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A MAL'TSEV GLANCE AT THE FIBRATION $(\ )_0 : Cat\mathbb{E} \to \mathbb{E}$ OF INTERNAL CATEGORIES

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Résumé. Nous montrons comment la fibration $(\ )_0 : Cat\mathbb{E} \to \mathbb{E}$ des catégories internes à $\mathbb{E}$ est munie de deux types de structures partiellement liées à des concepts protomodulaires et mal'tseviens, et nous en explicitons quelques conséquences. Cela mène, entre autres, à la notion de *catégorie Schreier-spéciale* qui détermine une sous-catégorie protomodulaire de chaque fibre $Cat_Y\mathbb{E}$.

Abstract. We show how the fibration $(\ )_0 : Cat\mathbb{E} \to \mathbb{E}$ associated with the internal categories in $\mathbb{E}$ is endowed with two kinds of structure which are partially dealing with Mal'tsev and protomodular concepts, and we make explicit some consequences. This leads, inter alia, to the notion of *Schreier special category* which determines a protomodular subcategory of any fiber $Cat_Y\mathbb{E}$.

Keywords. Mal'tsev and protomodular categories; split epimorphisms; internal categories and groupoids; connected, aspherical and affine groupoids, direction of aspherical affine groupoids, internal weak equivalence.

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Introduction

Given any finitely complete category $\mathbb{E}$, the fibration $(\ )_0 : Grd\mathbb{E} \to \mathbb{E}$ of internal groupoids in $\mathbb{E}$ is known to have a strong structural property [3]: any fiber $Grd_Y\mathbb{E}$ above the object Y in $\mathbb{E}$ is protomodular [3] and thus a Mal'tsev category, on the model of the fiber $Grd_1\mathbb{E}$ above the terminal object 1 which is nothing but the category $Gp\mathbb{E}$ of internal groups in $\mathbb{E}$. Nothing comparable did exist for the fibration $(\ )_0 : Cat\mathbb{E} \to \mathbb{E}$ of internal categories whose first structural properties were, strictly speaking, investigated in [2] more than thirty years ago, the notion of internal categories having been initiated, long before, in the pionnering work of C. Ehresmann [16].

But in [12, 13] was introduced a new structural aspect of the category Mon of monoids with the notion of Schreier split epimorphism and the associated notion of partial protomodularity (see Definition 3.1). Since Mon is nothing but the fiber Cat_1, the main aim of this work was to investigate whether it was possible to extend this result to any fiber Cat_Y and $Cat_Y\mathbb{E}$; in other words, this aim was to identify a class Σ_Y of split epimorphisms in the fibers $Cat_Y\mathbb{E}$ which would imply a partial protomodularity inside them. This is done in Section 3.1 with the extension of the notion of Schreier split epimorphism to internal functors, and this leads to the notion of *Schreier special category* (see Definition 4.4) which determines protomodular subcategories of the fibers $Cat_Y\mathbb{E}$.

Unexpectedly, a more global property of the fibration $(\ )_0 : Cat\mathbb{E} \to \mathbb{E}$ did emerge during this investigation, but, this time, related to the notion of partial mal'tsevness (see Theorem 1.4) from which a spectacular Mal'tsev type consequence is drawn: when $\mathbb{E}$ is regular, given any pullback in $Cat\mathbb{E}$:

$$\begin{tikzcd} X_\bullet \arrow[r, two heads, "x_\bullet"] \arrow[d, shift right, "f_\bullet"'] & X'_\bullet \arrow[d, shift right, "f'_\bullet"'] \\ Y_\bullet \arrow[r, two heads, "y_\bullet"'] \arrow[u, shift right, "s_\bullet"'] & Y'_\bullet \arrow[u, shift right, "s'_\bullet"'] \end{tikzcd}$$

where $y_\bullet$ is a fully faithful functor above a regular epimorphism y_0 in $\mathbb{E}$ and $(f'_\bullet, s'_\bullet)$ is any split epimorphism in a fiber of the fibration $(\ )_0$, then the upward square is necessarily a pushout, see Proposition 1.5.

Now, what is probably the most surprising in the whole process is the following observation: in the fiber $Cat_1 = Mon$, the notion of Schreier epimorphism is not intrinsic to Mon, clearly refereing to a non-homomorphic retraction of a given homomorphism. However, enlarging the definition to the whole category Cat, we get to a notion which is intrinsically bound to Cat, and even more surprisingly, it is intrinsically bound to its 2-*categorical nature*; and this, of course, remains valid for $Cat\mathbb{E}$, see Proposition 3.6.

Finally, in the last section, we shall complete the observations of [6, 8] about the affine groupoids (and their directions) in showing that they are stable under weak equivalences between groupoids; namely, under a specific class of functors which are cartesian with respect to the fibration $(\ )_0$.

The article is organized along the following lines: Section 1) is devoted to recalls about the basics on internal categories and culminates with the first structural observation, Theorem 1.4, from which we draw a short observation about the composition and the permution of some equivalence relations in the (non-regular) context of $Cat\mathbb{E}$. Section 2) introduces the structural concept of Mal'tsev fibration hidden behind this theorem and investigates its consequences. Section 3) is dealing with the partial protomodularity of the fibers $Cat_Y\mathbb{E}$ and culminates with the second structural observation: Theorem 3.5. Section 4) explicitely describes some consequences of this partial protomodularity. In particular, if the protomodular Schreier-core of $Cat_1 = Mon$ is the category Gp of groups, the protomodular Schreier-core of Cat_Y does not consist in the only groupoids with Y as set of objects, see Definition 4.4. All the results of this article as far as this point have been pre-published in [9]. Finally, Section 5) brings some precisions about the fibration $Grd\mathbb{E} \to \mathbb{E}$ relatively to the class of affine groupoids.

1. Internal categories

1.1 Basics

In this article any category $\mathbb{E}$ will be supposed finitely complete, and any pullback of an identity map will be chosen as being an identity map. We shall use a 3-truncated simplicial notation [21] for any internal category (including all the degeneracy maps which do not appear in the following diagram):

$$
\begin{tikzcd}
X_\bullet : & X_3 \arrow[r, shift left=4, "d_4^{X_\bullet}"] \arrow[r, shift left=2, "d_3^{X_\bullet}" description] \arrow[r] \arrow[r, shift right=2, "d_1^{X_\bullet}" description] \arrow[r, shift right=4, "d_0^{X_\bullet}"'] & X_2 \arrow[r, shift left=2, "d_2^{X_\bullet}"] \arrow[r, "d_1^{X_\bullet}" description] \arrow[r, shift right=2, "d_0^{X_\bullet}"'] & X_1 \arrow[r, shift left=2, "d_1^{X_\bullet}"] \arrow[r, shift right=2, "d_0^{X_\bullet}"'] & X_0 \arrow[l, "s_0^{X_\bullet}" description]
\end{tikzcd}
$$

where X_2 (resp. X_3) is obtained by the pullback of d_0 along d_1 (resp. d_0 along d_2), and for any internal functor as well. We denote by $Cat\mathbb{E}$ the category of internal categories in $\mathbb{E}$, and by $(\ )_0 : Cat\mathbb{E} \to \mathbb{E}$ the forgetful functor associating with any internal category $X_\bullet$ its "object of objects" X_0. The category $Cat\mathbb{E}$ is finitely complete since, by commutation of limits, it is easy to see that the finite limits in $Cat\mathbb{E}$ are built levelwise in $\mathbb{E}$. So, the forgetful functor $(\ )_0$ is left exact.

The functor $(\ )_0$ is actually a fibration whose cartesian maps are the internal fully faithful functors and whose maps in the fibers are the internal functors which are "identities on objects" (ido-functors for short).

It is clear that the fiber $Cat_1\mathbb{E}$ above the terminal object 1 is nothing but the category $Mon\mathbb{E}$ of internal monoids in $\mathbb{E}$. Any fiber $Cat_Y\mathbb{E}$ above an object Y, with $Y \neq 1$, has an initial object with the discrete equivalence relation Δ_Y and a terminal one with the indiscrete one ∇_Y. So, the left exact fully faithful functor $\nabla : \mathbb{E} \to Cat\mathbb{E}$ admits the fibration $(\ )_0$ as left adjoint and makes the pair $((\ )_0, \nabla)$ a *fibered reflection* in the sense of [2]. A functor $f_\bullet$ is then cartesian if and only if the following left hand side square is a pullback in $Cat\mathbb{E}$, or, equivalently the right hand side one is a pullback in $\mathbb{E}$:

$$\begin{tikzcd}
X_\bullet \arrow[r, "f_\bullet"] \arrow[d] & Y_\bullet \arrow[d] \\
\nabla_{X_\bullet} \arrow[r, "\nabla_{f_\bullet}"'] & \nabla_{Y_\bullet}
\end{tikzcd}
\qquad
\begin{tikzcd}
X_1 \arrow[r, "f_1"] \arrow[d, "{(d_0,d_1)}"'] & Y_1 \arrow[d, "{(d_0,d_1)}"] \\
X_0 \times X_0 \arrow[r, "f_0 \times f_0"'] & Y_0 \times Y_0
\end{tikzcd}$$

As for any left exact fibration, we get:

Proposition 1.1. *1) The cartesian functors are stable under pullbacks.*

2) Given any commutative square in $Cat\mathbb{E}$ where both $x_\bullet$ and $y_\bullet$ are cartesian functors:

$$\begin{tikzcd}
X_\bullet \arrow[r, "x_\bullet"] \arrow[d, "f_\bullet"'] & X'_\bullet \arrow[d, "f'_\bullet"] \\
Y_\bullet \arrow[r, "y_\bullet"'] & Y'_\bullet
\end{tikzcd}$$

then it is a pullback:
1) if and only if its image by $(\ )_0$ is a pullback
2) in particular when $f_\bullet$ and $f'_\bullet$ are ido-functors.

Given any ido-functor $f_\bullet : X_\bullet \to Y_\bullet$, the following left hand side pullback inside the fiber $Cat_Y\mathbb{E}$ above Y will be called its *kernel*; it only retains the endomorphisms in $X_\bullet$ which are sent on identities in $Y_\bullet$; the pullback on the right hand side, introducing the kernel of the terminal map in the fiber $Cat_Y\mathbb{E}$, only retains what is called the *endosome* $(\mathrm{End}X)_\bullet$ of the internal category $X_\bullet$, namely the internal monoid in the slice category $\mathbb{E}/Y$ consisting in the only endomorphisms of $X_\bullet$:

$$\begin{tikzcd}
(\mathrm{Ker}f)_\bullet \arrow[r, tail, "(\ker f)_\bullet"] \arrow[d, shift right] & X_\bullet \arrow[d, "f_\bullet"] \\
\Delta_Y \arrow[r, tail, "0_{Y_\bullet}"'] \arrow[u, shift right] & Y_\bullet
\end{tikzcd}
\qquad
\begin{tikzcd}
(\mathrm{End}X)_\bullet \arrow[r, tail, "(\epsilon X)_\bullet"] \arrow[d, shift right, "(\rho X)_\bullet"'] & X_\bullet \arrow[d] \\
\Delta_Y \arrow[r, tail, "0_Y"'] \arrow[u, shift right, "(\sigma X)_\bullet"'] & \nabla_Y
\end{tikzcd}$$

1.2 Natural transformations

We know that $Cat\mathbb{E}$ is actually underlying a 2-category with the notion of internal *natural transformations* between internal functors: in simplicial terms, they are just homotopies between the 3-truncated simplicial morphisms that are the internal functors. The cartesianness of the 1-arrows fits well with the 2-cells: an internal functor $f_\bullet : X_\bullet \to Y_\bullet$ is fully faithful if and only if, given any natural transformation $\gamma : f_\bullet.g_\bullet \Rightarrow f_\bullet.g'_\bullet$, there is a unique natural transformation $\bar{\gamma} : g_\bullet \Rightarrow g'_\bullet$ such that $\gamma_\bullet = f_\bullet.\bar{\gamma}_\bullet$. We have even better:

Proposition 1.2. *A split epimorphism $(f_\bullet, s_\bullet) : X_\bullet \rightleftarrows Y_\bullet$ in $Cat\mathbb{E}$ is cartesian if and only if it is a* strict left inverse equivalence*, namely it is such that there is a natural isomorphism $\gamma_\bullet : 1_{X_\bullet} \Rightarrow s_\bullet.f_\bullet$ satisfying $f_\bullet.\gamma_\bullet = 1_{f_\bullet}$ and $\gamma_\bullet.s_\bullet = 1_{s_\bullet}$ (which implies immediately $f_\bullet.s_\bullet = 1_{Y_\bullet}$):*

$$s_\bullet.f_\bullet \Leftarrow X_\bullet \underset{f_\bullet}{\overset{s^\bullet}{\rightleftarrows}} Y_\bullet$$

Proof. Suppose $f_\bullet : X_\bullet \to X_\bullet$ cartesian and split by $s_\bullet$. Accordingly, from the identity natural isomorphism between $f_\bullet$ and $f_\bullet = f_\bullet.s_\bullet.f_\bullet$, we get a natural isomorphism $\gamma : 1_{X_\bullet} \Rightarrow s_\bullet.f_\bullet$ such that $1_{f_\bullet} = f_\bullet.\gamma_\bullet$. From that we get $f_\bullet.(\gamma_\bullet.s_\bullet) = f_\bullet.s_\bullet = 1_{Y_\bullet}$, whence: $f_\bullet.s_\bullet = 1_{Y_\bullet}$.

