CAHIERS DE TOPOLOGIE ET GEOMETRIE DIFFERENTIELLE CATEGORIQUES

VOLUME LXIII-3

HERON'S FORMULA, AND VOLUME FORMS

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Résumé. La formule de Heron pour les aires des triangles (et pour d'autres simplexes) s'applique, dans une variété Riemannienne quelconque, aux simplexes qui sont infinitésimaux en un certain sens précis. Ceci conduit, dans la dimension supérieure, à une description géométrique de la forme volume sur la variété.

Abstract. Heron's formula for areas of triangles (and for other simplices) is applied in any Riemannian manifold, for simplices that are infinitesimal in a certain precise sense. This leads, in the top dimension, to a geometric description of the volume form of the manifold.

Keywords. Volume of simplices. Cayley-Menger determinant. Riemannian metric. Synthetic differential geometry.

Mathematics Subject Classification (2010). 14A25; 51K10; 51M25.

Introduction

The Greek geometers (Heron et al.) discovered a remarkable formula, expressing the area of a triangle in terms of the lengths of the three sides. Here, length and area are seen as non-negative numbers, which involves, in modern terms, formation of *absolute value* and *square root*. To express the notions and results involved without these non-smooth constructions, one can express the Heron Theorem in terms of the *squares* of the quantities in question: if $g(A, B)$ denotes the *square* of the length of the line segment given by A and B, the Heron formula says that the square of the area of the triangle ABC may be calculated by a simple algebraic formula out the three numbers $g(A,B)$, $g(A,C)$, and $g(B,C)$. Explicitly, the formula appears in (1) below. In modern terms, the formula is (except for a combinatorial constant -16^{-1}) the determinant of a certain symmetric 4×4 matrix constructed out of three numbers; see (2) below. This determinant, called the Cayley-Menger determinant, generalizes to simplices of higher dimensions, so that e.g. the square of the volume of a tetrahedron (3-simplex) $(ABCD)$ in space is given (except for a combinatorial constant) by the determinant of a certain 5×5 matrix constructed out of the six square lengths of the edges of the tetrahedron (by a formula already known by Piero della Francesca in the Renaissance).

The Heron formula is symmetric w.r.to permutations of the $k+1$ vertices of a k-simplex. Also, it does not refer to the vector space or affine structure of the ambient space.

The square lengths, square areas, square volumes etc. of the simplices can also be calculated by another well known and simple expression: namely as $(1/k!)^2$ times the Gram determinant of a certain $k \times k$ matrix constructed from the simplex, by choosing one of its vertices as origin. The Gram determinant itself expresses the square volume of the parallelepipedum spanned by k vectors in V that go from the origin to the remaining vertices.

An important difference between the two formulae is the $(k+1)!$-fold symmetry in the Heron formula, where the Gram formula is apriori only $k!$-fold symmetric, because of the special role of the chosen origin.

It is useful to think in terms of the quantities occurring as being quantities whose physical dimension is some power of length (measured in meter m, say), so that length is measured in m, area in m^2, square area in m^4, etc. Tangent vectors are not used in the following; they would have physical dimension of $m \cdot t^{-1}$ (velocity). The word *square-density* is used in any dimension. Square length, square area, and square volume are examples. The theory developed here was also attempted in my [8] (whose basis is the Gram method). I hope that the present account will be less ad hoc.

1. Cayley-Menger matrices

The basic idea for the construction of a square k-volume function goes, for the case $k = 2$, back to Heron of Alexandria (perhaps even to Archimedes); they knew how to express the square of the area of a triangle S (whether located in Euclidean 2-space or in a higher dimensional Euclidean space) in terms of an expression involving only the lengths a, b, c of the three sides:

$$area^2(S) = t \cdot (t-a) \cdot (t-b) \cdot (t-c)$$

where $t = \frac{1}{2}(a+b+c)$. Substituting for t, and multiplying out, one discovers ([3] 1.53) that all terms involving an odd number of any of the variables a, b, c cancel, and we are left with

$$area^2(S) = -16^{-1}(a^4 + b^4 + c^4 - 2a^2b^2 - 2a^2c^2 - 2b^2c^2), \qquad (1)$$

an expression that only involves the *squares* a^2, b^2 and c^2 of the lengths of the sides.

The expression in the parenthesis here may be written in terms of the determinant of a 4×4 matrix (described in (2) below). This provides a blueprint for how to generalize from 2-simplices (= triangles) to k-simplices, in terms of determinants of certain $(k+2) \times (k+2)$ matrices, "Cayley-Menger matrices/determinants"; they again only involve the square lengths of the $\binom{k+1}{2}$ edges of the simplex.

A k-simplex X in a space M is a $(k+1)$-tuple of points (vertices) $(x_0, x_1, \ldots, x_k)$ in M. If $g : M \times M \to R$ satisfies $g(x, x) = 0$ and $g(x, y) = g(y, x)$ for all x and y (like a metric $\mathrm{dist}(x, y)$, or its square), one may construct a $(k+2) \times (k+2)$ matrix $C(X)$ by the following recipe:

1) Take the $(k+1) \times (k+1)$ matrix $c(X)$ whose ijth entry is $g(x_i, x_j)$.

2) Enlarge this matrix $c(X)$ to a $(k+2) \times (k+2)$- matrix $C(X)$ by bordering it with $(0, 1, \ldots, 1)$ on the top and on the left.

