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AFFINE CONNECTIONS AND SECOND-ORDER AFFINE STRUCTURES

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Dedicated to my good friend Tom Rewwer on the occasion of his 35th birthday.

Résumé. Les variétés lisses ont toujours été intuitivement perçues comme étant des espaces munis d'une géométrie affine à l'échelle infinitésimale. Nous précisons cette notion en Géométrie Différentielle Synthétique en montrant que chaque variété est naturellement munie d'une structure d'espace infinitésimal affine, que nous interprétons comme l'action du clone des combinaisons affines sur une structure infinitésimale du premier ordre construite à partir du premier voisinage de la diagonale. Nous définissons une structure infinitésimale du second ordre basée sur le second voisinage de la diagonale et montrons que sur toute variété une connexion affine symétrique s'étend en une structure infinitésimale affine du second ordre en utilisant la bijection log-exp induite par la connexion.

Abstract. Smooth manifolds have been always understood intuitively as spaces with an affine geometry on the infinitesimal scale. We make this notion precise within Synthetic Differential Geometry by showing that every manifold carries a natural structure of an infinitesimally affine space, which we interpret as the action of the clone of affine combinations on a first-order infinitesimal structure constructed from the first neighbourhood of the diagonal. We define a second-order infinitesimal structure based on the second neighbourhood of the diagonal and show that on any manifold a symmetric affine connection extends to a second-order infinitesimally affine structure using the log-exp bijection induced by the connection.

Keywords. Infinitesimally affine space, higher-order infinitesimal structure, affine connection, Synthetic Differential Geometry.
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1. Introduction

A deeply rooted intuition about smooth manifolds is that of spaces that become linear spaces in the infinitesimal neighbourhood of each point. On the infinitesimal scale the geometry underlying a manifold is thus affine geometry. To make this intuition precise requires a good theory of infinitesimals as well as defining precisely what it means for two points on a manifold to be infinitesimally close. As regards infinitesimals we make use of *Synthetic Differential Geometry* (SDG) and adopt the neighbourhoods of the diagonal from Algebraic Geometry to define when two points are infinitesimally close. The key observations on how to proceed have been made by Kock in [5]: 1) The first neighbourhood of the diagonal exists on formal manifolds and can be understood as a symmetric, reflexive relation on points, saying when two points are infinitesimal neighbours, and 2) we can form affine combinations of points that are mutual neighbours.

It remains to make precise in which sense a manifold becomes a model of the theory of affine spaces. This has been done in [1]. Firstly, one abstracts from Kock's infinitesimal simplices of mutual infinitesimally neighbouring points to what is called an *infinitesimal structure*. (See also section 2 for a definition.) An infinitesimal structure serves then as the domain of definition for the operations of affine combinations. A space together with an infinitesimal structure (i-structure) and an action of the clone of affine operations on that infinitesimal structure is called an *infinitesimally affine space* (i-affine space).

Formal manifolds and affine schemes (considered as either duals of commutative rings, or C^∞-rings) are examples of i-affine spaces. The i-structures are generated by the first neighbourhood of the diagonal. In this paper we shall construct an i-structure from the second-order neighbourhood of the diagonal on R^n for a ring R satisfying the Kock-Lawvere axioms for second-order infinitesimals. The definition of this i-structure is guided by the requirement that it is preserved by all maps $f : R^n \to R^m$ (hence can be defined on formal manifolds as well) and that the affine structure of R^n restricts to an i-affine space on the second-order i-structure. Both of these hold true for the i-structure generated by the first neighbourhood of the diagonal. In contrast to the first neighbourhood of the diagonal the i-affine structure on the second-order neighbourhood is not preserved by all maps anymore. Therefore, whereas a manifold carries a second-order i-structure, an i-affine structure has to be imposed as an additional piece of data.

We show that any second-order i-affine structure on a manifold induces a symmetric affine connection, and, conversely, any symmetric affine connection extends to a second-order i-affine structure in such a way that the latter is of the same affine-algebraic form as the canonical connection on an affine space. The second-order i-affine structure is constructed by using the second-order $\log\text{-}\exp$ bijection induced by the connection as introduced by Kock in [5, chap. 8.2]. With the help of the $\log\text{-}\exp$ bijection the *infinitesimally linear* (i-linear) structure on the tangent space can be transported to the formal manifold. For affine combinations this is independent of the chosen base point and thus defines an i-affine structure on the second-order i-structure. The $\log\text{-}\exp$ bijection yields also a natural geometric interpretation of an affine combination as the geometric addition of geodesic line segments extending the familiar vector parallelogram construction from the affine plane to curved space.

2. Infinitesimally affine and linear spaces

We shall work mostly within naive axiomatic SDG, as it is done in [5], for example. Let A be a space. An *i-structure* on A amounts to give an n-ary relation $A\langle n\rangle$ for each $n \in \mathbb{N}$ that defines which n points in A are considered as being 'infinitesimally close' to each other.

Definition 2.1 (i-structure)**.** *Let A be a space. An* **i-structure** *on A is an $\mathbb{N}$-indexed family $n \mapsto A\langle n\rangle \subseteq A^n$ such that*

(1) $A\langle 1\rangle = A$, $A\langle 0\rangle = A^0 = 1$ (the 'one point' space, or terminal object)

(2) For every map $h : m \to n$ of finite sets and every $(P_1, \dots, P_n) \in A\langle n\rangle$ we have $(P_{h(1)}, \dots, P_{h(m)}) \in A\langle m\rangle$

The first condition is a normalisation condition. The second condition makes sure that the relations are compatible: if we have a family of points that are infinitesimally close to each other, then so is any subfamily of these points, or any family created from repetitions. In particular, we obtain that the $A\langle n\rangle$ are symmetric and reflexive relations. An n-tuple $(P_1, \ldots, P_n) \in A^n$ that lies in $A\langle n\rangle$ will be denoted by $\langle P_1, \ldots, P_n\rangle$ and we shall refer to these as *i-n-tuples*. A map $f : A \to X$ that maps i-n-tuples to i-n-tuples for each $n \in \mathbb{N}$, i.e. $f^n(A\langle n\rangle) \subseteq X\langle n\rangle$, is called an *i-morphism*.

