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A BASIS THEOREM FOR 2-RIGS AND RIG GEOMETRY

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Résumé. Un semi-anneau unitaire commutatif (ou *rig*, en abrégé) est *intégral* si $1 + x = 1$. Nous montrons que, de même que le classique 'gros topos' de Zariski associé à un corps algébriquement clos, le topos classifiant $\mathcal{Z}$ des rigs intégraux (réellement) locaux est pré-cohésif sur $\mathbf{Set}$. Le problème principal est de montrer que le morphisme géométrique canonique $\mathcal{Z} \to \mathbf{Set}$ est hyperconnexe essentiel et, encore comme dans le cas classique, le problème se réduit à certains résultats purement algébriques. L'hyperconnectivité est liée à une caractérisation inédite des rigs simples due à Schanuel. L'essentialité est un corollaire d'un analogue d'un 'théorème de la base' prouvée ici pour les rigs avec addition idempotente.

Abstract. A commutative unitary semi-ring (or *rig*, for short) is *integral* if $1 + x = 1$. We show that, just as the classical 'gros' Zariski topos associated to an algebraically closed field, the classifying topos $\mathcal{Z}$ of *(really) local* integral rigs is pre-cohesive over $\mathbf{Set}$. The main problem is to show that the canonical geometric morphism $\mathcal{Z} \to \mathbf{Set}$ is hyperconnected essential and, again as in the classical case, the problem reduces to certain purely algebraic results. Hyperconnectedness is related to an unpublished characterization of simple rigs due to Schanuel. Essentiality is a corollary of an analogue of a 'Basis Theorem' for rigs with idempotent addition proved here.

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Contents

1. Rig geometry

The present work is motivated by the claim (in the second paragraph of [11]) that some semi-combinatorial non-classical examples of cohesion can be handled in ways analogous to Grothendieck's algebraic geometry. More specifically, we are interested in the construction of 'gros' toposes from certain algebraic categories in a way that abstracts the classical construction of the 'gros' Zariski topos and related toposes. To motivate and outline the contents of the paper it is convenient to recall some of the details of that construction and one source of examples. We assume that the reader is familiar with some basic Topos Theory [12, 6], Lattice Theory and Commutative

Algebra. (Incidentally rings are to be understood as in and [2], i.e. commutative, with unit.)

Definition 1.1. A category $\mathcal{C}$ is called *extensive* if it has finite coproducts and the canonical functor $\mathcal{C}/X \times \mathcal{C}/Y \to \mathcal{C}/(X+Y)$ is an equivalence for any X, Y in $\mathcal{C}$.

For instance, every topos is extensive. In contrast, an additive category is extensive if and only if it is degenerate. If $\mathcal{C}$ is extensive, then so is the slice $\mathcal{C}/X$ for any X. See [3] and references therein.

An object X in an extensive category will be called *connected* if it is not initial and, for every coproduct diagram $X_0 \to X \leftarrow X_1$, either X_0 is initial (in which case $X_1 \to X$ is an isomorphism or X_1 is initial (in which case $X_0 \to X$ is an isomorphism). Roughly speaking, an object is connected if it is not empty and has no coproduct decompositions. An object in a topos is connected if and only if it has exactly two complemented subobjects.

A category is called *coextensive* if its opposite is extensive. If $\mathcal{A}$ is coextensive then, trivially by duality, $A/\mathcal{A}$ is coextensive for every A in $\mathcal{A}$, and an object in $\mathcal{A}^{\mathrm{op}}$ is connected if and only if it is directly indecomposable as an object of $\mathcal{A}$.

Let $\mathbf{Ring}$ be the category of rings.

Lemma 1.2. *The category* $\mathbf{Ring}$ *is coextensive. An object in* $\mathbf{Ring}^{\mathrm{op}}$ *is connected if and only if the corresponding ring has exactly two idempotents.*

Proof. This is well-known but let us sketch a proof. A useful characterization [3, Proposition 2.14] states that a category is extensive if and only if coproducts are universal and disjoint. The dual of this characterization may be applied directly to $\mathbf{Ring}$ as soon as we understand (direct) product decompositions there. Recall that if A is a ring and $e \in A$ is idempotent then the span $A[(1-e)^{-1}] \leftarrow A \to A[e^{-1}]$ is a product diagram. Moreover, this construction determines a bijection between direct decompositions (of A) and idempotents (in A). (If $A \cong B \times C$ is a direct product decomposition then the unique element in A corresponding to $(0, 1)$ is the associated idempotent.) It easily follows from this description of direct decompositions that products are codisjoint and couniversal (i.e. stable under pushout). □

If $\mathcal{C}$ is an extensive category then the finite families $(X_i \to X \mid i \in I)$ such that the induced $\sum_{i \in I} X_i \to X$ is an isomorphism form the basis of a

Grothendieck topology. If $\mathcal{C}$ is also small then the associated category of sheaves is a topos that we denote by $\mathfrak{G}\mathcal{C}$ and which is sometimes called the 'Gaeta' topos (of $\mathcal{C}$).

Recall that, for any small category $\mathcal{C}$, the Yoneda embedding $\mathcal{C} \to \widehat{\mathcal{C}}$ of $\mathcal{C}$ into the topos of presheaves on $\mathcal{C}$ preserves limits but not colimits. For instance, every representable object in $\widehat{\mathcal{C}}$ is connected so, if $\mathcal{C}$ has finite coproducts then the Yoneda embedding does not preserve them. On the other hand, if $\mathcal{C}$ is extensive then the Gaeta topology is subcanonical and the resulting full embedding $\mathcal{C} \to \mathfrak{G}\mathcal{C}$ preserves finite coproducts [11].

Recall also that a geometric morphism $f : \mathcal{E} \to \mathcal{S}$ is *essential* if its inverse image f^* has a left adjoint usually denoted by $f_! : \mathcal{E} \to \mathcal{S}$. For example, for any small category $\mathcal{C}$, the canonical geometric morphism $\widehat{\mathcal{C}} \to \mathbf{Set}$ is essential. On the other hand, if $\mathcal{C}$ is small and extensive then $\mathfrak{G}\mathcal{C} \to \mathbf{Set}$ need not be essential; although it is in some cases arising in Algebraic Geometry.

If $\mathcal{C}$ is an extensive category then the full subcategory of connected objects will be denoted by $\mathcal{C}_c \to \mathcal{C}$. The existence of finite coproduct decompositions guarantees that the Gaeta topos is essential as the next result shows.

Lemma 1.3. *Let $\mathcal{C}$ be small and extensive. If every object of $\mathcal{C}$ is a finite coproduct of connected objects then the canonical geometric morphism $\mathfrak{G}\mathcal{C} \to \mathbf{Set}$ is essential.*

Proof. If every object in $\mathcal{C}$ is a finite coproduct of connected objects then the Comparison Lemma [6, Theorem C2.2.3] can be applied and it implies that the restriction functor $\widehat{\mathcal{C}} \to \widehat{\mathcal{C}_c}$ restricts itself to an equivalence $\mathfrak{G}\mathcal{C} \to \widehat{\mathcal{C}_c}$. In other words, in this case, the Gaeta topos of $\mathcal{C}$ is a presheaf topos and so the canonical geometric morphism to $\mathbf{Set}$ is essential. $\square$

Let K be a ring and let $K/\mathbf{Ring}$ be the associated coextensive category of K-algebras. Let $(K/\mathbf{Ring})_{fp} \to K/\mathbf{Ring}$ be the full subcategory of finitely presentable K-algebras. The *category of affine K-schemes (of finite type)* is the opposite of the category $(K/\mathbf{Ring})_{fp}$ and, for brevity, it will be denoted by $\mathbf{Aff}_K$. As $(K/\mathbf{Ring})_{fp} \to K/\mathbf{Ring}$ is closed under finite colimits, $\mathbf{Aff}_K$ has finite limits.

Lemma 1.4. *If the ring K is Noetherian then $\mathbf{Aff}_K$ is extensive and every object is a finite coproduct of connected objects.*

Proof. It is enough to check that the subcategory $(K/\mathbf{Ring})_{fp} \to K/\mathbf{Ring}$ is not only closed under closed under finite colimits but also under finite products, so that the domain inherits coextensivity from the codomain.

Finitely generated K-algebras are closed under finite products for arbitrary K but, if K is Noetherian then Hilbert's Basis Theorem implies that finitely generated K-algebras are finitely presented so, in this case, $(K/\mathbf{Ring})_{fp} \to K/\mathbf{Ring}$ is closed under finite products. Also, a Noetherian K-algebra cannot have an infinite product decomposition. (We will review a proof in a more general context later.) $\square$

We stress the role of Noetherianity and Hilbert's Basis Theorem in the proof of Lemma 1.4. We will come back to the issue. We will see that Noetherianity is not necessary to prove extensivity of $\mathbf{Aff}_K$. On the other hand, the finite-coproduct-decomposition property does not hold in general.

The presheaf topos $\widehat{\mathbf{Aff}_K}$ is the classifier of K-algebras. It embeds (via Yoneda) the category of K-affine spaces and every object in $\widehat{\mathbf{Aff}_K}$ is a colimit of affine spaces. In this sense, $\widehat{\mathbf{Aff}_K}$ is a topos of 'K-schemes' but, it does not have the 'right' colimits. In particular, it does not have the right coproducts. Extensivity of $\mathbf{Aff}_K$ permits to solve this problem because we may consider the subtopos $\mathfrak{G}(\mathbf{Aff}_K) \to \widehat{\mathbf{Aff}_K}$ and the finite-coproduct preserving restricted Yoneda embedding $\mathbf{Aff}_K \to \mathfrak{G}(\mathbf{Aff}_K)$, into another topos of 'K-schemes' so to speak, but with better coproducts. (See also [11, Section 5] for a more conceptual discussion on the inexactness of affine schemes.)

Lemma 1.5. *If the ring K is Noetherian then the canonical geometric morphism $\mathfrak{G}(\mathbf{Aff}_K) \to \mathbf{Set}$ is essential.*

Proof. By Lemma 1.4, Lemma 1.3 is applicable to the case $\mathcal{C} = \mathbf{Aff}_K$. $\square$

Let $f : \mathfrak{G}(\mathbf{Aff}_K) \to \mathbf{Set}$ be the canonical geometric morphism. For general reasons, the direct image $f_* : \mathfrak{G}(\mathbf{Aff}_K) \to \mathbf{Set}$ sends X in $\mathfrak{G}(\mathbf{Aff}_K)$ to the set $f_*X = \mathfrak{G}(\mathbf{Aff}_K)(1, X) = X1$ of points of X. More explicitly, in this case, it sends a sheaf $X : (K/\mathbf{Ring})_{fp} \to \mathbf{Set}$ to the set $f_*X = XK$ where K is considered as the initial object of $(K/\mathbf{Ring})_{fp}$. In particular, if X is representable by A in $(K/\mathbf{Ring})_{fp}$ then

$$f_*X = (K/\mathbf{Ring})_{fp}(A, K) = (K/\mathbf{Ring})(A, K)$$

is the set of algebra morphisms from A to the base ring K.

As stressed in [10, Section II], even if we assume that K is a field, the leftmost adjoint $f_! : \mathfrak{G}(\mathbf{Aff}_K) \to \mathbf{Set}$ need not preserve finite products. (See also [16, Example 4.8].) This observation partially motivates the following axiomatization of a topos 'of spaces' over a topos 'of sets'.

Definition 1.6. A geometric morphism $p : \mathcal{E} \to \mathcal{S}$ is called *pre-cohesive* if the adjunction $p^* \dashv p_*$ extends to a string $p_! \dashv p^* \dashv p_* \dashv p^!$ of adjoint functors such that $p^*, p^! : \mathcal{S} \to \mathcal{E}$ are fully faithful, $p_! : \mathcal{E} \to \mathcal{S}$ preserves finite products and (Nullstellensatz) the canonical transformation $\theta : p_* \to p_!$ is epic.

