition 2.2. *The categories of Proposition 2.1 are equivalent to* **ChA**.

Proof. Let $i\colon A \hookrightarrow C$ be an object of **DCA**. Define X_A to be the image of

$$\mathit{Hom}(i, \mathsf{T})\colon \mathit{Hom}(C, \mathsf{T}) \to \mathit{Hom}(A, \mathsf{T}).$$

Thus X_A makes A into an object of **ChA**. Conversely, every object $A \in$ **ChA** gives rise to an embedding $A \hookrightarrow \mathrm{Hom}(X_A, \mathsf{T}_d)$ which maps $a \in A$ to $\chi \mapsto \chi(a)$. Thus $C := \mathrm{Hom}(X_A, \mathsf{T}_d)$ is a compact abelian group with character group *Hom*$(C, \mathsf{T}) \cong X_A$, and there is no non-zero character of C which annihilates A. Hence $A \subset C$ is dense. Thus **DCA** $\approx$ **ChA**. □

Remark. Alfsen and Fenstad [1] established an equivalence between totally bounded uniform structures and proximity spaces. Accordingly, there is a fifth description of **TBA** as a category of abelian groups with a compatible proximity structure. We leave it to the reader to carry this out.

Next we show that the full subcategory **LCA** of locally compact abelian groups (LCA groups for short) in **HAb** admits a full embedding into **TBA**. For any $A \in$ **LCA** there is a dense embedding $A \hookrightarrow \mathfrak{b}A$ into the Bohr compactification $\mathfrak{b}A := \mathrm{Hom}(\mathit{Hom}(A, \mathsf{T})_d, \mathsf{T})$ of A. By A^+ we denote the group A with the induced topology of $\mathfrak{b}A$. The following result is due to Trigos-Arrieta [67]. For convenience, we give a short proof.

Proposition 2.3. *The functor* $A \mapsto A^+$ *gives a full embedding* **LCA** $\hookrightarrow$ **TBA**.

Proof. For $A \in$ **LCA**, the group A^+ is totally bounded. So $A \mapsto A^+$ gives a faithful functor **LCA** $\hookrightarrow$ **TBA**. To show that it is full, let $f\colon A^+ \to B^+$ be a morphism in **TBA** with $A, B \in$ **LCA**. Then f extends uniquely to a morphism $f'\colon \mathfrak{b}A \to \mathfrak{b}B$ in **CA**. Let V be a neighbourhood of 0 in B. For any compact 0-neighbourhood K in A, the set $f(K) = f'(K)$ is compact in $\mathfrak{b}B$. By Glicksberg's theorem [40], $f(K)$ is compact in B. Hence $V \cap f(K)$ is a 0-neighbourhood in $f(K)$. Since $f'|_K$ is continuous, $f^{-1}(V) \cap K = f^{-1}(V \cap f(K)) \cap K$ is a 0-neighbourhood in A. Thus $A \mapsto A^+$ is full. □

3. The Morita category of LCA groups

Recall that an additive category is said to be *preabelian* [60] if it has kernels and cokernels. For a preabelian category $\mathscr{A}$, we call a sequence

$$A_0 \xrightarrow{a} A_1 \xrightarrow{b} A_2 \tag{4}$$

of morphisms *short exact* if $a = \ker b$ and $b = \operatorname{cok} a$. As usual, we depict kernels by tailed arrows $A_0 \rightarrowtail A_1$ and cokernels by two-head arrows $A_1 \twoheadrightarrow A_2$. A full subcategory $\mathscr{S}$ of $\mathscr{A}$ is said to be a *Serre subcategory* if for every short exact sequence (4), the middle term A_1 belongs to $\mathscr{S}$ if and only if the end terms A_0, A_2 belong to $\mathscr{S}$. If cokernels are stable under pullback and kernels are stable under pushout, $\mathscr{A}$ is said to be *quasi-abelian* [66]. By [64], Proposition 1 and Corollary 1, this implies that the short exact sequences form an exact structure in the sense of Quillen [61]. In [64], Section 8, we have shown that the category **LCA** is quasi-abelian.

For a preabelian category $\mathscr{A}$, let $\mathscr{A}_\circ$ denote the full subcategory of objects $A \in \mathscr{A}$ such that every monomorphism $A' \to A$ is a kernel. Similarly, we define $\mathscr{A}^\circ$ to be the full subcategory of objects $A \in \mathscr{A}$ such that every epimorphism $A \to A'$ is a cokernel. We call an object $P \in \mathscr{A}$ *projective* if for every short exact sequence (4), any morphism $P \to A_2$ factors through b. Similarly, $I \in \mathscr{A}$ is *injective* if each morphism $A_0 \to I$ factors through a. Let $\mathsf{S}_\circ\mathscr{A}$ denote the full subcategory of objects $P \in \mathscr{A}_\circ$ which are projective in $\mathscr{A}$, and let $\mathsf{S}^\circ\mathscr{A}$ be the full subcategory of objects $I \in \mathscr{A}^\circ$ which are injective in $\mathscr{A}$. We say that $f\colon A \to B$ is a $\circ$-*epimorphism* if every morphism $P \to B$ with $P \in \mathsf{S}_\circ\mathscr{A}$ factors through f. Similarly, f is $\circ$-*monic* if every morphism $A \to I$ with $I \in \mathsf{S}^\circ\mathscr{A}$ factors through f.

For example, $\mathbf{LCA}_\circ \approx \mathbf{Ab}$ and $\mathbf{LCA}^\circ \approx \mathbf{CA}$. Indeed, any object $A \in$ **LCA** determines a sequence of morphisms

$$\mathsf{Z}^{(H)} \xrightarrow{p} A \xrightarrow{i} \mathsf{T}^{H'} \tag{5}$$

with $H := \mathrm{Hom}(\mathsf{Z}, A)$ and $H' := \mathit{Hom}(A, \mathsf{T})$. The coimage $\mathrm{cok}(\ker p)$ of p is A_d, while the image $\ker(\mathrm{cok}\, i)$ of i is the Bohr compactification $\mathfrak{b}A = \overline{i(A)}$. Hence $A \in \mathbf{LCA}_\circ$ if and only if $A = A_d$ and $A \in \mathbf{LCA}^\circ$ if and only if A is a compact group. The objects in $\mathsf{S}_\circ\mathbf{LCA}$ are the free abelian groups $\mathsf{Z}^{(\kappa)}$, while $\mathsf{S}^\circ\mathbf{LCA}$ consists of the cofree compact groups T^κ. Besides these projectives and injectives in **LCA** there are only the vector groups R^n which are projective and injective. The following concept was introduced in [64]. Here we give a different formulation without using PI-varieties.