Two trivial examples of i-structures on A are the discrete and the indiscrete i-structure obtained by taking $A\langle n\rangle$ to be the diagonal Δ_n, respectively the whole A^n. The i-structures that are of main interest in SDG are the i-structures generated by the first neighbourhood of the diagonal (as relations). We call them *nil-square i-structures*. For example, let R be a ring[1]. Recall that

$$D(n) = \{(d_1, \ldots, d_n) \in R^n \mid d_i d_j = 0,\ 1 \leq i, j \leq n\}$$

On R^n the first neighbourhood of the diagonal is given by

$$\{(P_1, P_2) \mid P_2 - P_1 \in D(n)\}$$

This is a symmetric and reflexive relation and we can construct an i-structure from it: take the first neighbourhood of the diagonal as $R^n\langle 2\rangle$ and define the *nil-square i-structure* on R^n by

$$R^n\langle m\rangle = \{(P_1, \ldots, P_m) \mid (P_i, P_j) \in R^n\langle 2\rangle,\ 1 \leq i, j \leq m\}$$

This i-structure is thus *generated* by $R^n\langle 2\rangle$. Not all i-structures $A\langle -\rangle$ of interest need to be generated by $A\langle 2\rangle$. The second-order i-structure defined in section 3 is not, for example.

If the ring R satisfies the *Kock-Lawvere axiom*, that is for every $n \in \mathbb{N}$ and every map $t : D(n) \to R$ there are unique $a_0, \ldots, a_n \in R$ such that

$$t(d_1, \ldots, d_n) = a_0 + \sum_{j=1}^{n} a_j d_j, \qquad (d_1, \ldots, d_n) \in D(n),$$

[1]All rings are assumed to be commutative.

then every map $f : R^n \to R^m$ is an i-morphism of the nil-square i-structures. This is due to the following two facts: linear maps $R^n \to R^m$ map $D(n)$ to $D(m)$, and for $P_2 - P_1 \in D(n)$

$$f(P_2) - f(P_1) = \partial f(P_1)[P_2 - P_1] \tag{1}$$

where $\partial f(P_1)$ denotes the *derivative of* f at P_1. The stated property of linear maps can be checked by direct computation; the existence and uniqueness of the linear map $\partial f(P_1)$ are both a consequence of the Kock-Lawvere axiom.

The nil-square i-structure induces i-structures on subspaces $U \hookrightarrow R^n$ by restriction. For *formally open subspaces* $U \hookrightarrow R^n$, which are stable under infinitesimal perturbations at each point (see [4, I.17] or [1, def. 3.2.5] for a definition), each map $f : U \to R^m$ has a derivative; hence every map $f : U \to V$ between formally open subspaces is an i-morphism. Furthermore, it is possible to glue the i-structures on formally open subspaces together to get an i-structure on a formal manifold and show that every map between formal manifolds is an i-morphism. (See [4, prop. I.17.5] and [1, thm. 3.2.8] for proofs.)

Definition 2.2 (i-affine space)**.** *Let* $A\langle - \rangle$ *be an* i-structure *on* A*. Set* $\mathcal{A}(n) = \{(\lambda_1, ..., \lambda_n) \in R^n \mid \sum_{j=1}^n \lambda_j = 1\}$*. The space* A *is said to be an* **i-affine space** *(over* R*), if for every* $n \in \mathbb{N}$ *there are operations*

$$\mathcal{A}(n) \times A\langle n \rangle \to A, \qquad ((\lambda_1, \dots, \lambda_n), \langle P_1, \dots, P_n \rangle) \mapsto \sum_{j=1}^{n} \lambda_j P_j$$

satisfying the axioms

- *(*Neighbourhood*) Let* $\lambda^k \in \mathcal{A}(n)$*,* $1 \le k \le m$*. If* $\langle P_1, \dots, P_n \rangle \in A\langle n \rangle$ *then*
$$\Big(\sum_{j=1}^{n} \lambda_j^1 P_j, \dots, \sum_{j=1}^{n} \lambda_j^m P_j\Big) \in A\langle m \rangle$$
- *(*Associativity*) Let* $\lambda^k \in \mathcal{A}(n)$*,* $1 \le k \le m$*,* $\mu \in \mathcal{A}(m)$ *and* $\langle P_1, \dots, P_n \rangle \in A\langle n \rangle$*. We have*
$$\sum_{k=1}^{m} \mu_k \Big(\sum_{j=1}^{n} \lambda_j^k P_j\Big) = \sum_{j=1}^{n} \Big(\sum_{k=1}^{m} \mu_k \lambda_j^k\Big) P_j$$

(Note that the left-hand side is well-defined due to the neighbourhood axiom.)

- (Projection) *Let* $n \geq 1$ *and let* $e_k^n \in R^n$ *denote the* k*th standard basis vector for* $1 \leq k \leq n$. *For every* $\langle P_1, \dots, P_n \rangle \in A\langle n \rangle$ *it holds*

$$\sum_{j=1}^{n} (e_k^n)_j P_j = P_k$$

In particular, we have for $n = 1$ *that* $1P = P$, $P \in A$.

The neighbourhood axiom makes sure that we can compose affine combinations as we are used to, provided we are working over a fixed i-tuple. The associativity and projection axioms make sure the algebra of affine combinations follows the same rules as in all the R^n. A consequence of the neighbourhood axiom is that every i-tuple generates an affine space over R. This makes precise the statement that the geometry of the space A is affine on the infinitesimal scale.

It is not difficult to show by direct calculation that the affine space R^n satisfies the neighbourhood axiom for the nil-square i-structure making it an i-affine space[2]. Moreover, due to (1) it follows that every map $f : R^n \to R^m$ preserves not only the nil-square i-structure but the i-affine combinations as well. Each map f is an *i-affine map*.

The nil-square structure of R^n restricts to its formally open subspaces. Due to (1) all maps between formally open subspaces become i-affine maps for these i-structures. Like with the i-structures also the i-affine structures on formally open subspaces can be glued together to an i-affine structure on a formal manifold. All maps between formal manifolds become i-affine maps for these i-affine structures [1, thm. 3.2.8]. Any manifold in the sense of classical differential geometry is a formal manifold[3], so any manifold is an i-affine space and any smooth map between manifolds is i-affine.

[2] This is also a consequence of the more general [1, cor. 3.1.6 and 2.3.3].

[3] This is to be understood in the context of well-adapted models of SDG [3], where we have a fully faithful embedding of the category of smooth manifolds into a Grothendieck topos that admits a model of the Kock-Lawvere axioms. This embedding maps the real line $\mathbb{R}$ to R, analytical derivatives to derivatives in SDG and it maps open covers to covers by formally open spaces [3], [4, III.3].