The intuition is that $\mathcal{E}$ is a 'gros' topos over a topos $\mathcal{S}$ of 'sets'. (See [10], also [16, 14] and references therein.) So the objects of $\mathcal{E}$ are 'spaces' of some kind, $p^* : \mathcal{S} \to \mathcal{E}$ is the full subcategory of discrete spaces and its right adjoint $p_* : \mathcal{E} \to \mathcal{S}$ sends a space X to the set $p_* X$ of points of X. The leftmost adjoint $p_! : \mathcal{E} \to \mathcal{S}$ sends a space X to the set $p_! X$ of 'pieces' of X. The Nullstellensatz condition formulated above captures the idea that 'every piece has a point'. (In the presence of a string $p_! \dashv p^* \dashv p_* \dashv p^!$ with fully faithful $p^*, p^! : \mathcal{S} \to \mathcal{E}$, the Nullstellensatz is equivalent to $p : \mathcal{E} \to \mathcal{S}$ being *hyperconnected*, i.e. that both the unit an counit of $p^* \dashv p_*$ are monic [7].)

Proposition 1.7. *If K is an algebraically closed field then the essential geometric morphism $\mathfrak{G}(\mathbf{Aff}_K) \to \mathbf{Set}$ is pre-cohesive.*

Proof. We already know by Lemma 1.5 that $\mathfrak{G}(\mathbf{Aff}_K) \to \mathbf{Set}$ is essential. In fact, we know it is essential because $\mathfrak{G}(\mathbf{Aff}_K)$ is the topos of presheaves on the category of connected affine K-schemes. So it is enough to apply a characterization of the small categories whose associated presheaf topos is pre-cohesive over $\mathbf{Set}$ [7]: for a small category $\mathcal{D}$ whose idempotents split, the canonical $\widehat{\mathcal{D}} \to \mathbf{Set}$ is pre-cohesive if and only if $\mathcal{D}$ has a terminal object and every object has a point. (See also [16, Proposition 2.10].)

Let $\mathcal{C} = \mathbf{Aff}_K$ be the category of K-affine schemes. Since it has finite limits, idempotents split. Moreover, this property is inherited by the subcategory $\mathcal{C}_c$ of connected objects. As K is a field, it is directly indecomposable. Hence, the terminal object of $\mathbf{Aff}_K$ is connected and so $\mathcal{C}_c$ has a terminal object. Hilbert's Nullstellensatz implies that every object in $\mathcal{C}_c$ has a point. Then, by the result cited in the previous paragraph, $\mathfrak{G}(\mathbf{Aff}_K) = \widehat{\mathcal{C}_c} \to \mathbf{Set}$ is pre-cohesive. $\square$

If K is not algebraically closed then K-affine spaces still induce pre-cohesive geometric morphisms $\mathcal{E} \to \mathcal{S}$, but over a base $\mathcal{S}$ more informative than $\mathbf{Set}$ such as the Galois topos of the base field. See [10] and [16].

The classical 'gros' Zariski topos $\mathcal{Z}_K$ determined by a field K is a (non-presheaf) subtopos of $\mathfrak{G}(\mathbf{Aff}_K)$ and, if K is an algebraically closed field, the canonical geometric morphism $\mathcal{Z}_K \to \mathbf{Set}$ is pre-cohesive, but we need not go into that at this point.

So far we have used extensive categories to sketch some basic constructions in classical algebraic geometry which, in particular, produce a pre-cohesive topos $\mathfrak{G}(\mathbf{Aff}_K)$ over $\mathbf{Set}$ for any algebraically closed field. This sketch will prove useful to recall some of the material in [11] that motivates the original work in the present paper.

Definition 1.8. A *rig* is a set A equipped with two commutative monoid structures $(A, \cdot, 1)$ and $(A, +, 0)$ such that 'product distributes over addition' in the sense that $x \cdot 0 = 0$ and $x \cdot (y + z) = (x \cdot y) + (x \cdot z)$ for every $x, y, z \in A$.

The category of rigs and homomorphisms between them will be denoted by $\mathbf{Rig}$. Evidently, the category of rings may be seen as the full subcategory $\mathbf{Ring} \to \mathbf{Rig}$ of those rigs such that the underlying additive structure is a (necessarily Abelian) group. On the other hand, the category of (bounded) distributive lattices appears as the full subcategory $\mathbf{dLat} \to \mathbf{Rig}$ consisting of those rigs such that multiplication is idempotent and the equation $1 + x = 1$ holds [11, Section 8].

It is well-know that many of the constructions among rings have analogues for semi-rings and, in particular, for rigs. For instance, given a multiplicative submonoid $F \subseteq A$ of a rig A it is possible to construct the rig of fractions $A \to A[F^{-1}]$ much as in the case of rings. In particular, for $a \in A$ and $F = \{a^n \mid n \in \mathbb{N}\}$ we will write $A[a^{-1}]$ instead of $A[F^{-1}]$.

Lack of negatives implies that the treatment of idempotents is a little more subtle than in rings. An element b in a rig is called *Boolean* if there is a (necessarily unique) c such that $b + c = 1$ and $bc = 0$. In this case c may be called the *complement* of b. If b is Boolean then it is idempotent.

Proposition 1.9. *The category* $\mathbf{Rig}$ *is coextensive. An object in* $\mathbf{Rig}^{\mathrm{op}}$ *is connected if and only if the corresponding rig has exactly two Boolean elements.*

Proof. Essentially as in rings: direct product decompositions correspond to Boolean elements. More precisely, if A is a rig and $b \in A$ is Boolean (with complement c) then the canonical map $A \to A[b^{-1}] \times A[c^{-1}]$ is an isomorphism. Moreover, every direct product decomposition $A \to B \times C$ is determined as above by a unique Boolean element in A. The argument can be completed as in Lemma 1.2 which then may be seen as a corollary of the present result. □

As in the case of rings we may consider, for a rig K, the coextensive category $K/\mathbf{Rig}$ of *K-rigs* (or *K-algebras*). Notice that for a ring K, the canonical functor $K/\mathbf{Ring} \to K/\mathbf{Rig}$ is an equivalence. On the other hand, of special interest for us is the case of $K = 2$ the distributive lattice with two elements. In this case, $2/\mathbf{Rig} \to \mathbf{Rig}$ may be identified with the full subcategory of rigs with idempotent addition. For any rig K let $(K/\mathbf{Rig})_{fp} \to K/\mathbf{Rig}$ be the full subcategory of finitely presentable K-algebras.

Definition 1.10. The *category of affine K-spaces* is the opposite of the category $(K/\mathbf{Rig})_{fp}$ and it will be denoted by $\mathbf{Aff}_K$.

In Section 2 we prove that $\mathbf{Aff}_K$ is extensive, generalizing one of the two aspects of Lemma 1.4. Sections 3 to 5 culminate in the proof that the second aspect of Lemma 1.4 holds for the case $K = 2$. In other words, we prove that every object in $\mathbf{Aff}_2$ is a finite coproduct of connected objects. This requires the introduction of a suitable notion of Noetherian rig (Section 3) and related 'Basis Theorem' (Section 4) which is probably the main original result of the paper.

Sections 6 proves a Nullstellensatz for 2-rigs (essentially due to Schanuel) which is used in Section 7 to show an analogue of Proposition 1.7 for $K = 2$, namely, that the Gaeta topos of $\mathbf{Aff}_2$ is pre-cohesive over $\mathbf{Set}$. We also give a proof of the folk fact that the Gaeta topos classifies 2-rigs 'without Boolean elements' and that the generic model therein satisfies the Kock-Lawvere axiom for Synthetic Differential Geometry [8].

Our proof of the Nullstellensatz for 2-rigs involves another coextensive variety of rigs that we introduce below.

Definition 1.11. A rig A is *integral* if $1 + x = 1$ for every $x \in A$.

Without a name, integral rigs are briefly considered in [11, Section 8] where there the free integral rig I on one generator x is described as the order $\{0 < \ldots < x^n < \ldots < x^2 < x < x^0 = 1\}$ with the obvious multiplication and it is suggested that the spectrum of I can be visualized as an interval (not a lattice). It is also suggested that I may be viewed as an extended positive line by reading the structure logarithmically, suggesting a connection with tropical geometry.

The coextensive category $\mathbf{iRig}$ of integral rigs and morphisms between them is also studied in [4] where it is shown that, as in the Zariski representation of rings, every integral rig is the algebra of sections of a sheaf of *really local* integral rigs.

Let $\mathbf{iRig}$ be the category of integral rigs. Let $\mathbf{iRig}_{fp} \to \mathbf{iRig}$ be the full subcategory of finitely presentable integral rigs and let **iAff** be the opposite of $\mathbf{iRig}_{fp}$. Using the tools developed for the proof of the Nullstellensatz for 2-rigs we show in Section 8 that **iAff** is extensive and that the associated Gaeta topos is pre-cohesive over $\mathbf{Set}$. We also sketch a proof of the folk result that this topos classifies integral rigs without idempotents.

In Section 9 we recall the definition of really local integral rigs and show that the generic integral rig without idempotents is not really local in the Gaeta topos of **iAff**. In the classical case, the analogous fact that the generic ring without idempotents is not local may be seen as motivating the consideration of the Zariski topos. Section 10 proves that **iAff** has an analogue of the Zariski topology. This topology is proved to be subcanonical in Section 12. It is also proved there that the resulting topos is pre-cohesive and classifies really local integral rigs.

Altogether, the new Basis Theorem and Nullstellensatz for 2-rigs allow us to show that the classifying toposes of certain extensions of the theory of rigs with idempotent addition are 'gros' in the sense of Axiomatic Cohesion. It might be interesting to compare these with the various categories of 'tropical schemes' such as those in [5] and references therein.

As expected, much of the work reported below concerns ideals, so let us quickly recall a couple of basic facts in the context of rigs.

Definition 1.12. An *ideal* of a rig R is an additive submonoid $I \subseteq R$ such that for every $r \in R$ and $y \in I$, $ry \in I$.

If $a \in R$ then the subset $(a) = \{ra \mid r \in R\} \subseteq R$ is a *principal* ideal of the rig R. Ideals in rings coincide with the classical notion.

Every ideal $I \subseteq A$ determines the relation $\approx_I \subseteq A \times A$ defined by $x \approx_I y$ if and only if there are $i, j \in I$ such that $x + i = y + j$. It is straightforward to check that $\approx_I$ is a congruence.

Lemma 1.13. *For any ideal $I \subseteq A$ the quotient $q : A \to A/{\approx_I}$ is the universal map from A sending every element of I to 0. Also, the kernel $q^{-1}0 \subseteq A$ coincides with the ideal $\{x \in A \mid (\exists s \in I)(x + s \in I)\} \subseteq A$.*

Proof. The quotient $A \to A/{\approx_I}$ maps every $t \in I$ to 0. Now let $f : A \to B$ in $\mathbf{Rig}$ be such that $fI = \{0\}$. If $x \approx_I y$ then there are $t, t' \in I$ such that $x + t = y + t'$ in A and so $fx = f(x + t) = f(y + t') = fy$.

Finally, $qx = 0$ if and only if $x \approx_I 0$ in A. This holds if and only if there are $s, s' \in I$ such that $x + s = 0 + s' = s'$. In turn, this is equivalent to the existence of an $s \in I$ such that $x + s \in I$. □

Naturally, the quotient of A by $\approx_I$ will be denoted by $A \to A/I$. Its kernel will be called the *saturation* of I and will be denoted by $\overline{I} \subseteq A$. Of course, $I \subseteq \overline{I} \subseteq A$. The ideal I will be called *saturated* if $I = \overline{I}$ as ideals of A. Notice that, in a ring, every ideal is saturated.

Lemma 1.14. *If $b \in A$ is a Boolean element (with complement c) of the rig A then $A \to A[b^{-1}]$ and $A \to A/(c)$ coincide in the sense that each has the universal property of the other.*

Proof. A Boolean element is invertible if and only if its complement is 0. □

2. The extensive category of affine K-schemes

Fix a rig K. The purpose of the present section is to show that $\mathbf{Aff}_K$ is extensive. We actually show that the subcategory $(K/\mathbf{Rig})_{fp} \to K/\mathbf{Rig}$, which is closed under finite colimits, is also closed under finite products and therefore the domain inherits coextensivity from the codomain. (This may be a folk fact but we have not found it in the literature. It is certainly classical for the case of Noetherian rings K. See Lemma 1.4.)