Definition 3.1. We define a *Morita category* to be a preabelian category with the following properties:

(a) Each object $A \in \mathscr{A}$ admits a $\circ$-epimorphism $P \to A$ and a $\circ$-monomorphism $A \to I$ with $P \in \mathsf{S}_\circ\mathscr{A}$ and $I \in \mathsf{S}^\circ\mathscr{A}$.

(b) Any $\circ$-epic $\circ$-monomorphism is invertible.

Proposition 3.2. *The category* **LCA** *is a Morita category.*

Proof. Condition (a) follows immediately by (5). Condition (b) follows by a theorem of Kaplansky and Glicksberg ([40], Corollary 2.4). □

A morphism $e\colon A \to A'$ in an arbitrary category is said to be *left orthogonal* to $m\colon B \to B'$ (and m is said to be *right orthogonal* to e) if for each commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \downarrow{\scriptstyle e} & & \downarrow{\scriptstyle m} \\ A' & \xrightarrow{f'} & B' \end{array} \tag{6}$$

there exists a morphism $h\colon A' \to B$ with $he = f$ and $mh = f'$. The crucial condition (b) of Definition 3.1 implies:

Proposition 3.3. *For a Morita category $\mathscr{A}$, the $\circ$-epimorphisms are left orthogonal to the $\circ$-monomorphisms.*

Proof. Let (6) be a commutative diagram such that e is $\circ$-epic and m is $\circ$-monic. The pushout

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \downarrow{\scriptstyle e} & & \downarrow{\scriptstyle p} \\ A' & \xrightarrow{g} & C \end{array}$$

gives rise to a morphism $h\colon C \to B'$ with $hg = f'$ and $hp = m$. Since $(g\ p)\colon A' \oplus B \to C$ is a cokernel, any morphism $c\colon P \to C$ with $P \in \mathsf{S}_\circ\mathscr{A}$ factors through $(g\ p)$. So there are morphisms $u\colon P \to A'$ and $v\colon P \to B$ with $c = gu + pv$. Since u factors through e, this implies that c factors through p. Thus p is $\circ$-epic. Since m factors through p, we infer that p is also a $\circ$-monomorphism. Thus p is invertible, so that $p^{-1}g$ gives the desired diagonal in (6). □

Another consequence of Definition 3.1(b) is the following

Proposition 3.4. *Every Morita category $\mathscr{A}$ is quasi-abelian. In particular, $\mathscr{A}_\circ$ and $\mathscr{A}^\circ$ are abelian Serre subcategories of $\mathscr{A}$.*

Proof. Let (6) be a pullback with a cokernel f'. With $k := \ker f$, this implies that $ek = \ker f'$ and $f' = \operatorname{cok} ek$. Let $c\colon A \twoheadrightarrow C$ be the cokernel of k. Then $f = rc$ for some $r\colon C \to B$. Since f' is $\circ$-epic, it follows that r is

$\circ$-epic. On the other hand, let $i\colon C \to I$ be a morphism with $I \in \mathsf{S}^\circ\mathscr{A}$. Since $\binom{e}{f}$ is a kernel, ic factors through $\binom{e}{f}$. So there are morphisms $u\colon A' \to I$ and $v\colon B \to I$ with $ic = ue + vf$. Hence $uek = 0$, which yields a morphism $u'\colon B' \to I$ with $u = u'f'$. Thus $ic = u'f'e + vf = u'mf + vf$, which yields $i = (u'm + v)r$. So r is $\circ$-monic and $\circ$-epic, hence invertible. By symmetry, this implies that $\mathscr{A}$ is quasi-abelian. The assertions on $\mathscr{A}_\circ$ and $\mathscr{A}^\circ$ hold by [64], Proposition 8. $\square$

Corollary 1. *Let $\mathscr{A}$ be a Morita category. Every $\circ$-epimorphism in $\mathscr{A}$ is epic, and every $\circ$-monomorphism is monic.*

Proof. By [64], Corollary 1 of Proposition 1, every morphism $f\colon A \to B$ in $\mathscr{A}$ has a factorization $f = ie$ into an epimorphism e and a kernel i. Assume that f is $\circ$-epic. Then i is $\circ$-epic and $\circ$-monic, hence invertible. So f is epic. By symmetry, this proves the corollary. $\square$

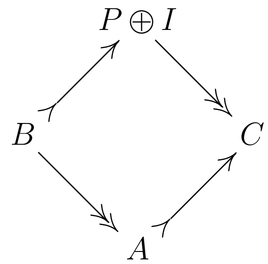
If an abelian category $\mathscr{A}$ has enough projectives, the projective objects form a full subcategory $\mathscr{P}$ with $\mathscr{A} \approx \mathbf{mod}(\mathscr{P})$, see [64], Section 3 ([4], III.1) for the definition of $\mathbf{mod}(\mathscr{P})$. If $\mathscr{P}$ is skeletally small, $\mathbf{mod}(\mathscr{P})$ can be identified with the category of finitely presented additive functors $\mathscr{P}^{\mathrm{op}} \to \mathbf{Ab}$. Similarly, if $\mathscr{A}$ has enough injectives, making up a full subcategory $\mathscr{I}$, we have $\mathscr{A} \approx \mathbf{com}(\mathscr{I}) := \mathbf{mod}(\mathscr{I}^{\mathrm{op}})^{\mathrm{op}}$.

Corollary 2. *Let $\mathscr{A}$ be a Morita category. Then $\mathscr{A}_\circ \approx \mathbf{mod}(\mathsf{S}_\circ\mathscr{A})$ and $\mathscr{A}^\circ \approx \mathbf{com}(\mathsf{S}^\circ\mathscr{A})$.*

Proof. Since $\mathscr{A}_\circ$ is an abelian Serre subcategory of $\mathscr{A}$, every short exact sequence (4) in $\mathscr{A}_\circ$ is short exact in $\mathscr{A}$. Hence $\mathsf{S}_\circ\mathscr{A}$ consists of projective objects in $\mathscr{A}_\circ$. By Corollary 1, the objects $P \in \mathsf{S}_\circ\mathscr{A}$ provide enough projective objects in $\mathscr{A}_\circ$. By symmetry, this proves the claim. $\square$

Definition 3.5. Let $\mathscr{C}$ be a full subcategory of a Morita category $\mathscr{A}$. We say that $\mathscr{C} \hookrightarrow \mathscr{A}$ is a $\circ$-*embedding* and that $\mathscr{C}$ is a $\circ$-*subcategory* of $\mathscr{A}$ if $\mathscr{C}$ is closed with respect to kernels and cokernels such that $\mathscr{C}_\circ = \mathscr{A}_\circ$ and $\mathscr{C}^\circ = \mathscr{A}^\circ$.