Affine schemes (considered as either duals of commutative rings, or C^∞-rings) become examples of i-affine spaces over their respective nil-square i-structure [1, cor. 2.3.3 and 3.1.6]. Every morphism of affine schemes becomes an i-morphism. Affine C^∞-schemes, for example, form a category of spaces generalising smooth manifolds. Besides manifolds the category fully faithfully embeds locally closed subsets of Euclidean space with smooth maps between them [7, prop. 1.5]. This provides us with a wealth of examples of i-affine spaces. Furthermore, i-affine spaces are surprisingly well-behaved under taking colimits of the underlying spaces [1, chap. 2.6], [2]. This and their algebraic nature makes them a suitable type of space to study geometric notions based on infinitesimals.

Besides i-affine spaces we shall also consider *i-linear spaces*. The definition is almost identical to that of i-affine spaces; the main difference being that an i-linear space has a constant, the zero vector 0.

Definition 2.3 (i-linear space). *Let $V\langle - \rangle$ be an* i-structure *on V. Set $\mathcal{L}(n) = R^n$, $n \in \mathbb{N}$. The space V is said to be an* **i-linear space** *(over R), if for every $n \in \mathbb{N}$ there are operations*

$$\mathcal{L}(n) \times V\langle n \rangle \to V, \qquad ((\lambda_1, \dots, \lambda_n), \langle v_1, \dots, v_n \rangle) \mapsto \sum_{j=1}^{n} \lambda_j v_j$$

where we denote the constant $\mathcal{L}(0) \times V\langle 0 \rangle \cong 1 \to V$ by 0. These operations satisfy the axioms

- *(*Neighbourhood*) Let $\lambda^k \in \mathcal{L}(n)$, $1 \leq k \leq m$. If $\langle v_1, \dots, v_n \rangle \in V\langle n \rangle$ then*

$$\Big(\sum_{j=1}^{n} \lambda_j^1 v_j, \dots, \sum_{j=1}^{n} \lambda_j^m v_j \Big) \in V\langle m \rangle$$

- *(*Associativity*) Let $\lambda^k \in \mathcal{L}(n)$, $1 \leq k \leq m$, $\mu \in \mathcal{L}(m)$ and $\langle v_1, \dots, v_n \rangle \in V\langle n \rangle$. We have*

$$\sum_{k=1}^{m} \mu_k \Big(\sum_{j=1}^{n} \lambda_j^k v_j \Big) = \sum_{j=1}^{n} \Big(\sum_{k=1}^{m} \mu_k \lambda_j^k \Big) v_j$$

  and for $0 \in \mathcal{L}(n)$ and $0 \in V$

$$\sum_{j=1}^{n} 0\, v_j = 0$$

- *(*Projection*) Let $n \geq 1$ and let $e_k^n \in R^n$ denote the kth standard basis vector for $1 \leq k \leq n$. For every $\langle v_1, \ldots, v_n \rangle \in A\langle n \rangle$ it holds*

$$\sum_{j=1}^{n} (e_k^n)_j v_j = v_k$$

In particular, we have for $n = 1$ that $1v = v$, $v \in V$.

The existence of the constant $0 \in V$ implies that for any $\langle v_1, \ldots, v_n \rangle \in V\langle n \rangle$ we have $\langle 0, v_1, \ldots, v_n \rangle \in V\langle n+1 \rangle$. This follows from combining the associativity axiom

$$\sum_{j=1}^{n} 0\, v_j = 0$$

with the projection and neighbourhood axioms. In particular, 0 has to be infinitesimally close to any other vector $v \in V$, which has a major implication on the size of V.

An example of an i-linear space is $D(n) \subset R^n$ with the restriction (= pullback) of the nil-square i-structure and the R-linear structure on R^n. More generally, for any *KL vector space* V the space

$$D(V) = \{v \in V \mid \phi[v]^2 = 0 \text{ for any bilinear map } \phi : V^2 \to R\}$$

becomes an i-linear space with the i-structure and R-linear structure induced by V. Writing $\phi[v]^\ell$ for an ℓ-linear map ϕ means that we evaluate ϕ on the ℓ-tuple $(v, \ldots, v)$. Indeed, recall that an R-vector space V is called KL if it satisfies the Kock-Lawvere axiom [4] for all maps $t : D(n) \to V$ and $n \in \mathbb{N}$. For the case of $V = R^n$ we have $D(V) = D(n)$ [5, prop. 1.2.2] and, like R^n, each KL vector space V carries a nil-square i-structure generated by

$$\{(v_1, v_2) \mid v_2 - v_1 \in D(V)\}$$

A KL vector space is called *finite-dimensional* if $V \cong R^n$ for some $n \in \mathbb{N}$. It follows from (1) that any map $f : V \to W$ between finite-dimensional KL vector spaces satisfying $f(0) = 0$ is an i-morphism preserving the linear combinations and thus restricts to an *i-linear map* $D(V) \to D(W)$.

[4] As with R, a KL vector space maybe required to satisfy more axioms from the Kock-Lawvere axiom scheme based on the context. See [5, chap. 1.3], for example.

An important class of examples of i-linear spaces is given by the subsequent general construction. For an i-affine space A and $P \in A$ we define the *monad* around P

$$\mathfrak{M}(P) = \{Q \in A \mid \langle P, Q\rangle \in A\langle 2\rangle\}$$

as the set of points infinitesimally close to P. It carries a natural i-structure

$$\mathfrak{M}(P)\langle n\rangle = \{(Q_1, \dots, Q_n\rangle) \in \mathfrak{M}(P)^n \mid \langle P, Q_1, \dots, Q_n\rangle \in A\langle n+1\rangle\}$$

Using the i-affine structure of A we can define a natural action of $\mathcal{L}(n)$ for each $n \in \mathbb{N}$

$$\begin{aligned} &\mathcal{L}(n) \times \mathfrak{M}(P)\langle n\rangle \to \mathfrak{M}(P), \\ &\qquad ((\lambda_1, \dots, \lambda_n), \langle Q_1, \dots, Q_n\rangle) \mapsto \big(1 - \sum_{j=1}^{n} \lambda_j\big)P + \sum_{j=1}^{n} \lambda_j Q_j \end{aligned}$$

making $\mathfrak{M}(P)$ into an i-linear space with P the zero vector. Any i-affine map f induces an i-linear map

$$f : \mathfrak{M}(P) \to \mathfrak{M}(f(P))$$

This is just the familiar construction of a vector space from an affine space for a given base point P re-phrased in infinitesimal algebra. Indeed, in the case of A being an affine space with the indiscrete i-structure we have $\mathfrak{M}(P) = A$, i-affine maps are precisely the affine maps and the base-point dependency of this construction disappears. We shall denote the action of $\lambda \in \mathcal{L}(n)$ on $\langle Q_1, \dots, Q_n\rangle \in \mathfrak{M}(P)\langle n\rangle$ by

$$P + \sum_{j=1}^{n} \lambda_j (Q_j - P)$$

As $D(V) = \mathfrak{M}(0)$ for a KL vector space V equipped with the nil-square i-structure the monad construction subsumes the first class of examples.