The full subcategory $(K/\mathbf{Rig})_{fp} \to K/\mathbf{Rig}$ contains the terminal object because it may be presented as $K/(1)$ where (1) is the principal ideal generated by 1. So we are interested in sufficient conditions for the subcategory to be closed under finite products. By [15, Proposition 3.6] it is enough to check that the product of two finitely generated free K-rigs is finitely presented.

The free K-rig on a set S may be identified with the rig of polynomials $K[S]$ with coefficients in K and 'variables' in S. Let S and T be two finite sets. The product $K[S] \times K[T]$ is easily seen to be (finitely) generated by $(1,0)$, $(0,1)$, $(s,0)$ for any $s \in S$ and $(0,t)$ for any $t \in T$. To prove that the product is finitely presented we need to be more detailed so consider the free K-rig $K[S+T+\{\sigma,\tau\}]$. Let $L : K[S+T+\{\sigma,\tau\}] \to K[S]$ be the unique morphism of K-rigs such that $Ls = s$ for every $s \in S$, $Lt = 0$ for every $t \in T$, $L\sigma = 1$ and $L\tau = 0$. The morphism L sends a polynomial $p(S,T,\sigma,\tau) \in K[S+T+\{\sigma,\tau\}]$ to $p(S,0,1,0) \in K[S]$. Similarly, we let $R : K[S+T+\{\sigma,\tau\}] \to K[T]$ be the unique morphism of K-rigs such that $Rs = 0$, $Rt = t$, $R\sigma = 0$ and $R\tau = 1$.

Lemma 2.1. *The map $\langle L, R\rangle : K[S+T+\{\sigma,\tau\}] \to K[S] \times K[T]$ is surjective.*

Proof. The map $\langle L, R\rangle$ sends σ to $(1,0)$, τ to $(0,1)$, $s \in S$ to $(s,0)$ and $t \in T$ to $(0,t)$. □

Lemma 2.1 is just another way of saying that finite products of finitely generated free K-rigs are finitely generated. It remains to show that congruence determined by the quotient $\langle L, R\rangle$ is finitely generated.

Lemma 2.2. *The following elements of $K[S+T+\{\sigma,\tau\}]$*

1. *st for every $s \in S$ and $t \in T$,*
2. *$t\sigma$ for every $t \in t$,*
3. *$s\tau$ for every $s \in S$,*
4. *$\sigma\tau$*

are in the kernel of $\langle L, R\rangle$. Also, $\langle L, R\rangle(\sigma+\tau) = 1 \in K[S] \times K[T]$.

Proof. Notice that $\langle L, R\rangle(t \cdot \sigma) = (0 \cdot 1, t \cdot 0) = (0,0) \in K[S] \times K[T]$ for $t \in T$ and $\langle L, R\rangle(\sigma+\tau) = (1+0, 0+1) = (1,1)$. We leave the details for the reader. □

Let $\approx$ be the congruence on $K[S+T+\{\sigma,\tau\}]$ generated by the relations

$$st \approx t\sigma \approx s\tau \approx \sigma\tau \approx 0 \quad \sigma+\tau \approx 1$$

for $s \in S$ and $t \in T$. We stress that, as S and T are finite, the congruence $\approx$ is finitely generated. By Lemma 2.2 there exists a unique morphism $\Gamma : K[S+T+\{\sigma,\tau\}]/\!\approx \to K[S]\times K[T]$ such that the following diagram commutes

$$\begin{tikzcd} K[S+T+\{\sigma,\tau\}] \arrow[r] \arrow[dr, "\langle L,R\rangle"'] & K[S+T+\{\sigma,\tau\}]/\!\approx \arrow[d, "\Gamma"] \\ & K[S]\times K[T] \end{tikzcd}$$

and Γ is surjective because $\langle L,R\rangle$ is so by Lemma 2.1.

Proposition 2.3. *For any rig K the subcategory $(K/\mathbf{Rig})_{fp} \to K/\mathbf{Rig}$ is closed under finite products and it is therefore coextensive.*

Proof. We continue the argument preceding the statement. It remains to show that Γ is injective. For brevity let $W = K[S+T+\{\sigma,\tau\}]/\!\approx$.

As σ and τ complement each other in W, they are Boolean and therefore idempotent. Together with the first four items of Lemma 2.2 we deduce that every element of W is of the form $k + p(S) + q(T) + k_\sigma\sigma + k_\tau\tau$ with $p(S) \in K[S]$ and $p(0) = 0$, $q(T) \in K[T]$ and $q(0) = 0$, and $k, k_\sigma, k_\tau \in K$. Moreover, as $k = k(\sigma+\tau) = k\sigma + k\tau$ we conclude that every element of W is of the form

$$p(S) + q(T) + k_\sigma\sigma + k_\tau\tau$$

with $p(S) \in K[S]$ and $p(0) = 0$, $q(T) \in K[T]$ and $q(0) = 0$, and $k_\sigma, k_\tau \in K$.

Let $p'(S) + q'(S) + k'_\sigma\sigma + k'_\tau\tau$ be another element of W in the same 'normal form' and assume that Γ sends them both to the same thing. That is,

$$(p(S) + k_\sigma, q(T) + k_\tau) = (p'(S) + k'_\sigma, q'(T) + k'_\tau)$$

in $K[S]\times K[T]$. Then $p(S) = p'(S)$, $k_\sigma = k'_\sigma$, $q(T) = q'(T)$ and $k_\tau = k'_\tau$. Hence,

$$p(S) + q(T) + k_\sigma\sigma + k_\tau\tau = p'(S) + q'(S) + k'_\sigma\sigma + k'_\tau\tau$$

in W completing the proof that Γ is injective. $\square$

Corollary 2.4. *The category $\mathbf{Aff}_K$ is extensive for any rig K.*

3. Noetherian rigs

In this section we introduce a notion of Noetherianity for rigs involving saturated ideals as defined in Section 1 and which abstracts the standard notion for rings.

Let A be a rig.

Lemma 3.1. *If $I_0 \subseteq I_1 \subseteq \ldots$ is a sequence of saturated ideals of A then so is the union $I = \bigcup_{n\in\mathbb{N}} I_n$.*

Proof. Let $x \in A$ and $s \in I$ be such that $x + s \in I$. Then there are $m, n \in \mathbb{N}$ such that $s \in I_m$ and $x + s \in I_n$. Then $s, x + s \in I_{m+n}$ and, as S_{m+n} is saturated, $x \in I_{m+n} \subseteq I$ by Lemma 1.13. □

For any family $(x_s \in A \mid s \in S)$ there is a least ideal containing the elements in that family. It is called the ideal *generated* by the family. Its elements are those of the form $\sum_{i\in I} a_i x_i$ for some finite subset $I \subseteq S$ and $a_i \in A$ for each $i \in I$. An ideal of A is *finitely generated* if it is generated by finite family. We next introduce something less standard.

Definition 3.2. A saturated ideal is *essentially finitely generated* if it is the saturation of a finitely generated ideal.

Of course, a finitely generated saturated ideal is essentially finitely generated. In the case of rings the converse holds because ideals of rings are saturated.

Lemma 3.3. *The following are equivalent:*

1. *Every sequence $I_0 \subseteq I_1 \subseteq \ldots$ of saturated ideals of A is stationary; that is, there is an $m \in \mathbb{N}$ such that $I_m = I_n$ for every $n \geq m$.*

2. *Every saturated ideal $I \subseteq A$ is essentially finitely generated.*

Proof. (Just as in the classical case, but taking the necessary precautions to deal with saturation.) Assume that the first item holds and, for the sake of contradiction, let $I \subseteq A$ be a saturated ideal that is not essentially finitely generated. Choose an element $s_0 \in I$, let $S_0 \subseteq A$ be the ideal generated by s_0 and let $\overline{S_0}$ be the saturation which is, of course, essentially finitely generated. Certainly, $\overline{S_0} \subseteq I$ but, as I is not essentially finitely generated, there is an $s_1 \in I$ such that $s_1 \notin \overline{S_0}$. Let $S_1 \subseteq A$ be the ideal generated by s_0, s_1. Then $\overline{S_0} \subset \overline{S_1} \subset I$. Again, there must exist an $s_2 \in I$ such that $s_2 \notin \overline{S_1}$ and continuing with this process we obtain a sequence $\overline{S_0} \subset \overline{S_1} \subset \ldots$ of saturated ideals of A that is not stationary; a contradiction.

Conversely, assume that the second item holds. The union $I = \bigcup_{n\in\mathbb{N}} I_n$ is a saturated ideal by Lemma 3.1 so, by hypothesis, it is essentially finitely generated. Let $(g_s \in I \mid s \in S)$ be a finite family generating an ideal J such that $\overline{J} = I$. Then there are $m_s \in \mathbb{N}$ such that $g_s \in I_{m_s}$. As the set S is finite, $g_s \in I_m$ for $m = \sum_{t\in S} m_t$ and every $s \in S$, so $J \subseteq I_m$. Then $I = \overline{J} \subseteq \overline{I_m} = I_m$ so $I_m = I$. $\square$

Although congruences of rigs are not in bijective correspondence with ideals, the following terminology seems fair.

Definition 3.4. A rig A will be called *Noetherian* if it satisfies the equivalent conditions of Lemma 3.3. Also, a rig is *strongly Noetherian* if every saturated ideal in it is finitely generated.

Of course, strongly Noetherian implies Noetherian; and the converse holds for rings. So a ring is Noetherian in the present 'rig sense' if and only if it is Noetherian in the classical sense.

The following lemmas will be needed later and are simple variations of standard facts about Noetherian rings. The proofs are also variations that take saturation into account. (Recall that, in algebraic categories, regular epimorphisms coincide with surjections.)

Lemma 3.5. *If $A \to B$ is a regular epi in $\mathbf{Rig}$ and A is Noetherian then so is B.*

Proof. Let $f : A \to B$ be a map in $\mathbf{Rig}$. For any ideal $I \subseteq B$, the inverse image $f^{-1}I \subseteq A$ is an ideal. Moreover, if I is saturated then so is $f^{-1}I$. Also, if $J \subseteq B$ is another ideal and $I \subseteq J$ then $f^{-1}I \subseteq f^{-1}J$. Hence, every ascending sequence $I_1 \subseteq I_2 \subseteq \ldots$ of saturated ideals of B determines an ascending sequence $f^{-1}I_1 \subseteq f^{-1}I_2 \subseteq \ldots$ of saturated ideals of A. As A is Noetherian, this sequence is stationary. So, to complete the proof, it is enough to prove the following lemma: For $I \subseteq J$ ideals of B such $f^{-1}I = f^{-1}J$, if f is surjective then $I = J$. In turn, it is enough to show that $J \subseteq I$. So let $b \in J$. As f is surjective, $b = fa$ for some $a \in A$. Then $a \in f^{-1}J = f^{-1}I$ so $b = fa \in I$. $\square$

Lemma 3.6. *If the rig A is Noetherian then it is a finite product of directly indecomposable rigs.*

Proof. Assume that A is not a finite direct product of directly indecomposable rigs. Then A is not directly indecomposable so $A = A_0 \times A'_0$ for non-terminal A_0, A'_0. Moreover, either A_0 or A_1 is not a finite direct product of directly indecomposable rigs. Without loss of generality we can assume that A_0 is not. By Lemma 1.14 the projection $A \to A_0$ is the quotient by a saturated ideal $I_0 \subseteq A$. (Indeed, an ideal generated by Boolean element.) Moreover, the ideal is strict because A'_0 is not terminal. By our current assumption, $A_0 = A_1 \times A'_1$ for non-terminal A_1 and A'_1. Again, we may assume that A_1 is not directly indecomposable and let $I_1 \subset A$ be the strict saturated ideal whose quotient is the composite projection $A \to A_0 \to A_1$. Also, $I_0 \subset I_1$ as ideals of A. Repeating the process we obtain a non-stationary sequence $I_0 \subset I_1 \subset \ldots$ of saturated ideals of A, contradicting Noetherianity of A. $\square$

4. The lower Basis Theorem

Every commutative monoid determines a pre-order on its underlying set. In particular, addition in a rig induces a pre-order. In more detail, let A be a rig and declare, for every $x, y \in A$, that $x \leq y$ if and only if there is a $d \in A$ such that $x + d = y$. We sometimes call this the 'canonical pre-order' of A. It is easy to check that addition and multiplication are monotone with respect to the canonical pre-order.