Thus any $\circ$-subcategory $\mathscr{C}$ of a Morita category $\mathscr{A}$ is a Morita category. If $i\colon C \rightarrowtail A$ is a kernel of a morphism $f\colon A \to B$ in $\mathscr{A}$ with $A \in \mathscr{C}$, there is a monomorphism $j\colon B \to I$ with $I \in \mathsf{S}^\circ\mathscr{A}$. Hence $i = \ker jf$, and thus $C \in \mathscr{C}$. By [64], Proposition 1, it follows that any Morita category $\mathscr{A}$ admits a smallest $\circ$-subcategory $\underset{\sim}{\mathscr{A}}$, consisting of the subquotients of objects $P \oplus I$ with $P \in \mathsf{S}_\circ\mathscr{A}$ and $I \in \mathsf{S}^\circ\mathscr{A}$ (cf. [64], Proposition 21). By a *subquotient* of $C \in \mathscr{A}$ we mean an object S which admits a kernel $S \rightarrowtail A$ and a cokernel $C \twoheadrightarrow A$ (equivalently, a cokernel $B \twoheadrightarrow S$ and a kernel $B \rightarrowtail C$). So the objects A of $\underset{\sim}{\mathscr{A}}$ are related to pairs $P \in \mathsf{S}_\circ\mathscr{A}$ and $I \in \mathsf{S}^\circ\mathscr{A}$ as follows:



In the category of LCA groups, this subcategory is of special importance. It consists of the groups for which the projective-injective object R does not occur as a direct summand:

Proposition 3.6. *The smallest $\circ$-subcategory of* **LCA** *consists of the locally compact abelian groups which admit an open compact subgroup.*

Proof. By [43], Theorem 24.30, every LCA group is of the form $\mathsf{R}^n \oplus A$ such that A admits an open compact subgroup. As R is projective and injective, it can be removed from **LCA** to get a $\circ$-subcategory $\mathscr{A}$ which consists of the objects A with an open compact subgroup C. In other words, A admits a short exact sequence

$$C \overset{i}{\rightarrowtail} A \overset{p}{\twoheadrightarrow} D$$

with a discrete abelian group D. So there exists a kernel $j\colon C \rightarrowtail \mathsf{T}^\kappa$ for some cardinal κ. The pushout of i along j gives a split short exact sequence $\mathsf{T}^\kappa \rightarrowtail \mathsf{T}^\kappa \oplus D \twoheadrightarrow D$ and a kernel $A \rightarrowtail \mathsf{T}^\kappa \oplus D$. Thus A is a subquotient of an object of the form $\mathsf{T}^\kappa \oplus \mathsf{Z}^{(\lambda)}$. □

4. The ample closure

In [64], Morita categories are introduced as a special class of PI-categories, which implies that every Morita category $\mathscr{A}$ admits a largest Morita category with $\mathscr{A}$ as a $\circ$-subcategory. In this section, we give a direct and more symmetric construction of the *ample closure* of a Morita category, and determine it for the category of LCA groups.

Definition 4.1. A Morita category $\mathscr{A}$ is said to be *ample* [64] if every morphism $f \in \mathscr{A}$ admits a factorization $f = me$ with a $\circ$-epimorphism e and a $\circ$-monomorphism m.

Theorem 4.2. *Every Morita category $\mathscr{A}$ admits a $\circ$-embedding into an ample Morita category $\widetilde{\mathscr{A}}$ which is unique up to equivalence.*

Proof. Let $\mathscr{A}$ be a Morita category, and let $\mathscr{C}$ be the category of monic epimorphisms $r\colon D \to C$ in $\mathscr{A}$ with $D \in \mathscr{A}_\circ$ and $C \in \mathscr{A}^\circ$. Morphisms in $\mathscr{C}$ are commutative squares

$$\begin{array}{ccc} D & \xrightarrow{\;f\;} & D' \\ {\scriptstyle r}\downarrow & & \downarrow{\scriptstyle r'} \\ C & \xrightarrow{\;g\;} & C'. \end{array} \tag{7}$$

The kernel $r_0\colon D_0 \to C_0$ of such a morphism is obtained by taking the kernel $k\colon D_0 \rightarrowtail D$ of f in $\mathscr{A}_\circ$. Since $\mathscr{A}$ is quasi-abelian, the monomorphism rk admits a factorization $rk = \ell r_0$ with a kernel ℓ and a monic epimorphism $r_0\colon D_0 \to C_0$. By Proposition 3.4, the object C_0 belongs to $\mathscr{A}^\circ$. It is easily verified that the so constructed $r_0 \to r$ is a kernel of the morphism (7) in $\mathscr{C}$. Thus, by symmetry, $\mathscr{C}$ is preabelian.

To any $A \in \mathscr{A}$, there is a $\circ$-epimorphism $p\colon P \to A$ with $P \in \mathsf{S}_\circ\mathscr{A}$ and a $\circ$-monomorphism $i\colon A \to I$ with $I \in \mathsf{S}^\circ\mathscr{A}$. Since $\mathscr{A}$ is quasi-abelian, there are factorizations $p = r_A q$ and $i = jr^A$ with a cokernel q, a kernel j, and monic epimorphisms $r_A\colon A_\circ \to A$ and $r^A\colon A \to A^\circ$. By Proposition 3.4, $A_\circ \in \mathscr{A}_\circ$ and $A^\circ \in \mathscr{A}^\circ$. For any morphism $f\colon A \to B$ in $\mathscr{A}$, there is a morphism $f'\colon P \to B_\circ$ with $fp = r_B f'$ because r_B is $\circ$-epic. Since $r_B f'$ annihilates the kernel of q, it follows that f' annihilates the kernel of q. So f' factors through q, which shows that fr_A factors through r_B. By symmetry, this gives a faithful additive functor $\mathscr{A} \to \mathscr{C}$ which maps A to $r^A r_A$. Up to isomorphism, $r_A\colon A_\circ \to A$ is uniquely determined as a $\circ$-epic monomorphism with $A_\circ \in \mathscr{A}_\circ$. To show that the functor $\mathscr{A} \to \mathscr{C}$ is full, let a morphism $r^A r_A \to r^B r_B$ be given by a commutative diagram

$$\begin{array}{ccccc} A_\circ & \xrightarrow{r_A} & A & \xrightarrow{r^A} & A^\circ \\ \Big\downarrow f & & & & \Big\downarrow g \\ B_\circ & \xrightarrow{r_B} & B & \xrightarrow{r^B} & B^\circ \end{array}$$

By Proposition 3.3, r_A is left orthogonal to r^B. Hence there is a unique morphism $h\colon A \to B$ with $hr_A = r_B f$ and $r^B h = gr^A$. So we have a full embedding $\mathscr{A} \hookrightarrow \mathscr{C}$.