3. Second-order infinitesimal structures

The important examples of i-structures so far have all been the nil-square i-structures, which are constructed from the first neighbourhood of the diagonal. In this section we wish to define an i-structure $A_2 = A_2\langle - \rangle$ on $A = R^n$ such that $A_2\langle 2 \rangle$ is the second neighbourhood of the diagonal

$$\{(P_1, P_2) \mid P_2 - P_1 \in D_2(n)\}$$

where $D_2(n)$ is the space of second-order infinitesimals

$$D_2(n) = \{(d_1, \ldots, d_n) \in R^n \mid \text{any product of three } d_j \text{ vanishes}\}$$

The i-structure $A_2\langle - \rangle$ shall satisfy

1) All maps $f : R^n \to R^m$ become i-morphisms for the respective second-order i-structures on R^n and R^m

2) The affine space $A = R^n$ becomes an i-affine space over $A_2\langle - \rangle$.

To be able to study 1) we assume henceforth that R is a $\mathbb{Q}$-algebra that satisfies the Kock-Lawvere axiom for $D_2(n)$ with $n \geq 1$.[5] This amounts to say that each map $t : D_2(n) \to R$ is a polynomial function for a uniquely determined polynomial in $R[X_1, \ldots, X_n]$ of total degree ≤ 2, i.e.

$$t(d_1, \ldots, d_n) = a_0 + \sum_{j=1}^{n} a_j d_j + \sum_{1 \leq j \leq k \leq n} a_{jk} d_j d_k$$

for uniquely determined $a_j \in R$ and $a_{jk} \in R$. An important consequence is that every map $f : A \to R^m$ has a *Taylor representation*

$$f(P) - f(Q) = \partial f(Q)[P - Q] + \frac{1}{2}\partial^2 f(Q)[P - Q]^2$$

for $P - Q \in D_2(n)$. Here $\partial^2 f(Q)$ stands for the second derivative of f at Q, which is a symmetric bilinear map $(R^n)^2 \to R^m$. The following characterisation of $D_2(n)$ in [5, prop. 1.2.2] will be useful

$$D_2(n) = \{d \in R^n \mid \phi[d]^3 = 0 \text{ for all trilinear } \phi : (R^n)^3 \to R\}$$

[5]This requirement is not an overly restrictive one. For example, in a well-adapted model, where R is taken to be the embedding of the smooth manifold $\mathbb{R}$, the $\mathbb{R}$-algebra R satisfies the whole Kock-Lawvere axiom scheme [3, thm. 4.5], [7, prop. V.7.2].

It allows us to define $D_2(V)$ for any R-linear space V. Let $V \cong R^n$ be a finite-dimensional KL vector space[6]. We define $\mathrm{DN}_2(V)$ to be the space

$$\mathrm{DN}_2(V) = \{(v_1, v_2, v_3) \in \mathrm{D}_2(V)^3 \mid \text{For any trilinear map } \phi : V^3 \to R,\ \phi[v_1, v_2, v_3] = 0\}$$

In the subsequent definition and discussion we will use $A = V$ to mean the (affine) space induced by the R-linear space V.

Definition 3.1 (Second-order i-structure on R^n)**.** *Let $A = V \cong R^n$ be a finite-dimensional KL vector space. We define the* **second-order i-structure** *A_2 on A by*

(1) $A_2\langle 1\rangle = A$, $A_2\langle 0\rangle = A^0 = 1$

(2) For $m \geq 2$

$$A_2\langle m\rangle = \{(P_1, \ldots, P_m) \in A^m \mid (P_{i_1} - P_{j_1}, P_{i_2} - P_{j_2}, P_{i_3} - P_{j_3}) \in \mathrm{DN}_2(V), \textit{ for all } i_\ell, j_\ell \in \{1, \ldots, m\}, 1 \leq \ell \leq 3\}$$

From the definition it follows readily that A_2 is indeed an i-structure and that

$$A_2\langle 2\rangle = \{(P_1, P_2) \in A^2 \mid P_2 - P_1 \in D_2(n)\}$$

is the second neighbourhood of the diagonal, as desired.

The following two results show that the second-order i-structure A_2 is natural and makes any finite-dimensional KL vector space V into an i-affine space.

Theorem 3.2. *Every map $f : V \to W$ between two finite-dimensional KL vector spaces is an i-morphism for the respective second-order i-structures.*

Proof. Let $\langle P_1, \ldots, P_n\rangle \in V_2\langle n\rangle$ for an index $n \geq 2$. We have to show

$$\langle f(P_1), \ldots, f(P_n)\rangle \in W_2\langle n\rangle$$

[6] Note that V is also assumed to satisfy the respective V-valued Kock-Lawvere axiom for all $D_2(n)$.

By definition this amounts to show

$$\phi[f(P_{i_1}) - f(P_{j_1}), f(P_{i_2}) - f(P_{j_2}), f(P_{i_3}) - f(P_{j_3})] = 0$$

for all $i_\ell, j_\ell \in \{1, \ldots, n\}$, $1 \leq \ell \leq 3$ and any trilinear form ϕ on W. Since each $P_{i_\ell} - P_{j_\ell} \in D_2(V)$ we can apply Taylor expansion

$$f(P_{i_\ell}) - f(P_{j_\ell}) = \partial f(P_{j_\ell})[P_{i_\ell} - P_{j_\ell}] + \frac{1}{2}\partial^2 f(P_{j_\ell})[P_{i_\ell} - P_{j_\ell}]^2$$

Substituting each $f(P_{i_\ell}) - f(P_{j_\ell})$ with its respective Taylor expansion in ϕ and applying multilinearity to expand the three sums yields a sum of multilinear forms on V of the order 3 or higher with arguments being combinations of $P_{i_\ell} - P_{j_\ell}$ for $i_\ell, j_\ell \in \{1, \ldots, n\}$, $1 \leq \ell \leq 3$. Because of $\langle P_1, \ldots, P_n \rangle \in A_2\langle n \rangle$ each such multilinear form evaluates to 0, hence does the sum. This shows that

$$\phi[f(P_{i_1}) - f(P_{j_1}), f(P_{i_2}) - f(P_{j_2}), f(P_{i_3}) - f(P_{j_3})] = 0$$

as required. We conclude that f is an i-morphism as claimed. □

Theorem 3.3. *The affine structure on the KL vector space $A = V$ restricts to the second-order i-structure A_2, making A_2 an i-affine subspace of the affine space A (equipped with the indiscrete i-structure).*

Proof. To show A_2 an i-affine subspace of A it suffices to show that the affine operations on A satisfy the neighbourhood axiom for A_2.