Conversely, suppose we have a left inverse equivalence given by a natural isomorphism $\gamma_\bullet : 1_{X_\bullet} \Rightarrow s_\bullet.f_\bullet$. Starting with any natural transformation $\tau_\bullet : f_\bullet.g_\bullet \Rightarrow f_\bullet.g'_\bullet$, the natural transformation $\bar{\tau}_\bullet = (\gamma_\bullet g'_\bullet)^{-1}.s_\bullet\tau_\bullet.\gamma_\bullet g_\bullet : g_\bullet \Rightarrow g'_\bullet$ is the unique one such that $f_\bullet.\bar{\tau}_\bullet = \tau_\bullet$ ☐

In a way, the above proposition shows how the cartesian split epimorphisms in $Cat\mathbb{E}$ capture, as soon as level 1, a hidden invertible aspect of the 2-categorical level of the category $Cat\mathbb{E}$.

1.3 The regular context

In this section we shall suppose that $\mathbb{E}$ is a regular category [1]. We shall recall the effect of this further property on the fibration $(\ )_0$, see [2]. The category $Cat\mathbb{E}$ is certainly not itself a regular category; we already know how much characterizing the regular epimorphisms in Cat is complicated. However we can assert about $(\ )_0$ two very interesting and strong properties:

Proposition 1.3. *Let $\mathbb{E}$ be a regular category. Then:*
1) any fiber $Cat_Y\mathbb{E}$ is a regular category;
2) any cartesian functor $f_\bullet : X_\bullet \to Y_\bullet$ above a regular epimorphism f_0 in $\mathbb{E}$ is a pullback stable regular epimorphism in $Cat\mathbb{E}$.

Proof. 1) Since $\mathbb{E}$ is regular, the regular epimorphisms are stable under products in $\mathbb{E}$, and in any slice category $\mathbb{E}/Y$. So, the regular epimorphisms $f_\bullet : X_\bullet \to Y_\bullet$ in the fiber $Cat_Y\mathbb{E}$ are levelwise epimorphisms in $\mathbb{E}$: namely they are such that $f_0 = 1_Y$, and the pair (f_1, f_2) is a pair of regular epimorphisms. So, the fiber $Cat_Y\mathbb{E}$ is immediately a regular one.

2) From the above characterization of cartesian maps, when f_0 is a regular epimorphism in $\mathbb{E}$, so is f_1 as a pullback of the regular epimorphism $f_0 \times f_0$. From that, by commutation of limits, so is f_2. So, again, we get a levelwise regular epimorphism is $\mathbb{E}$. Now, let $h_\bullet : X_\bullet \to Z_\bullet$ be any functor annihilating the kernel equivalence relation $R[f_\bullet]$. This implies that h_0 and h_1 annihilate the kernel equivalence relations $R[f_0]$ and $R[f_1]$. Since f_0 and f_1 are regular epimorphisms in $\mathbb{E}$, we get unique factorizations $\bar{h}_0 : Y_0 \to Z_0$ and $\bar{h}_1 : Y_1 \to Z_1$ such that $\bar{h}_0.f_0 = h_0$ and $\bar{h}_1.f_1 = h_1$. Since f_1 is a regular epimorphism, the pair $(\bar{h}_0, \bar{h}_1)$ produces a morphism between the underlying reflexive graphs. On the other hand, this pair induces a map $\bar{h}_2 : Y_1 \times_0 Y_1 \to Z_1 \times_0 Z_1$. We check that $(\bar{h}_0, \bar{h}_1)$ is actually underlying an internal functor (i.e. internally respects the composition of morphisms) by composition with the regular epimorphism f_2. So, the functor $f_\bullet : X_\bullet \to Y_\bullet$ is a regular epimorphism in $Cat\mathbb{E}$. Clearly, when $\mathbb{E}$ is a regular category, this kind of functor is stable under pullbacks in $Cat\mathbb{E}$. $\square$

1.4 First structural observation

We reached our first (from two) main structural observation (which was already made for the category $Grd\mathbb{E}$ of groupoids in $\mathbb{E}$ (Proposition 1 in [6])):

Theorem 1.4. *Any commutative square of split epimorphisms in $Cat\mathbb{E}$:*

$$\begin{tikzcd} X_\bullet \arrow[r, shift left, "x_\bullet"] \arrow[d, shift right, "f_\bullet"'] & X'_\bullet \arrow[l, shift left, "\sigma_\bullet"] \arrow[d, shift right, "f'_\bullet"'] \\ Y_\bullet \arrow[u, shift right, "s_\bullet"'] \arrow[r, shift left, "y_\bullet"] & Y'_\bullet \arrow[l, shift left, "\tau_\bullet"] \arrow[u, shift right, "s'_\bullet"'] \end{tikzcd}$$

where both $x_\bullet$ and $y_\bullet$ are cartesian and both $f_\bullet$ and $f'_\bullet$ are ido-functors is such that the pair $(s_\bullet, \sigma_\bullet)$ of subobjects of $X_\bullet$ in $Cat\mathbb{E}$ is jointly extremally epic, or, in other words, is such that their supremum as subobjects of $X_\bullet$ is nothing but $1_{X_\bullet}$.

Proof. First observe that, according to Proposition 1.1, this square is necessarily a pullback in $Cat\mathbb{E}$. On the other hand, thanks to the Yoneda embedding, it is enough to check the assertion in Cat. According with the notations of the previous proposition, consider, for any map ϕ, the following commutative diagram in the category $X_\bullet$:

$$\begin{tikzcd} a \arrow[r, "\gamma_\bullet(a)"] \arrow[d, "\phi"'] & \sigma_\bullet x_\bullet(a) \arrow[d, "\sigma_\bullet x_\bullet(\phi)"] \\ b \arrow[r, "\gamma_\bullet(b)"'] & \sigma_\bullet x_\bullet(b) \end{tikzcd}$$

Since the isomorphism $\gamma_\bullet(a)$ comes from $1_{x_\bullet(a)}$ and the pair $(s_\bullet, s'_\bullet)$ is a pair of ido-functors (i.e. the objects of $X_\bullet$ (resp. $X'_\bullet$) and $Y_\bullet$ (resp. $Y'_\bullet$) coincide), this isomorphism is nothing but the image by $s_\bullet$ of the isomorphism $\gamma_\bullet(a)$ in the category $Y_\bullet$. Consequently any subcategory $U_\bullet$ of $X_\bullet$ containing $Y_\bullet$ and $X'_\bullet$ contains $\gamma_\bullet(a)$, $\gamma_\bullet(b)$ and $\sigma_\bullet x_\bullet(\phi)$; so, it contains ϕ. $\square$

Inspired by the knowledge of the Mal'tsev processes, we get the following:

Proposition 1.5. *Let $\mathbb{E}$ be a regular category. Any pullback in $Cat\mathbb{E}$:*

$$\begin{tikzcd} X_\bullet \arrow[r, two heads, "x_\bullet"] \arrow[d, shift right, "f_\bullet"'] & X'_\bullet \arrow[d, shift right, "f'_\bullet"'] \\ Y_\bullet \arrow[r, two heads, "y_\bullet"'] \arrow[u, shift right, "s_\bullet"'] & Y'_\bullet \arrow[u, shift right, "s'_\bullet"'] \end{tikzcd}$$

where $y_\bullet$ is a cartesian regular epimorphism and $(f'_\bullet, s'_\bullet)$ a split epimorphic ido-functor is such that the upward square is a pushout.

Proof. Apply Proposition 2.4 below. □

1.5 A short remark about the composition of relations in $Cat\mathbb{E}$

The category Cat (and a fortiori $Cat\mathbb{E}$ even if $\mathbb{E}$ is regular) is not a regular one; so it is not possible in general to compose internal relations. Let us call *ido* (resp. *cartesian*) equivalence relation any equivalence relation $(d_0, d_1) : R_\bullet \rightrightarrows X_\bullet$ such that d_0 (and thus d_1) is an ido-functor (resp. a cartesian one).

Proposition 1.6. *Given any category $\mathbb{E}$, let $(R_\bullet, S_\bullet)$ be any pair of an ido and a cartesian equivalence relation on $X_\bullet$ in $Cat\mathbb{E}$. Then:*
1) $R_\bullet \cap S_\bullet = \Delta_{X_\bullet}$;
2) $R_\bullet$ and $S_\bullet$ are composable and permute.

Proof. By the Yoneda embedding it is enough to show that in Set. The first point is a consequence of the fact that if a parallel pair $(\phi, \psi) : x \rightrightarrows x' \in X_\bullet$ is in $R_\bullet \cap S_\bullet$, the fact that $d_0^{S_\bullet}$ is cartesian implies $\phi = \psi$. Now consider the square construction $R_\bullet \square S_\bullet$ given by the largest double equivalence relation on $X_\bullet$ produced from $R_\bullet$ and $S_\bullet$ in $Cat\mathbb{E}$:

$$\begin{tikzcd} R_\bullet \square S_\bullet \arrow[r, shift left=2] \arrow[r, shift right=2] \arrow[d, shift right=2] \arrow[d, shift left=2] & S_\bullet \arrow[l] \arrow[d, shift right=2, "d_0^{S_\bullet}"'] \arrow[d, shift left=2, "d_1^{S_\bullet}"] \\ R_\bullet \arrow[u] \arrow[r, shift left=2, "d_1^{R_\bullet}"] \arrow[r, shift right=2, "d_0^{R_\bullet}"'] & X_\bullet \arrow[u] \arrow[l] \end{tikzcd}$$

Then 1) implies that the canonical factorization $R_\bullet \Box S_\bullet \to R_\bullet \times_{X_\bullet} S_\bullet$ to the following pullback is a monomorphism:

$$\begin{tikzcd}
R_\bullet \times_{X_\bullet} S_\bullet \arrow[r, shift left] \arrow[d, shift right] & S_\bullet \arrow[l, shift left] \arrow[d, shift right, "d_0^{S_\bullet}"'] \\
R_\bullet \arrow[r, shift left, "d_1^{R_\bullet}"] \arrow[u, shift right] & X_\bullet \arrow[l, shift left] \arrow[u, shift right]
\end{tikzcd}$$

So, according to Theorem 1.4, it is an isomorphism, whence 2). $\square$

2. Mal'tsev fibration

In this section, we shall make explicit some formal aspects of the previous property of the fibered reflection $((\ )_0, \nabla)$.

A category $\mathbb{E}$ is said to be a *Mal'tsev* one [14, 15], when any reflexive relation is an equivalence relation. This is a categorical characterization of the Mal'tsev varieties, namely those ones which produce a Mal'tsev term, i.e. a ternary term p satisfying $p(x, y, y) = x = p(y, y, x)$ [19]. In [4] was produced the following characterization:

Theorem 2.1. *For any category $\mathbb{E}$, the following conditions are equivalent:*
1) $\mathbb{E}$ is a Mal'tsev category;
2) given any pullback of split epimorphisms in $\mathbb{E}$:

$$\begin{tikzcd}
X \arrow[r, shift left, "x"] \arrow[d, shift right, "f"'] & X' \arrow[l, shift left, "\sigma"] \arrow[d, shift right, "f'"'] \\
Y \arrow[r, shift left, "y"] \arrow[u, shift right, "s"'] & Y' \arrow[l, shift left, "\tau"] \arrow[u, shift right, "s'"']
\end{tikzcd}$$

the pair (s, σ) of subobjects of X is jointly extremally epic.