In particular, each object $r\colon D \to C$ in $\mathscr{C}$ gives rise to a commutative diagram

$$\begin{array}{ccccc} D & = & D & \longrightarrow & C_\circ \\ \Big\downarrow r^D & & \Big\downarrow r & & \Big\downarrow r_C \\ D^\circ & \longrightarrow & C & = & C \end{array} \tag{8}$$

where r^D and r_C can be viewed as objects of the full subcategories $\mathscr{A}_\circ$ and $\mathscr{A}^\circ$, respectively. Furthermore, the morphism $r^D \to r$ is monic and epic, which shows that $\mathscr{C}_\circ$ is equivalent to a full subcategory of $\mathscr{A}_\circ$. Conversely, let (7) be a monomorphism in $\mathscr{C}$ with $r' = r^{D'}$. Then f is monic in $\mathscr{A}$, hence a kernel. Therefore, $gr = r'f$ is $\circ$-monic, which shows that r is isomorphic to an object in $\mathscr{A}_\circ$. On the other hand, there is a factorization $gr = js$ with a kernel j and a monic epimorphism s. Thus $s\colon D \to C''$ is again $\circ$-monic, and $C'' \in \mathscr{A}^\circ$. Hence g is a kernel. Now let $D' \twoheadrightarrow D''$ be the cokernel of f. Then it follows immediately that r is the kernel of the induced morphism $r' \to r^{D''}$. Thus $\mathscr{C}_\circ \approx \mathscr{A}_\circ$. In particular, this shows that $\mathscr{A} \subset \mathscr{C}$ is closed with respect to kernels. By symmetry, $\mathscr{C}^\circ \approx \mathscr{A}^\circ$, and $\mathscr{A} \subset \mathscr{C}$ is closed with respect to cokernels.

So the diagram (8) implies that $\mathscr{C}$ satisfies condition (a) of Definition 3.1. (It is easily checked that $\mathsf{S}_\circ\mathscr{A}$ consists of the projective objects in $\mathscr{C}$.) To verify (b), consider a morphism (7) in $\mathscr{C}$ which is $\circ$-monic and $\circ$-epic. Choose a kernel $i\colon C \rightarrowtail I$ with $I \in \mathsf{S}^\circ\mathscr{A}$. Then there exists a morphism $j\colon D \to I_\circ$ with $r_I j = ir$. Hence i factors through g. By [64], Proposition 2, this implies that g is a kernel. By duality, f is a cokernel. Since r annihilates the kernel of f, there exists a morphism $h\colon D' \to C$ with $hf = r$ and

$gh = r'$. Therefore, f is a monic cokernel, hence invertible. Similarly, g is invertible, which proves that $\mathscr{C}$ is a Morita category such that $\mathscr{A}$ is equivalent to a $\circ$-subcategory.

To show that $\mathscr{C}$ is ample, let (7) be any morphism $\varphi \in \mathscr{C}$. Taking the image of f and g, we obtain a factorization $\varphi = \mu\varepsilon$ where ε is given by a commutative diagram (7) with cokernels f, g, and μ is given by such a commutative diagram with kernels f, g. Thus, by symmetry, it is enough to verify that a morphism (7) is $\circ$-epic whenever f and g are cokernels. Assuming this, let $f' \colon P \to D'$ and $g' \colon P^\circ \to C'$ be morphisms with $P \in \mathsf{S}_\circ\mathscr{A}$ and $r'f' = g'r^P$. Then $f' = fh$ for some $h \colon P \to D$. Since $C \in \mathscr{A}^\circ$, we find a morphism $h' \colon P^\circ \to C$ with $rh = h'r^P$. Thus $g'r^P = r'f' = r'fh = gh'r^P$, which yields $g' = gh'$. So the morphism (7) is $\circ$-epic.

We set $\widetilde{\mathscr{A}} := \mathscr{C}$. To prove the uniqueness statement, let $\mathscr{A} \hookrightarrow \mathscr{B}$ be a $\circ$-embedding into an ample Morita category $\mathscr{B}$. Then the above constructed ample category $\mathscr{C}$ is the same for both $\mathscr{A}$ and $\mathscr{B}$. Since $\mathscr{B}$ is ample, any object $r \colon D \to C$ in $\mathscr{C}$ satisfies $r = me$ with a $\circ$-epimorphism e and a $\circ$-monomorphism m. Hence $\mathscr{B} \approx \mathscr{C}$. □

The ample category $\widetilde{\mathscr{A}}$ will be called the *ample closure* of $\mathscr{A}$. By the preceding proof, we have

Corollary 1. *Let $\mathscr{A}$ be a Morita category. For any object $A \in \mathscr{A}$ there are monic epimorphisms $r_A \colon A_\circ \to A$ and $r^A \colon A \to A^\circ$ with $A_\circ \in \mathscr{A}_\circ$ and $A^\circ \in \mathscr{A}^\circ$ such that r_A is $\circ$-epic and r^A is $\circ$-monic. Up to isomorphism, the morphisms r_A and r^A are unique.*

Recall that a *factorization system* [36, 37] in a category is given by a pair $(\mathscr{E}, \mathscr{M})$ of morphism classes, containing the isomorphisms and closed under composition, such that the morphisms $e \in \mathscr{E}$ are left orthogonal to the morphisms $m \in \mathscr{M}$ and each $f \in \mathscr{A}$ admits a factorization $f = me$ with $m \in \mathscr{M}$ and $e \in \mathscr{E}$.

Corollary 2. *For a Morita category $\mathscr{A}$, the following are equivalent:*

(a) *Every $\circ$-embedding into a Morita category is an equivalence.*

(b) *The $\circ$-epimorphisms and $\circ$-monomorphisms form a factorization system.*

(c) *$\mathscr{A}$ is ample.*

Proof. (a) $\Leftrightarrow$ (c) follows by Theorem 4.2. Proposition 3.3 yields the equivalence (b) ⇔ (c). □

Corollary 3. *The category* **TBA** *of totally bounded abelian groups is the ample closure of the category* **LCA** *of locally compact abelian groups.*

Proof. By Proposition 2.1, **TBA** is equivalent to the category **DCA**. By the proof of Theorem 4.2, $\mathbf{DCA} \approx \widetilde{\mathbf{LCA}}$. □

As is well known, the category **LCA** is neither complete nor cocomplete [53, 45].