Both $c(X)$ and $C(X)$ have 0s down the diagonal and are symmetric, by the two assumption about g. For the case $k = 2$, $C(X)$ is depicted here, writing $g(ij)$ for $g(x_i, x_j)$ for brevity; note $g(01) = g(10)$ etc., so that the matrix is symmetric.

$$\begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & g(01) & g(02) \\ 1 & g(10) & 0 & g(12) \\ 1 & g(20) & g(21) & 0 \end{bmatrix} \tag{2}$$

(The indices of the rows and columns are most conveniently taken to be $-1, 0, 1, 2$.)

This is the Cayley-Menger[1] matrix $C(X)$ for the simplex X, and its determinant is its Cayley-Menger determinant. Heron's formula then says that the value of this determinant is, modulo the "combinatorial" factor -16^{-1}, the square of the *area* of a triangle with vertices x_0, x_1, x_2, as expressed in terms of squares $g(x_i, x_j)$ of the *distances* between them. Similarly for (square-) volumes of higher dimensional simplices. Note that no coordinates are used in the construction of this matrix/determinant.

The general formula is that the square of the volume of a k-simplex is $-(-2)^{-k} \cdot (k!)^{-2}$ times the determinant of C, e.g. for $k = 1, 2$, and 3, the factors are 2^{-1}, -16^{-1}, and 288^{-1}, respectively.

Proposition 1.1. *The Cayley-Menger determinant for a k-simplex is invariant under the $(k+1)!$ symmetries of the vertices of the simplex.*

Proof. Interchanging the vertices x_i and x_j has the effect of first interchanging the ith and jth column, and then interchanging the ith and jth row of the new matrix. Each of these changes will change the determinant by a factor -1. □

2. Square volumes in coordinates

2.1 Heron's formula

We shall now work in the space R^n, with its standard metric. So the square of the distance between x and y is $\sum_{i=1}^{n}(x_i - y_i)^2$. This is the matrix product $(x-y)^T \cdot (x-y)$, where elements in R^n are identified with $n \times 1$ matrices (column matrices), and where $(-)^T$ denotes transposition of matrices. The

[1] We shall sometimes use the acronym "CM" for "Cayley-Menger".

displayed matrix product is therefore a 1×1 matrix, i.e. an element of R. For a symmetric $n \times n$ matrix G, we may more generally consider the matrix product $(x-y)^T \cdot G \cdot (x-y)$ (if G is the identity $n \times n$ matrix I, we retrieve the standard metric). Then the function $g(x,y) := (x-y)^T \cdot G \cdot (x-y)$ has the two properties $g(x,x) = 0$ and $g(x,y) = g(y,x)$, which was all we needed to describe the Cayley-Menger determinant (the g thus defined may not be a square-metric in any reasonable sense. This would require that G is positive definite; we return to this in Section 5.)

We denote by heron_G the "square volume" for k-simplices in R^n, when calculated using such G for the entries in the Cayley-Menger determinant.

2.2 Gram's formula

For a k-tuple of vectors $(y_1, \ldots, y_k)$ in R^n, one may form the Gram determinant: first form the $n \times k$ matrix Y whose k columns are the y_js. Then form the (symmetric) $k \times k$ matrix obtained as the matrix product $Y^T \cdot Y$, where Y^T denotes the transpose of Y. So the ijth entry in $Y^T \cdot Y$ is the inner product $y_i \cdot y_j$. Let us write

$$\mathrm{Gram}(Y) := \det(Y^T \cdot Y)$$

for the determinant of this $k \times k$ matrix. The significance of this determinant is that it describes the square of the volume of the parallelepipedum spanned by the k vectors y_j. Therefore the square $\mathrm{gram}(Y)$ of the volume of the simplex spanned by these vectors is smaller, it is

$$\mathrm{gram}(Y) = (k!)^{-2} \cdot \mathrm{Gram}(Y).$$

Let Y be as above, and let G be a symmetric $n \times n$ matrix. Then we may instead of $Y^T \cdot Y$ consider the (symmetric) $k \times k$ matrix given by $Y^T \cdot G \cdot Y$, and write

$$\mathrm{Gram}_G(Y) := \det(Y^T \cdot G \cdot Y),$$

thus for the $n \times n$ identity matrix I, $\mathrm{Gram}_I(Y) = \mathrm{Gram}(Y)$.

Proposition 2.1. *If an $n \times n$ matrix G can be written $G = H^T \cdot H$ for an $n \times n$ matrix H, then (G is symmetric and) we have for any $n \times k$ matrix Y that*

$$\mathrm{Gram}_G(Y) = \mathrm{Gram}_I(H \cdot Y),$$

(and hence also $\mathrm{gram}_G(Y) = \mathrm{gram}_I(H \cdot Y)$*).*

Proof. We have for $\mathrm{Gram}_G(Y)$ the following calculation

$$\det(Y^T \cdot G \cdot Y) = \det(Y^T \cdot H^T \cdot H \cdot Y) = \det((H \cdot Y)^T \cdot (H \cdot Y))$$

which is $\mathrm{Gram}(H \cdot Y)$ (i.e. $\mathrm{Gram}_I(H \cdot Y)$). $\square$

Remark. We note that if R denotes the real numbers, then the existence of an invertible matrix H with $H^T \cdot H = G$ is equivalent to G being positive definite in the standard sense, see e.g. Proposition 6 in [10] XI.4.

2.3 Comparison formula

For R^n, it makes sense to compare the values of the Heron and Gram formulas for square volume of a k-simplex $X = (x_0, x_1, \dots, x_k)$. For $j = 1, \dots, k$, let y_j denote the vector $x_j - x_0 \in R^n$, and let $Y = (y_1, \dots, y_k)$ denote the resulting $n \times k$ matrix. Let $C = C(X)$ denote the $(k+2) \times (k+2)$ matrix ((Heron-) Cayley-Menger) arising from the square distances between the vertices, as described above, and let $Y^T \cdot Y$ be the Gram $k \times k$ matrix of the simplex, likewise described above. There is a known relation between their determinants

$$-(-2)^{-k} \det(C) = \det(Y^T \cdot Y). \tag{3}$$

For a proof, see reference [4].