So, it is legitimate to introduce the following:

Definition 2.2. *A fibration $U : \mathbb{C} \to \mathbb{D}$ is said to be a Mal'tsev fibration when it it is left exact and such that any square of split epimorphism:*

$$\begin{tikzcd}
X \arrow[r, shift left, "x"] \arrow[d, shift right, "f"'] & X' \arrow[l, shift left, "\sigma"] \arrow[d, shift right, "f'"'] \\
Y \arrow[r, shift left, "y"] \arrow[u, shift right, "s"'] & Y' \arrow[l, shift left, "\tau"] \arrow[u, shift right, "s'"']
\end{tikzcd}$$

where both x and y are cartesian maps and both f and f' are inside a fiber is such that the pair (s, σ) of subobjects of X is jointly extremally epic,

Now, according to Theorem 1.4, our first structural observation becomes: *the fibered reflection $((\ )_0, \nabla)$ is a Mal'tsev one*. On the model of what happens for Mal'tsev categories we get the following characterization we shall need later on:

Lemma 2.3. *A left exact fibration $U : \mathbb{C} \to \mathbb{D}$ is a Mal'tsev one if and only if, for any square of split epimorphisms where y is cartesian and f' inside a fiber:*

$$\begin{tikzcd} W \arrow[r, shift left, "\check{x}"] \arrow[d, shift right, "\check{f}"'] & X' \arrow[l, shift left, "\check{\sigma}"] \arrow[d, shift right, "f'"'] \\ Y \arrow[u, shift right, "\check{s}"'] \arrow[r, shift left, "y"] & Y' \arrow[u, shift right, "s'"'] \arrow[l, shift left, "\tau"] \end{tikzcd}$$

the induced factorization $(\check{f}, \check{\sigma}) : W \to X$ is an extremal epimorphism.

We can now easily generalize a well-known Mal'tsev type process with the following:

Proposition 2.4. *Let $U : \mathbb{C} \to \mathbb{D}$ be a Mal'tsev fibration. Suppose, in addition, that $\mathbb{D}$ is a regular category and that any cartesian map in $\mathbb{C}$ above a regular epimorphism in $\mathbb{D}$ is a regular epimorphism in $\mathbb{C}$. Then:*
1) this class Θ of regular epimorphisms in $\mathbb{C}$ is stable under pullbacks;
2) given such a regular epimorphism $h : Y \twoheadrightarrow Y'$ and any pullback in $\mathbb{C}$:

$$\begin{tikzcd} X \arrow[r, two heads, "g"] \arrow[d, shift right, "f"'] & X' \arrow[d, shift right, "f'"'] \\ Y \arrow[u, shift right, "s"'] \arrow[r, two heads, "h"'] & Y' \arrow[u, shift right, "s'"'] \end{tikzcd}$$

where (f', s') is a split epimorphism inside a fiber, the upward square is a pushout.

Accordingly, pulling back the split epimorphisms in the fibers along the regular epimorphism h in $\mathbb{C}$ is a fully faithful process.

Proof. The first point is straightforward as soon as the fibration is left exact. Now, since h is a cartesian regular epimorphism in $\mathbb{C}$, so is g. Consider any pair $(\bar{g}, k)$ of maps in $\mathbb{C}$:

$$\begin{tikzcd}
 & & & Z \\
R[g] \arrow[r, shift left=2, "d_0^g"] \arrow[r, shift right=2, "d_1^g"'] \arrow[d, shift right, "R(f)"'] & X \arrow[l] \arrow[r, two heads, "g"] \arrow[urr, "\bar{g}"] \arrow[d, shift right, "f"'] & X' \arrow[ur, dotted, "\gamma"] \arrow[d, shift right, "f'"'] & \\
R[h] \arrow[r, shift left=2, "d_0^h"] \arrow[r, shift right=2, "d_1^h"'] \arrow[u, shift right, "R(s)"'] & Y \arrow[l] \arrow[r, two heads, "h"'] \arrow[u, shift right, "s"'] & Y' \arrow[u, shift right] \arrow[uur, "k"'] &
\end{tikzcd}$$

such that $h.k = \bar{g}.s$ $(*)$. Complete the diagram with the horizontal kernel equivalence relations. Now, it is clear that we shall get the desired dotted factorization γ if and only if the map $\bar{g}$ coequalizes the pair (d_0^g, d_1^g). The left hand side squares are pullbacks, since so is the right hand side one. Accordingly, the pair $(R(s), s_0^g)$ is jointly extremally epic. So, the coequalization in question can be checked by composition with s_0^g (straightforward) and with $R(s)$, which is a direct consequence of $(*)$.

The pulling back in question is clearly faithful since it is pulling back along pullback stable regular epimorphisms. As for the fullness, consider the following diagram where the two quadrangles are pullbacks of split epimorphisms in the fibers along the cartesian regular epimorphism h in $\mathbb{C}$ and where m is any morphism of split epimorphisms:

$$\begin{tikzcd}
X \arrow[rr, two heads, "g"] \arrow[dr, "m"] \arrow[dd, shift right, "f"'] & & X' \arrow[dr, dotted, "n"] \arrow[dd, shift right, "f'"'] & \\
 & \bar{X} \arrow[rr, two heads, "\bar{q}"] \arrow[dl, shift right, "\bar{f}"'] & & \bar{X}' \arrow[dl, shift right] \\
Y \arrow[uu, shift right, "s"'] \arrow[ur, shift right, "\bar{s}"'] \arrow[rr, two heads, "h"'] & & Y' \arrow[uu, shift right, "s'"'] \arrow[ur, shift right, "\bar{s}'"'] &
\end{tikzcd}$$

The commutative square of split epimorphims being a pullback, the upward square towards X' is a pushout; so, the map m produces the desired dotted factorization n. $\square$

When we have a cartesian split epimorphism the result if even stronger:

Proposition 2.5. *Let $U : \mathbb{C} \to \mathbb{D}$ be a Mal'tsev fibration. Then, given any cartesian split epimorphism $(h, t) : V \rightleftarrows W$ in $\mathbb{C}$, pulling back along it* *the split epimorphisms in the fibers of U is a process which is "saturated on subobject".*

Proof. This means that, given any subobject m of $(\bar{f}', \bar{s}') = h^*(\bar{f}, \bar{s})$ in the fiber above $U(V)$:

$$\begin{tikzcd}
X' \arrow[dr, "m"] \arrow[dd, shift right, "f'"'] & & & \\
& \bar{X}' \arrow[rr, two heads, "\bar{h}"] \arrow[dl, shift right, "\bar{f}'"'] & & \bar{X} \arrow[dl, shift right, "\bar{f}"'] \\
V \arrow[uu, shift right, "s'"'] \arrow[ur, shift right, "\bar{s}'"'] \arrow[rr, two heads, "h"'] & & W \arrow[ur, shift right, "\bar{s}"'] &
\end{tikzcd}$$

there is a subobject n of $(\bar{f}, \bar{s})$ in the fiber above $U(W)$ such that $h^*(n) = m$. For that, complete the diagram with the kernel equivalence relations $R[h]$, $R[\bar{h}]$ and $R[\bar{h}.m]$. The factorization $R(m)$ between the two last ones is then a monomorphism.

$$\begin{tikzcd}
R[\bar{h}.m] \arrow[rr, shift left=2, "\delta_0"] \arrow[rr, shift right=2, "\delta_1"'] \arrow[dr, "R(m)"] \arrow[dd, shift right, "R(f')"'] & & X' \arrow[ll] \arrow[dr, "m"] \arrow[dd, shift right, "f'"'] & & & \\
& R[\bar{h}] \arrow[rr, shift left=2, "d_0^{\bar{h}}"] \arrow[rr, shift right=2, "d_1^{\bar{h}}"'] \arrow[dl, shift right] & & \bar{X}' \arrow[ll] \arrow[rr, two heads, "\bar{h}"] \arrow[dl, shift right, "\bar{f}'"'] & & \bar{X} \arrow[dl, shift right, "\bar{f}"'] \\
R[h] \arrow[uu, shift right, "R(s')"'] \arrow[ur, shift right] \arrow[rr, shift left=2, "d_0^h"] \arrow[rr, shift right=2, "d_1^h"'] & & V \arrow[ll] \arrow[uu, shift right, "s'"'] \arrow[ur, shift right, "\bar{s}'"'] \arrow[rr, two heads, "h"'] & & W \arrow[ur, shift right, "\bar{s}"'] &
\end{tikzcd}$$

The left hand side quadrangles indexed by 0 and 1 are pullbacks since so is the right hand side one. In the context of a Mal'tsev fibration, the left hand side commutative vertical squares are pullbacks as well: indeed, since $R(m)$ is a monomorphism, it is also the case for the factorization τ of the left hand side vertical square indexed by 0 to the pullback of (f', s') along the split epimorphism (d_0^h, s_0^h); but this τ is an extremal epimorphism as well by Lemma 2.3, since the fibration is a Mal'tsev one; so this factorization τ is an isomorphism, and the vertical left hand side square indexed by 0 is a pullback.

Now, the following downward left hand side diagram is underlying a discrete fibration between equivalence relations:

$$\begin{tikzcd}
R[\bar{h}.m] \arrow[r, shift left=2, "\delta_0"] \arrow[r, shift right=2, "\delta_1"'] \arrow[d, shift right, "R(f')"'] & X' \arrow[l] \arrow[r, dotted, two heads, "h'"] \arrow[d, shift right, "f'"'] & X \arrow[l, shift left=2, "t'"] \arrow[d, shift right, "f"'] \\
R[q] \arrow[r, shift left=2, "d_0^h"] \arrow[r, shift right=2, "d_1^h"'] \arrow[u, shift right, "R(s')"'] & V \arrow[l] \arrow[r, two heads, "h"] \arrow[u, shift right, "s'"'] & W \arrow[l, shift left=2, "t"] \arrow[u, shift right, "s"']
\end{tikzcd}$$

Then, by Lemma 2.2 in [7], the above right hand side pullback along t gives to t' a retraction h' above h which is the quotient of the equivalence relation $R[\bar{h}.m]$; accordingly, $n = t^*(m)$ produces the monomorphism we were looking for. $\square$

3. Schreier split epimorphims in the fibers $Cat_Y\mathbb{E}$

3.1 Schreier split epimorphims

It is clear that all the previous results concerning the fibered reflection $(\ )_0 : Cat\mathbb{E} \to \mathbb{E}$ remain valid for its restriction $(\ )_0 : Grd\mathbb{E} \to \mathbb{E}$ to the internal groupoids in $\mathbb{E}$. One main structural fact estalished for this last fibration is the following one: any fiber $Grd_Y\mathbb{E}$ is protomodular [3], as is $Grd_1\mathbb{E} = Gp\mathbb{E}$ the category of internal groups in $\mathbb{E}$. This property does not hold in the fibers $Cat_Y\mathbb{E}$, see $Mon\mathbb{E}$. The aim of this section is to show that, however, any fiber $Cat_Y\mathbb{E}$ keeps one fraction of this property: it is only protomodular with respect to a certain class Σ_Y of split epimorphisms, on the model of what is shown for the categories Mon of monoids in [12] and $Mon\mathbb{E}$ of internal monoids in $\mathbb{E}$ in [13].

In the category Mon of monoids, a split epimorphism $(f, s) : A \rightleftarrows B$ is said to be a *Schreier* one if, for any $b \in B$, the map μ_b: $\mathrm{Ker}\, f \to f^{-1}(b)$ defined by $\mu_b(k) = k \cdot s(b)$ is bijective [20]. This defines a class Σ of split epimorphims in Mon which makes this category Σ-protomodular [12, 10], where:

Definition 3.1. *A pointed category $\mathbb{E}$ is said to be Σ-protomodular, when,* *given any split epimorphism $(f, s) \in \Sigma$, the following pullback:*

$$\begin{tikzcd} \mathrm{Ker} f \arrow[r, tail, "\ker f"] \arrow[d, shift right] & X \arrow[d, shift right, "f"'] \\ 1 \arrow[r, tail, "0_Y"'] \arrow[u, shift right] & Y \arrow[u, shift right, "s"'] \end{tikzcd}$$

is such that the pair $(s, \ker f)$ of subobjects of X is jointly extremally epic, namely is such that the supremum of this pair of subobjects is 1_X.