An ideal $I \subseteq A$ is called *lower-closed* if $x \leq y \in I$ implies $x \in I$. We stress an obvious corollary of Lemma 1.13: lower-closed implies saturated.

For example, the canonical pre-order of a ring is codiscrete (in the sense that $x \leq y$ for every x, y) so the only lower-closed ideal in a ring is that containing 1. On the other hand, the canonical pre-order of a distributive lattice (considered as a rig) coincides with the lattice.

Fix a rig K.

Lemma 4.1. *If $I \subseteq K[x]$ is a lower closed ideal then every element of I is a sum of monomials in I. Hence, I is generated by the monomials in I.*

Proof. If the polynomial $\sum_{i=0}^{m} k_i x^i$ is in I then, by lower-closedness, I contains $k_i x^i$ for each $0 \leq i \leq m \in \mathbb{N}$. $\square$

In the next auxiliary result the reader may recognize a trick used in the classical proof Hilbert's Basis Theorem. It is no accident.

Lemma 4.2. *If K is such that every lower-closed ideal is finitely generated then for every lower-closed ideal $I \subseteq K[x]$ there is an $n \in \mathbb{N}$ such that I is generated by monomials of degree at most n.*

Proof. By Lemma 4.1 it is enough to check that every monomial in I may be expressed as a linear combination (with coefficients in $K[x]$) of monomials in I of some bounded degree.

Let $L \subseteq K$ be the subset consisting of 0 together with the leading coefficients of polynomials in I. The subset $L \subseteq K$ is clearly an ideal and it is also lower-closed. To see this assume that $a \leq b \in L$. Then there is a polynomial $f = bx^n + (\text{lower terms})$ in I. So $ax^n \leq bx^n \leq f \in I$ and, as I is lower closed, $ax^n \in I$. Hence, $a \in L$ so L is indeed lower-closed.

By hypothesis, there is a finite family $(\kappa_s \in K \mid s \in S)$ spanning L. For each κ_s there exists a polynomial $f_s \in I$ that has κ_s as leading coefficient. Let n be the largest degree of any of the f_s's. Multiplying the polynomials f_s with suitable powers of x we obtain polynomials $g_s \in I$ all of the same degree n and each g_s with leading coefficient κ_s. As I is lower closed, $\kappa_s x^n \in I$ for every $s \in S$.

Let $m \geq n$ and $ax^m \in I$. Then a is a linear combination, with coefficients in K, of $(\kappa_s \mid s \in S)$. So ax^m is a linear combination, with coefficients in $K[x]$, of the polynomials $\kappa_s x^n \in I$. Hence, every monomial in I is a linear combination, with coefficients in $K[x]$, of the monomials in I of degree strictly less than n; as we needed to prove. $\square$

We can now mimic the classical proof of Hilbert's Basis Theorem but using lower-closedness of the ideals involved instead of the existence of negatives.

Theorem 4.3 (The lower Basis Theorem)**.** *If K is such that every lower-closed ideal is finitely generated then every lower-closed ideal of $K[x]$ is finitely generated.*

Proof. Let $I \subseteq K[x]$ be a lower-closed ideal. By Lemma 4.2 there is an $n \in \mathbb{N}$ such that I is generated by the monomials in I of degree at most n. For each $m \leq n$ let $L_m \subseteq K$ be the subset consisting of 0 and all coefficients of monomials of degree m in I. As before, L_m is a lower-closed ideal in K so, by hypothesis, it is generated by a finite family $(\kappa_{m,s} \mid s \in S_m)$. Then every monomial of degree m may be expressed as a linear combination (with coefficients in K, actually) of the monomials $\kappa_{m,s} x^m$. Then every monomial of degree at most n is a linear combination of the finite family of monomials $(\kappa_{m,s} x^m \mid s \in S_m, m \leq n)$. So the same family generates the ideal I. $\square$

5. The 2-Basis Theorem

Let 2 be the initial distributive lattice. For any rig A there is at most one rig morphism $2 \to A$ so the forgetful functor $2/\mathbf{Rig} \to \mathbf{Rig}$ is full as well as faithful. The objects in the subcategory may be identified with the rigs whose addition is idempotent. Of course, from this perspective, the initial object of $2/\mathbf{Rig}$ is 2. Also, idempotence of addition implies that the canonical preorder is anti-symmetric so, for any 2-rig A, we will picture $(A, +, 0)$ as a join-semilattice.

Lemma 5.1. *If A is a 2-rig and $I \subseteq A$ is an ideal then the following hold:*

1. *For every $x, y \in A$, $x \approx_I y$ if and only if there is a $k \in I$ such that $x + k = y + k$.*
2. *The ideal I is saturated if and only if it is lower closed.*

Proof. By the definition of $\approx_I$, $x \approx_I y$ if and only if there are $i, j \in I$ such that $x + i = y + j$. In this case,

$$x + i + j = x + i + i + j = y + j + i + j = y + i + j$$

so we may take $k = i + j$.

Assume that I is saturated. If $x \leq y \in I$ then $x + y = y$ so, by saturation, $x \in I$. On the other hand, if I is lower closed then it is trivially saturated. $\square$

Hence, for 2-rigs, we may reformulate strong Noetherianity as follows.

Proposition 5.2. *A 2-rig is strongly Noetherian if and only if every lower closed ideal is finitely generated.*

Proof. Recall that a rig A is *strongly Noetherian* if every saturated ideal is finitely generated. So the statement follows immediately from the second item of Lemma 5.1. $\square$

Combining Theorem 4.3 and Proposition 5.2 we obtain the following.

Corollary 5.3 (The 2-Basis-Theorem)**.** *If K is a strongly Noetherian 2-rig then so is $K[x]$.*

As in the classical case, a simple induction implies that free 2-rigs on a finite set of generators are strongly Noetherian.

Corollary 5.4. *Finitely generated 2-rigs are Noetherian.*

Proof. Follows from the previous remark and Lemma 3.5. $\square$

Corollary 5.4 and Lemma 3.6 imply the following.

Corollary 5.5. *Every finitely generated 2-rig is a finite product of directly indecomposable finitely generated 2-rigs.*

We don't know if finitely generated implies finitely presentable for 2-rigs so we are forced to state the following separately.

Corollary 5.6. *Every finitely presentable 2-rig is a finite product of directly indecomposable finitely presentable 2-rigs.*

Proof. If A is a finitely presentable 2-rig then it is finitely generated so, by Corollary 5.5, $A = \prod_{s\in S} A_s$ for a finite set S and A_s directly indecomposable for every $s \in S$. By Lemma 1.14, the projection $A \to A_s$ is the quotient by a principal ideal. Hence, as A is finitely presentable, so is A_s. $\square$

We can now deduce an analogue of Lemma 1.4.

Corollary 5.7. *Every object in the extensive* $\mathbf{Aff}_2$ *is a finite coproduct of connected objects.*

6. Integral rigs and a Nullstellensatz for 2-rigs

Section 4 in [11] attributes to Schanuel the result that the only simple rigs are fields and the distributive lattice 2. We prove here a weaker statement with an argument that is more convenient for our purposes.

Let A be a rig and let $F \subseteq A$ be a multiplicative submonoid.

For $x, y \in A$ we write $x \leq_F y$ if there is an $u \in F$ such that $x \leq uy$. In this case we may say that u *witnesses* that $x \leq_F y$. The relation $\leq_F$ on the set A is reflexive because $1 \in F$ and it is transitive because if $u, v \in F$ witness that $x \leq_F y$ and $y \leq_F z$ respectively then uv witnesses that $x \leq_F z$. Hence, $\leq_F$ is a pre-order.

Write $x \approx_F y$ if both $x \leq_F y$ and $y \leq_F x$. As $\leq_F$ is a pre-order, $\approx_F$ is an equivalence relation. We next give a sufficient condition for it to be a congruence.

Lemma 6.1. *If $1 + F \subseteq F \subseteq A$ then $\approx_F$ is a congruence on A. In this case, the quotient $A/\approx_F$ is a 2-rig and, it is trivial if and only if A is a ring.*

Proof. We have already seen that the relation $\approx_F$ is an equivalence relation. For $a, b, c, d \in A$ assume that $a \leq_F b$ is witnessed by $u \in F$ and that $c \leq_F d$ is witnessed by $v \in F$. Then uv witnesses that $ac \leq_F bd$.

Assume from now on that $1 + F \subseteq F$. We claim that if $x \in A$ and $a \leq_F b$ then $x + a \leq_F x + b$. By hypothesis there is an $u \in F$ such that $a \leq ub$ so

$$x + a \leq x + ub \leq x + (b + ux) + ub = (x + b) + u(x + b) = (1 + u)(x + b)$$

and hence $x + a \leq_F x + b$, so the claim is proved.

Using the claim one easily shows that if $a \leq_F b$ and $x \leq_F y$ then also $a + x \leq_F b + y$. It follows that $\approx_F$ is a congruence.

Trivially, $1 \leq_F 1 + 1$ and, since $1 + 1 \in F$ by hypothesis, the inequality $1 + 1 \leq (1 + 1)1$ implies $1 + 1 \leq_F 1$. So $1 \approx_F 1 + 1$ and hence the quotient $A/\approx_F$ is a 2-rig.

Assume now that $0 = 1$ in the quotient $A/\approx_F$. That is, $0 \approx_F 1$ in A. Equivalently, $0 \leq_F 1$ and $1 \leq_F 0$. One of the conjuncts holds trivially and the other is equivalent to $1 \leq 0$. So the quotient is terminal if and only if $1 \leq 0$ in A. $\square$

For instance, we recall Schanuel's construction [17] of the left adjoint to the full inclusion $2/\mathbf{Rig} \to \mathbf{Rig}$. The rig $\mathbb{N}$ of natural numbers with the usual addition and multiplication is initial in $\mathbf{Rig}$. That is, for any rig A there exists a unique $\nabla : \mathbb{N} \to A$ in $\mathbf{Rig}$. The subset $F = \{\nabla n \mid 1 \leq n\} \subseteq A$ satisfies the hypotheses of Lemma 6.1. The induced pre-order on A satisfies: $a \leq_F b$ if and only if there is an $1 \leq n \in \mathbb{N}$ such that $a \leq nb$. The quotient by $\approx_F$ is denoted by $\dim : A \to D(A)$ and is universal from A to the inclusion $2/\mathbf{Rig} \to \mathbf{Rig}$. This construction suggests something more general.

Lemma 6.2. *Let $1 + F \subseteq F$ so that $\approx_F$ is a congruence by Lemma 6.1. If $1 \leq_F u$ for every $u \in F$, then the quotient $A \to A/{\approx_F}$ is the universal morphism sending $F \subseteq A$ to 1 in the codomain.*

Proof. Trivially $u \leq_F 1$ for every $u \in F$. As $1 \leq_F u$ by hypothesis, $1 \approx_F u$ for every $u \in F$ so the quotient $A \to A/{\approx_F}$ sends $F \subseteq A$ to the unit 1 in the codomain. Now let $f : A \to B$ in $\mathbf{Rig}$ be such that $fu = 1$ for every $u \in F$. As $1 + 1 \in F$, B is a 2-rig. If $a \leq_F b$ then $a \leq ub$ for some $u \in F$. Then $fa \leq (fu)(fb) = fb$. So, if $a \approx_F b$ then $fa \leq fb$ and $fb \leq fa$ and, as B is a 2-rig, $fa = fb$. Hence, f factors uniquely through the quotient $A \to A/{\approx_F}$. □

Recall that $\mathbf{iRig} \to \mathbf{Rig}$ is the variety of rigs determined by the equation $1 + x = 1$. We next describe the left adjoint to $\mathbf{iRig} \to \mathbf{Rig}$.