Note that the left hand hand side in (3) does not make use of the algebraic structure of R^n, but only on the (square-) distance function (arising from the inner product). This flexibility will be crucial when we later on consider Riemannian manifolds.

We denote the square volume of a simplex X, as calculated in terms of the Cayley-Menger matrix C, by $\mathrm{heron}(X)$, and denote the square volume of the corresponding parallelepipedum, as calculated by Gram's method, by $\mathrm{Gram}(X)$ So by dividing (3) by $(k!)^2$, we have

$$\mathrm{heron}(X) = \mathrm{gram}(X) \tag{4}$$

We described at the end of Section 2.1 how one may modify the Heron expression using a symmetric $n \times n$ matrix G, so one may ask whether the G-modified $\mathrm{heron}_G(X)$ equals $\mathrm{gram}_G(X)$? This holds if G is the identity $n \times n$ matrix, by (4).

Proposition 2.2. *Let $X = (x_0, \ldots, x_k)$ be a k-simplex in R^n. If G is positive definite, in the sense that $G = H^T \cdot H$ for some square matrix H, then* $\mathrm{heron}_G(X) = \mathrm{gram}_G(X)$.

Proof. The submatrix $c(X)$ of $C(X)$ for calculating $\mathrm{heron}_G(X)$ is the $(k+1) \times (k+1)$ matrix whose i, j entry is the G-square distance between x_i and x_j, i.e. it is

$$(x_i - x_j)^T \cdot G \cdot (x_i - x_j) = (x_i - x_j)^T \cdot H^T \cdot H \cdot (x_i - x_j)$$

$$= (H \cdot (x_i - x_j))^T \cdot (H \cdot (x_i - x_j)) = (H \cdot x_i - H \cdot x_j)^T \cdot (H \cdot x_i - H \cdot x_j)$$

which is the i, j entry in the CM matrix for the simplex $H \cdot X$. We conclude that $\mathrm{heron}_G(X) = \mathrm{heron}(H \cdot X)$. By (4), $\mathrm{heron}(H \cdot X) = \mathrm{gram}(H \cdot X)$, which in turn is $\mathrm{gram}_G(X)$ by Proposition 2.1. $\square$

Remark. In terms of physical dimensions alluded to in the Introduction: volume of a k-simplex has dimension m^k, so its square volume has dimension $(m^k)^2$; the entries $g(x_i, x_j)$ in the Cayley-Menger matrix have physical dimension m^2, and expanding its determinant, all terms are products of k copies of these entries. (The entries 0 and 1 in the top line and left column in the matrix are "pure" quantities, i.e. of dimension m^0). So the value of the determinant is of physical dimension $(m^2)^k$. The Heron formula is then meaningful in the sense that it equates quantities of dimension $(m^2)^k$ and $(m^k)^2$.

In particular, the comparison between the square volumes of a k-simplex, as calculated by Heron-Cayley-Menger and by Gram, which is a consequence of (3), is dimensionally meaningful; both have physical dimension m^{2k}.

2.4 The terms in the Cayley-Menger determinant

Given a k-simplex $X = (x_0, \ldots, x_k)$ (in a space M, with a "square distance" function $g(x, y)$, as in Section 1). Consider the CM determinant $C(X)$ as described by the recipe in Section 1. The terms of this determinant have $k + 2$ factors; but two of these factors are 1 (expand after top row, and then after leftmost column, and use that the top left entry is 0). So each of the terms of the CM determinant is a product of k factors placed in a k-element pattern S in the $(k+1) \times (k+1)$- matrix $c(X)$; and S has no entries in the diagonal, since the CM matrix has 0s in the diagonal. We want to describe and classify the patterns that occur: To place a set of k chess rooks on an $m \times m$ chess board ($k \leq m$), so that no one of them can beat another one, can be expressed: no two of them are placed in the same row, and no two of them are placed in the same column. Let us for brevity call such a k-element set S of positions in an $m \times m$ matrix a *rook pattern*.

We conclude that the terms in a CM matrix for a k-simplex are named by k-element rook patterns S (containing no diagonal entries) in $\{0 \ldots, k\} \times \{0, \ldots, k\}$; the term named by such S is $\pm \prod_{(i,j) \in S} g(x_i, x_j)$.

Each such pattern S gives rise to an oriented graph with $k+1$ vertices $0, 1, \ldots k$, and with an edge from i to j ($i \neq j$) if $(i,j) \in S$. Hence this graph has k edges. Also, for every vertex i, there is a most one edge with i as domain, and at most one edge with i as codomain.

For such oriented graphs with $k+1$ vertices, there are two alternatives (mutually exclusive): 1) the graph is *singular*, in the sense that there is some closed path in the graph; 2) there is a path of length k, passing through each of the $k+1$ vertices; we call such rook-pattens and their graphs *non-singular*. Note that for a non-singular path, there are exactly two extreme vertices, and $k-1$ intermediate vertices.

Looking at the classical Heron formula (1), the three first terms are named by singular graphs, the three (really six) last terms are named by non-singular graphs.