This definition extends the notion of pointed *protomodular category* [3] where the previous property holds for any split epimorphism in $\mathbb{E}$, the major examples of such categories being the category $Gp = Grd_1$ of groups. Recall the following observation which produces a global characterization of the Schreier split epimorphisms in Mon [12]:

Proposition 3.2. *A split epimorphism $(f, s) : X \rightleftarrows Y$ in Mon is a Schreier one if and only if there is a set-theoretical retraction $q : X \to \mathrm{Ker} f$ to the homomorphic inclusion* $\ker f$*:*

$$\begin{tikzcd} \mathrm{Ker} f \arrow[r, tail, "\ker f"'] \arrow[d, shift right] & X \arrow[d, shift right, "f"'] \arrow[l, dotted, bend right, "q"'] \\ 1 \arrow[r, tail, "0_Y"'] \arrow[u, shift right] & Y \arrow[u, shift right, "s"'] \end{tikzcd}$$

which, in addition, is such that, for all $x \in X$, we get $q(x) \cdot sf(x) = x$ and, for all $(k, y) \in \mathrm{Ker} f \times Y$, we get $q(k \cdot s(y)) = k$.

From that, it is not difficult to extend this definition to any fiber $Cat_Y\mathbb{E}$:

Definition 3.3. *Let $(f_\bullet, s_\bullet) : X_\bullet \rightleftarrows Y_\bullet$ be any split epimorphic ido-functor in the fiber $Cat_Y\mathbb{E}$. It is called a Schreier split epimorphism when there is a retraction $q_1 : X_1 \to (\mathrm{Ker} f)_1$ of $(\ker f)_1$ in $\mathbb{E}$:*

$$\begin{tikzcd} (\mathrm{Ker} f)_1 \arrow[r, tail, "(\ker f)_1"'] \arrow[d, shift right] & X_1 \arrow[d, shift right, "f_1"'] \arrow[l, dotted, bend right, "q_1"'] \\ Y \arrow[r, tail, "s_0^{Y_\bullet}"'] \arrow[u, shift right] & Y_1 \arrow[u, shift right, "s_1"'] \end{tikzcd}$$

such that: 1) $d_0^{X_\bullet}.(\ker f)_1.q_1 = d_1^{X_\bullet}$*, 2)* $d_1^{X_\bullet}.(s_1 f_1, q_1) = 1_{X_1}$ *and:*
3) $q_1.d_1^{X_\bullet}.(s_1 \times_Y (\ker f)_1) = p_0^K : (\mathrm{Ker} f)_1 \times_Y Y_1 \to (\mathrm{Ker} f)_1$.

The set-theoretical translations of these three equations are:
1) for any map $\phi : a \to b$ in the category $X_\bullet$, we get an endomap map: $q_1(\phi) : b \to b$ in $(Kerf)_1$ such that $\phi = q_1(\phi).s_1 f_1(\phi)$;
2) for any pair (ψ, α) of arrows in $Y_1 \times (\mathrm{Ker} f)_1$ with $d_1(\psi) = d_0(\alpha) = d_1(\alpha)$, we get: $q_1(\alpha.s_1(\psi)) = \alpha$.

In other words, we get a Schreier split epic ido-functor when, given any pair (a, b) of objects in the category $X_\bullet$, the monoid of endomorphisms on b belonging to the kernel of this functor $f_\bullet$ produces a special kind of action on the subsets of $Hom(a, b)$ whose elements have a same image by $f_\bullet$, an action which will be more precisely understood in Proposition 3.6. We shall denote by Σ_Y the class of Schreier split epimorphisms in the fiber $Cat_Y\mathbb{E}$. This class has good stability properties:

Proposition 3.4. *In* $Cat_Y\mathbb{E}$*, the class* Σ_Y *is stable under pullbacks and under composition. It is "point-congruous": namely, it is stable under products and under finite limits inside the category* $Pt(Cat_Y\mathbb{E})$ *of split epimorphims in* $Cat_Y\mathbb{E}$*.*

Here is now our second structural fact:

Theorem 3.5. *Any fiber* $Cat_Y\mathbb{E}$ *is* Σ_Y*-protomodular, i.e. it is such that, given any Schreier split epic ido-functor* $(f_\bullet, s_\bullet) : X_\bullet \to Y_\bullet$*, the following pullback in* $Cat_Y\mathbb{E}$*:*

$$\begin{tikzcd}
(\mathrm{Ker} f)_\bullet \arrow[r, tail, "(\ker f)_\bullet"] \arrow[d, shift right] & X_\bullet \arrow[d, shift right, "f_\bullet"'] \\
\Delta_Y \arrow[r, tail, "0_{Y_\bullet}"'] \arrow[u, shift right] & Y_\bullet \arrow[u, shift right, "s_\bullet"']
\end{tikzcd}$$

makes jointly extremally epic the pair $(s_\bullet, (\ker f)_\bullet)$ *of subobjects of* $X_\bullet$*.*

Proof. Thanks to the Yoneda embedding, we are allowed to check it in the set-theoretical environment, and it is quasi-immediate. Consider the following diagram in the category $X_\bullet$ for any map $\phi : a \to b$:

$$\begin{tikzcd} a \arrow[r, "s_\bullet f_\bullet(\phi)"] \arrow[d, equal] & b \arrow[d, "q_1(\phi)"] \\ a \arrow[r, "\phi"'] & b \end{tikzcd}$$

It is commutative by the axiom 1, so that any subcategory of $X_\bullet$ containing $(\ker f)_\bullet$ and $s_\bullet$ is equal to $X_\bullet$. □

3.2 Back again to the 2-categorical nature of $Cat\mathbb{E}$

In this section, we shall show how, actually, the notion of Schreier split epimorphisms is, again in a hidden way, related to the 2-categorical dimension of $Cat\mathbb{E}$. More than that, this notion, which is not intrinsic to $Mon\mathbb{E}$ by the presence of the non-homomorphic retraction q, becomes intrinsic, and more precisely intrinsic to this 2-dimensional aspect, when it is contextualized in $Cat\mathbb{E}$.

A functor $f_\bullet : X_\bullet \to Y_\bullet$ in Cat is a *cofibration*, when, given any pair $(a, \psi) \in X_0 \times Y_1$ with $d_0(\phi) = f_0(a)$, there is a universal map with domain a above it in X_1, which is called the *cocartesian* map above ψ. It is a *split cofibration* when the choice of these universal maps is enforced. In [22], it is shown that the split (co-)fibrations above $Y_\bullet$ in $Cat\mathbb{E}$ are clearly internally defined, as being the algebras of a left exact monad on the slice category $Cat\mathbb{E}/Y_\bullet$ which explicitely uses the natural transformations (via the notion of "comma category") and, therefore, is wholly based on the 2-categorical nature of $Cat\mathbb{E}$.

Now consider a ido-functor $f_\bullet : X_\bullet \to Y_\bullet$; if, in addition it is a split cofibration in Cat, the choice of the cocartesian maps determines an actual functorial splitting $s_\bullet : Y_\bullet \to X_\bullet$ of $f_\bullet$ which is such that any map $s_1(\psi)$ is the chosen cocartesian above ψ.

Proposition 3.6. *Let $(f_\bullet, s_\bullet) : X_\bullet \rightleftarrows Y_\bullet$ be a split epimorphism in a fiber $Cat_Y\mathbb{E}$. The following conditions are equivalent:*
1) it is a Schreier split epimorphism in $Cat_Y\mathbb{E}$;
2) it is an internal split ido-cofibration in $Cat\mathbb{E}$.

Proof. Let us check it in the set-theoretical environment. Suppose it is a Schreier split epimorphism. Start with a map $\psi : a \to b \in Y_1$; we are going to show that $s_1(\psi) : a \to b$ is a cocartesian map above ψ. So, let $\phi : a \to c$ be a map in X_1 with a factorization $f_1(\phi) = \chi.\psi$ in Y_1. Then, with the map $q_1(\phi).s_1(\chi) : b \to c$, we get the unique map in X_1 such that $\phi = (q_1(\phi).s_1(\chi)).s_1(\psi)$ and $f_1(q_1(\phi).s_1(\chi)) = \chi$.

Conversely suppose it is an internal split ido-cofibration. Since $s_1(\psi)$ is cocartesian, it determines, for any map $\phi : a \to b \in X_1$ above ψ a unique factorization $q_1(\phi) : b \to b$ such that $f_1(q_1(\phi)) = 1_a$ and $q_1(\phi).s_1(\psi) = \phi$; the first equality insuring that $q_1(\phi)$ is in $(\mathrm{Ker} f)_1$ and the second one insuring Axiom 1). Axiom 2) is then straighforward. $\square$

By duality, let us define by Σ_Y^{op} the class of split ido-fibrations in $Cat_Y\mathbb{E}$. It is clear that this class is stable under pullback, point-congruous and makes the fiber $Cat_Y\mathbb{E}$ a Σ_Y^{op}-protomodular category as well.

4. Outcomes of the partial protomodularity of $Cat_Y\mathbb{E}$

So, any fiber $Cat_Y\mathbb{E}$ inherites all the properties of a Σ-protomodular category, see [10]. Here, we shall develop some of them. The first one is the following:

Proposition 4.1 ([10]). *Any fiber $Cat_Y\mathbb{E}$ is a Σ_Y-Mal'tsev category; i.e. when the split epimorphism $(f'_\bullet, s'_\bullet)$ is a Schreier one, any pullback of split epimorphisms in $Cat_Y\mathbb{E}$:*

$$\begin{tikzcd} X_\bullet \arrow[r, shift left, "x_\bullet"] \arrow[d, shift right, "f_\bullet"'] & X'_\bullet \arrow[l, shift left, "\sigma_\bullet"] \arrow[d, shift right, "f'_\bullet"'] \\ Y_\bullet \arrow[r, shift left, "y_\bullet"] \arrow[u, shift right, "s_\bullet"'] & Y'_\bullet \arrow[l, shift left, "\tau_\bullet"] \arrow[u, shift right, "s'_\bullet"'] \end{tikzcd}$$

is such that the pair $(s_\bullet, \sigma_\bullet)$ of subobjects of $X_\bullet$ is jointly extremally epic.

Then let us introduce the following:

Definition 4.2. *Let Σ be any class of split epimorphims in a category $\mathbb{E}$.*

A reflexive relation R on an object X: $\begin{tikzcd} R \arrow[r, shift left=2, "d_1^R"] \arrow[r, shift right=2, "d_0^R"'] & X \arrow[l, "s_0^R" description] \end{tikzcd}$ *is said to be a Σ-one, when the split epimorphism (d_0^R, s_0^R) is in Σ, a morphism $f : X \to Y$ is* *said to be Σ-special when its kernel equivalence relation $R[f]$ is a Σ-one. An object X is said to be Σ-special when the terminal map $\tau_X : X \to 1$ is Σ-special or, equivalently, when its indiscrete equivalence relation ∇_X is a Σ-one. The same kind of definition can be extended to reflexive graphs, and to internal categories and groupoids.*

Warning: a split Σ-special morphism belongs to Σ, but the converse is not necessarilty true: a split epimorphism belonging to Σ is not necessarily Σ-special. In Mon, the natural preorder on the commutative monoid $\mathbb{N}$ of intergers is an emblematic example of a reflexive and transitive Schreier reflexive relation [12]. More generally, any preorder on a group G provides us with an example of a reflexive and transitive Schreier relation in Mon. We shall denote by $SPoMon$ the category of Schreier preordered monoids and order-preserving homomorphisms.

Theorem 4.3 ([10])**.** *Suppose the class Σ is stable under pullback and $\mathbb{E}$ is a Σ-Mal'tsev category. Then any reflexive (resp. reflexive and symmetric) Σ-relation is transitive (resp. an equivalence relation). When, in addition, Σ is point congruous, the full subcategory SP/Y of the slice category $\mathbb{E}/Y$ whose objects are the Σ-special morphims is a Mal'tsev category. It is the case, in particular of the Σ-core, namely the full subcategory $\Sigma\mathbb{E}_\sharp$ of $\mathbb{E}$ whose objects are Σ-special.*

When $\mathbb{E}$ is Σ-protomodular and Σ is point congruous, then SP/Y and $\Sigma\mathbb{E}_\sharp$ are protomodular.