Let $\lambda^i \in \mathcal{A}(n)$ for $1 \leq i \leq m$ and $\langle P_1, \ldots, P_n \rangle \in A_2\langle n \rangle$. We have to show

$$\big\langle \sum_{j=1}^{n} \lambda_j^1 P_j, \ldots, \sum_{j=1}^{n} \lambda_j^m P_j \big\rangle \in A_2\langle m \rangle$$

Let ϕ be a trilinear form on V and $i_\ell, j_\ell \in \{1, \ldots, m\}$ for all $1 \leq \ell \leq 3$. Using $\sum_{j=1}^{n} \lambda_j^i = 1$ for all $1 \leq i \leq m$ yields

$$\phi\big[\sum_{i=1}^{n} \lambda_i^{i_1} P_i - \sum_{j=1}^{n} \lambda_j^{j_1} P_j, \sum_{i=1}^{n} \lambda_i^{i_2} P_i - \sum_{j=1}^{n} \lambda_j^{j_2} P_j, \sum_{i=1}^{n} \lambda_i^{i_3} P_i - \sum_{j=1}^{n} \lambda_j^{j_3} P_j\big]$$
$$= \phi\big[\sum_{i,j=1}^{n} \lambda_i^{i_1}\lambda_j^{j_1}(P_i - P_j), \sum_{i,j=1}^{n} \lambda_i^{i_2}\lambda_j^{j_2}(P_i - P_j), \sum_{i,j=1}^{n} \lambda_i^{i_3}\lambda_j^{j_3}(P_i - P_j)\big]$$

Applying the trilinearity of ϕ yields a sum of trilinear forms with arguments being combinations of $P_{i_\ell} - P_{j_\ell}$ for $i_\ell, j_\ell \in \{1, \ldots, n\}$, $1 \leq \ell \leq 3$, which all evaluate to zero by assumption. We conclude

$$\langle \sum_{j=1}^{n} \lambda_j^1 P_j, \ldots, \sum_{j=1}^{n} \lambda_j^m P_j \rangle \in A_2\langle m \rangle$$

as required. □

The definitions of the second-order i-structure A_2 together with theorems 3.2 and 3.3 can be generalised to a formally open subspace A of R^n directly. This allows us to glue together the second-order i-structures to a second-order i-structure on a formal manifold and all maps between formal manifolds will preserve that structure.

Theorem 3.4. *Let A be a formal manifold.*

(i) A carries a unique i-structure A_2 with the universal property that any map $f : A \to M$ is an i-morphism, if and only if it is an i-morphism on the charts of A.

(ii) All maps between formal manifolds become i-morphisms for the respective second-order i-structures.

Proof. (i) (Essentially, this part is theorem 2.6.19 in [1] applied to the i-structure only. See also [2].) We consider a *chart* $\iota : U \hookrightarrow A$ of A, i.e. a formally open subspace of A that is also a formally open subspace of R^n. Pulling back the second-order i-structure on R^n yields a second-order i-structure U_2. For each $n \geq 1$ we define $A_2\langle n \rangle$ as the join of the images of $U_2\langle n \rangle$ over all the charts. It is easy to see that this yields an i-structure on A with the desired universal property.

(ii) Let $f : A \to M$ be a map between two formal manifolds equipped with the second-order i-structure as defined in (i) and $\langle P_1, \ldots, P_n \rangle \in A_2\langle n \rangle$. By construction there is an A-chart $\iota : U \hookrightarrow A$, $\phi : U \hookrightarrow R^n$, and $\langle x_1, \ldots, x_n \rangle \in U_2\langle n \rangle$ such that $\iota(x_\ell) = P_\ell$, $1 \leq \ell \leq n$.

We also find an M-chart $j : V \hookrightarrow M$ containing $f(P_1)$. Pulling back j along f yields a formally open subspace $f^*j : f^{-1}(V) \hookrightarrow M$, which becomes a chart after taking the intersection with ι

$$\iota^* f^* j : U \cap f^{-1}(V) \hookrightarrow A, \quad (\iota^* f^* j)^* \phi : U \cap f^{-1}(V) \hookrightarrow R^n$$

(Recall that formally open subspaces are stable under pullback.) Let $W = U \cap f^{-1}(V)$. The restriction of $f : W \to V$ is a map between formally open subspaces of R^n and R^m, respectively, and thus an i-morphism by theorem 3.2 and the constructions of W_2 and V_2. Since $x_1 \in W \subset U$ and W is a formally open subspace of U, we find $\langle x_1, \dots, x_n \rangle \in W_2\langle n \rangle$ and hence $\langle f(x_1), \dots, f(x_n) \rangle \in V_2\langle n \rangle$; but this implies that

$$\langle f(P_1), \dots, f(P_n) \rangle = \langle j(f(x_1)), \dots, j(f(x_n)) \rangle \in M_2\langle n \rangle$$

and that f is an i-morphism as claimed.

□

Remark 3.5. Instead of forming the union over all charts in the construction of A_2 in the proof of part (i), it is sufficient to consider the union over a covering family, i.e. an *atlas* of A. Moreover, f is an i-morphism if and only if all its restriction to the charts of the atlas are i-morphisms.

Indeed, any chart of $\iota : U \hookrightarrow A$ can be covered by restrictions of charts of the chosen atlas, which are formally open subspaces of both A and some R^n. The same argument as presented in the proof of (ii) above shows that ι is an i-morphism when applied to U and the charts of the atlas.