Let $\uparrow 1 \subseteq A$ be the upper-closed multiplicative submonoid of the elements in A above 1. The relation $\approx_{\uparrow 1}$ is a congruence by Lemma 6.1. Denote the associated quotient $A/{\approx_{\uparrow 1}}$ by LA.

Proposition 6.3. *The quotient $A \to LA$ is universal from A to* $\mathbf{iRig} \to \mathbf{Rig}$ *and the resulting left adjoint L :* $\mathbf{Rig} \to \mathbf{iRig}$ *preserves finite products.*

Proof. Evidently, $1 + x \in \uparrow 1 \subseteq A$ for all x so, by Lemma 6.2, the quotient $A \to LA$ sends $1 + x \in A$ to $1 \in LA$ for every $x \in A$; so LA is integral. To prove that the quotient $A \to LA$ is universal let R be an integral rig and let $f : A \to R$ be a rig homomorphism. Then $fu = 1$ for every $1 \leq u \in A$, so f factors through $A \to LA$ by Lemma 6.2.

Let $L : \mathbf{Rig} \to \mathbf{iRig}$ be the resulting left adjoint and denote the unit by η. Let A, B be rigs and let γ be the unique map such that the following diagram

$$\begin{tikzcd} A \times B \arrow[r, "\eta"] \arrow[dr, "\eta\times\eta"'] & L(A \times B) \arrow[d, "\gamma"] \\ & LA \times LB \end{tikzcd}$$

commutes in $\mathbf{Rig}$. Then γ is surjective so we need only prove that it is monic. Let $(a, b), (a', b') \in A \times B$ and assume that $\gamma(\eta(a, b)) = \gamma(\eta(a', b'))$. Then both $\eta a = \eta a'$ and $\eta b = \eta b'$. Hence $a \approx_{\uparrow 1} a'$ in A and $b \approx_{\uparrow 1} b'$ in B. That is, $a \leq_{\uparrow 1} a'$ and $a' \leq_{\uparrow 1} a$ in A and also $b \leq_{\uparrow 1} b'$ and $b' \leq_{\uparrow 1} b$ in B. Let $1 \leq u \in A$ witness that $a \leq_{\uparrow 1} a'$ and $1 \leq v \in B$ witness that $b \leq_{\uparrow 1} b'$. Then $(1, 1) \leq (u, v) \in A \times B$ and $(a, b) \leq (ua', vb') = (u, v)(a', b')$. Hence $(a, b) \leq_{\uparrow 1} (a', b')$ in $A \times B$. Similarly, $(a', b') \leq_{\uparrow 1} (a, b)$ so $(a, b) \approx_{\uparrow 1} (a', b')$ as we needed to show. $\square$

The inclusion $\mathbf{iRig} \to \mathbf{Rig}$ factors through the right adjoint inclusion $2/\mathbf{Rig} \to \mathbf{Rig}$. The left adjoint to the factorization $\mathbf{iRig} \to 2/\mathbf{Rig}$ is just the restriction of the left adjoint $L : \mathbf{Rig} \to \mathbf{iRig}$. Hence, we may deduce the following result that will be needed later.

Corollary 6.4. *The left adjoint to* $\mathbf{iRig} \to 2/\mathbf{Rig}$ *preserves finite products.*

Combining the integral reflection described above with some of the material in [4] we arrive at the promised weak version of Schanuel's result.

Proposition 6.5 (Nullstellensatz)**.** *For any non-trivial 2-rig A there is a map $A \to 2$.*

Proof. By hypothesis and Lemma 6.1, the codomain of the unit $A \to LA$ is not trivial. Consider now the variety $\mathbf{dLat} \to \mathbf{iRig}$. The left adjoint $L' : \mathbf{iRig} \to \mathbf{dLat}$ is described explicitly in [4, Lemma 4.3] which also implies that the unit $LA \to L'(LA)$ is local (in the sense that it reflects 1) so the distributive lattice $L'(LA)$ is non-trivial. Classical lattice theory then implies the existence of a map $L'(LA) \to 2$, so we have a composite rig morphism $A \to LA \to L'(LA) \to 2$. $\square$

Corollary 6.6 (Nullstellensatz)**.** *Every connected object in* $\mathbf{Aff}_2$ *has a point.*

7. The Gaeta topos of $\mathbf{Aff}_2$

We can now apply standard topos theory to construct a topos 'of spaces' embedding the category of affine 2-spaces in such a way that finite coproducts are preserved.

Theorem 7.1. *The Gaeta topos of* $\mathbf{Aff}_2$ *is pre-cohesive over sets.*

Proof. Exactly as in Proposition 1.7. By Corollary 5.6 the Gaeta topos of $\mathbf{Aff}_2$ is equivalent to the topos of presheaves on the category of connected affine 2-schemes and every every connected affine 2-scheme has a point by Corollary 6.6. □

Theorem 7.1 and the related Proposition 6.5 show that the rig 2 has certain typical properties of algebraically closed fields.

As suggested in [11], standard techniques allow us to give a presentation of the geometric theory classified by the topos of Theorem 7.1. We give details below.

Proposition 7.2. *The Gaeta topos of* $\mathbf{Aff}_2$ *classifies the extension of the theory of 2-rigs presented by the following sequents.*

$$\begin{array}{rcl} 0 = 1 & \vdash & \bot \\ (x+y=1) \wedge (xy=0) & \vdash_{x,y} & [(x=1)\wedge(y=0)] \vee [(x=0)\wedge(y=1)] \end{array}$$

In other words, this $\mathfrak{G}(\mathbf{Aff}_2)$ *classifies 'Boolean-free' 2-rigs.*

Proof. First let us give a dual description of the basis for the Gaeta topology in $(2/\mathbf{Rig})_{fp}$. Our knowledge of products in $2/\mathbf{Rig}$ implies that a Gaeta cocover on an (f.p.) 2-rig A is a finite family $(A \to A[a_i^{-1}] \mid i \in I)$ of maps in $(2/\mathbf{Rig})_{fp}$ such that $a_i a_j = 0$ for every $i, j \in I$ and the ideal $\langle a_i \mid i \in I \rangle$ generated by the a_i's is trivial in the sense that it contains 1. In this case, for brevity, we will also say that the family $(a_i \mid i \in I)$ covers A.

On the other hand, there is a more or less general procedure to exhibit an explicit site for the classifier of Boolean-free 2-rigs. See, for example, [6, Proposition D3.1.10]). Roughly speaking, one first constructs the classifier for the restricted (algebraic) theory presented by the equations and then forces the remaining axioms by imposing a Grothendieck topology. In the present case, the classifier for the theory of 2-rigs may be described as the topos $[(2/\mathbf{Rig})_{fp}, \mathbf{Set}] = \widehat{\mathbf{Aff}_2}$ of functors $(2/\mathbf{Rig})_{fp} \to \mathbf{Set}$; and the classifier of Boolean-free 2-rigs may be obtained as the sheaf topos associated to the least Grothendieck topology on $\mathbf{Aff}_2$ 'forcing' the coherent sequents in the statement. More explicitly, the classifying topos for idempotent 2-rigs may be described as the topos of sheaves on the site $(\mathbf{Aff}_2, J)$ where J is the least Grothendieck topology 'containing' the cocover

$$\begin{tikzcd} 2[x,y]/(xy=0, x+y=1) \arrow[r] \arrow[d] & 2[x,y]/(x=0,y=1) \cong 2 \\ 2 \cong 2[x,y]/(x=1,y=0) & \end{tikzcd}$$

and the empty cocover on the terminal object. The explicit dual description of the Gaeta topology in the first paragraph implies that the two cocovers generating J are in the basis for the Gaeta topology. So J is included in the Gaeta topology. On the other hand, any binary cocover

$$A/(v) \longleftarrow A \longrightarrow A/(u)$$

with $uv = 0$ and $u + v = 1$ in the Gaeta basis appears as the pushout, along the map $2[(x+y)^{-1}, xy] \to A$ that sends x to u and y to v, of the main coverage generating J. A simple inductive argument as in [12, Lemma VIII.6.2] implies that all the non-empty Gaeta cocovers are in J. Hence, the Gaeta topology is included in J. Altogether, the two topologies are the same. $\square$

It is well-known that for any ring K, the classifier of K-algebras (i.e. the presheaf topos $[(K/\mathbf{Ring})_{fp}, \mathbf{Set}]$) and some of its subtoposes are models of Synthetic Differential Geometry [8, Part III]. Folklore says that this also holds for arbitrary rigs. We end this section with a sketch of the proof that one of the key axioms of SDG holds in the Gaeta topos of 2.

Let $R = (2/\mathbf{Rig})_{fp}(2[x], -)$ in $\mathfrak{G}(\mathbf{Aff}_2)$ be the generic Boolean-free 2-rig. Let the following diagram be a pullback

$$\begin{tikzcd} D \arrow[rr] \arrow[d] & & 1 \arrow[d, "0"] \\ R \arrow[r, "\Delta"'] & R \times R \arrow[r, "\cdot"'] & R \end{tikzcd}$$

or, alternatively, define $D = \{x \in R \mid x^2 = 0\} \subseteq R$ using the internal language of $\mathfrak{G}(\mathbf{Aff}_2)$. The composite

$$R \times R \times D \xrightarrow{id_R \times \cdot} R \times R \xrightarrow{+} R$$

transposes to a map $R \times R \to R^D$.

Proposition 7.3 (The KL-axiom holds in the Gaeta topos of 2)**.** *The canonical map $R \times R \to R^D$ is an isomorphism in $\mathfrak{G}(\mathbf{Aff}_2)$.*

Proof. For any rig A, the universal morphism $A \to A[\epsilon]$ in $\mathbf{Rig}$ adding an element ϵ of square-zero may be built as usual by taking the additive monoid $A \times A$ equipped with multiplication $(a, a')(b, b') = (ab, ab' + a'b)$ and $\epsilon = (0, 1)$ as selected element of square 0. If we let $a = (a, 0) \in A[\epsilon]$ then every element of $A[\epsilon]$ is of the form $a + b\epsilon$. The object D is representable by $2[\epsilon]$ and the subobject $D \to R$, as a cosieve in $(2/\mathbf{Rig})_{fp}$, is generated by the map $2[x] \to 2[\epsilon]$ sending x to ϵ. (Notice that the pullback defining D could be taken in $\mathbf{Aff}_2$.) The object R^D, as a functor $(2/\mathbf{Rig})_{fp} \to \mathbf{Set}$, sends A in the domain to the underlying set $A \times A$ of $A[\epsilon]$. The canonical map $R \times R \to R^D$, at stage A, sends the ordered pair $(a, b) \in (R \times R)A = A \times A$ to $a + b\epsilon \in (R^D)A = A[\epsilon]$. $\square$

The resulting differential geometry in $\mathfrak{G}(\mathbf{Aff}_2)$ should be an interesting pursuit. See also [11, Section 1].

8. The extensive category of Affine i-schemes

Let $\mathbf{iRig}_{fp} \to \mathbf{iRig}$ be the full subcategory of finitely presentable integral rigs.

Corollary 8.1. *The full subcategory* $\mathbf{iRig}_{fp} \to \mathbf{iRig}$ *is closed under products and it is therefore coextensive.*

Proof. Let $F[S]$ be the free integral rig generated by the set S. As in Proposition 2.3 we need only show that if S and T are finite then $F[S] \times F[T]$ is finitely presented. By Proposition 2.3 again there are finite sets U, V and a coequalizer

$$2[V] \rightrightarrows 2[U] \longrightarrow 2[S] \times 2[T]$$

in the category $2/\mathbf{Rig}$. The reflection $L : 2/\mathbf{Rig} \to \mathbf{iRig}$ preserves finite products by Corollary 6.4 so it sends the coequalizer above to the coequalizer

$$L(2[V]) \rightrightarrows L(2[U]) \longrightarrow L(2[S]) \times L(2[T])$$

in $\mathbf{iRig}$. As $L(2[W]) = FW$ for any set W, the result follows. $\square$

Naturally, we introduce the following.