In Mon, the core associated with the class of Schreier split epimorphisms is the protomodular subcategory Gp of groups [12]. In this section, inter alia, we shall characterize the objects of the protomodular core associated with the class Σ_Y in the fiber $Cat_Y\mathbb{E}$.

First, given any category $Y_\bullet$, its terminal map in the fiber $Y_\bullet \to \nabla_Y$ is a monomorphism if and only if $Y_\bullet$ is a preorder on Y. Since any monomorphism is necessarily Σ_Y-special, then any internal preorder on Y, seen as an internal category in $\mathbb{E}$, lies in the Σ_Y-core.

A reflexive relation R on a category $X_\bullet$ is given by a reflexive relation on each $Hom(a, b)$ which is stable under composition in $X_\bullet$. *It is a Schreier reflexive relation* if and only if for any pair (ψ, χ) of parallel arrows between a and b, we have $\psi R\chi$ if and only if there is a unique map $\phi : b \to b$ such that $1_b R\phi$ and $\chi = \phi.\psi$.

Accordingly, an equivalence relation R on a category $X_\bullet$ is a Schreier one if and only if:
1) for any object a, the class $\bar{1}_a$ is a subgroup of End_a;
2) the left action of the group $\bar{1}_b$ on any $Hom(a,b) \neq \emptyset$ is free and the classes of the equivalence relation R on $Hom(a,b)$ coincide with the orbits of this action. Whence the following definition which characterizes those categories which are in the protomodular Σ_Y-core (resp. the Σ_Y^{op}-core):

Definition 4.4. *An internal category $Y_\bullet$ is called Schreier special (respectively Schreier opspecial) when:*
1) the endosome $(EndY)_1 \rightleftarrows Y$ is a group in $\mathbb{E}/Y$; in other words, any endomorphism in $Y_\bullet$ is an automorphism;
2) the natural left action (resp. right action) of this group on the object $d_1^{Y_\bullet} : Y_1 \to Y$ (resp. $d_0^{Y_\bullet} : Y_1 \to Y$) in the slice category $\mathbb{E}/Y$ is simply transitive in this slice category.

Accordingly, as expected, the groupoids are Schreier-special (respectively Schreier opspecial) categories, but they are not the only ones, since, as we just saw, such is any preorder. On the other hand, the core, being protomodular, is a Mal'tsev category. Accordingly there is an intrinsic notion of *affine object* when this object is endowed with a (unique) internal Mal'tsev operation. A Schreier special category is an affine object in the protomodular core if and only if the group defined in 1) is an abelian one. In this way, any preorder appears as an affine Schreier special category.

We shall now briefly introduce some easy processes to produce Schreier split epimorphisms and Schreier special categories in the set-theoretical context.

Let Y be any set and $Y_\bullet$ any category with Y as set objects. Then, by the Grothendiek construction, any functor $F_\bullet : Y_\bullet \to Mon$ produces a Schreier split epimorphism $(U_\bullet, S_\bullet) : (FY)_\bullet \rightleftarrows Y_\bullet$ where the set objects of $(FY)_0$ is Y, and where a map $a \to b$ in $(FY)_1$ is given by a pair (ψ, α) with $\psi : a \to b \in Y_1$ and $\alpha \in F(b)$ and where the composition in the category $(FY)_\bullet$ is given by: $(\psi', \alpha').(\psi, \alpha) = (\psi'.\psi, \alpha' \cdot F(\psi')(\alpha))$. The identity map on the object a in $(FY)_0 = Y$ is given by $(1_a, 0_{F(a)})$; then we set $U_\bullet(\psi, \alpha) = \psi$ and $S_\bullet(\psi) = (\psi, 0_{F(b)})$.

Now, in the same way, starting with any functor $\bar{F}_\bullet : Y_\bullet \to SPoMon$, and denoting by $F_\bullet : Y_\bullet \to Mon$ the associated functor which forgets the preorders, then $\bar{F}_\bullet$ produces a Schreier preorder on the category $(FY)_\bullet$, by $(\psi,\alpha) \leq (\psi',\alpha')$ if and only if $\psi' = \psi$ and $\alpha \leq \alpha'$. This is the case, in particular, when $\bar{F}$ is chosen as the constant functor on the monoid $\mathbb{N}$.

Proposition 4.5. *Let $Y_\bullet$ be a Schreier special category and $F_\bullet : Y_\bullet \to Gp$ any functor, then the category $(FY)_\bullet$ is a Schreier special category as well. Accordingly, when $\mathbb{T}$ is a preorder on the set Y, any category $(F\mathbb{T})_\bullet$ of this kind is a Schreier special category. It is affine as soon as the functor F takes its values in the category Ab of abelian groups.*

Proof. Since $Y_\bullet$ is a Schreier special category, any $Hom_{Y_\bullet}(a,a)$ is a group. On the other hand, the restriction of the functor $F_\bullet$ to this group produces a group homomorphism $F_a : Hom_{Y_\bullet}(a,a) \to Aut(F(a))$. Then, being nothing but the semi-direct product $F(a) \rtimes Hom_{Y_\bullet}(a,a)$, any $Hom_{(FY)_\bullet}(a,a)$ is a group.

Now let $((\psi,x),(\chi,y))$ be a parallel pair of morphims between a and b in $(FY)_\bullet$. Since $Y_\bullet$ is a Schreier special category, there is a unique inversible map $\alpha : b \to b$ such that $\alpha.\psi = \chi$. Then the map $(\alpha, y \cdot F(\alpha)(x^{-1}))$ in $Hom_{(FY)_\bullet}(b,b)$ is the unique one such that: $(\alpha, y\cdot F(\alpha)(x^{-1})).(\psi,x) = (\chi,y)$. The last assertion is straightforward once recalled that any preorder is an affine Schreier special category. □

In this way, any group homomorphism $h : G \to G'$, seen as a functor: $\{0 \to 1\} \to Gp$ gives rise to a Schreier special category with two objects which is neither a preorder, nor a groupoid. It is an affine object in the protomodular core when both G and G' are abelian.

We shall close this section by just recalling two important consequences of the Σ_Y-protomodularity of the fibers $Cat_Y\mathbb{E}$, and we shall refer to [10] and [5] for the details:
1) any regular epic Σ_Y-special ido-functor $f_\bullet : X_\bullet \twoheadrightarrow Y_\bullet$ in $Cat_Y\mathbb{E}$ is the cokernel of its kernel;
2) there is, in $Cat_Y\mathbb{E}$, an intrinsic notion of *abelian* Σ_Y-equivalence relation R. When, in addition, the ground category $\mathbb{E}$ is exact, we can associate with any Σ_Y-special ido-extension $f_\bullet : X_\bullet \twoheadrightarrow Y_\bullet$ having an abelian kernel equivalence relation an internal abelian group $A_\bullet \rightleftarrows Y_\bullet$ in $(Cat_Y\mathbb{E})/Y_\bullet$ called the *direction* of this extension $f_\bullet$. Furthermore the set $Ext_{A_\bullet}(Y_\bullet)$ of isomorphism classes of such extensions above $Y_\bullet$ with a given direction $A_\bullet \rightleftarrows Y_\bullet$

is canonically endowed with an abelian group structure via a construction generalizing the classical Baer sum.

5. Some p recisions about the fibration $Grd\mathbb{E} \to \mathbb{E}$

When $\mathbb{E}$ is regular, the base-changes of the fibration $Cat\mathbb{E} \to \mathbb{E}$ along a regular epimorphism $f : X \twoheadrightarrow Y$ in $\mathbb{E}$ are fully faithful, and thus conservative, since the cartesian maps above regular epimorphisms in $\mathbb{E}$ are pullback stable regular epimorphisms in $Cat\mathbb{E}$; it is a fortiori the case for the same kind of base-changes of the fibration $Grd\mathbb{E} \to \mathbb{E}$.

In this section we shall show that, for $Grd\mathbb{E}$, the base-changes along a morphism $h : U \to X$ in $\mathbb{E}$ partially keep a conservative aspect, provided that the objects U and X have same support, and the ground category $\mathbb{E}$ is efficiently regular (and a fortiori exact), see Theorem 5.7 and Proposition 5.13 below. For that we need the following:

Definition 5.1 ([8]). *A category $\mathbb{E}$ is said to be efficiently regular when it is regular and such that any equivalence relation T on an object X which is a subobject $i : T \rightarrowtail R[f]$ of an effective equivalence relation by an effective monomorphism (i.e. an equalizer) i is itself effective.*

This notion is clearly stable under slicing and coslicing. The categories $GpTop$ and $AbTop$ of topological groups and topological abelian groups are examples of efficiently regular categories which are not exact ones.

5.1 Connected and aspherical internal groupoids

From now on, in this section, we shall suppose that the category $\mathbb{E}$ is at least regular. In such a category, the support of an object X is the subobject $J \rightarrowtail 1$ determined by the canonical decomposition of the terminal map $\tau_X : X \to 1$. Accordingly, an object X is said to have a global support when the terminal map $\tau_X : X \to 1$ is a regular epimorphism.

Then, since any fiber $Grd_Y\mathbb{E}$ is a regular category as well, any groupoid has a support in its fiber: $X_\bullet \twoheadrightarrow SuppX_\bullet \rightarrowtail \nabla_{X_0}$, and this support is an equivalence relation in $\mathbb{E}$. Let us recall the following:

Definition 5.2 ([6]). *A groupoid $X_\bullet$ is said to be connected when it has a global support in the fiber $Grd_{X_0}\mathbb{E}$; it is said to be aspherical, when, in* *addition, it object of objects X_0 has a global support in $\mathbb{E}$. When the equivalence relation $SuppX_\bullet$ has a quotient map $\gamma_{X_\bullet}$ in $\mathbb{E}$, its codomain, denoted by $\pi_0(X_\bullet)$, is called the internal object of the "connected components" of the groupoid $X_\bullet$. Then clearly $\gamma_{X_\bullet}$ is the coequalizer of the pair $(d_0^{X_\bullet}, d_1^{X_\bullet})$ in $\mathbb{E}$.*

In Set a groupoid $X_\bullet$ is aspherical when $X_0 \neq \emptyset$ and the groupoid $X_\bullet$ is connected in the usual sense. The connected groupoids are stable under the base-changes f^* of the fibration $Grd\mathbb{E} \to \mathbb{E}$. Aspherical ones are stable only when the domain of f has a global support.

5.2 Affine internal groupoids

Now, observe that, given any internal groupoid $X_\bullet$ in $\mathbb{E}$, the object $(d_0^{X_\bullet}, d_1^{X_\bullet}) : X_1 \to X_0 \times X_0$ in the slice category $\mathbb{E}/(X_0 \times X_0)$ (which is nothing but the level 1 of the terminal functor $X_\bullet \to \nabla_X$ in the fiber $Grd_{X_0}\mathbb{E}$) is canonically endowed with an *associative Mal'tsev operation* p defined (in set-theoretical terms) by $p(\phi, \chi, \psi) = \phi.\chi^{-1}.\psi$ for any triple of parallel maps in $X_\bullet$. This Mal'tsev operation will be a keypoint in the development below. Observe moreover that: $(*)$ $p(\phi.\beta, \chi.\beta, \psi.\beta) = p(\phi, \chi, \psi).\beta$ and $p(\gamma.\phi, \gamma.\chi, \gamma.\psi) = \gamma.p(\phi, \chi, \psi)$.

Definition 5.3. *The groupoid $X_\bullet$ is said to be* affine *in $\mathbb{E}$ when the ternary operation p is autonomous or, equivalently, when p is underlying an internal functor $p_\bullet : X_\bullet \times_0 X_\bullet \times_0 X_\bullet \to X_\bullet$ in the fiber $Grd_{X_0}\mathbb{E}$.*

The same notion was introduced under the name of *abelian* groupoid in [6], but we do prefer now *affine*, since the above second assertion exactly means that $X_\bullet$ is an affine object in the protomodular (whence Mal'tsev) fiber $Grd_{X_0}\mathbb{E}$. Any equivalence relation is an affine groupoid.

