CAHIERS DE TOPOLOGIE ET
GEOMETRIE DIFFERENTIELLE
CATEGORIQUES

VOLUME LXIV-1 (2023)

LIMITS OF TOPOLOGICAL SPACES AS ENRICHED CATEGORIES

Derek Scott Cook and Ittay Weiss

Résumé. Le lien entre les espaces métriques et la topologie repose sur certaines propriétés du treillis des réels non négatifs. En raison de la célèbre observation de Lawvere selon laquelle la théorie des espaces métriques est une branche de la théorie des catégories enrichies, il est naturel d'étudier dans quelle mesure le lien avec la topologie survit lors de l'enrichissement dans d'autres petits cosmos. En même temps, l'essor récent des techniques topologiques en science des données soulève la question de savoir quelles propriétés théoriques du treillis des réels non négatifs jouent un rôle vital, dans le but d'axiomatiser ces propriétés afin d'améliorer l'applicabilité des techniques au-delà métrique classique. Nous considérons ces deux motivations comme les faces théorique et applicable d'une même médaille mathématique. Nous identifions une large classe de quantales comme réponse commune aux deux questions, et utilisons les résultats pour présenter une construction de limites d'espaces qui est classiquement équivalente à la construction topologique, mais qui a un potentiel constructif différent.

Abstract. The link between metric spaces and topology relies on various lattice theoretic properties of the non-negative reals. Due to Lawvere's famous observation that metric space theory is a branch of enriched category theory, it is natural to study the extent to which the link with topology survives when enriching in other small cosmoses. At the same time, the recent flourish of topological techniques in data science raises the question of which lattice theoretic properties of the non-negative reals play a vital role, with the aim of axiomatising just those properties in order to enhance the applicability of techniques beyond the classical metric setting. We view these two motiva-

tions as the theoretical and applicable sides of the same mathematical coin. We pinpoint a wide class of quantales as the common answer to the two questions, and use the results to present a construction of limits of spaces that is classically equivalent to the topological one, but has constructively different potential.
Keywords. quantale, quantale enrichment, generalised metric space, constructive complete distributivity, induced topology, topological data analysis.
Mathematics Subject Classification (2010). 18D20, 18F75, 18B35, 54E35.

1. Introduction and motivation

It is well known ([20]) that it is fruitful to view a metric space as a small category enriched in the monoidal category $[0,\infty]$, with an arrow $x \to y$ precisely when $x \geq y$, and monoidal structure provided by addition. Any aspect of metric space theory thus becomes a source of enriched categorical investigation. Our interest is in continuity for a function $f\colon X \to Y$ between two sets, each of which is the set of objects of an enriched category. A suitable category $\mathcal{V}$ to enrich in is a *cosmos*, i.e., a symmetric closed monoidal cocomplete category ([19]). When $\mathcal{V}$ is also small, it is canonically equivalent to a complete lattice. Such a lattice is then precisely a commutative quantale Q. Equivalently, a (commutative) quantale is a (commutative) monoid object in the category $\mathbf{CJLat}$ of complete join lattices with respect to its tensor product (see [17] and [10]). Let $\mathbf{cQnt}$ be the category of commutative quantales with morphisms the join preserving monoidal functors. In more detail, a quantale Q is a complete lattice with joins $\bigvee$, meets $\bigwedge$, bottom element $\bot$, and top element $\top$. It is equipped with a monoidal product, namely an associative operation $\cdot$ with a two-sided unit 1, and it distributes over arbitrary joins, i.e.,

$$x \cdot \bigvee S = \bigvee\{x \cdot s \mid s \in S\} \quad \text{and} \quad \bigvee S \cdot x = \bigvee\{s \cdot x \mid s \in S\}$$

for all $x \in Q$ and $S \subseteq Q$. A morphism $f\colon Q \to Q'$ is a monoidal functor, again necessarily strict, that is also a complete join homomorphism. The quantale is *affine* if its monoidal unit is the top element. It is commutative when its monoidal product is symmetric (necessarily strictly so). Recall from [11] that a complete lattice L is *constructively completely distributive* (CCD) if $\bigvee\colon \mathcal{D}(L) \to L$, as a functor from the lattice of down-closed subsets of L, admits a left adjoint $\Downarrow(-)$. Explicitly, $y \in \Downarrow(x)$ precisely when, for all subsets $S \subseteq Q$, the condition $x \leq \bigvee S$ implies that $y \leq s$ for some $s \in S$. We then say that y is *totally below* x, and write $y \lll x$. The CCD condition then amounts to

$$x = \bigvee \Downarrow(x),$$

for all $x \in L$. We shall use the auxiliary notation

$$\downarrow(x) = \{y \in L \mid y \leq x\}, \quad \downarrow\!\!\!\!-(x) = \{y \in L \mid x < y\}, \quad \text{and} \quad \Uparrow(x) = \{y \in L \mid y > x\}$$

for elements in a lattice.