Proposition 4.3. *The ample category* **TBA** *of totally bounded abelian groups is complete and cocomplete.*

Proof. Using Proposition 2.1, we consider the equivalent category **DCA**. Since **DCA** is preabelian, it is enough to show that **DCA** has products and coproducts. Thus let (r_i) be a family of objects in **DCA**. So the $r_i \colon D_i \to C_i$ are dense embeddings of discrete groups D_i into compact groups C_i. Thus $r := \prod r_i$ is a dense embedding $r \colon \prod D_i \to \prod C_i$, and it is easily verified that r is a product of the r_i in **DCA**. Since **DCA** is equivalent to **DS**(**Ab**) by Proposition 2.1, the category **DCA** has a natural duality which maps an object $r \colon D \to C$ to $Hom(r, \mathsf{T}) \colon Hom(C, \mathsf{T}) \to Hom(D, \mathsf{T})$. Thus **DCA** is cocomplete. □

5. Duality

By a *duality* of a category $\mathscr{C}$ we mean an involution $\mathscr{C}^{\mathrm{op}} \to \mathscr{C}$. The Pontryagin-van Kampen theorem ([59, 68]; [43], Theorem 24.8) states that $A \mapsto Hom(A, \mathsf{T})$ is a duality of **LCA** if the continuous dual of an LCA group is endowed with the compact-open topology.

Extensions of Pontryagin duality have been investigated by many authors, e. g., [48, 49, 53, 58, 69, 70, 71, 18, 22, 31, 32, 72, 52, 8, 62, 55, 5, 41, 39, 3]. Kaplan [48] proved that products of LCA groups are *Pontryagin-reflexive* (*P-reflexive* for short) in the sense that the natural map to the bidual is a topological isomorphism. He raised the problem to determine the class of all P-reflexive topological abelian groups. Freundlich-Smith [35] proved that the additive group of a real Banach space or a reflexive locally convex space is P-reflexive. In 1976, Venkataraman [71] proposed a solution to Kaplan's problem which contains a wrong statement of [69], similar to the incorrect characterization of P-reflexive locally convex spaces in [52] (see [5], Section 8). Correct solutions are given in [41] and [39], respectively. Deviating from the compact-open topology, Binz [15, 16, 17, 18] and Butzmann [21, 22, 23] studied duality within a class of abelian convergence groups [30, 51] and convergence vector spaces.

Note that abelian convergence groups form a *closed category* [54], that is, a symmetric monoidal category $\mathscr{A}$ with internal hom-objects $\mathit{Hom}(A, B)$. An object D of $\mathscr{A}$ is said to be *dualizing* if the natural morphism $A \to \mathit{Hom}(\mathit{Hom}(A, D), D)$ is invertible for each object A. A closed category with a dualizing object D is said to be $*$*-autonomous* [10].

Morita [56] and Roeder [63] proved that Pontryagin duality is essentially unique as a duality of the category **LCA**. More generally, we have the following

Theorem 5.1. *The ample closure of* **LCA** *is a complete and cocomplete* $*$*-autonomous category with an essentially unique duality which extends the Pontryagin duality of* **LCA**.

Proof. Any duality $X \mapsto \widehat{X}$ of $\mathscr{A} := \widetilde{\mathbf{LCA}}$ maps $\mathscr{A}_\circ = \mathbf{Ab}$ to $\mathscr{A}^\circ = \mathbf{CA}$. Up to isomorphism, **Ab** contains a unique indecomposable projective object Z. Hence $\widehat{\mathsf{Z}} \cong \mathsf{T}$. So the duality restricts to the Pontryagin duality on **Ab**. Identifying $\widetilde{\mathbf{LCA}}$ with **DCA**, the objects of this category are given as monic epimorphisms $r\colon D \to C$ with $D \in \mathbf{Ab}$ and $C \in \mathbf{CA}$. Now the Pontryagin-dual of $r_C\colon C_d \to C$ is the Bohr compactification $r^{\widehat{C}}\colon \widehat{C} \to \mathfrak{b}\widehat{C}$. Since $\circ$-epimorphisms correspond to $\circ$-monomorphisms under the duality, this shows that the dual of the morphism $r\colon D \xrightarrow{r} C_d \xrightarrow{r_C} C$ in **DCA** coincides with the Pontryagin-dual of this map in **LCA**. So the dual of the object $r \in \mathbf{DCA}$ is given by the Pontryagin-dual $\widehat{r}\colon \widehat{C} \to \widehat{D}$ of the morphism $r \in \mathbf{LCA}$. Thus $r \mapsto \widehat{r}$ is the unique duality of $\mathbf{DCA} \approx \widetilde{\mathbf{LCA}}$.

Now let $A \in \mathbf{LCA}$ be given. By dualizing the sequence $A_d \to A \to \mathfrak{b}A$, we obtain a sequence $\mathit{Hom}(\mathfrak{b}A, \mathsf{T}) \to \mathit{Hom}(A, \mathsf{T}) \to \mathrm{Hom}(A_d, \mathsf{T})$ with $\mathit{Hom}(A, \mathsf{T})_d = \mathit{Hom}(\mathfrak{b}A, \mathsf{T})$. Furthermore, Pontryagin duality implies that $\mathit{Hom}(A, \mathsf{T}) \to \mathrm{Hom}(A_d, \mathsf{T})$ is the Bohr compactification of $\mathit{Hom}(A, \mathsf{T})$. Thus $\mathit{Hom}(A, \mathsf{T})$ coincides with the dual of A in **DCA**.

To show that $\mathbf{TBA} \approx \widetilde{\mathbf{LCA}}$ is a closed category, we endow the group $\mathit{Hom}(A, B)$ of continuous homomorphisms for $A, B \in \mathbf{TBA}$ with the weak topology, induced by the embedding $\mathit{Hom}(A,B) \hookrightarrow B^A$, and write $\mathit{Hom}^\sigma(A,B)$ for this group. Thus $\mathit{Hom}^\sigma(A,B) \in$ **TBA**. For $B := \mathsf{T}$, this gives the weak dual $A^\sigma := \mathit{Hom}^\sigma(A,\mathsf{T})$, in accordance with the duality in **DCA**. Every biadditive map $\beta\colon A \times B \to C$ with LCA groups A, B, C corresponds to a group homomorphism $\beta_\ell\colon A \to \mathrm{Hom}(B,C)$, which maps A into $\mathit{Hom}^\sigma(B,C)$ if and only if the partial map $\beta(a,-)$ is continuous for all $a \in A$. The embedding $\mathit{Hom}^\sigma(B,C) \hookrightarrow C^B$ shows that β_ℓ is continuous if and only if the partial maps $\beta(-,b)$ are continuous for all $b \in B$. Together with the embeddings

$$\mathit{Hom}^\sigma(A,\mathit{Hom}^\sigma(B,C)) \hookrightarrow \mathit{Hom}^\sigma(B,C)^A \hookrightarrow (C^B)^A \cong C^{A\times B},$$

this gives a topological isomorphism

$$\mathit{Hom}^\sigma(A,\mathit{Hom}^\sigma(B,C)) \cong \mathit{Hom}^\sigma(B,\mathit{Hom}^\sigma(A,C)).$$

In particular,

$$\mathit{Hom}^\sigma(A^\sigma,B^\sigma) \cong \mathit{Hom}^\sigma(B,A^{\sigma\sigma}) \cong \mathit{Hom}^\sigma(B,A).$$