Proposition 5.4. *Given any fully faithful (=cartesian) functor $f_\bullet : X_\bullet \to Z_\bullet$, the groupoid $X_\bullet$ is affine as soon as so is $Z_\bullet$. If, in addition, $f_\bullet$ is split, $X_\bullet$ is affine if and only if so is $Z_\bullet$. When the category $\mathbb{E}$ is regular, the same equivalence holds for any cartesian regular epimorphism $f_\bullet$.*

Proof. The first point is straightforward, and the second too, since when a cartesian functor $f_\bullet$ is split, its splitting $s_\bullet$ is cartesian as well. Let us go to the third one. Let $f_\bullet : X_\bullet \twoheadrightarrow Z_\bullet$ be any cartesian regular epimorphism and $X_\bullet$ an affine groupoid. Then consider the following diagram in $\mathbb{E}$:

$$\begin{tikzcd}
R_2[(d_0,d_1)] \arrow[r, two heads, "R_2(f_1)"] \arrow[d, shift right=4, "d_2"'] \arrow[d, "d_1"] \arrow[d, shift left=4, "d_0"] \arrow[dd, bend right=60, "p_X"'] & R_2[(d_0,d_1)] \arrow[d, shift right=4, "d_2"'] \arrow[d, "d_1"] \arrow[d, shift left=4, "d_0"] \arrow[dd, bend left=60, "p_Z"] \\
R[(d_0,d_1)] \arrow[r, two heads, "R(f_1)"] \arrow[d, shift right=4, "d_1"'] \arrow[d, shift left=4, "d_0"] & R[(d_0,d_1)] \arrow[d, shift right=4, "p_1"'] \arrow[d, shift left=4, "p_0"] \\
X_1 \arrow[r, two heads, "f_1"] \arrow[u] \arrow[d, "(d_0,d_1)"'] & Z_1 \arrow[u] \arrow[d, "(d_0,d_1)"] \\
X_0 \times X_0 \arrow[r, two heads, "f_0 \times f_0"'] & Z_0 \times Z_0
\end{tikzcd}$$

The lower commutative square is a pullback since $f_\bullet$ is cartesian. Accordingly so are the upper ones. Since f_0 is a regular epimorphism, so are f_1 and the factorizations $R(f_1)$ and $R_2(f_1)$. Clearly the Mal'tsev operations p commute with any functor $f_\bullet$. Now, when p_X is autonomous (=$X_\bullet$ affine), so is p_Z since $R_2(f_1)$ is a regular epimorphism; so, $Z_\bullet$ is affine. □

5.3 The direction of an affine aspherical groupoid

In Set, given an affine groupoid $X_\bullet$, all the maps $\phi : x \to x'$ produce the same group homomorphism $\alpha \mapsto \phi.\alpha.\phi^{-1}$ between the groups End_x and $End_{x'}$. When, in addition, this groupoid is aspherical all these groups End_x are isomorphic; so, by the choice of an object x_0 and of a map $\phi_x : x \to x_0$ for all x, we get a canonical equivalence of categories $X_\bullet \simeq End_{x_0}$; in this way, the (=any) abelian group End_{x_0} becomes a meaningful invariant of this affine aspherical groupoid $X_\bullet$.

We are now going to recall from [6] how to define this invariant, called the *direction* of the aspherical affine groupoid $X_\bullet$, in an internal way. For that, we shall need the following kind of *anatomical decomposition of what is an internal groupoid* $X_\bullet$ which will consist in showing that *the upper horizontal part of the following diagram is again an internal groupoid* resulting of what we shall call:

the canonical action of the groupoid $X_\bullet$ on its endosome $(EndX)_\bullet$:

$$\begin{tikzcd}[column sep=large, row sep=large]
X_1 \times_0 R[(d_0^{X_\bullet}, d_1^{X_\bullet})] \arrow[r, shift left=2, "\bar{d}_2^{X_\bullet}"] \arrow[r, "\bar{d}_1^{X_\bullet}" description] \arrow[r, shift right=2, dotted, "\bar{d}_0^{X_\bullet}"'] \arrow[d, shift right, "X_1 \times_0 d_0^R"'] & R[(d_0^{X_\bullet}, d_1^{X_\bullet})] \arrow[r, shift left=2, "\delta_1^{X_\bullet}"] \arrow[r, shift right=2, dotted, "\delta_0^{X_\bullet}"'] \arrow[d, shift right, "d_0^R"'] & (EndX)_1 \arrow[l, "\sigma_0^{X_\bullet}" description] \arrow[r, dotted, two heads, "q"] \arrow[d, shift right, "(\rho X)_1"'] & dX_\bullet \arrow[d, shift right] \\
X_1 \times_0 X_1 \arrow[u, shift right, "X_1 \times_0 s_0^R"'] \arrow[r, shift left=2, "d_2^{X_\bullet}"] \arrow[r, "d_1^{X_\bullet}" description] \arrow[r, shift right=2, "d_0^{X_\bullet}"'] & X_1 \arrow[u, shift right, "s_0^R"'] \arrow[r, shift left=2, "d_1^{X_\bullet}"] \arrow[r, shift right=2, "d_0^{X_\bullet}"'] & X_0 \arrow[l, "s_0^{X_\bullet}" description] \arrow[u, shift right, "(\sigma X)_1"'] \arrow[r, "\tau_X"'] & 1 \arrow[u, shift right, "0"']
\end{tikzcd}$$

Proof. In the central part of this diagram, given any parallel pair $(\phi, \psi) \in R[(d_0^{X_\bullet}, d_1^{X_\bullet})]$ of morphisms of $X_\bullet$, we set $\delta_1^{X_\bullet}(\phi, \psi) = \psi.\phi^{-1}$. It is then straightforward to check that the downward square with horizontal maps indexed by 1 is a pullback, and then that $s_0^{X_\bullet}$ induces the retraction of $\delta_1^{X_\bullet}$ defined by $\sigma_0^{X_\bullet}(\alpha) = (1_{d_0(\alpha)}, \alpha)$. Being a pullback, this square transfers the group structure in the slice category $\mathbb{E}/X_0$ given by the endosome $(EndX)_\bullet$ to a group structure on the object d_0^R in $\mathbb{E}/X_1$ which is given by $(\phi, \psi) \circ (\phi, \chi) = (\phi, p(\psi, \phi, \chi))$.

Now let us set: $\delta_0^{X_\bullet}(\phi, \psi) = \phi^{-1}.\psi$. Then the downward and upward squares with horizontal maps indexed by 0 do commute as well, as shown by the following diagram:

$$\begin{tikzcd}
x \arrow[loop left, "\phi^{-1}.\psi"] \arrow[r, shift left, "\phi"] \arrow[r, shift right, "\psi"'] & x' \arrow[loop right, "\psi.\phi^{-1}"]
\end{tikzcd}$$

This makes the map $\delta_0^{X_\bullet}$ satisfy the first axiom of an action of the groupoid $X_\bullet$ on the split epimorphism $((\rho X)_1, (\sigma X)_1)$. Furthermore this map $\delta_0^{X_\bullet}$ respects the group laws of the slice categories since:
$\delta_0^{X_\bullet}(\phi, \psi) \circ \delta_0^{X_\bullet}(\phi, \chi) = (\phi^{-1}.\psi) \circ (\phi^{-1}.\chi)) = \phi^{-1}.\psi.\phi^{-1}.\chi = \phi^{-1}.p(\psi, \phi, \chi)$;
while: $\delta_0^{X_\bullet}((\phi, \psi) \circ (\phi, \chi)) = \delta_0^{X_\bullet}(\phi, p(\psi, \phi, \chi)) = \phi^{-1}.p(\psi, \phi, \chi)$.
So, the map $\delta_0^{X_\bullet}$ satisfies the first axiom of an action of the groupoid $X_\bullet$ on the endosome group $(EndX)_\bullet$.

We are going to show now that it satisfies the second axiom of an action, namely that, in the diagram above, we get $\delta_0^{X_\bullet}.\bar{d}_0^{X_\bullet} = \delta_0^{X_\bullet}.\bar{d}_1^{X_\bullet}$, where $\bar{d}_0^{X_\bullet}(\gamma, (\phi, \psi)) = (\gamma, \phi^{-1}.\psi.\gamma)$. So:
$\delta_0^{X_\bullet}.\bar{d}_0^{X_\bullet}(\gamma, (\phi, \psi)) = \delta_0^{X_\bullet}(\gamma, \phi^{-1}.\psi.\gamma) = \gamma^{-1}.\phi^{-1}.\psi.\gamma$;
and: $\delta_0^{X_\bullet}.\bar{d}_1^{X_\bullet}(\gamma, (\phi, \psi)) = \delta_0^{X_\bullet}(\phi.\gamma, \psi.\gamma) = \gamma^{-1}.\phi^{-1}.\psi.\gamma$.

This makes the upper horizontal diagram a groupoid (we shall denote by $\overline{EndX}_\bullet$) since the action in question is produced above a groupoid. So, the vertical downward and upward maps produce a split epic functor. This functorial situation is coherent with the vertical group structure on $(EndX)_\bullet$:
since $\bar{d}_0^{X_\bullet}((\gamma,(\phi,\psi)) \circ (\gamma,(\phi,\psi'))) = \bar{d}_0^{X_\bullet}(\gamma,(\phi,p(\psi,\phi,\psi'))$
$= (\gamma,\phi^{-1}.p(\psi,\phi,\psi').\gamma)$;
and: $\bar{d}_0^{X_\bullet}(\gamma,(\phi,\psi)) \circ \bar{d}_0^{X_\bullet}(\gamma,(\phi,\psi')) = (\gamma,\phi^{-1}.\psi.\gamma) \circ (\gamma,\phi^{-1}.\psi'.\gamma)$
$= (\gamma, p(\phi^{-1}.\psi.\gamma,\gamma,\phi^{-1}.\psi'.\gamma)) = (\gamma, p(\phi^{-1}.\psi.\gamma,\phi^{-1}.\phi.\gamma,\phi^{-1}.\psi'.\gamma))$.
By the identities $(*)$ given at the beginning of tis section, we get the desired equality between the two previous terms. $\square$

The quickest way to make emerge this anatomical decomposition is to see it as underlying the double category whose double arrows are the commutative squares in $X_\bullet$:

$$\begin{tikzcd}
x \arrow[r, "\phi"] \arrow[d, "\alpha"'] & x' \arrow[d, "\beta"] \\
x \arrow[r, "\phi"'] & x'
\end{tikzcd}$$

(which can be identified up to isomorphism with the parallel pair $(\phi,\beta.\phi) \in R[(d_0^{X_\bullet}, d_1^{X_\bullet})]$) and where the vertical and horizontal compositions are clear. These double arrows are the arrows of the groupoid $\overline{EndX}_\bullet$ having the endomap α as domain and the endomap β as codomain.

Now, let us suppose that, in addition, the groupoid $X_\bullet$ is connected, and let us consider the right hand side quadrangled pullback above the projection $p_1 : X_0 \times X_0 \to X_0$:

$$\begin{tikzcd}
 & X_0 \times (EndX)_1 \arrow[dd, shift right] \arrow[dr, shift left, "X_0 \times p_1"] \arrow[dr, dashed, bend left, "\check{p}_0"] & & \\
R[(d_0^{X_\bullet}, d_1^{X_\bullet})] \arrow[ur, two heads, "\partial"] \arrow[rr, shift left=2, "\delta_1^{X_\bullet}"] \arrow[rr, shift right=2, "\delta_0^{X_\bullet}"'] \arrow[dd, shift right, "d_0^R"'] & & (EndX)_1 \arrow[ll] \arrow[ul, shift left] \arrow[r, dotted, two heads, "q"] \arrow[dd, shift right, "(\rho X)_1"'] & dX_\bullet \arrow[dd, shift right] \\
 & X_0 \times X_0 \arrow[uu, shift right] \arrow[dr, shift left, "p_1"] \arrow[dr, shift right=2, "p_0"'] & & \\
X_1 \arrow[uu, shift right, "s_0^R"'] \arrow[ur, two heads] \arrow[rr, shift left=2, "d_1^{X_\bullet}"] \arrow[rr, shift right=2, "d_0^{X_\bullet}"'] & & X_0 \arrow[ll] \arrow[ul] \arrow[uu, shift right, "(\sigma X)_1"'] \arrow[r, two heads] & 1 \arrow[uu, shift right, "0"']
\end{tikzcd}$$

Since, in the lower left hand side part of this diagram, the square indexed by 1 is a pullback the factorization $\partial : R[(d_0^{X_\bullet}, d_1^{X_\bullet})] \to X_0 \times (EndX)_1$ induced by the regular epimorphism $(d_0^{X_\bullet}, d_1^{X_\bullet}) : X_1 \twoheadrightarrow X_0 \times X_0$ is a regular epimorphism as well; as such ∂ is the quotient of its kernel equivalence relation $R[\partial]$. In set-theoretical terms we get $\partial(\phi,\psi) = (d_0(\psi), d_1^{X_\bullet}(\phi,\psi)) = (d_0(\phi), d_1^{X_\bullet}(\phi,\psi))$. So, in categorical terms, we get $\partial = (d_0^{X_\bullet}.d_0^R, \delta_1^{X_\bullet}) = (d_0^{X_\bullet}.d_1^R, \delta_1^{X_\bullet})$.