3. Differential forms and square densities

In this Section, we work in the context of synthetic differential geometry (SDG); this is a category $\mathcal{E}$ (with suitable properties, say a topos, but less will do for the present note), together with a basic commutative ring object $R \in \mathcal{E}$, the "number line", satisfying certain axioms. In such context, one derives a notion of n-dimensional *manifold* M; this means objects which locally are diffeomorphic[2] to R^n. Since we shall only consider local issues,

[2]the maps in the category $\mathcal{E}$ are termed *smooth*, and an isomorphism in $\mathcal{E}$ is therefore termed a *diffeomorphism*

we shall use the term manifold for any object which *admits* a open inclusion $M \to R^n$ (called a *chart*), but no such chart is part of the structure of M.

In such M, one may define, for each $r = 0, 1, 2, \ldots$ a binary (reflexive symmetric) relation $M_{(r)} \subseteq M \times M$. For x and y (generalized[3]) elements of M, we write $x \sim_r y$ if $(x, y) \in M_{(r)} \subseteq M \times M$. For R^n itself, the relation $\sim_r$ may be described in terms of (generalized) elements as follows: ,

$x \sim_r y$ *iff for any* $r + 1$*-linear function* $\phi : R^n \times \ldots \times R^n \to R$*, we have*

$$\phi(x - y, \ldots, x - y) = 0. \tag{5}$$

In particular: if $x \sim_2 y$, then any trilinear $\phi : R^n \times R^n \times R^n \to R$, vanishes on $(x - y, x - y, x - y)$. (Here "linear" means "R-linear").

It can be proved in the context of SDG that the relation $\sim_r$ is preserved and reflected by local diffeomorphisms of R^n and hence, via charts from R^n, $\sim_r$ makes sense for arbitrary n-dimensional manifolds M, but is independent of the choice of chart. (That $\sim_r$ is well defined, independent of of the chart chosen, is a version of Ehresmann's theory of jets, [5].)

One has that $x \sim_r y$ implies $x \sim_{r+1} y$. We are in the present paper only interested in the case $r = 0, 1, 2$ (where $x \sim_0 x$ is equivalent to $x = y$, since on R^n, there are sufficiently many 1-linear $R^n \to R$, e.g. the n projections).

In particular we consider, for a natural number k, the object of r-infinitesimal k-simplices in M, meaning the subobject of $M \times M \times \ldots \times M$ ($k+1$ times) consisting of $k + 1$-tuples $(x_0, x_1, \ldots, x_k)$ of elements of M with $(x_i, x_j) \in M_{(r)}$ for all $i, j = 0, 1, \ldots, k$; such a $k + 1$-tuple, we shall call an r*-infinitesimal* k*-simplex*; the x_is are the *vertices* of the simplex.

For $r = 1$ and $r = 2$, we shall consider certain maps from the object of r-infinitesimal k-simplices $(x_0, \ldots, x_k)$ in M to R, namely (smooth!) maps which have the property that they vanish if $x_i = x_j$ for some $i \neq j$. For $r = 1$, combinatorial differential k forms ω have this property. (In the context of SDG, such maps are automatically alternating with respect to the $(k + 1)!$ permutations of the x_is, see [7] Theorem 3.1.5.)

[3] We use the well known "synthetic" language to express constructions in categories $\mathcal{E}$ with finite limits, in "elementwise" terms. Recall that a generalized element of an object M in a category $\mathcal{E}$ is just an arbitrary map in $\mathcal{E}$ with codomain M; see e.g. [6] II.1, [11] V.5, or [12] 1.4.

For $r = 2$, such maps have not been considered much[4], except for the case where $k = 1$, where 1-square densities g (pseudo-Riemannian metrics), in the combinatorial sense (recalled after Definition 3.3 below), are examples of such maps; for this case, we think of $g(x_0, x_1)$ as the square of the distance between x_0 and x_1. For manifolds M, we have

Proposition 3.1. *Given $g : M_{(2)} \to R$ with $g(x, x) = 0$ for all x. Then g is symmetric iff it vanishes on $M_{(1)} \subseteq M_{(2)}$.*

Proof. In a chart $M \cong R^n$, consider, for fixed x, the degree ≤ 2 part of the Taylor expansion of g around x. Then g is given as

$$g(x, y) = C(x) + \Omega(x; x - y) + (x - y)^T \cdot G(x) \cdot (x - y),$$

where $C(x)$ is a constant, Ω is linear in the argument after the semicolon, and $G(x)$ is a symmetric $n \times n$ matrix . To say that g vanishes on the diagonal $M_{(0)}$ (i.e. $g(x, x) = 0$ for all x) is equivalent to saying that $C(x) = 0$ for all x. We now compare $g(x, y)$ and $g(y, x)$; we claim

$$(x - y)^T \cdot G(x) \cdot (x - y) = (y - x)^T \cdot G(y) \cdot (y - x). \tag{6}$$

For, Taylor expanding from x the $G(y)$ on the right hand side, gives that the difference between the two sides is $(y - x) \cdot dG(x; y - x) \cdot (y - x)$ which is trilinear in $y - x$, and therefore vanishes, since $x \sim_2 y$. So we have that if C vanishes, then g is symmetric; vice versa, if g is symmetric, its restriction to $M_{(1)}$ is likewise symmetric, and (being a differential 1-form), it is alternating, so the Ω-part vanishes, which in coordinate free terms says: $g(x, y) = 0$ for $x \sim_1 y$. $\square$

(For the number line R, $(x_0, x_1) \in R_{(2)}$ iff $(x_0 - x_1)^3 = 0$, and the map g given by $g(x_0, x_1) := (x_0 - x_1)^2$ is a map as described in the Proposition. In fact, it is the restriction of the standard "square-distance" function $R \times R \to R$.)

So we recall, respectively pose, the following definitions, corresponding to $r = 1$ and $r = 2$. Let M be a manifold.

[4]For $r = 2$ and $k = 1$, such things were in [7] 8.1 called "quadratic differential forms".