Definition 8.2. The *category of affine i-schemes* is the (extensive) opposite of $\mathbf{iRig}_{fp}$ and it will be denoted by **iAff**.

We next show that the Gaeta topos of **iAff** is pre-cohesive using the same techniques that we used for a $\mathbf{Aff}_2$.

Corollary 8.3. *Every finitely generated integral rig is Noetherian.*

Proof. It is clear from the definition of integral rig that $\mathbf{iRig}$ is a variety of 2-rigs so it follows from classical universal algebra that the full subcategory $\mathbf{iRig} \to 2/\mathbf{Rig}$ is regular epireflective and closed under regular quotients and directed unions [1, Corollary 10.21].

By regular epireflectivity every integral rig freely generated by a set of generators is a quotient of free 2-rig freely generated by the same set. If the generating set is finite then the free 2-rig is Noetherian by Lemma 5.4, so the free integral rig is also Noetherian by Lemma 3.5. Lemma 3.5 also implies that finitely generated integral rigs are Noetherian. $\square$

Just as in Corollaries 5.5 and 5.6 we may deduce the next result.

Corollary 8.4. *Every finitely presentable integral rig is a finite product of directly indecomposable finitely presentable integral rigs.*

It is plausible that these finite direct decomposition results may be lifted to other algebraic categories equipped with a suitable functor to $2/\mathbf{Rig}$ or to $\mathbf{iRig}$ such as those discussed in [4], but we will not pursue that here.

Theorem 8.5. *The Gaeta topos of* **iAff** *is pre-cohesive over sets and classifies the extension of the theory of 2-rigs presented by the following sequents.*

$$\begin{array}{rcl} 0 = 1 & \vdash & \bot \\ (x + y = 1) \wedge (xy = 0) & \vdash_{x,y} & [(x = 1) \wedge (y = 0)] \vee [(x = 0) \wedge (y = 1)] \end{array}$$

In other words, this Gaeta topos classifies 'Boolean-free' integral rigs.

Proof. To prove that the topos is pre-cohesive proceed as in Proposition 1.7 (or Theorem 7.1). By Corollary 8.4 the Gaeta topos of **iAff** is equivalent to the topos of presheaves on the category of connected affine i-schemes and every connected affine i-scheme has a point by Corollary 6.6.

Also, the Gaeta topology in **iAff** has the same dual description made explicit in Proposition 7.2. (See [4].) Then the same argument used in 7.2 proves the present result. □

9. Really local integral rigs

A rig A (in a topos, with subobject classifier Ω) is *really local* if the characteristic map $A \to \Omega$ of the subobject of (multiplicatively) invertible elements of A is a rig morphism when Ω is considered equipped with its canonical distributive lattice structure [9].

In an integral rig the unit 1 is the only invertible element. It is then easy to check [4, Lemma 6.2] that an integral rig (in a topos $\mathcal{E}$) is really local if and only if it satisfies the following sequents

$$\begin{array}{rcl} 0 = 1 & \vdash & \bot \\ x + y = 1 & \vdash_{x,y} & (x = 1) \vee (y = 1) \end{array}$$

in the internal logic of $\mathcal{E}$. Notice that this sequents imply those in Theorem 8.5.

Notice also that if R is an integral rig in a topos $\mathcal{E}$ then the sequent

$$(x = 1) \vee (y = 1) \ \vdash_{x,y} \ x + y = 1$$

holds, but the witnessing inclusion

$$\{(x, y) \mid (x = 1) \vee (y = 1)\} \subseteq \{(x, y) \mid x + y = 1\}$$

of subobjects of $R \times R$ need not be an isomorphism, so R need not be really local. Something similar happens in the classical context: the generic idempotent-free $\mathbb{C}$-algebra is not local (in the classical sense) in the complex Gaeta topos; on the other hand, the same object, as an algebra in the Zariski subtopos, is local; indeed, it is the generic local $\mathbb{C}$-algebra.

Lemma 9.1. *The generic Boolean-free integral rig R is not really local.*

Proof. By Theorem 8.5 the classifier of Boolean-free integral rigs is the Gaeta topos of **iAff** and the generic object therein is the 'affine line' representable by the free integral rig on one generator. To check if R is really local in the Gaeta topos $\mathfrak{G}(\textbf{iAff})$ it is convenient to present the topos as that of presheaves on connected objects. Let $\mathbf{iRig}_{fpi} \to \mathbf{iRig}_{fp}$ be the full subcategory of directly indecomposable (finitely presentable) integral rigs so that $\mathfrak{G}(\textbf{iAff})$ may be identified with the functor category $[\mathbf{iRig}_{fpi}, \mathbf{Set}]$.

Let $\mathbf{i} : \mathbf{Set} \to \mathbf{iRig}$ be the left adjoint to the forgetful functor and let $\mathbf{i}[x]$ be the free integral rig on one generator so that the representable object $R = \mathbf{iRig}_{fpi}(\mathbf{i}[x], -)$ in $\mathfrak{G}(\textbf{iAff})$ is the generic Boolean-free integral rig.

The 'affine plane' $R \times R$ in $\mathfrak{G}(\textbf{iAff})$ is representable by the free integral rig $\mathbf{i}[x, y]$ on two generators so the subobject $\{(x, 1) \mid x \in R\} \subseteq R \times R$ in $\mathfrak{G}(\textbf{iAff})$, which is the same thing as the monic $id \times 1 : R \times 1 \to R \times R$, is the cosieve in $\mathbf{iRig}_{fpi}$ generated by the map $\mathbf{i}[x, y] \to \mathbf{i}[x, y, y^{-1}] \cong \mathbf{i}[x]$ that sends x to x, and y to 1. Similarly for $\{(1, y) \mid y \in R\} \subseteq R \times R$. Hence, the subobject $\{(x, y) \mid (x = 1) \vee (y = 1)\} \subseteq R \times R$ is the cosieve in $\mathbf{iRig}_{fpi}$ generated by the span $\mathbf{i}[x] \leftarrow \mathbf{i}[x, y] \to \mathbf{i}[y]$.

On the other hand, the subobject $\{(x, y) \mid x + y = 1\} \subseteq R \times R$ in the topos $\mathfrak{G}(\textbf{iAff})$ is the cosieve in $\mathbf{iRig}_{fpi}$ generated by the quotient morphism $\mathbf{i}[x, y] \to \mathbf{i}[x, y]/(x + y = 1)$. So it is enough to show that this quotient does not factor through $\mathbf{i}[x, y] \to \mathbf{i}[x]$ or $\mathbf{i}[x, y] \to \mathbf{i}[y]$; but this is easy. $\square$

Loosely speaking, although $\mathfrak{G}(\textbf{iAff})$ has the 'right' coproducts, the colimit (join)

$$\{(x, 1) \mid x \in R\} \vee \{(1, y) \mid y \in R\}$$

of subobjects of $R \times R$ is not 'right' in $\mathfrak{G}(\textbf{iAff})$ (or in **iAff**) but we can correct it by a considering a suitable subtopos. Indeed, the least subtopos of $\mathfrak{G}(\textbf{iAff})$ forcing the inclusion

$$\{(x, y) \mid (x = 1) \vee (y = 1)\} \subseteq \{(x, y) \mid x + y = 1\}$$

to become an isomorphism is the topos of sheaves on **iAff** for the least Grothendieck topology containing the Gaeta coverage and also the sieve (co)generated by the span

$$\mathbf{i}[x] \longleftarrow \mathbf{i}[x, y]/(x + y = 1) \longrightarrow \mathbf{i}[x]$$

in $\textbf{iAff}^{\mathrm{op}} = \mathbf{iRig}_{fp}$. In the classical case over the complex numbers the analogous construction results in the Zariski topos.

10. The 'Zariski' topos of the theory of integral rigs

Let $\mathcal{C}$ be a category with finite limits and equipped with a distinguished integral rig R. For any finite family $(f_i : X \to R \mid i \in I)$ we denote the composite

$$X \xrightarrow{\langle f_i | i \in I \rangle} R^I \xrightarrow{\sum_{i \in I}} R$$

by $\bigoplus_{i \in I} f_i : X \to R$. The family is said to *cocover* X if the diagram below

$$\begin{tikzcd} X \arrow[r] \arrow[d, "\langle f_i | i \in I \rangle"'] \arrow[dr, "\bigoplus_{i \in I} f_i"] & 1 \arrow[d, "1"] \\ R^I \arrow[r, "\sum_{i \in I}"'] & R \end{tikzcd}$$

commutes.

A finite family $(u_i : U_i \to X \mid i \in I)$ of maps in $\mathcal{C}$ is said to *cover* X if there is a cocovering family $(f_i : X \to R \mid i \in I)$ such that the following diagram is a pullback

$$\begin{tikzcd} U_i \arrow[r] \arrow[d, "u_i"'] & 1 \arrow[d, "1"] \\ X \arrow[r, "f_i"'] & R \end{tikzcd}$$

for every $i \in I$. Notice that all the maps in a covering family must be monic. One easily sees that isomorphisms cover and that covers are stable under pullback.

Different properties of R will determine different properties of covers. Rather than pursuing this idea in the abstract we are going to concentrate on the case $\mathcal{C} = \mathbf{iAff}$ equipped with the integral rig R therein determined by the free integral rig $\mathbf{i}[x]$ on one generator (considered as an object in $\mathbf{iAff}^{\mathrm{op}} = \mathbf{iRig}_{fp}$).

If A is in $\mathbf{iRig}_{fp}$ and X is the corresponding object in $\mathbf{iAff}$ then a map $X \to R$ in $\mathbf{iAff}$ is a map $\mathbf{i}[x] \to A$ in $\mathbf{iRig}_{fp}$; that is, an element in A. So a family $(f_i : X \to R \mid i \in I)$ may be identified with a family $(a_i \mid i \in I)$ of elements in A. The map $\bigoplus_{i \in I} f_i : X \to R$ corresponds to $\sum_{i \in I} a_i \in A$. Hence, the family $(f_i : X \to R \mid i \in I)$ cocovers the object X if and only if $\sum_{i \in I} a_i = 1 \in A$. In this case we say that $(a_i \mid i \in I)$ *cocovers* A.

Lemma 10.1. *A finite family $(a_i \mid i \in I)$ cocovers A if and only if the ideal generated by the family contains $1 \in A$.*

Proof. The generated ideal contains the unit 1 if and only if there is a family $(b_i \mid i \in I)$ such that $\sum_{i\in I} a_i b_i = 1$. In an integral rig this holds if and only if $1 \leq \sum_{i\in I} a_i b_i \leq \sum_{i\in I} a_i$. □

Again, let $\mathbf{i}[x] \to A$ in $\mathbf{iRig}_{fp}$ be the unique map determined by $a \in A$ and let $X \to R$ be the corresponding map in **iAff**. Any pullback in **iAff** as on the left below

$$\begin{tikzcd} U \arrow[r] \arrow[d] & 1 \arrow[d, "1"] \\ X \arrow[r] & R \end{tikzcd} \qquad \begin{tikzcd} \mathbf{i}[x] \arrow[r] \arrow[d] & 2 \arrow[d] \\ A \arrow[r] & A[a^{-1}] \end{tikzcd}$$

corresponds to a pushout in $\mathbf{iRig}_{fp}$ as on the right above, where the top map sends x to 1 and the left map sends x to $a \in A$. Hence, a finite family of $(u_i : U_i \to X \mid i \in I)$ of maps in $\mathcal{C}$ *covers* X if and only if there is a cocover $(a_i \mid i \in I)$ of A such that the map in $\mathbf{iRig}_{fp}$ corresponding to u_i has the universal property of $A \to A[a_i^{-1}]$ for each $i \in I$.