Note that theorem 3.4 does not extend to the i-affine structures. Maps are not going to preserve the i-affine structure on U_2 for a formally open subspace $U \hookrightarrow R^n$, in general. Only special classes of maps will have that property. Indeed, the Taylor expansion of a map f to second order contains quadratic terms, in general, hence can only preserve affine combinations up to quadratic terms. Therefore, unlike R^n a formal manifold does *not* carry a canonical i-affine structure on its canonical second-order i-structure.

This is in contrast to the nil-square i-structure on R^n, where the i-affine structure is preserved by all maps and therefore induces a canonical i-affine structure on the canonical nil-square i-structure of a formal manifold [1, thm. 3.2.8].

4. Affine connections and second-order i-affine structures

In differential geometry affine connections on a manifold come in three equivalent notions: a geometric notion of parallel transport of tangent vectors along paths, and two algebraic notions; that of a covariant derivative on vector fields and the horizontal subbundle of the iterated tangent bundle. In SDG we can study these notions from the infinitesimal viewpoint by either using tangent vectors [6], [5], which are 'infinitesimal paths' $t : D \to A$ in SDG, or using points [5].

Befitting our consideration of the infinitesimal algebra of points we shall consider Kock's affine connection *for points* as defined in [5, chap. 2.3][7]. It is based on the idea of completing three points P, Q, S to a parallelogram $PQRS$. Here $\langle P, Q\rangle$ and $\langle P, S\rangle$ are first-order neighbours, but Q and S don't need to be. The resulting point R is a first-order neighbour of Q and of S, hence it is a second-order neighbour of P. If we follow [5] and denote the point R by $\lambda(P, Q, S)$ then an *affine connection* (on points) λ is a map mapping a triple (P, Q, S) with $\langle P, Q\rangle, \langle P, S\rangle \in A\langle 2\rangle$ to a point $\lambda(P, Q, S)$ such that

$$\lambda(P, Q, P) = Q$$
$$\lambda(P, P, S) = S$$

These properties are sufficient to derive the other nil-square neighbourhood relationships [5, chap. 2.3]. An affine connection is called *symmetric*, if

$$\lambda(P, Q, S) = \lambda(P, S, Q)$$

For $A = R^n$ a symmetric affine connection is induced by its affine structure

$$\lambda(P, Q, S) = Q + S - P$$

Geometrically, this corresponds to the addition of vectors using parallel transport to construct a vector parallelogram at P. In fact, any i-affine structure on A_2 induces a symmetric affine connection in this way.

[7] Note that all the other notions of affine connection can be derived from that of a pointwise affine connection.

Proposition 4.1. *Let A be a formal manifold that admits an i-affine structure on A_2, then A admits a symmetric affine connection on points.*

Proof. We wish to define the symmetric affine connection λ by

$$\lambda(P, Q, S) := Q + S - P$$

where the right hand side denotes the i-affine combination in A_2. For this to be well-defined we need to show $\langle P, Q, S\rangle \in A_2\langle 3\rangle$. We work in a chart. First note that $Q - P, S - P, Q - S \in D_2(n)$. Let ϕ be a trilinear map. We find

$$\phi[Q-P, S-P, Q-S] = \phi[Q-P, S-P, Q-P]-\phi[Q-P, S-P, S-P] = 0$$

as the two trilinear maps on the right hand side are quadratic in $Q - P \in D(n)$, respectively in $S - P \in D(n)$. This is sufficient to show $\langle P, Q, S\rangle \in A_2\langle 3\rangle$. The defining properties showing λ an affine connection are immediate consequence of the algebra of affine combinations. □

We wish to show the converse, i.e. that any symmetric affine connection λ on a formal manifold A extends to a second-order i-affine structure. Our strategy is to construct the i-affine structure on A_2 by transporting the second-order i-affine structure from the tangent space

$$T_P A = \{t \in A^D \mid t(0) = P\}$$

to the manifold A using the second-order $\log$-$\exp$ bijection induced by λ as defined in [5, chap. 8.2].

To begin with note that each tangent vector $t \in T_P A$ is an i-morphism and hence factors through the monad $\mathfrak{M}(P)$ induced by the nil-square i-structure on A. Moreover, any n tangent vectors $t_1, \ldots, t_n \in T_P A$ satisfy

$$\langle t_1(d), \ldots, t_n(d)\rangle \in \mathfrak{M}(P)\langle n\rangle, \qquad \forall d \in D$$

(See [5, chap. 4.2] or [1, chap. 3.3.2] for the details for $n = 2$, which implies the general case.) The R-linear structure on $T_P A$ is obtained pointwise from the i-linear structure on $\mathfrak{M}(P)$ making $T_P A$ a finite-dimensional KL vector space over R [5, chap. 4.2].

Using the i-linear structure on $\mathfrak{M}(P)$ for each $Q \in \mathfrak{M}(P)$ we can define a tangent vector $\log_P(Q) \in T_P A$ by $\log_P(Q)(d) = P + d\,(Q - P)$. This yields an i-linear map

$$\log_P : \mathfrak{M}(P) \to D(T_P A), \qquad Q \mapsto \log_P(Q)$$

The map $\log_P$ has the *exponential map* $\exp_P$ as an inverse [5, thm. 4.3.2], which is given in a chart $U \hookrightarrow A$ as

$$\exp_P(t) = P + v$$

Here $v \in D(n) \subset R^n$ is the *principal part* of $t \in D(T_P A)$ considered in U; that is the unique vector v satisfying $t(d) = P + d\,v$, for all $d \in D$.

As a finite-dimensional KL vector space $T_P A$ is an i-affine space for the second-order i-structure by theorem 3.3. This makes $D_2(T_p A)$, which is the monad for the second-order i-structure at the zero vector, into an i-linear space. On the other hand, the second-order i-structure on A induces a monad $\mathfrak{M}_2(P)$. Kock has shown that using the symmetric affine connection λ we can extend the log-exp bijection to a bijection

$$\log_P : \mathfrak{M}_2(P) \to D_2(T_P A), \qquad \exp_P : D_2(T_P A) \to \mathfrak{M}_2(P)$$

This bijection can now be used to transport the i-linear structure to $\mathfrak{M}_2(P)$. Since this can be done for any point $P \in A$, we can use it to define an action of $\mathcal{A}(n)$ on A_2. We state our main result:

Theorem 4.2. *Let A be a formal manifold and λ a symmetric affine connection on A.*

(1) The second-order log-exp *bijection induced by the connection λ defines an i-affine structure on A_2 by*

$$\mu \cdot \langle P_1, \ldots, P_n \rangle = \exp_{P_1}\Big(\sum_{j=1}^{n} \mu_j \, \log_{P_1}(P_j)\Big) \tag{2}$$

where $\langle P_1, \ldots, P_n \rangle \in A_2\langle n \rangle$ and $\mu \in \mathcal{A}(n)$.