Definition 3.2. *A (combinatorial)* differential k-form *on M is an R-valued function ω on the object of 1-infinitesimal k-simplices in M, which is alternating with respect to the $(k+1)!$ permutations of the vertices of the simplex.*

So ω vanishes on simplices where two vertices are equal.

For the category $\mathcal{E}$ of affine K-schemes, combinatorial differential forms were studied and applied in [2]; here R is the scheme represented by the algebra of polynomials in one variable over K. The commutative ring representing the objects of 1-infinitesimal simplices is described explicitly. It is a version of the construction of the module of Kaehler differentials.

Definition 3.3. *A* k-square-density *on M is an R-valued function on the object of 2-infinitesimal k-simplices in M, which is symmetric with respect to the $(k+1)!$ permutations of the vertices of the simplex, and which vanishes on simplices where two vertices are equal.*

Note that for $k = 1$, Proposition 3.1 gives that 1-square densities (square lengths) g have the property that they vanish not just on $M_{(0)}$ (the diagonal), but also on $M_{(1)}$: $g(x, y) = 0$ if $x \sim_1 y$.

I apologize for the following proliferation of terminology:
1-square density = differential quadratic form = pseudo-Riemannian metric (where "differential quadratic form" was the term used in [7], Section 8.1).

3.1 k-square-densities heron_g from 1-square-densities g

Given a 1-square-density g. We shall argue that the Cayley-Menger determinants, using this g, for 2-infinitesimal simplices $(x_0, \ldots, x_k)$, define a k-square-density. We already argued (Proposition 1.1) that these determinants are symmetric: the value does not change when interchanging x_i and x_j. We have to argue for the vanishing condition required. If $x_i = x_j$, then $g(x_i, x_m) = g(x_j, x_m)$ for all m, and this implies that the ith and jth rows in the Cayley-Menger matrix are equal, which implies that the determinant is 0.

3.2 k-square-densities from differential k-forms

Essentially this is the process of *squaring* (in R) the values, so it is tempting to denote the square-density which we are aiming for, by ω^2. Precisely: we get a well defined k-square-density out of a differential k-form by a two step procedure: 1) to *extend* the given k form ω to a suitable function $\overline{\omega}$, to allow as inputs not just 1-infinitesimal k-simplices, but also also certain 2-infinitesimal k-configurations; and then 2) *squaring* $\overline{\omega}$ valuewise.

Given a combinatorial k-form ω on M. In a coordinate chart R^n, it may be expressed (as in [7] 3.1) in terms of a function $\Omega : M \times (R^n)^k \to R$ which is k-linear and alternating in the last k arguments,

$$\omega(x_0, x_1, \ldots, x_k) = \Omega(x_0; x_0 - x_1, x_0 - x_2, \ldots, x_0 - x_k). \tag{7}$$

The right hand side is defined without restrictions in the $x_0 - x_i$s. Let us denote it $\overline{\omega}$.

Proposition 3.4. *The valuewise square $\overline{\omega}^2$, when applied to 2-infinitesimal k-simplices, is a k-square density.*

Proof. It clearly vanishes if two vertices are equal, since Ω, hence $\overline{\omega}$, have this property. For the $(k+1)!$-fold symmetry: interchanging x_i and x_j (for $i, j \geq 1$) gives a sign change in the value of $\overline{\omega}$, since Ω is alternating in the last k arguments. So squaring the value gives no change. For interchange of x_0 and x_i for $i \geq 1$, a more delicate argument is needed: We shall only do the case $k = 1$. First, we have by a Taylor expansion from x_0

$$\Omega(x_1; x_0 - x_1) = \Omega(x_0; x_0 - x_1) + d\Omega(x_0; x_1 - x_0; x_0 - x_1)$$

$$+ \text{ a term } d^2\Omega(x_0; \ldots), \text{ trilinear in } x_1 - x_0.$$

The trilinear term vanishes, because $x_1 \sim_2 x_0$. Now we square, and get

$$\Omega(x_1; x_0 - x_1)^2 = \Omega(x_0; x_0 - x_1)^2 + 2 \cdot \Omega(x_0; x_0 - x_1) \cdot d\Omega(x_0; x_1 - x_0, x_0 - x_1)$$

$$+ \text{ a term } (d\Omega(x_0; \ldots))^2, \text{ quadrilinear in } x_1 - x_0.$$

The quadrilinear term vanishes because $x_1 \sim_2 x_0$, but also the term $\Omega \cdot d\Omega$ vanishes, because it is trilinear in $x_1 - x_0$. So we get

$$\Omega(x_1; x_0 - x_1)^2 = \Omega(x_0; x_0 - x_1)^2 = \Omega(x_0; x_1 - x_0)^2,$$

as desired. $\square$

We shall prove that the square density constructed is independent of the choice of chart used for constructing it. The unicity can be formulated without reference to any coordinate chart. To formulate it, let us introduce an auxiliary terminology: a function $\overline{\omega}$ from the set of $(k+1)$-tuples $(x_0, x_1, \ldots, x_k)$ with $x_0 \sim_2 x_i$ for $i = 1, \ldots, k$ we call an *extended form*, if it takes value 0 if two of its arguments are equal. Such an extended form restricts to a function on the set of 1-infinitesimal k-simplices, and hence it makes sense to say that $\overline{\omega}$ extends a given (combinatorial) differential k-form ω. We shall then prove the coordinate free assertion:

Proposition 3.5. *If two extended k-forms $\overline{\omega}$ and $\overline{\omega}'$ extend the same differential k-form ω, then $\overline{\omega}^2 = \overline{\omega}'^2$.*