Altogether, already familiar with the (trivial) duality $\mathbf{iAff} = \mathbf{iRig}_{fp}^{\mathrm{op}}$, we may say a *(co)cover* of A in $\mathbf{iRig}_{fp}$ is a finite family of universal maps $(A \to A[a_i^{-1}] \mid i \in I)$ such that $\sum_{i\in I} a_i = 1 \in A$.

In order to continue our study of (co)covers it is convenient to have a concrete construction the universal maps inverting elements in integral rigs.

Let A be an integral rig and $F \subseteq A$ be a multiplicative submonoid. Let $A \to A[F^{-1}]$ be the universal map in **iRig** inverting all the elements of F; in other words, sending all the elements of F to 1. For $x, y \in A$ write $x \mid_F y$ if there is a $w \in F$ such that $wx \leq y$. (Notice the similarity with $\leq_F$ in Section 6; but notice also that, as A is integral, the condition "$1 \leq_F u$ for every $u \in F$" in Lemma 6.2 only holds if F is trivial.) Write $x \equiv_F y$ if $x \mid_F y$ and $y \mid_F x$. Lemma 3.4 in [4] shows that $\equiv_F$ is a congruence and that the quotient $A \to A/{\equiv_F}$ has the universal property of $A \to A[F^{-1}]$.

Lemma 10.2. *The map $A \to A[F^{-1}]$ inverts $a \in A$ if and only if there exists $w \in F$ such that $w \leq a$. Also, the object $A[F^{-1}]$ is terminal if and only if $0 \in F$.*

Proof. The universal map inverts a if and only if $1 \equiv_F a$ if and only if $1 \mid_F a$ and $a \mid_F 1$. One of the conjuncts is trivial and the other is equivalent to the existence of a $w \in F$ such that $w \leq a$.

Also, $A[F^{-1}]$ is terminal if and only if $0 \equiv_F 1$ if and only if $0 \leq_F 1$ and $1 \leq_F 0$. Again, one of the conjuncts is trivial and the other is equivalent to the existence of an $w \in F$ such that $w \leq 0$. □

Taking $a \in A$ and $F = \{a^n \mid n \in \mathbb{N}\} \subseteq A$ we obtain $A \to A[a^{-1}]$. By Lemma 10.2 this map inverts $b \in A$ if and only if there is an $n \in \mathbb{N}$ such that $a^n \leq b$.

We have already observed in the abstract setting that isomorphisms cover and that covers are stable under pullback. So naturally we now concentrate on compositions of (co)covers. In order to carry out the arguments we introduce a small piece of notation. For $a \in A$ we write $\frac{(-)}{a} : A \to A[a^{-1}]$ for the universal map so that, for $b \in A$, the resulting element in $A[a^{-1}]$ is denoted by $\frac{b}{a} \in A[a^{-1}]$. For instance, a straightforward argument using universal properties shows the following.

Lemma 10.3. *For any $a, b \in A$, the composite maps*

$$A \to A[a^{-1}] \to A[a^{-1}][(\frac{b}{a})^{-1}] \quad \text{and} \quad A \to A[b^{-1}] \to A[b^{-1}][(\frac{a}{b})^{-1}]$$

have the universal property of $A \to A[(ab)^{-1}]$ in $\mathbf{iRig}$ *and the following square is a pushout*

$$\begin{tikzcd} A \arrow[r] \arrow[d] & A[b^{-1}] \arrow[d] \\ A[a^{-1}] \arrow[r] & A[(ab)^{-1}] \end{tikzcd}$$

in $\mathbf{iRig}$.

Those familiar with the usual presentation of the Zariski topos will recognize the following auxiliary fact.

Lemma 10.4. *If the family $(\frac{b_j}{a} \mid j \in J)$ covers $A[a^{-1}]$ then there exists a $1 \leq k \in \mathbb{N}$ such that $a^k \leq \sum_{j \in J} ab_j$ in A.*

Proof. By hypothesis we have $\frac{\sum_j b_j}{a} = \sum_j \frac{b_j}{a} = 1$ so, as explained above (Lemma 10.2), there is an $n \in \mathbb{N}$ such that $a^n \leq \sum_j b_j$, so $a^{n+1} \leq \sum_j ab_j$. $\square$

We can now prove that (co)covers compose.

Lemma 10.5. *If $(a_i \mid i \in I)$ covers A and, for each $i \in I$, $(\frac{b_{i,j}}{a_i} \mid j \in J_i)$ covers $A[a_i^{-1}]$ then $(a_i b_{i,j} \mid i \in I, j \in J_i)$ covers A.*

Proof. By hypothesis $\sum_{i \in I} a_i = 1$ and, by Lemma 10.4 above, there is a $1 \leq k_i \in \mathbb{N}$ such that $a_i^{k_i} \leq \sum_j a_i b_{i,j}$ for each $i \in I$. So, by [4, Lemma 4.1],

$$1 = \left(\sum_{i \in I} a_i \right)^{\prod_{i \in I} k_i} \leq \sum_{i \in I} a_i^{k_i} \leq \sum_{i \in I} \sum_{j \in J_i} a_i b_{i,j}$$

as we needed to show. $\square$

We summarize what we have obtained so far in this section.

Proposition 10.6. *The cocovering families in $\mathbf{iRig}_{fp}$ form the basis for a Grothendieck topology on* **iAff** *and the resulting topos of sheaves classifies really local integral rigs.*

Proof. We observed that identities cover and that covers are stable under pullback. Lemma 10.5 proves that covers compose. We therefore have a basis and the resulting topos of sheaves. We occasionally refer to it as the 'Zariski' basis.

An argument analogous to that of Proposition 7.2 (and Theorem 8.5) establishes the classifying role of the topos of sheaves. In more detail one shows that the topology generated be the sequents stated in the beginning of the section coincides with the topology generated by the Zariski basis. To sketch the idea in more detail let $\mathbf{i} : \mathbf{Set} \to \mathbf{iRig}$ be the left adjoint to the forgetful functor. For efficiency we use some familiar notational tricks so, for example we write $\mathbf{i}[(x+y)^{-1}]$ instead of $\mathbf{i}[x,y][(x+y)^{-1}]$. Consider the span

$$\mathbf{i}[x^{-1}, y] \longleftarrow \mathbf{i}[(x+y)^{-1}] \longrightarrow \mathbf{i}[x, y^{-1}]$$

in $\mathbf{iRig}_{fp}$ induced by the sequent $x + y = 1 \vdash_{x,y} (x = 1) \vee (y = 1)$. Clearly, the pair $x, y \in \mathbf{i}[(x + y)^{-1}]$ cocovers. Similarly, the empty family cocovers the terminal algebra. That is, the topology generated by the sequents is included in the Zariski topology. Finally, one checks that these to covers generate the Zariski basis. □

The basis on **iAff** described in this section may be called the 'Zariski' basis. (We stress the evident fact that, as in the classical case over fields, the Zariski basis contains the Gaeta basis.) The topos of sheaves for the Zariski basis on **iAff** will be denoted by $\mathcal{Z}$.

Remark 10.7 (On the representation of integral rigs)**.** Let R be the generic really local integral rig in $\mathcal{Z}$. The results in [4] imply that for any integral rig A there exists a spatial topos $\Gamma : \mathcal{E}_A \to \mathbf{Set}$ and a geometric morphism $\mathcal{O}_A : \mathcal{E}_A \to \mathcal{Z}$ over $\mathbf{Set}$ such that the algebra $\Gamma(\mathcal{O}_A^* R)$ of global sections of the sheaf $\mathcal{O}_A^* R$ of really local integral rigs is isomorphic to A. Compare with the classical Zariski representation of rings.

11. 'Zariski' covers of connected objects

In order to show that the Zariski topos of Section 10 is locally connected (over $\mathbf{Set}$) we will present a locally connected site for it. Local connectedness of the site will follow from the main result of the present section which proves, roughly speaking, that the Zariski basis on **iAff** is well behaved with respect to connectedness. We first need an algebraic result concerning covering families.

Lemma 11.1. *Let A be an integral rig and let the finite family $(a_i \in A \mid i \in I)$ cover A. Then, for any family $(k_i \in \mathbb{N} \mid i \in I)$, $(a_i^{k_i} \in A \mid i \in I)$ covers A.*

Proof. A standard argument using the multinomial theorem. In more detail, if we let $k = I \cdot \max_{i \in I} k_i$ then

$$1 = \left(\sum_{i \in I} a_i \right)^k = \sum_{i \in I} u_i({a_i}^{k_i})$$

for some family $(u_i \mid i \in I)$ of elements of A. Hence, $1 \leq \sum_{i \in I} a_i^{k_i}$. □

The next result has a more geometric flavour.

Lemma 11.2. *Let X be connected in* **iAff** *and let the subobjects $u : U \to X$, $v : V \to X$ form a Zariski cover of X. If u and v are disjoint then either U is initial or V is initial.*

Proof. We argue on the algebraic side. Let $a, b \in A$ cover a directly indecomposable integral rig A. By Lemma 10.3 the cointersection of $A \to A[a^{-1}]$ and $A \to A[b^{-1}]$ is the universal $A \to A[(ab)^{-1}]$.

If the cointersection $A[(ab)^{-1}]$ is terminal then, by Lemma 10.2, there is an $n \in \mathbb{N}$ such that $(ab)^n = a^n b^n = 0$. Also, by Lemma 11.1, $a^n + b^n = 1$. So, as A is directly indecomposable by hypothesis, we may, without loss of generality, assume that $a^n = 1$ and $b^n = 0$. Then $A[b^{-1}]$ is terminal by Lemma 10.2. $\square$

The following variant will be useful.

Lemma 11.3. *Let X be connected in* **iAff** *and let the subobjects $u : U \to X$, $v : V \to X$ form a Zariski cover of X. If U, V are non-initial in* **iAff** *then there is a point in the intersection $u \wedge v$. Equivalently, there are points $1 \to U$ and $1 \to V$ such that the following diagram*

$$\begin{tikzcd} 1 \arrow[r] \arrow[d] & V \arrow[d, "v"] \\ U \arrow[r, "u"'] & X \end{tikzcd}$$

commutes in **iAff**.

Proof. By Lemma 11.2 the intersection is not empty so it is a finite coproduct of connected objects. Hence, a point in the intersection exists by the Nullstellensatz for 2-rigs. $\square$

A subobject $U \to X$ in **iAff** is *basic* if the corresponding map in $\mathbf{iRig}_{fp}$ is of the form $A \to A[a^{-1}]$ for some $a \in A$. The next result shows that finite families of basic subobjects (of a common object) have a kind of 'join'.

Lemma 11.4. *If $(u_i : U_i \to X \mid i \in I)$ is a finite family of basic subobjects in* **iAff** *then there is a basic subobject $u : U \to X$ such that the following hold:*

1. *For every $i \in I$, $u_i \leq u$ as subobjects of X.*

2. *Every point of U factors through one of the inclusions $U_i \to U$ of the previous item.*

Proof. We argue on the algebraic side. We have a finite family

$$(A \to A[a_i^{-1}] \mid i \in I)$$

in $\mathbf{iRig}_{fp}$ (corresponding to the family of subobjects in the statement). Let $a = \sum_{i \in I} a_i$ and consider the map $A \to A[a^{-1}]$. Its universal property implies, for each $i \in I$, the existence of a unique map $A[a^{-1}] \to A[a_i^{-1}]$ such that the left triangle below

$$\begin{tikzcd}
A \arrow[r] \arrow[dr] & A[a^{-1}] \arrow[d] \arrow[dr, "f"] & \\
& A[a_i^{-1}] \arrow[r, "f'"'] & 2
\end{tikzcd}$$

commutes in $\mathbf{iRig}_{fp}$, so the first item is proved. To prove the second item let f be a map as in the right above. Then $\sum_{i \in I} f a_i = 1 \in 2$ and, as 2 is really local, there is an $i \in I$ such that $f a_i = 1$. So there is an f' such that the right triangle above commutes. $\square$

The next result is a 'Zariski analogue' of a familiar property of open covers of connected topological spaces.