Hence

$$\begin{aligned}\mathit{Hom}^\sigma(A,\mathit{Hom}^\sigma(B,C)) &\cong \mathit{Hom}^\sigma(A,\mathit{Hom}^\sigma(C^\sigma,B^\sigma))\\ &\cong \mathit{Hom}^\sigma(C^\sigma,\mathit{Hom}^\sigma(A,B^\sigma))\\ &\cong \mathit{Hom}^\sigma(\mathit{Hom}^\sigma(A,B^\sigma)^\sigma,C),\end{aligned}$$

and thus $A \otimes B := \mathit{Hom}^\sigma(A,B^\sigma)^\sigma \cong \mathit{Hom}^\sigma(B,A^\sigma)^\sigma$ yields the desired isomorphism

$$\mathit{Hom}^\sigma(A\otimes B,C) \cong \mathit{Hom}^\sigma(A,\mathit{Hom}^\sigma(B,C)).$$

Note that the associativity of $\otimes$ follows by this formula. With the dualizing object T, the category **TBA** is $*$-autonomous. By Proposition 4.3, it is complete and cocomplete. □

Remarks. 1. Theorem 5.1 suggests an alternative proof of classical Pontryagin duality. Since LCA groups are topological k-spaces, and $\circ$-epimorphisms are transformed into $\circ$-monomorphisms under duality, one only has to verify that subsets K of $\mathit{Hom}(A,\mathsf{T})$ which are compact in $\mathrm{Hom}(A_d,\mathsf{T})$ are compact in the compact-open topology of $Hom(A, \mathsf{T})$. Using the Arzelà-Ascoli theorem for LCA groups ([7], Theorem 4), it suffices to show that K is evenly continuous [50]. Using [58], Lemma 2.1, this is easily verified.

2. Note that the unique duality of **TBA** is not the Pontryagin duality in **TBA** (see [3], Example 2.6). For example, let B be the image of $\mathsf{Z} \hookrightarrow \mathfrak{b}\mathsf{Z}$, with the induced topology. By Glicksberg's theorem [40], the compact subsets of B are finite. So the Pontryagin duals of B and Z are topologically isomorphic. Since Z is not totally bounded, this shows that B is not P-reflexive. By [24], Theorem 2, the image of a dense embedding $\mathsf{Z} \hookrightarrow \mathsf{T}$ with the induced topology is not P-reflexive either.

3. It is natural to ask how far Theorem 5.1 can be extended to the category of (not necessarily abelian) locally compact groups. A first step would be to describe the category of compact groups, which is semi-abelian in the sense of [47], within the category of all locally compact groups, in analogy to the subcategory $\mathscr{A}^\circ$ of a quasi-abelian category. Such a study will probably require concepts beyond those discussed in [20].

Let **KAb** be the category of topological abelian groups which *respect compactness* [62], that is, weakly compact subsets are compact. This category contains the nuclear groups [9] and more generally, the locally quasi-convex Schwartz groups ([6], Theorem 4.4). Glicksberg's theorem [40] states that every LCA group belongs to **KAb**. For a while, it was believed ([70], Theorem 1.1) that **KAb** also contains the category **PAb** of P-reflexive Hausdorff abelian groups. This was disproved by Remus and Trigos-Arrieta [62] who provided a class of locally convex spaces as counterexamples, including the additive group of the separable Hilbert space $\ell^2(\mathbb{R})$. More precisely, it is known [35] that the additive group of a reflexive locally convex space X is P-reflexive. By [62], Theorem 1.4, such a group belongs to **KAb** if and only if X is a *Montel space* [65]. This leads to the category **PKAb** $:=$ **PAb** $\cap$ **KAb** which was introduced by Trigos-Arrieta ([67], 1.8) and studied in [62], Section 2. By [62], Corollary 2.2, the full subcategory **PKAb** of **PAb** is closed with respect to products, which shows that **PKAb** strictly contains **LCA**.

Theorem 5.2. *The category* **PKAb** *of P-reflexive Hausdorff abelian groups respecting compactness admits a duality preserving full embedding into the ample closure of* **LCA**.

Proof. Let A be a Hausdorff abelian group in **PAb**. By [2], Theorem 1, A has a Bohr compactification $r^A\colon A \to \mathfrak{b}A$ with $\mathfrak{b}A \cong \mathrm{Hom}(\mathit{Hom}(A,\mathsf{T})_d,\mathsf{T})$ [46, 27], so that every character $A \to \mathsf{T}$ factors uniquely through r^A. Since A has enough characters, r^A is a dense embedding. Dually, there is a continuous bijection $r_A\colon A_d \to A$. We associate the object $r^A r_A \in$ **DCA** to A. This gives a faithful additive functor $F\colon$ **PAb** $\to$ **DCA**.

We show first that F respects duality. Indeed, $\mathit{Hom}(A,\mathsf{T})_d = \mathit{Hom}(\mathfrak{b}A,\mathsf{T})$. With $X := \mathit{Hom}(A,\mathsf{T})$, this shows that $r_X\colon X_d \to X$ is given by

$$\mathit{Hom}(r^A,\mathsf{T})\colon \mathit{Hom}(\mathfrak{b}A,\mathsf{T}) \to \mathit{Hom}(A,\mathsf{T}).$$

Hence $r^X\colon X \to \mathfrak{b}X$ is given by $\mathit{Hom}(r_A,\mathsf{T})\colon \mathit{Hom}(A,\mathsf{T}) \to \mathrm{Hom}(A_d,\mathsf{T})$. So the duality of **DCA** restricts to the duality of **PAb**.

To show that the restriction $F_k\colon$ **PKAb** $\to$ **DCA** of F is full, let $FA \to FB$ be a morphism in **DCA** with $A, B \in$ **PKAb**. This gives a commutative diagram

$$\begin{array}{ccccc} A_d & \xrightarrow{r_A} & A & \xrightarrow{r^A} & \mathfrak{b}A \\ \Big\downarrow{\scriptstyle f} & & & & \Big\downarrow{\scriptstyle g} \\ B_d & \xrightarrow{r_B} & B & \xrightarrow{r^B} & \mathfrak{b}B. \end{array} \tag{9}$$