Proof. We have to prove that

$$\overline{\omega}^2(x_0, x_1, \ldots, x_k) = \overline{\omega}'^2(x_0, x_1, \ldots, x_k),$$

for any 2-infinitesimal k-simplex $(x_0, x_1, \ldots, x_k)$. It suffices prove it for $M = R^n$ and with $x_0 = 0$. In this case $\overline{\omega}$ and $\overline{\omega}'$ are functions Ω and $\Omega' : D_2(n) \times \ldots \times D_2(n) \to R$ (k factors in the product). Here $D_2(n) \subseteq R^n$ has for its (generalized) elements $x \in R^n$ with $x \sim_2 0$. By the basic axiom scheme of SDG, the ring A of functions $D_2(n) \to R$ is of the form $A = A_0 \oplus A_1 \oplus A_2$, with A_0 consisting of the constant functions $R^n \to R$, A_1 of the linear functions $R^n \to R$, and A_2 of the (homogeneous) quadratic functions $R^n \to R$. This A is a graded ring (only non-zero in degrees 0,1 and 2). The ideal of functions vanishing on 0 is $A_1 \oplus A_2 \subseteq A$. So the ideal of functions $(D_2(n))^k \to R$, which vanish if at least one of its arguments is 0, is the k-fold tensor product of $(A_1 \oplus A_2)$,

$$(A_1 \oplus A_2)^{\otimes k} \subseteq A^{\otimes k}. \tag{8}$$

The ring $A^{\otimes k}$ is k-graded, with e.g. the multidegree $(1, \ldots, 1)$ consisting of the k-linear functions $(R^n)^k \to R$

By assumption, both Ω and Ω' belong to the ideal (8). The assumption that both Ω and Ω' restrict to the same differential k-form ω implies that Ω and Ω' agree in their component of multidegree $(1, \ldots, 1)$ (this component being the coordinate expression of ω). Thus $\Omega' = \Omega + \theta$, with θ of multidegree $\geq (1, \ldots, 1)$ and of total degree $\geq k + 1$. The required equation is, in these terms, that $(\Omega+\theta)^2 = \Omega^2$, and this is a simple "counting degrees"-argument in the k-graded ring A^k:

$$(\Omega+\theta)^2 = \Omega^2 + 2\Omega\cdot\theta + \theta^2. \tag{9}$$

Here, θ^2 has total degree $\geq 2\cdot(k+1) \geq 2k+1$, which is 0 since A^k is 0 in total degrees $> 2k$; and θ is a linear combination of terms of multidegree of the form $(1,1,\ldots,1+p,\ldots 1)$ for $p \geq 1$, so $\theta\cdot\omega$ is a linear combination of terms of multidegree

$$(1,1,\ldots,1+p,\ldots,1) + (1,1,\ldots,1,\ldots,1) = (2,2,\ldots,2+p,\ldots,2)$$

which is of total degree $2k+p \geq 2k+1$. So the two last terms in (9) are 0, and this proves the Proposition. $\square$

Because of the Proposition, there is a well-defined "squaring" process, leading from differential k-forms to k-square-densities on a manifold M: extend the form ω, and square the result. It is natural to denote this square density by ω^2, with the understanding that it means $\overline{\omega}^2$ for any extended form $\overline{\omega}$, extending ω.

4. Variable metric tensor

We consider a manifold M. A finite sequence of points

$$\tilde{x} = (x_0, x_1, \ldots, x_k)$$

in M, which are consecutive 2-neigbours, i.e. $x_i \sim_2 x_{i+1}$ for $i = 0,\ldots k-1$, we shall for simplicity call a *path* of length k. We call $x_0,\ldots,x_k$ a *closed* path if $x_0 = x_k$. If M is provided with a pseudo-Riemannian metric g, we shall, for a path $\tilde{x}$ of length k, be interested in products of the form

$$g(\tilde{x}) := g(x_0,x_1)\cdot g(x_1,x_2)\cdot\ldots\cdot g(x_{k-1},x_k). \tag{10}$$

For any chart $M \subseteq R^n$, the metric g is described by a variable "metric tensor" G: i.e. by a family of (symmetric) $n\times n$ matrices $G(x)$, for $x \in M$, and varying smoothly with x; more precisely, G is a map in $\mathcal{E}$ from M to the (finite dimensional) vector space W of $n \times n$ matrices over R. And g is expressed in terms of G: for $x \sim_2 y$ in M

$$g(x, y) = (x - y)^T \cdot G(x) \cdot (x - y) \tag{11}$$

(which equals $(y - x)^T \cdot G(y) \cdot (y - x)$ by (6)). The displayed product (10) is then calculated as the iterated matrix product

$$\begin{aligned} g(\tilde{x}) &= G(\tilde{x}) = \\ &= [(x_0 - x_1) \cdot G(x_0) \cdot (x_0 - x_1)] \cdot [(x_1 - x_2)^T \cdot G(x_1) \cdot (x_1 - x_2)] \cdot \\ &\qquad \cdot \ldots \cdot [(x_{k-1} - x_k)^T \cdot G(x_{k-1}) \cdot (x_{k-1} - x_k)]. \end{aligned} \tag{12}$$

(The square brackets are only inserted for readability; mathematically, they are redundant, by associativity of matrix multiplication.)