Lemma 5.5. *In any category $\mathbb{E}$, the kernel equivalence relation $R[\partial]$ is nothing but the Chasles equivalence relation $Ch[p]$ on the object $R[(d_0^{X_\bullet}, d_1^{X_\bullet})]$ associated with the associative Mal'tsev operation $p : R_2[(d_0^{X_\bullet}, d_1^{X_\bullet})] \to X_1$. We get the inclusion $R[\partial] \subset R[\delta_0^{X_\bullet}]$ if and only if the groupoid $X_\bullet$ is affine.*

Proof. Given any associative Mal'tsev operation p on a set X, recall from [5] that it induces an equivalence relation $Ch[p]$ on $X \times X$ defined by $(x,z)Ch[p](x',z')$ if $x' = p(x,z,z')$. Thanks to the Yoneda embedding it is enough to check our assertion in Set. Starting with two pairs (ϕ,ψ) and (ϕ',ψ') of parallel arrows in $X_\bullet$, such that $\partial(\phi,\psi) = \partial(\phi',\psi')$, these pairs have same domain and codomain, and are such that $\psi.\phi^{-1} = \psi'.(\phi')^{-1}$. This last point holds if and only: if $\phi' = \phi.\psi^{-1}.\psi' = p(\phi,\psi,\psi')$; whence our first point.

Two pairs (ϕ,ψ) and (ϕ',ψ') of parallel arrows in the groupoid $X_\bullet$ are such that $\delta_0^{X_\bullet}(\phi,\psi) = \delta_0^{X_\bullet}(\phi',\psi')$ if and only if $\phi^{-1}.\psi = (\phi')^{-1}.\psi'$, namely $\phi' = \psi'.\psi^{-1}.\phi = p(\psi',\psi,\phi)$. So that $R[\partial] \subset R[\delta_0^{X_\bullet}]$ if and only $p(\phi,\psi,\psi')$ $= p(\psi',\psi,\phi)$, namely if and only if the groupoid $X_\bullet$ is affine. $\square$

Proposition 5.6. *Let $\mathbb{E}$ be a regular category and $X_\bullet$ an internal connected affine groupoid. Then, in the above diagram, there is a factorization $\check{p}_0$ above p_0 such that $\check{p}_0.\partial = \delta_0^{X_\bullet}$. The pair $(\check{p}_0, X_0 \times p_1)$ is then underlying the equivalence relation $Supp\overline{End}X_\bullet$, and we get a discrete fibration $Supp\overline{End}X_\bullet \to \nabla_{X_0}$.*

Suppose now $\mathbb{E}$ is efficiently regular, and $X_\bullet$ is aspherical. Then the equivalence relation $Supp\overline{End}X_\bullet$ admits a quotient $dX_\bullet$ which makes the right hand side square a pullback and provides $dX_\bullet$ with an internal abelian group structure. This abelian group $dX_\bullet$ is called the direction *of the aspherical affine groupoid $X_\bullet$.*

This construction functorially extends to any ido-functor $f_\bullet : X_\bullet \to Z_\bullet$ between aspherical affine groupoids in $\mathbb{E}$ in a way which makes the following *square a pullback:*

$$
\begin{tikzcd}
(EndX)_\bullet \arrow[r, two heads, "q_{X_\bullet}"] \arrow[d, "(Endf)_\bullet"'] & dX_\bullet \arrow[d, "df_\bullet"] \\
(EndZ)_\bullet \arrow[r, two heads, "q_{Z_\bullet}"'] & dZ_\bullet
\end{tikzcd}
$$

Accordingly, $df_\bullet$ is an isomorphism if and only if so is $(Endf)_\bullet$

Proof. By the previous lemma, as soon as $X_\bullet$ is an affine groupoid, we get $R[\partial] \subset R[\delta_0^{X_\bullet}]$. If $X_\bullet$ is connected, the map ∂ is a regular epimorphism, whence the factorization $\check{p}_0$ in question. Since the quadrangled square indexed by 1 is a pullback and ∇_{X_0} is a relation, so is $(\check{p}_0, X_0 \times p_1)$. Accordingly, this relation is underlying the equivalence relation $Supp\overline{End}X_\bullet$ which, then, is endowed with a discrete fibration $Supp\overline{End}X_\bullet \to \nabla_{X_0}$. Accordingly the inclusion $i : Supp\overline{End}X_\bullet \rightarrowtail ((\rho X)_1)^{-1}(\nabla_{X_0})$ is split in $\mathbb{E}$ and consequently a regular monomorphism in $\mathbb{E}$, so that, when $\mathbb{E}$ is efficiently regular, $Supp\overline{End}X_\bullet$ has a quotient $dX_\bullet$ which, by the Barr-Kock Theorem in regular categories, makes the above right hand square a pullback.

It remains to show that $dX_\bullet$ is endowed with an abelian group structure. First, it is clear that when the groupoid $X_\bullet$ is affine the endosome group $(EndX)_\bullet$ is abelian. Now consider the following extention of the previous diagram by the kernel equivalence relations of the vertical maps:

$$
\begin{tikzcd}[row sep=large, column sep=huge]
R[d_0^R] \arrow[r, shift left=2, "R(\delta_1^{X_\bullet})"] \arrow[r, shift right=2, "R(\delta_0^{X_\bullet})"'] \arrow[d, shift right=3, "d_1"'] \arrow[d, shift left=3, "d_0"] \arrow[d, bend left=50, "\circ"] & R[(\rho X)_1] \arrow[l] \arrow[r, two heads, "R(q)"] \arrow[d, shift right=3, "d_1"'] \arrow[d, shift left=3, "d_0"] \arrow[d, bend left=50, "\circ"] & dX_\bullet \times dX_\bullet \arrow[d, shift right=3, "p_1"'] \arrow[d, shift left=3, "p_0"] \arrow[d, dotted, bend left=50, "\circ"] \\
R[(d_0^{X_\bullet}, d_1^{X_\bullet})] \arrow[u] \arrow[r, shift left=2, "\delta_1^{X_\bullet}"] \arrow[r, shift right=2, "\delta_0^{X_\bullet}"'] \arrow[d, shift right=2, "d_0^R"'] & (EndX)_1 \arrow[u] \arrow[l] \arrow[r, two heads, "q"] \arrow[d, shift right=2, "(\rho X)_1"'] & dX_\bullet \arrow[u] \arrow[d, shift right=2] \\
X_1 \arrow[u, shift right=2, "s_0^R"'] \arrow[r, shift left=2, "d_1^{X_\bullet}"] \arrow[r, shift right=2, "d_0^{X_\bullet}"'] & X_0 \arrow[l] \arrow[u, shift right=2, "(\sigma X)_1"'] \arrow[r, two heads] & 1 \arrow[u, shift right=2, "0"']
\end{tikzcd}
$$

Since the lower right and side square is a pullback, such are the upper right hand side ones; and consequently the factorization $R(q)$ is a regular epimorphism. We showed that the maps $\delta_0^{X_\bullet}$ and $\delta_1^{X_\bullet}$ did respect the group structures $\circ$. Accordingly they produce the right hand side vertical dotted factorization

$\circ$ which gives the abelian group structure to $dX_\bullet$. The last assertion is then straightforward. $\square$

We can now assert the first result we were aiming to:

Theorem 5.7. *Let $\mathbb{E}$ be an efficiently regular category and $h : U \to X$ a morphism such that U and X have same support in $\mathbb{E}$. Given any ido-functor $f_\bullet : X_\bullet \to Z_\bullet$ between connected affine groupoids, if $h^*(f_\bullet) : h^*(X_\bullet) \to h^*(Z_\bullet)$ is an isomorphism, then so is $f_\bullet$.*

Proof. Let $h = m.\bar{h}$ be the canonical decomposition of h into a regular epimorphism and a monomorphism. Since, by Proposition 1.3.2, the base change $\bar{h}^*$ is certainly conservative, it is enough to prove our assertion when $m : U \rightarrowtail X$ is a monomorphism.

Let us denote by $J \rightarrowtail 1$ the common support of U and X. This J is the common quotient of the equivalence relations ∇_X and ∇_U. In particular the connected groupoids $X_\bullet$ and $Z_\bullet$ in $\mathbb{E}$ become aspherical groupoids in the category $\mathbb{F} = \mathbb{E}/J$ which is efficiently regular as well. All the diagrams in $\mathbb{E}$ involved by our assertion actually lie in $\mathbb{F}$ and they are preserved by the left exact forgetful functor $\mathbb{F} \to \mathbb{E}$ which is obviously conservative and which preserves and reflects the regular epimophisms. So, we can now work without any restriction in the category $\mathbb{F}$.

We know that any fiber $Grd_Y\mathbb{F}$ is protomodular [3]. This fiber being regular as well, an ido-functor $f_\bullet : X_\bullet \to Z_\bullet$ between connected groupoids is an isomorphism if and only if the functor $(Endf)_\bullet$ is itself an isomorphism. If, moreover, the two groupoids are aspherical and affine, then, thanks to the previous proposition, this condition is equivalent to: $df_\bullet$ is an isomorphism. So, our assumption is equivalent to: $dm^*(f_\bullet)$ is an isomorphism. Now, consider the following diagram in $\mathbb{F}$, where U and X have a global support:

$$\begin{tikzcd}
(End(m^*X))_1 \arrow[r, tail, "\tilde{m}"] \arrow[d, shift right, "(\rho(m^*X))_1"'] & (EndX)_1 \arrow[r, two heads, "q"] \arrow[d, shift right, "(\rho X)_1"'] & dX_\bullet \arrow[d, shift right] \\
U \arrow[r, tail, "m"] \arrow[u, shift right, "(\sigma(m^*X))_1"'] \arrow[rr, bend right, "\tau_U"'] & X \arrow[r, two heads, "\tau_X"] \arrow[u, shift right, "(\sigma X)_1"'] & 1 \arrow[u, shift right, "0"']
\end{tikzcd}$$

Since the map $m^*(X_\bullet) \rightarrowtail X_\bullet$ is cartesian, the left hand side square is a pullback. And since U has a global support, we get $dm^*X_\bullet = dX_\bullet$; so, we get $dm^*f_\bullet = df_\bullet$ as well. Accordingly, $df_\bullet$ is an isomorphism, which implies that so is $f_\bullet$. □

5.4 Internal weak equivalences and affine groupoids

Let $f_\bullet : X_\bullet \to Z_\bullet$ be any functor in $Grd\mathbb{E}$. Consider the following left and side pullback in $\mathbb{E}$:

$$\begin{tikzcd} Af_0 \arrow[r, "\phi_0"] \arrow[d, shift right, "\delta_0"'] & Z_1 \arrow[r, "d_1^{Z_\bullet}"] \arrow[d, shift right, "d_0^{Z_\bullet}"'] & Z_0 \\ X_0 \arrow[r, "f_0"'] \arrow[u, shift right, "\sigma_0"'] & Z_0 \arrow[u, shift right, "s_0^{Z_\bullet}"'] & \end{tikzcd}$$

Definition 5.8. *Let $\mathbb{E}$ be a regular category. An internal functor $f_\bullet : X_\bullet \to Z_\bullet$ is said to be essentially surjective when the upper horizontal map $d_1^{Z_\bullet}.\phi_0$ is a regular epimorphism in $\mathbb{E}$. It is said to be a weak equivalence, when, in addition, it is fully faithful (i.e. $(\ )_0$-cartesian).*

Given any essentially surjective functor $f_\bullet : X_\bullet \to Z_\bullet$, the objects X_0 and Z_0 have necessarily same support. The essentially surjective functors (resp. the weak equivalences) are stable under composition; when $g_\bullet.f_\bullet$ is essentially surjective, so is $g_\bullet$.