We show that the unique map $h\colon A \to B$ with $hr_A = r_B f$ is continuous. Let K be a compact subset of A. Then $r^B h(K) = gr^A(K)$ is compact. Hence $h(K)$ is compact in B. Thus h maps compact sets to compact sets. Dualizing the diagram (9) leads to a map $\widehat{g}\colon \mathit{Hom}(\mathfrak{b}B,\mathsf{T}) \to \mathit{Hom}(\mathfrak{b}A,\mathsf{T})$ which induces a map $\widehat{h}\colon \mathit{Hom}(B,\mathsf{T}) \to \mathit{Hom}(A,\mathsf{T})$ with $\widehat{h}(\chi) = \chi h$. For a 0-neighbourhood W in T, consider the 0-neighbourhood ($K \subset A$ compact)

$$U(K,W) := \{\chi \in \mathit{Hom}(A,\mathsf{T}) \mid \chi(K) \subset W\}$$

in $\mathit{Hom}(A,\mathsf{T})$. Then

$$\chi \in \widehat{h}^{-1}(U(K,W)) \Leftrightarrow \chi h(K) \subset W \Leftrightarrow \chi \in U(h(K),W),$$

which yields $\widehat{h}^{-1}(U(K,W)) = U(h(K),W)$. Thus $\widehat{h}$ is continuous. By duality, this implies that h is continuous. So the functor F_k is a full embedding. □

Remark. The first part of the proof yields a duality preserving faithful functor **PAb** → **DCA**. Without the counterexamples to [70], Theorem 1.1, this functor would be a full embedding. The failure of this, analysed in [62], led to the category **PKAb** which is more well-behaved with respect to P-duality.

Conversely, the proof of Theorem 5.2 shows that **PKAb** is the largest full subcategory of **PAb** which allows a full embedding into the ample closure of **LCA**. Namely, if $A \in$ **PAb** does not respect compactness, the totally bounded image A^+ of r^A gives a commutative diagram

$$\begin{array}{ccccc} A^+_d & \xrightarrow{r_{A^+}} & A^+ & \xrightarrow{r^{A^+}} & \mathfrak{b}A^+ \\ \| & & \downarrow h & & \| \\ A_d & \xrightarrow{r_A} & A & \xrightarrow{r^A} & \mathfrak{b}A \end{array}$$

with a non-continuous bijection h. So the compact-open topology becomes inadequate beyond **PKAb**. In topological terms, Theorem 5.2 implies that a weakly continuous homomorphism between topological groups in **PKAb** is continuous. For LCA groups this was proved by Trigos-Arrieta ([67], Theorem 1.2).

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References

[1] E. M. Alfsen, J. E. Fenstad: On the equivalence between proximity structures and totally bounded uniform structures, Math. Scand. 7 (1959), 353-360

[2] E. M. Alfsen, P. Holm: A note on compact representations and almost periodicity in topological groups, Math. Scand. 10 (1962), 127-136

[3] S. Ardanza-Trevijano, M. J. Chasco, X. Domínguez, M. Tkachenko: Precompact noncompact reflexive Abelian groups, Forum Math. 24 (2012), no. 2, 289-302

[4] M. Auslander: The representation dimension of artin algebras, Queen Mary College Mathematics Notes (1971), Republished in Selected works of Maurice Auslander, Amer. Math. Soc., Providence, 1999

[5] L. Außenhofer: Contributions to the duality theory of abelian topological groups and to the theory of nuclear groups, Dissertationes Math. (Rozprawy Mat.) 384 (1999), 113 pp.

[6] L. Außenhofer: On the Glicksberg theorem for locally quasi-convex Schwartz groups, Fund. Math. 201 (2008), no. 2, 163-177.

[7] R. W. Bagley, J. S. Yang: On k-spaces and function spaces, Proc. Amer. Math. Soc. 17 (1966), 703-705

[8] W. Banaszczyk: Additive subgroups of topological vector spaces, Lecture Notes in Mathematics, 1466, Springer-Verlag, Berlin, 1991

[9] W. Banaszczyk, E. Martín-Peinador: The Glicksberg theorem on weakly compact sets for nuclear groups, Papers on general topology and applications (Amsterdam, 1994), 34-39, Ann. New York Acad. Sci., 788, New York Acad. Sci., New York, 1996

[10] M. Barr: $*$-autonomous categories and linear logic, Math. Structures Comput. Sci. 1 (1991), no. 2, 159-178

[11] M. Barr: $*$-autonomous categories, revisited, J. Pure Appl. Algebra 111 (1996), no. 1-3, 1-20

[12] M. Barr: The Chu construction: history of an idea, Theory Appl. Categ. 17 (2006), No. 1, 10-16

[13] M. Barr, H. Kleisli: On Mackey topologies in topological abelian groups, Theory Appl. Categ. 8 (2001), 54-62

[14] R. Beattie, H.-P. Butzmann: Continuous duality of limits and colimits of topological abelian groups, Appl. Categ. Structures 16 (2008), no. 4, 535-549

[15] E. Binz, H.-P. Butzmann, K. Kutzler: Über den c-Dual eines topologischen Vektorraumes, Math. Z. 127 (1972), 70-74

[16] E. Binz: Continuous convergence on $C(X)$, Lectures Notes in Mathematics, Vol. 469, Springer-Verlag, Berlin-New York, 1975

[17] E. Binz: Charaktergruppen von Gruppen von S^1-wertigen stetigen Funktionen, Categorical topology (Proc. Conf., Mannheim, 1975), pp. 43-92, Lecture Notes in Math., Vol. 540, Springer, Berlin, 1976

[18] E. Binz: On an extension of Pontryagin's duality theory, General topology and its relations to modern analysis and algebra, IV (Proc. Fourth Prague Topological Sympos., Prague, 1976), Part A, pp. 1-20, Lecture Notes in Math., Vol. 609, Springer, Berlin, 1977

[19] E. Binz, H. H. Keller: Funktionenräume in der Kategorie der Limesräume, Ann. Acad. Sci. Fenn. Ser. A I No. 383 (1966), 1-21

[20] F. Borceux, D. Bourn: Mal'cev, protomodular, homological and semi-abelian categories, Mathematics and its Applications, 566, Kluwer Academic Publishers, Dordrecht, 2004

[21] H.-P. Butzmann: Über die c-Reflexivität von $\mathscr{C}_c(X)$, Comment. Math. Helv. 47 (1972), 92-101

[22] H.-P. Butzmann: Pontrjagin-Dualität für topologische Vektorräume, Arch. Math. 28 (1977), no. 6, 632-637

[23] H.-P. Butzmann: Duality theory for convergence groups, Topology Appl. 111 (2001), 95-104

[24] M. J. Chasco: Pontryagin duality for metrizable groups, Arch. Math. 70 (1998), no. 1, 22-28

[25] M. J. Chasco, D. Dikranjan, E. Martín-Peinador: A survey on reflexivity of abelian topological groups, Topology Appl. 159 (2012), no. 9, 2290-2309

[26] M. J. Chasco, E. Martín-Peinador: An approach to duality on abelian precompact groups, J. Group Theory 11 (2008), no. 5, 635-643

[27] Hsin Chu: Compactification and duality of topological groups, Trans. Amer. Math. Soc. 123 (1966), 310-324