Lemma 4.1. *If $\tilde{x}$ is a closed path, then $g(\tilde{x}) = 0$.*

Proof. It suffices to prove this in a chart. In a given chart, g is represented by symmetric matrices $G(x)$, as described above. Using the charts, let a_i be the vector $x_i - x_{i-1}$ for $i = 1, \ldots, k$. Since $x_k = x_0$, we have $a_1 + \ldots + a_k = 0$, so a_k is a linear combination of the a_is, for $i < k$. Then the last factor $[a_k^T \cdot G(x_{k-1}) \cdot a_k]$ in the above product is a linear combination of terms $a_i^T \cdot G(x_{k-1}) \cdot a_j$ with $i < k$ and $j < k$. But among the remaining factors in the product for $G(\tilde{x})$, we have $a_i \cdot G(x_{i-1}) \cdot a_i$, so altogether, a_i appears trilinearly in the corresponding term, and so vanishes since $a_i \sim_2 0$. $\square$

We shall derive some further properties for products of the form (10). With notation as in the previous proof, the product (12) takes the form

$$g(\tilde{x}) = [a_1^T \cdot G(x_0) \cdot a_1] \cdot \ldots \cdot [a_k^T \cdot G(x_{k-1}) \cdot a_k].$$

We write $\overline{G}(\tilde{x})$ for the similar expression, but with all the $G(x_i)$s replaced by $G(x_0)$ where x_0 is the *first* vertex of the path $\tilde{x}$. If $\tilde{x}$ is a path of length k, we get a path of length $k - 1$ by omitting the first of the vertex of the path. Let us denote this truncated path by $|\tilde{x}$. Thus in $\overline{G}(|\tilde{x})$, the constant matrix used is $G(x_1)$ because the first vertex of $|\tilde{x}$ is x_1.

Lemma 4.2. *For any path $\tilde{x}$, $g(\tilde{x}) = \overline{G}(\tilde{x})$.*

Proof. By induction of the length k of the path. The assertion is clearly true for $k = 1$. Assume that it holds for $k - 1$. We use the expression (12) for $g(\tilde{x})$. Then

$$g(\tilde{x}) = (x_0 - x_1)^T \cdot G(x_0) \cdot (x_0 - x_1) \cdot \overline{G}(|\tilde{x}), \tag{13}$$

by the induction assumption, used for the path $|\tilde{x}$. So $\overline{G}(|\tilde{x})$ is a matrix product containg the matrix $G(x_1)$ as a factor $k - 1$ times. By Taylor expansion of the function $G : M \to W$, from x_0 in the direction $x_1 - x_0$, we get

$$G(x_1) = G(x_0) + dG(x_0; x_1 - x_0) + \tfrac{1}{2}d^2G(x_0; x_1 - x_0, x_1 - x_0)$$

in the vector space W of $n \times n$ matrices. In each of these factors in $\overline{G}(|\tilde{x})$, we substitute the Taylor expansion exhibited; and then multiply by $(x_0 - x_1)^T \cdot G(x_0) \cdot (x_0 - x_1)$ to arrive at (13). However, this latter factor is quadratic in $(x_0 - x_1)$. Since $x_0 \sim_2 x_1$, all terms in $\overline{G}(|\tilde{x})$ containing a factor linear or quadratic in $x_1 - x_0$, like $dG(x_0; x_1 - x_0)$, get annihilated by being multiplied by $(x_0 - x_1)^T \cdot G(x_0) \cdot (x_0 - x_1)$, since then in whole product, $x_1 - x_0$ appears in a trilnear way and $x_0 \sim_2 x_1$. So now the matrices $G(x_1)$ have been replaced by $G(x_0)$, and then we have $\overline{G}(\tilde{x})$. This proves the Lemma. □

The same argument, using truncation in the other end (the vertex x_k) of the path, gives that one may also uniformly use $G(x_k)$ instead of the varying $G(x_i)$s. In fact, we may equally well pick any fixed x_j instead of either $G(x_0)$ or $G(x_k)$. For, split the path into two paths $(x_0, \ldots, x_j)$ and $(x_j, \ldots, x_k)$, and pick for the first of these two paths its end vertex x_j, and for the second of the two paths, pick its initial vertex x_j; then use the Lemma 4.2 for paths of length j and of length $k - j$, respectively. Summarizing, the Lemma may be strengthened, and formulated in a more complete way:

Lemma 4.3. *For any path $\tilde{x}$ of length k, for any fixed $j = 1, \ldots, k$, and for any chart, we have $G(\tilde{x}) = \overline{G}(\tilde{x})$, where the $G(x)$ is the expression for g in the chart, and where $\overline{G}$ uses $G(x_j)$ uniformly, (i.e. in (12), all the $G(x_i)$ are replaced by $G(x_j)$).*

Remark. The argument simplifies for the case of "restricted" 2-infinitesimal k-simplices, as considered by [1], since there one has that each of the individual $g(x_i, x_j)$ in a simplex $(x_0, \ldots, x_k)$ may be calculated by using $G(x_0)$.

In preliminary versions of the present note, I only considered the restricted simplices; but the value of the Heron-Cayley-Menger formula on such simplices is probably not enough for characterizing the volume form, which is our aim.

Now recall from Subsection 2.4 that the terms in the CM determinant $C(X)$ for a k-simplex $X = (x_0, \ldots, x_k)$ are named by k-element rook patterns S in the $(k+1) \times (k+1)$ matrix $c(X)$ of square-distances $g(x_i, x_j)$, and that each of these rook patterns gives rise to a graph. Now we concentrate on 2-infinitesimal k-simplices. Given a rook pattern whose graph is singular, i.e. contains a closed path. Then the corresponding product of the $g(x_i, x_j)$s is 0, by Lemma 4.1. So we need only be interested in rook patterns S whose corresponding graph is a non-singular. The corresponding product of k terms is, possibly after renumbering, of the form as displayed in (10). And for such, Lemma 4.2 implies that we, in a chart, may calculate $g(\tilde{x}) = G(\tilde{x})$ by using instead $\overline{G}(\tilde{x})$, that is, we may uniformly use $G(x_0)$ (or any other fixed $G(x_j)$) instead of the varying $G(x_i)$s.