(2) The i-affine structure on A_2 as defined in (1) is an extension of λ in the sense that

$$\lambda(P, Q, S) = (-1, 1, 1)\langle P, Q, S \rangle$$

for all $(P, Q, S) \in A^3$ such that $\langle P, Q \rangle, \langle P, S \rangle \in A\langle 2 \rangle$.

Proof. (1) We shall proceed in two steps. First we note that $\log_P$ (and hence $\exp_P$) are i-morphisms. Indeed, any chart mapping P to $0 \in R^n$ induces a bijection $\mathfrak{M}_2(P) \cong D_2(n)$. Since $T_P A$ is a finite-dimensional KL vector space any map $D_2(n) \to T_P A$ has a unique extension to a map $R^n \to T_P A$; but any such map must be an i-morphism by theorem 3.2. Due to the construction of A_2 from charts in theorem 3.4(i), as well as the definition of the induced i-structures on the respective monads, $\log_P$ must be an i-morphism.

With this and the $\log$-$\exp$ bijection we can see that the i-structure $\mathfrak{M}_2(P)\langle - \rangle$ becomes an i-linear space for each $P \in A$. Moreover, the action of each $\mathcal{L}(n)$ is given by the same formula as in (2). In the next step we need to show that the action as defined in (2) is independent of the base point P.

Lemma 4.3. *Let* $\mu \in \mathcal{A}(n)$ *and* $\langle Q, P, P_1, \dots, P_n \rangle \in A_2\langle n+2 \rangle$*, then*

$$\exp_P\Big(\sum_{j=1}^{n} \mu_j \, \log_P(P_j)\Big) = \exp_Q\Big(\sum_{j=1}^{n} \mu_j \, \log_Q(P_j)\Big)$$

With the base point independence all the three axioms of an i-affine structure follow from the respective axioms of the i-linear structure on $\mathfrak{M}_2(P)$ for a suitably chosen $P \in A$ (for example, choosing the first point in the respective i-tuple). To conclude the proof of (1) it remains to show lemma 4.3.

Proof. (Lemma) It is sufficient to show this in a chart U. In U we find

$$\lambda(P, Q, S) = Q + S - P + \Gamma_P[Q - P, S - P]$$

for a symmetric bilinear map Γ_P [5, chapter 2.3], which we will refer to as the *connection symbol* of the connection λ as it is done in [5][8]. The second-order extensions $\log_P$ and $\exp_P$ have the following local

[8]Note that Γ_P is the *negative* of the classically defined connection symbol of a covariant derivative. In [5] the connection symbol is referred to as the *Christoffel symbol*.

representations [5, chap. 8.2]

$$
\begin{aligned}
\log_P(Q)(d) &= P + d\big((Q-P) - \frac{1}{2}\Gamma_P[Q-P]^2\big) \\
\exp_P(t) &= P + v + \frac{1}{2}\Gamma_P[v]^2
\end{aligned}
$$

for $Q \in \mathfrak{M}_2(P) \subset U$ and $t \in D_2(T_PU)$ with principal part $v \in D_2(n)$. We use this to derive a local formula for the second-order i-affine combination

$$
\begin{aligned}
&\exp_P\Big(\sum_{j=1}^n \mu_j \, \log_P(P_j)\Big) \\
&\quad = P + \sum_{j=1}^n \mu_j \, \big((P_j - P) - \frac{1}{2}\Gamma_P[P_j - P]^2\big) \\
&\qquad + \frac{1}{2}\Gamma_P\Big[\sum_{j=1}^n \mu_j \, \big((P_j - P) - \frac{1}{2}\Gamma_P[P_j - P]^2\big)\Big]^2
\end{aligned}
$$

In the next step we expand the last connection symbol. Since $\langle P, P_1, \ldots, P_n\rangle \in U\langle n{+}1\rangle$ all the multilinear occurrences of order three and four in the $P_j - P$ vanish. Using $\sum_{j=1}^n \mu_j = 1$ the above expression thus simplifies to the local representation

$$
\begin{aligned}
&\exp_P\Big(\sum_{j=1}^n \mu_j \, \log_P(P_j)\Big) \\
&\quad = \sum_{j=1}^n \mu_j P_j + \frac{1}{2}\Big(\Gamma_P\big[\sum_{j=1}^n \mu_j \, P_j - P\big]^2 - \sum_{j=1}^n \mu_j \Gamma_P[P_j - P]^2\Big)
\end{aligned}
\tag{3}
$$

The respective local representation for the base point Q is obtained by replacing P with Q in the above equation.

The base point independence is equivalent to the identity

$$\begin{aligned}\Gamma_P\big[\sum_{j=1}^{n}\mu_j\,P_j-P\big]^2-\sum_{j=1}^{n}\mu_j\Gamma_P[P_j-P]^2\\=\Gamma_Q\big[\sum_{j=1}^{n}\mu_j\,P_j-Q\big]^2-\sum_{j=1}^{n}\mu_j\Gamma_Q[P_j-Q]^2\end{aligned}\tag{4}$$

We use that $Q-P\in D_2(n)$ and represent $\Gamma_Q=\Gamma_{P+(Q-P)}$ using a Taylor expansion of order two

$$\begin{aligned}\Gamma_Q\big[\sum_{j=1}^{n}\mu_j\,P_j-Q\big]^2&=\Gamma_P\big[\sum_{j=1}^{n}\mu_j\,P_j-Q\big]^2\\&\quad+\partial\Gamma_P\big[Q-P\big]\big[\sum_{j=1}^{n}\mu_j\,P_j-Q\big]^2\\&\quad+\frac{1}{2}\partial^2\Gamma_P\big[Q-P\big]^2\big[\sum_{j=1}^{n}\mu_j\,P_j-Q\big]^2\end{aligned}$$

Due to $\langle Q,P,P_1,\dots,P_n\rangle\in U_2\langle n+2\rangle$ this simplifies to

$$\Gamma_Q\big[\sum_{j=1}^{n}\mu_j\,P_j-Q\big]^2=\Gamma_P\big[\sum_{j=1}^{n}\mu_j\,P_j-Q\big]^2\tag{5}$$