The 2-category $Cat\mathbb{E}$ is actually a strongly representable 2-category in the sense of [17]: namely, for any internal category $Z_\bullet$, there is a universal natural transformation with codomain $Z_\bullet$:

$$\begin{tikzcd} ComZ_\bullet \arrow[r, shift left=2, "(\bar\delta_0^Z)_\bullet"] \arrow[r, shift right=2, "(\bar\delta_1^Z)_\bullet"'] \arrow[r, phantom, "\Downarrow"] & Z_\bullet \end{tikzcd}$$

where $(ComZ)_0$ is Z_1 and $(ComZ)_1$ is the internal object of the "commutative squares" in $Z_\bullet$, i.e. it is obtained as the object $R[d_1^{Z_\bullet}]$ determined by the kernel equivalence relation of the map $d_1^{Z_\bullet} : Z_2 \to Z_1$ in $\mathbb{E}$. We get a common section $(\bar\sigma_0^Z)_\bullet$ of the pair $((\bar\delta_0^Z)_\bullet, (\bar\delta_1^Z)_\bullet)$ from the identity natural transformation $1_{Z_\bullet} \Rightarrow 1_{Z_\bullet}$. Internal groupoids are characterized among internal categories by the following:

Lemma 5.9. *[2] An internal category $Z_\bullet$ is groupoid if and only if the split epimorphism $((\bar\delta_0^Z)_\bullet, (\bar\sigma_0^Z)_\bullet)$ (resp. $((\bar\delta_1^Z)_\bullet, (\bar\sigma_0^Z)_\bullet)$) is cartesian in $Cat\mathbb{E}$.*

Now consider the following left hand side pullback in $Grd\mathbb{E}$:

$$\begin{tikzcd} Af_\bullet \arrow[r, "\phi_\bullet"] \arrow[d, shift right, "(\bar\delta_0^f)_\bullet"'] & ComZ_\bullet \arrow[r, "(\bar\delta_1^Z)_\bullet"] \arrow[d, shift right, "(\bar\delta_0^Z)_\bullet"'] & Z_\bullet \\ X_\bullet \arrow[r, "f_\bullet"'] \arrow[u, shift right, "(\bar\sigma_0^f)_\bullet"'] & Z_\bullet \arrow[u, shift right, "(\bar\sigma_0^Z)_\bullet"'] & \end{tikzcd}$$

Straightforward are the following observations:

Lemma 5.10. *1) The split epimorphism $((\bar\delta_0^f)_\bullet, (\bar\sigma_0^f)_\bullet)$ is cartesian. 2) We get $f_\bullet = ((\bar\delta_1^Z)_\bullet.\phi_\bullet).(\bar\sigma_0^f)_\bullet$. Accordingly, 3) the functor $f_\bullet$ is cartesian if and only if so is the functor $(\bar\delta_1^Z)_\bullet.\phi_\bullet$.*

When the category $\mathbb{E}$ is an exact one, any equivalence relation $SuppX_\bullet$ has a quotient which is nothing but the internal object $\pi_0(X_\bullet)$ of the "connected components" of the groupoid $X_\bullet$. This construction produces a left adjoint to the fully faithful functor $\Delta : \mathbb{E} \to Grd\mathbb{E}$. Basic is the following:

Lemma 5.11. *Let $\mathbb{E}$ be an exact category. Given any parallel pair $(f_\bullet, g_\bullet)$ of functors between groupoids, we get $\pi_0(f_\bullet) = \pi_0(g_\bullet)$ as soon as we have a natural transformation $\alpha : f_\bullet \Rightarrow f'_\bullet$.*

Proof. Given any natural transformation $\alpha : f_\bullet \Rightarrow g_\bullet$, the map $\alpha_0 : X_0 \to Z_1$ underlying this natural transformation is such that $d_0^{Z_\bullet}.\alpha_0 = f_0$ and $d_1^{Z_\bullet}.\alpha_0 = g_0$, so that the coequalizer $\gamma_{Z_\bullet} : Z_0 \twoheadrightarrow \pi_0(Z_\bullet)$ of the pair $(d_0^{Z_\bullet}, d_1^{Z_\bullet})$ coequalizes the pair (f_0, g_0) as well, which implies $\pi_0(f_\bullet) = \pi_0(g_\bullet)$. $\square$

More meaningful are following ones:

Lemma 5.12. *Let $\mathbb{E}$ be an exact category. 1) When the functor $f_\bullet$ is fully faithful (=cartesian), then $\pi_0(f_\bullet)$ is a monomorphism. 2) The functor $f_\bullet$ is essentially surjective if and only if $\pi_0(f_\bullet)$ is a regular epimorphism. So, when $f_\bullet$ is a weak equivalence, then $\pi_0(f_\bullet)$ is an isomorphism.*

Proof. When $f_\bullet$ is fully faithful, then $SuppX_\bullet = f_0^{-1}(SuppZ_\bullet)$, so that $\pi_0(f_\bullet)$ is a monomorphism.

It is clear that, as soon as f_0 is a regular epimorphism, $\pi_0(f_\bullet)$ is a regular epimorphism. Suppose $f_\bullet$ is essentially surjective. The functor $\phi_\bullet : Af_\bullet \to ComZ_\bullet$ determines a natural transformation $\alpha : f_\bullet.(\bar\delta_0^f)_\bullet \Rightarrow (\bar\delta_1^X)_\bullet.\phi_\bullet$. So, we get $\pi_0(f_\bullet).\pi_0((\bar{\delta}_0^f)_\bullet) = \pi_0((\bar{\delta}_1^Z)_\bullet.\phi_\bullet)$. Now, when $f_\bullet$ is essentially surjective, the map $\pi_0((\bar{\delta}_1^Z)_\bullet.\phi_\bullet)$ is a regular epimorphism since $d_1^{Z_\bullet}.\phi_0$ is a regular epimorphism; accordingly, so is $f_\bullet$. Conversely, suppose that the map $\pi_0(f_\bullet)$ is a regular epimorphism. Then consider the following diagram built from the vertical functor $f_\bullet$:

$$\begin{tikzcd}
X_1 \arrow[rr, two heads, "\eta_{X_\bullet}"] \arrow[dd, "f_1"'] & & SuppX_\bullet \arrow[rr, shift left=2, "d_1"] \arrow[rr, shift right=2, "d_0"'] \arrow[dd, "Supp(f_\bullet)"'] & & X_0 \arrow[ll] \arrow[rr, two heads, "q_{X_\bullet}"] \arrow[dd, "f_0"] & & \pi_0(X_\bullet) \arrow[dd, "\pi_0(f_\bullet)"] \\
& Af_0 \arrow[rr, "\bar{\eta}"] \arrow[dl, "\phi_0"] & & \Sigma \arrow[rr, dotted, two heads, "\bar{d}_1"'] \arrow[dl, "\psi"] \arrow[ur, "\bar{d}_0"] & & P \arrow[dl, "\phi"] \arrow[ur] & \\
Z_1 \arrow[rr, two heads, "\eta_{Z_\bullet}"'] & & SuppZ_\bullet \arrow[rr, shift left=2, "d_1"] \arrow[rr, shift right=2, "d_0"'] & & Z_0 \arrow[ll] \arrow[rr, two heads, "q_{Z_\bullet}"'] & & \pi_0(Z_\bullet)
\end{tikzcd}$$

where ϕ is the pullback of the regular epimorphism $\pi_0(f_\bullet)$ along the regular epimorphism $q_{Z_\bullet}$, so that ϕ is a regular epimorphism; and where ψ is the pullback of f_0 along d_0. These pullbacks produce a factorization $\bar{d}_1 : \Sigma \to P$ above d_1, which by commutation of limits makes the upper upward right hand side quadrangle a pullback as well. Accordingly, since $q_{X_\bullet}$ is a regular epimorphism, so is $\bar{d}_1$. Finally, let ϕ_0 be the pullback of ψ along the regular epimorphism $\eta_{Z_\bullet}$ so that: 1) $\bar{\eta}$ is a regular epimorphism and 2) the map $\bar{d}_0.\bar{\eta}$ is nothing but the map δ_0 of the diagram defining Af_0 in the definition of an essentially surjective functor. Now $d_1^{Z_\bullet}.\phi_0 = d_1.\eta_{Z_\bullet}.\phi_0 = d_1.\psi.\bar{\eta} = \phi.\bar{d}_1.\bar{\eta}$, where these three last maps are regular epimorphisms. So, $d_1^{Z_\bullet}.\phi_0$ is a regular epimorphism, and the functor $f_\bullet$ is essentially surjective. □

Whence the second important result we were aiming to:

Proposition 5.13. *Let $\mathbb{E}$ be an exact category. Consider any weak equivalence $f_\bullet : X_\bullet \to Z_\bullet$. Then $X_\bullet$ is affine if and only if $Z_\bullet$ is affine. In this case these groupoids are both aspherical in the slice category $\mathbb{E}/Q$ where $Q = \pi_0(Z_\bullet)$, and they have same direction in $\mathbb{E}/Q$.*

Accordingly, when $Z_\bullet$ is affine, then the groupoid $Z_\bullet$ is an equivalence relation if and only if so is $X_\bullet$.

Proof. In any category $\mathbb{E}$, when $f_\bullet : X_\bullet \to Z_\bullet$ is fully faithful, then $X_\bullet$ is affine (resp. an equivalence relation) as soon as so is $Z_\bullet$.

Conversely, suppose that $X_\bullet$ is affine. According to Lemma 5.10.1, so is the groupoid $Af_\bullet$. When $f_\bullet$ is fully faithful, so is the functor $\psi_\bullet = (\bar{\delta}_1^Z)_\bullet.\phi_\bullet$. But ψ_0 is a regular epimorphism in $\mathbb{E}$. Then, according to Proposition 5.4, the groupoid $Z_\bullet$ is affine, since so is $Af_\bullet$.

Since $f_\bullet$ is a weak equivalence, $\pi_0(f_\bullet)$ is an isomorphism, and consequently all the diagrams involved by our situation lie in the slice category $\mathbb{E}/\pi_0(Z_\bullet)$ where the groupoids $X_\bullet$ and $Z_\bullet$ become aspherical; so that, being affine, they get a direction in this exact slice category. It remains to show that these directions are the same. For that consider the following diagram:

$$\begin{tikzcd}
(EndX)_1 \arrow[r, "(Endf)_1"] \arrow[d, shift right, "(\rho X)_1"'] & (EndZ)_1 \arrow[r, two heads, "q_Z"] \arrow[d, shift right, "(\rho Z)_1"'] & dZ_\bullet \arrow[d, shift right] \\
X_0 \arrow[r, "f_0"] \arrow[u, shift right, "(\sigma X)_1"'] \arrow[rr, two heads, bend right] & Z_0 \arrow[r, two heads] \arrow[u, shift right, "(\sigma Z)_1"'] & \pi_0(Z_\bullet) \arrow[u, shift right, "0"']
\end{tikzcd}$$

The right hand side square is a pullback by definition. The left hand side one is a pullback since $f_\bullet$ is fully faithful. So the whole rectangle is a pullback; and since the long lower horizontal map is a regular epimorphism, this rectangle defines the direction of the aspherical affine groupoid $X_\bullet$. Accordingly $dX_\bullet = dZ_\bullet$.

Now, suppose $Z_\bullet$ is affine; then $X_\bullet$ is affine. Saying that $X_\bullet$ is an equivalence relation is saying that its direction in the slice category $\mathbb{E}/\pi_0(Z_\bullet)$ is trivial, namely the terminal object. According to the first part of the proposition, so is the direction of $Z_\bullet$, which, in turn, means that $Z_\bullet$ is an equivalence relation. $\square$

The Theorem 5.7 and the last part of the previous proposition are of particular interest in the Mal'tsev and Gumm categories, where any groupoid is affine, see [8], [18] and [11].

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