So in the CM determinant for a 2-infinitesimal k-simplex, the terms are named by non-singular rook patterns and so each of the terms may be calculated by expressions (10) for paths; and in any chart, this expression my be calculated as asserted in Lemma 4.3, by picking arbitrarily any x_j in the path. But each non-singular paths of length k in a k simplex passes through all the vertices of the simplex, say x_0.

We conclude that for calculating the square volume of a 2-infinitesimal k simplex $X = (x_0, \ldots, x_k)$, all the factors $g(x_i, x_j)$ in all the terms in the CM determinant may, in any chart, may be replaced by $(x_i - x_j)^T \cdot G(x_0) \cdot (x_i - x_j)$.

From the Lemma, we conclude, for any variable metric tensor G:

Proposition 4.4. *Given a 2-infinitesimal k-simplex $X = (x_0, \ldots, x_k)$. Then* $(\mathrm{heron}_g(X) =)\, \mathrm{heron}_G(X) = \mathrm{heron}_{G(x_0)}(X)$.

For, any non-singular path of length k contains all the $k+1$ vertices, in particular they all contain x_0 (although not necessarily as first or last vertex), so we may, by Lemma 4.3, for each non-singular path, pick the constant matrix $G(x_0)$ for the calculation.

Combining with the comparison in (3), we get

Proposition 4.5. *Given a coordinate patch $M \subseteq R^n$ and a 2-infinitesimal k-simplex $X = (x_0, x_1, \dots, x_k)$ in M. Then*

$$\mathrm{heron}_g(X) = \mathrm{gram}_{G(x_0)}(X).$$

5. Volume form

The volume form is a differential n-form that may be defined on an n-dimensional manifold M equipped with a *positive definite* metric g. (Since we only consider here open subspaces $M \subseteq R^n$, we need not mention the usual orientability requirement for M.) We take "positive definite" in the sense, which for individual symmetric matrices was described in the Remark at the end of Subsection 2.2; but now, for variable metric tensor $G : M \to W$, we require there exists another $H : M \to W$ in $\mathcal{E}$ with $G(x) = H^T(x) \cdot H(x)$ for all $x \in M$. (This is a property of g which does not depend on the chart.) However, the smoothness of such H, which implicitly is assumed here, is for the real C^∞-case with positive definite G, probably not automatic.)

Recall from the last lines of Section 3 the notation ω^2 for the square k-volume constructed out of a differential k-form ω:

Theorem 5.1. *Assume that g is a Riemannian metric on an n-dimensional manifold $M \subseteq R^n$. Then there exists on M a differential n-form ω such that* heron_g *and ω^2 agree on all 2-infinitesimal n-simplices; such ω deserves the name a* volume form *for g.*

Proof. Since the data and assertions in the statement do not depend on the choice of a coordinate chart, it suffices to prove the assertion in an arbitrary chart. So assume that M is identified with an open subspace of R^n and that G is given in terms of the positive definite $n \times n$ matrices $G(x)$ (for $x \in M$) (i.e. $G : M \to W$), with $G(x) = H(x)^T \cdot H(x)$ for all $x \in M$, with $H : M \to W$ smooth. Now consider the extended n-form $\overline{\omega}$, whose value on a 2-infinitesimal n-simplex $X = (x_0, \dots, x_n)$ is given by the by the formula

$$\overline{\omega}(X) := \frac{\det(H(x_0))}{n!} \cdot \det(Y) \tag{14}$$

where Y denote the $n \times n$ matrix with $y_i := x_i - x_0$ as its ith column. Now $\det G(x_0) = \det(H(x_0))^2$ by the product and transposition rules for determinants. Therefore squaring the defining equality (14) for $\overline{\omega}$ gives

$$\overline{\omega}^2(X) = \frac{\det G(x_0)}{n!^2} \cdot (\det Y)^2 = \frac{1}{n!^2} \det(Y^T \cdot G(x_0) \cdot Y) \qquad (15)$$

for any 2-infinitesimal n-simplex $X = (x_0, \ldots, x_n)$ using again the product rule and transposition rules for determinants. By definition of Gram, the equation continues

$$= \frac{1}{n!^2} \mathrm{Gram}_{G(x_0)}(X) = \mathrm{heron}_{G(x_0)}(X) = \mathrm{heron}_G(X),$$

using the Heron-Gram comparison Proposition 2.2 and Proposition 4.5. This proves the existence of the claimed differential n-form. □

Since $\det G(x_0) = \det(H(x_0)^T \cdot H(x_0)) = \det(H(x_0))^2$ by the product rule for determinants, $\det H(x_0)$ is a square root of $\det G(x_0)$, so except for the ambiguity of square roots, the formula derived here for "the" volume form may be written in the familiar form

$$\frac{\sqrt{\det(G(x_0))}}{n!} \cdot \det(y_1, \ldots, y_n).$$

We would like to have a uniqueness statement for volume form. This requires more structure or assumptions on the basic ring R, namely a positivity notion such that an invertible element in R is positive iff it is a square iff it is a square of a positive element.

Also, one would require that M is oriented, in the sense that there is given an n-form δ on M such that every other n-form on M is of the form $f \cdot \delta$ for a unique $f : M \to R$; this is redundant with the simplifying assumption we have made that M is an open subspace of R^n (where determinant formation provides the desired δ).

Under these circumstances, one may prove that there among the volume forms on M, there is a unique one of the form $f \cdot \delta$ with $f : M \to R$ positive, (meaning that f has only positive values).

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February 2022
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