In the same vein we obtain

$$\sum_{j=1}^{n}\mu_j\Gamma_Q\big[P_j-Q\big]^2=\sum_{j=1}^{n}\mu_j\Gamma_P\big[P_j-Q\big]^2\tag{6}$$

Expanding

$$\Gamma_P\big[\sum_{j=1}^{n}\mu_j\,P_j-Q\big]^2=\Gamma_P\big[\sum_{j=1}^{n}\mu_j\,P_j-P+(P-Q)\big]^2$$

yields

$$\begin{aligned}\Gamma_P\big[\sum_{j=1}^{n}\mu_j\,P_j-Q\big]^2&=\Gamma_P\big[\sum_{j=1}^{n}\mu_j\,P_j-P\big]^2+\Gamma_P\big[P-Q\big]^2\\&\quad+2\,\Gamma_P\big[\sum_{j=1}^{n}\mu_j\,P_j-P,\;P-Q\big]\\&=\Gamma_P\big[\sum_{j=1}^{n}\mu_j\,P_j-P\big]^2+\Gamma_P\big[P-Q\big]^2\\&\quad+2\sum_{j=1}^{n}\mu_j\Gamma_P\big[P_j-P,\;P-Q\big],\end{aligned}$$

where we have used $\sum_{j=1}^{n}\mu_j=1$ in the last step. Expanding

$$\Gamma_P\big[P_j-Q\big]^2=\Gamma_P\big[P_j-P+(P-Q)\big]^2$$

yields

$$\begin{aligned}\Gamma_P\big[P_j-Q\big]^2&=\Gamma_P\big[P_j-P\big]^2+\Gamma_P\big[P-Q\big]^2\\&\quad+2\,\Gamma_P\big[P_j-P,\;P-Q\big]\end{aligned}$$

and thus

$$\begin{aligned}\sum_{j=1}^{n}\mu_j\Gamma_P\big[P_j-Q\big]^2&=\sum_{j=1}^{n}\mu_j\Gamma_P\big[P_j-P\big]^2+\Gamma_P\big[P-Q\big]^2\\&\quad+2\sum_{j=1}^{n}\mu_j\Gamma_P\big[P_j-P,\;P-Q\big]\end{aligned}$$

Combining equations (5) and (6) with the above expansions yields equation (4) and thus establishes the independence of (2) from the chosen base point. □

(2) It remains to show that λ agrees with the affine combination of the second-order i-affine structure as given by (2) for $\mu=(-1,1,1)$. As

shown in proposition 4.1 we have $\langle P, Q, S\rangle \in A_2\langle 3\rangle$. We consider everything in a chart U. Combining equations (2) and (3) we get

$$(-1,1,1)\cdot\langle P,Q,S\rangle = -P+Q+S+\frac{1}{2}\big(\Gamma_P[Q-P+S-P]^2 - \Gamma_P[Q-P]^2 - \Gamma_P[S-P]^2\big)$$

Expanding the symmetric bilinear map Γ_P thus results in

$$(-1,1,1)\cdot\langle P,Q,S\rangle = \lambda(P,Q,S)$$

as claimed.

□

5. Conclusion

An action of (the clone of) affine combinations on an i-structure is an algebraic model that makes precise the long-standing idea of differential geometry and of calculus that a (smooth) space has a geometry that is affine at the infinitesimal scale. These algebraic structures have been extracted by the author from Kock's work [4], [5]. The author has then generalised and studied them as infinitesimal models of algebraic theories in [1].

Within the framework of Synthetic Differential Geometry, in particular within the algebraic and well-adapted models of SDG there is a wealth of examples of i-affine spaces besides that of smooth and formal manifolds. This means that *the same* infinitesimal constructs and *the same* algebra of infinitesimals can be applied much more widely and beyond the context of (smooth) manifolds. However, so far (almost) all these examples have been based on the nil-square i-structure only[9].

In this paper we have shown that besides the canonical nil-square i-structure, a formal manifold carries a natural second-order i-structure (theorem 3.4). The affine structure on R^n induces an i-affine structure on the second-order i-structure (theorem 3.3). In contrast to the nil-square i-affine structure the second-order i-affine structure is not preserved by all maps

[9] The only exception has been the pointwise i-affine structure on function spaces studied in [1, chap. 3.3.2].

$R^n \to R^m$, and is hence not natural anymore. However, as we have shown for formal manifolds, there is a correspondence between symmetric affine connections (on points) and second-order i-affine structures (theorem 4.2). This provides us with a first example that a higher-order i-affine structure can be obtained from the data of a higher-order geometric structure on a formal manifold. Moreover, the $\log$-$\exp$ bijection yields also a natural geometric interpretation of an affine combination as the geometric addition of geodesic line segments extending the familiar vector parallelogram construction from the affine plane to curved space.

Does a manifold admit 3rd and higher-order i-affine structures? Firstly, it is possible and straight forward to generalise the construction of a second-order i-structure on R^n to kth-order i-structures that are preserved by all maps $R^n \to R^m$; the idea being that any $(k+1)$-linear occurrences of difference vectors formed from an i-n-tuple has to vanish. Due to the general gluing theorems in [1] it is then possible to show that any formal manifold carries a natural kth-order i-structure. Theorem 3.3 generalises to kth-order i-structures, too, but like with the second-order i-affine structure, kth-order i-affine structures are not preserved by maps $R^n \to R^m$.

As regards the construction of a 3rd-order i-affine structure on formal manifolds the author has obtained two results pointing towards the problem being more intricate than anticipated. Firstly, assuming the existence of a 3rd-order $\log$-$\exp$ bijection theorem 4.2 does not seem to generalise to 3rd-order i-affine structures. However, assuming that the formal manifold is a retract of a formally open subset of some R^n it seems possible to project the 3rd-order i-affine structure of R^n to the manifold. Understanding this discrepancy as well as the geometric obstruction responsible for the failure of the $\log$-$\exp$ bijection in order three is subject to current research.

Does a symmetric affine connection determine an i-affine structure uniquely? We have shown that a symmetric affine connection extends to a second-order i-affine structure on formal manifolds. What we have not addressed is the question whether the second-order i-affine structure is uniquely determined by the connection, or, if not, what structure parametrises the possible freedom of choice.

The author was able to show that in a *well-adapted model* each smooth manifold A carries *only one* i-affine structure on the first-order i-structure A_1. Studying the uniqueness of second-order i-affine structures is current work in progress.

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