[28] W. W. Comfort, K. A. Ross: Topologies induced by groups of characters, Fund. Math. 55 (1964), 283-291

[29] W. W. Comfort, S. Hernández, F. J. Trigos-Arrieta: Relating a locally compact abelian group to its Bohr compactification, Adv. Math. 120 (1996), no. 2, 322-344

[30] C. H. Cook, H. R. Fischer: Uniform convergence structures, Math. Ann. 173 (1967), 290-306

[31] B. J. Day: An extension of Pontryagin duality, Bull. Austral. Math. Soc. 19 (1978), no. 3, 445-456

[32] B. J. Day: On Pontryagin duality, Glasgow Math. J. 20 (1979), no. 1, 15-25

[33] J. Dieudonné: La dualité dans les espaces vectoriels topologiques, Ann. Sci. École Norm. Sup. 59, (1942), 107-139

[34] H. R. Fischer: Limesräume, Math. Ann. 137 (1959), 269-303

[35] M. Freundlich-Smith: The Pontrjagin duality theorem in linear spaces, Ann. of Math. 56 (1952), 248-253

[36] P. J. Freyd, G. M. Kelly: Categories of continuous functors, I., J. Pure Appl. Algebra 2 (1972), 169-191

[37] P. J. Freyd, G. M. Kelly: Erratum: "Categories of continuous functors, I", J. Pure Appl. Algebra 4 (1974), 121

[38] J. Galindo, M. Tkachenko, M. Bruguera, C. Hernández: Reflexivity in precompact groups and extensions, Topology Appl. 163 (2014), 112-127

[39] F. Garibay Bonales, F. J. Trigos-Arrieta, R. Vera Mendoza: A characterization of Pontryagin-van Kampen duality for locally convex spaces, Topology Appl. 121 (2002), no. 1-2, 75-89

[40] I. Glicksberg: Uniform boundedness for groups, Canadian J. Math. 14 (1962), 269-276

[41] S. Hernández: Pontryagin duality for topological abelian groups, Math. Z. 238 (2001), no. 3, 493-503

[42] S. Hernández, F. J. Trigos-Arrieta: Group duality with the topology of precompact convergence, J. Math. Anal. Appl. 303 (2005), no. 1, 274-287

[43] E. Hewitt, K. A. Ross: Abstract Harmonic Analysis I, Berlin-Heidelberg-New York, 1979

[44] N. Hoffmann, M. Spitzweck: Homological algebra with locally compact abelian groups, Adv. Math. 212 (2007), no. 2, 504-524

[45] K. H. Hofmann, S. A. Morris: Locally compact products and coproducts in categories of topological groups, Bull. Austral. Math. Soc. 17 (1977), no. 3, 401-417

[46] P. Holm: On the Bohr compactification, Math. Ann. 156 (1964), 34-46

[47] G. Janelidze, L. Márki, W. Tholen: Semi-abelian categories, J. Pure Appl. Algebra 168 (2002), no. 2-3, 367-386

[48] S. Kaplan: Extensions of the Pontrjagin duality, I, Infinite products, Duke Math. J. 15 (1948), 649-658

[49] S. Kaplan: Extensions of the Pontrjagin duality, II, Direct and inverse sequences, Duke Math. J. 17 (1950), 419-435

[50] J. L. Kelley: General topology, D. Van Nostrand Company, Inc., Toronto-New York-London, 1955

[51] D. C. Kent: On convergence groups and convergence uniformities, Fund. Math. 60 (1967), 213-222

[52] S.-H. Kye: Pontryagin Duality in Real Linear Topological Spaces, Chinese J. Math. 12 (1984), 129-136

[53] H. Leptin: Zur Dualitätstheorie projektiver Limites abelscher Gruppen, Abh. Math. Sem. Univ. Hamburg 19 (1955), 264-268

[54] S. Mac Lane: Categories for the Working Mathematician, New York - Heidelberg - Berlin 1971

[55] E. Martín Peinador: A reflexive admissible topological group must be locally compact, Proc. Amer. Math. Soc. 123 (1995), no. 11, 3563-3566

[56] K. Morita: Duality for modules and its applications to the theory of rings with minimum condition, Sci. Rep. Tokyo Kyoiku Daigaku Sect. A 6 (1958), 83-142

[57] M. Moskowitz: Homological algebra in locally compact abelian groups, Trans. Amer. Math. Soc. 127 (1967), 361-404

[58] N. Noble: k-groups and duality, Trans. Amer. Math. Soc. 151 (1970), 551-561

[59] L. Pontryagin: The theory of topological commutative groups, Ann. of Math. 35 (1934), no. 2, 361-388

[60] N. Popescu: Abelian Categories with Applications to Rings and Modules, London - New York 1973

[61] D. Quillen: Higher algebraic K-theory, I, in: Algebraic K-theory, I: Higher K-theories (Proc. Conf., Battelle Memorial Inst., Seattle, Wash., 1972), pp. 85-147, Lecture Notes in Math. 341, Springer, Berlin, 1973

[62] D. Remus, F. J. Trigos-Arrieta: Abelian groups which satisfy Pontryagin duality need not respect compactness, Proc. Amer. Math. Soc. 117 (1993), no. 4, 1195-1200

[63] D. W. Roeder: Functorial characterizations of Pontryagin duality, Trans. Amer. Math. Soc. 154 (1971), 151-175

[64] W. Rump: Almost Abelian Categories, Cahiers de topologie et géométrie différentielle catégoriques XLII (2001), 163-225

[65] H. H. Schaefer: Topological vector spaces, Graduate Texts in Mathematics, Vol. 3, Springer-Verlag, New York-Berlin, 1971

[66] J.-P. Schneiders: Quasi-abelian categories and sheaves, Mém. Soc. Math. France 1999, no. 76

[67] F. J. Trigos-Arrieta: Continuity, boundedness, connectedness and the Lindelöf property for topological groups, J. Pure Appl. Algebra 70 (1991), no. 1-2, 199-210

[68] E. R. van Kampen: Locally bicompact abelian groups and their character groups, Ann. of Math. 36 (1935), no. 2, 448-463

[69] R. Venkataraman: Extensions of Pontryagin duality, Math. Z. 143 (1975), no. 2, 105-112

[70] R. Venkataraman: Compactness in abelian topological groups, Pacific J. Math. 57 (1975), no. 2, 591-595

[71] R. Venkataraman: A characterization of Pontryagin duality, Math. Z. 149 (1976), no. 2, 109-119

[72] R. Venkataraman: Coproduct in the category of groups satisfying Pontryagin duality, Monatsh. Math. 96 (1983), no. 4, 311-315

[73] A. Weil: Sur les espaces à structure uniforme et sur la topologie générale, Publ. Math. Univ. Strasbourg, Hermann, Paris, 1937

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