Proposition 11.5. *Let $(u_i : U_i \to X \mid i \in I)$ be a Zariski cover of X in* **iAff** *such that U_i is not initial for each $i \in I$. If X is connected then, for every $k, l \in I$ there exists a sequence $k = i_0, i_1, \ldots, i_n = l \in I$ and a commutative diagram as below.*

$$\begin{tikzcd}
& 1 \arrow[dl] \arrow[dr] & & 1 \arrow[dl] \arrow[dr] & & \cdots & & 1 \arrow[dl] \arrow[dr] & \\
U_{i_0} \arrow[drrrrr, bend right, "u_{i_0}"'] & & U_{i_1} \arrow[drrr, bend right, "u_{i_1}"] & & U_{i_2} \arrow[dr, bend right, "u_{i_2}"] & \cdots & U_{i_{n-1}} \arrow[dl, bend left, "u_{i_{n-1}}"'] & & U_{i_n} \arrow[dlll, bend left, "u_{i_n}"] \\
& & & & & X & & &
\end{tikzcd}$$

Proof. Fix $k \in I$ and let $J \subseteq I$ be the subset of those $l \in I$ such that there is a sequence $k = i_0, i_1, \ldots, i_n = l \in I$ and a diagram as in the statement. Let $u : U \to X$ be the basic subobject determined as in Lemma 11.4 by the family $(u_j : U_j \to X \mid j \in J)$. Similarly, Let $v : V \to X$ be the basic subobject determined by the complement $J' \subseteq I$ of $J \subseteq I$. It is not difficult to check that u, v cover X. Assume for the sake of contradiction that J' is not-empty. Then V is not initial so Lemma 11.3 implies the existence of a point in the intersection of u and v. Lemma 11.4 implies that the same point is in U_j for some $j \in J$ and in $U_{j'}$ for some $j' \in J'$. Then $j' \in J$, which is absurd. Hence J' is empty. □

12. The 'Zariski' topos is pre-cohesive

In Section 10 we equipped **iAff** with the basis of a 'Zariski' topology and showed that the resulting topos $\mathcal{Z}$ of sheaves classifies really local integral rigs. In this section we show that this basis is subcanonical and that the canonical geometric morphism $\mathcal{Z} \to \mathbf{Set}$ is pre-cohesive. (Again, the general strategy is analogous to that of the classical case.)

Lemma 12.1. *Zariski covers in* **iAff** *are jointly epic.*

Proof. We argue on the algebraic side. We prove that if A is a integral rig and the finite family $(a_i \in A \mid i \in I)$ covers A then $(A \to A[a_i^{-1}] \mid i \in I)$ is a jointly monic family of maps.

Let $x, y \in A$ be such that $\frac{x}{a_i} = \frac{y}{a_i}$ in $A[a_i^{-1}]$ for each $i \in I$. Then, there is an $m \in \mathbb{N}$ such that $a_i^m x \leq y$ and $a_i^m y \leq x$ for each $i \in I$. As the family $(a_i \in A \mid i \in I)$ covers, so does $(a_i^m \in A \mid i \in I)$ by Lemma 11.1. That is, $1 = \sum_{i \in I} a_i^m$. Then

$$x = \sum_{i \in I} a_i^m x \leq \sum_{i \in I} y = y$$

and, similarly, $y \leq x$. □

It follows that representable objects in $\widehat{\mathbf{iAff}}$ are separated.

Proposition 12.2. *The 'Zariski' topology on* **iAff** *is subcanonical.*

Proof. Recall that we denote the left adjoint to the forgetful functor by $\mathbf{i} : \mathbf{Set} \to \mathbf{iRig}$ so that the free integral rig on one generator may be denoted by $\mathbf{i}[x]$. We first prove that $R = \mathbf{iRig}_{fp}(\mathbf{i}[x], -) : \mathbf{iRig}_{fp} \to \mathbf{Set}$ is a sheaf. Let $(a_i \mid i \in I)$ cover A. By Lemma 10.3, a family $(\frac{x_i}{a_i} \in A[a_i^{-1}] \mid i \in I)$ is compatible with the cover if, for every $i, j \in I$, $\frac{x_i}{a_i a_j} = \frac{x_j}{a_i a_j} \in A[(a_i a_j)^{-1}]$. By Lemma 10.2 above there is, for each $i, j \in I$, an $m_{i,j} \in \mathbb{N}$ such that $(a_i a_j)^{m_{i,j}} x_i \leq x_j$ and $(a_i a_j)^{m_{i,j}} x_j \leq x_i$. If we let m be the largest of the $m_{i,j}$'s then we get that $(a_i a_j)^m x_i \leq x_j$ and $(a_i a_j)^m x_j \leq x_i$. So, if we let $x = \sum_{i \in I} a_i^m x_i$ then, clearly $a_j^m x_j \leq x$ for every $j \in I$ and also

$$a_j^m x = \sum_{i \in I} (a_i a_j)^m x_i \leq \sum_{i \in I} x_j = x_j$$

so $\frac{x_j}{a_j} = \frac{x}{a_j}$ in $A[a_j^{-1}]$. In other words, the compatible family has an amalgamation. This amalgamation is unique by Lemma 12.1.

Sheaves are closed under finite limits and every object in **iAff** is the equalizer of a parallel pair of maps between finite powers of R. As R is a sheaf, the result follows. $\square$

We next show that the canonical geometric morphism $\mathcal{Z} \to \mathbf{Set}$ is pre-cohesive. It is enough to provide a locally connected site for $\mathcal{Z}$, but the one we have on **iAff** is not. The rest of the section is devoted to find one.

Since we have presented our toposes using bases for Grothendieck topologies it is convenient have a version of the Comparison Lemma in terms of these. The following is surely folklore.

Let $\mathcal{C}$ be a small category equipped with the basis K for a Grothendieck topology. Let $\mathcal{D} \to \mathcal{C}$ be a full subcategory of $\mathcal{C}$. We say that the subcategory is *(K-)dense* if for every C in $\mathcal{C}$ there is a K-cover $(D_i \to C \mid i \in I)$ in $\mathcal{C}$ with D_i in $\mathcal{D}$ for every $i \in I$. For D in $\mathcal{D}$ let $K'D \subseteq KD$ be the set of K-covers $(D_i \to D \mid i \in I)$ such that D_i in $\mathcal{D}$ for every $i \in I$.

Lemma 12.3. *With the notation above, if $\mathcal{D} \to \mathcal{C}$ is K-dense then K' is the basis for a Grothendieck topology on $\mathcal{D}$ and the obvious restriction functor induces an equivalence* $\mathrm{Sh}(\mathcal{C}, K) \to \mathrm{Sh}(\mathcal{D}, K')$.

Proof. It is easy to check that isomorphisms K'-cover and that K'-covers compose. Assume now that $(f_i : D_i \to D \mid i \in I)$ is a K'-cover. So it is a K-cover and then, for any $g : E \to D$ in $\mathcal{D}$, there exists a K-cover $(g_j : C_j \to E \mid j \in J)$ such that for every $j \in J$ there is an $i_j \in I$ such that gg_j factors through f_{i_j}. As $\mathcal{D}$ is K-dense there is, for each $j \in J$, a K-cover $(h_{j,k} : B_{j,k} \to C_j \mid k \in J_j)$ with $B_{k,j}$ in $\mathcal{D}$ for every $k \in J_j$. The composite family $(g_j h_{j,k} : B_{j,k} \to E \mid j \in J, k \in J_j)$ is a K-cover and, as all the domains are in $\mathcal{D}$, it is also a K'-cover. Moreover, for every $j \in J$ and $k \in J_j$ the map $gg_j h_{j,k}$ factors through f_{i_j}. Altogether, we have shown that K' is the basis of a Grothendieck topology.

Let $\overline{K}$ be the Grothendieck topology generated by K. That is, a sieve on C in $\mathcal{C}$ is $\overline{K}$-covering if and only if it contains all the maps in a K-covering family. Density of $\mathcal{D}$ in the 'basis sense' of the statement easily implies that $\mathcal{D}$ is $\overline{K}$-dense in the sense of the Comparison Lemma, so restriction along $\mathcal{D} \to \mathcal{C}$ induces an equivalence $\mathrm{Sh}(\mathcal{C}, \overline{K}) \to \mathrm{Sh}(\mathcal{D}, L)$ where L is the topology on $\mathcal{D}$ induced by $\overline{K}$ (in the sense of the Comparison Lemma). It remains to show that the basis K' generates the topology L.

A sieve S in $\mathcal{D}$ on an object D is L-covering if and only if the generated sieve $\overline{S} = \{fg \mid f : D' \to D \text{ in } S, g : C \to D' \text{ in } \mathcal{C}\}$ in $\mathcal{C}$ is $\overline{K}$-covering. That is, if and only if $\overline{S}$ contains the maps in a K-covering family F on D. Composing F with the special covers provided by density (in the 'basis sense'), as in the paragraph above, we obtain that $\overline{S}$ contains the maps in a K'-cover. In other words, every L-covering sieve is $\overline{K'}$-covering where $\overline{K'}$ is the topology generated by K'. Conversely, if a sieve S on D is $\overline{K'}$-covering then it contains the maps in a K'-covering family. As every K'-covering family is K-covering, $\overline{S}$ is $\overline{K}$-covering and hence, S is L-covering. □

We may now prove the main result of the section.

Theorem 12.4. *The classifier of really local integral rigs is pre-cohesive over sets.*

Proof. By [7, Proposition 1.4] it is enough to provide a connected and locally connected site of definition for $\mathcal{Z}$ such that every object in the site has a point. Let K be the 'Zariski' basis on **iAff** introduced in Section 10. As the Zariski basis contains the Gaeta basis and every object in **iAff** is a finite coproduct of connected objects (Corollary 8.4), the full subcategory $\mathbf{iAff}_c \to \mathbf{iAff}$ of connected objects is K-dense. Lemma 12.3 implies that $\mathcal{Z} = \mathrm{Sh}(\mathbf{iAff}, K)$ is equivalent to $\mathrm{Sh}(\mathbf{iAff}_c, K')$ where K' is the restriction of K. The category $\mathbf{iAff}_c$ has a terminal object (because 2 is directly indecomposable in $\mathbf{iRig}$). That is, the site $(\mathbf{iAff}_c, K')$ is connected. Also, every object has a point by Proposition 6.5. Finally, the site is locally connected by Proposition 11.5. $\square$

Altogether, as in the classical space, the classifier $\mathcal{Z} \to \mathbf{Set}$ of really local integral rigs is pre-cohesive; the Yoneda embedding restricts to a full inclusion $\mathbf{iAff} \to \mathcal{Z}$ that sends Zariski covers to jointly epimorphic families so, in particular, it preserves finite coproducts.

Recent unpublished work on integral rigs by Jipsen and Spada on subdirectly irreducible integral rigs suggests that it is possible to calculate level ϵ of the pre-cohesive toposes $\mathfrak{G}(\mathbf{iAff})$ and $\mathcal{Z}$ as in the classical complex case discussed in [14].

On the other hand, if we let R be the generic really local integral rig then, although the subobject $D = \{x \in R \mid x^2 = 0\} \to R$, is non-trivial, the exponential R^D is not isomorphic to $R \times R$. In other words, the Kock-Lawvere axiom for SDG does not hold. At present it is not clear to the author if this is a drawback or an opportunity for interesting variants of the KL-axiom. Also in contrast with the classical case, the topos of simplicial sets is a subtopos of $\mathcal{Z}$. So there is a full inclusion $\widehat{\Delta} \to \mathcal{Z}$ and, for every X in $\mathcal{Z}$ (an 'i-scheme'), a universal map $X \to SX$ towards a simplicial set. Intuitively, the inverse image $\mathcal{Z} \to \widehat{\Delta}$ is a 'combinatorial realization' analogous to the classical 'geometric realizations' or, perhaps, it is more similar to the 'combinatorial truncations' $\widehat{\Delta} \to \widehat{\Delta_n}$ induced by the inclusions $\Delta_n \to \Delta$ for each $n \in \mathbb{N}$. See Corollary 7.5 in [13].

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