Let Q_1 and Q_2 be quantales, X a small Q_1-category, Y a small Q_2-category, and $f\colon X \to Y$ a function between the underlying sets of objects. We say that f is *continuous* at $x \in X$ if for all $\varepsilon \lll \top$ in Q_2 there exists a $\delta \lll \top$ in Q_1 such that

$$\delta \lll X(x,y) \implies \varepsilon \lll Y(fx, fy).$$

Expectantly, we say that f is *continuous* if f is continuous at all points $x \in X$. It is easily seen that the identity function from a small Q-category to itself is continuous, and that the composition of continuous functions is continuous. Therefore, if Γ is a class of quantales, we obtain the category $\Gamma\mathbf{Cat}_{\text{cont}}$ consisting of all small Q-categories, where Q is of class Γ, with all continuous functions as morphisms. For instance, if $\Gamma = \{[0,\infty]\}$, then $\Gamma\mathbf{Cat}_{\text{cont}}$ is the category of all Lawvere metric spaces with morphisms all functions satisfying the usual cauchy condition of continuity.

A quantale is a *value quantale* ([12]) when it is affine, its underlying lattice is CCD, and $\Downarrow(\top)$ is closed under finite joins. The following result was noted in [32, 7].

Theorem 1.1. *Let* F *be the class of all value quantales. The open ball topology functor* $\mathcal{O}\colon \mathsf{F}\mathbf{Cat}_{\text{cont}} \to \mathbf{Top}$ *is an equivalence of categories. In fact, it is the unique equivalence between these categories as concrete categories.*

As a consequence, there arises a translation mechanism between topology and the language of enriched categories. Given any topological concept, one may ask whether it is captured enriched categorically in a natural fashion. For instance, since $\mathbf{Top}$ is complete, so is $\mathsf{F}\mathbf{Cat}_{\text{cont}}$. If X and Y are two objects of $\mathsf{FCat}_{\text{cont}}$, their product may be computed as $\Phi((\mathcal{O}(X)\times\mathcal{O}(Y)))$, where the product is computed in $\mathbf{Top}$ and $\Phi\colon \mathbf{Top} \to \mathsf{FCat}_{\text{cont}}$ is any choice of an equivalency. Of course, this is not what we mean by capturing products enriched categorically. Instead, we insist on treating X and Y as enriched categories and remain firmly within the category $\mathsf{FCat}_{\text{cont}}$, without passing to the equivalent $\mathbf{Top}$. Doing so, suppose that X and Y are enriched, respectively, in the quantales Q_1 and Q_2. Suppose further that a suitable coproduct quantale $Q = Q_1 \coprod Q_2$ exists. It is then natural to define the Q-enriched category $X \times Y$ with $\text{ob}(X \times Y) = \text{ob}(X) \times \text{ob}(Y)$ and

$$(X \times Y)((x,y),(x',y')) = X(x,x') \otimes Y(y,y')$$

where, for elements $x \in Q_1$ and $y \in Q_2$, we write $x \otimes y$ for the element $\iota_1(x) \cdot \iota_2(y) \in Q_1 \coprod Q_2$, namely the monoidal product in the coproduct $Q_1 \coprod Q_2$, of the canonical injections of x and y in it. We will show that this construction is legitimate and that it results in the categorical product of X and Y in $\mathsf{FCat}_{\text{cont}}$. A fortiori, the open ball topology $\mathcal{O}(X \times Y)$ must coincide with the usual product topology of $\mathcal{O}(X)$ and $\mathcal{O}(Y)$. We will treat all small limits in this enriched categorical sense.

1.1 The plan of the article

The rest of the introduction discusses the foundational aspect of the approach above to topological data analysis, and quickly surveys related work. This is a potential application we see to the approach we present, but the presentation itself is of independent interest. Section 2 leads to the identification of topological quantales, namely lattice-theoretic conditions that guarantee that the classical link between metric spaces and topology extends to the quantale enrichment case. Section 3 develops the infrastructure of coproducts of commutative quantales required for the main result. Throughout the paper, and particularly in that section, we pay attention to the constructive validity of the results. Section 4 then presents the construction of all small limits, deliberately by the use of essentially metric techniques. The proof can be seen as a very elementary $\epsilon - \delta$ style proof. However, its correctness rests upon the precise lattice theoretic machinery developed earlier. In particular, the construction of limits of spaces is as constructive as the lattices that are used for metrising each of the ingredient spaces.

1.2 Connection to topological data analysis

In topological data analysis (TDA) the starting point is a point cloud, which is nothing but a finite metric space. Of course, in practice all data is finite, but mathematically the restriction to finite metric spaces is artificial. Dropping it, the starting point of TDA is a metric space. Equivalently, the starting point of TDA is a small $[0, \infty]$-enriched category. The techniques used are then topological. Topology is by design blind to small perturbations in the presentation of a metric space, and this precisely leads to robustness in the analysis of the data to various types of contamination (see [3]). TDA techniques typically result in what is known as a bar code; a combination of the blindness of topology together with the rigidity of the metric presentation of the problem. Stated more formally, the algorithm is performed on the metric space X and not on its open ball topology $\mathcal{O}(X)$. The metric presentation is crucial.

A categorical understanding of TDA, as begun in [4], must handle the tension (see [29]) between the metric presentation of the problem and the topological techniques. In particular, any topological technique that can be used for TDA must allow the scale ε to affect the computation. This is often achieved by converting the given metric space with a chosen scale ε, into purely topological form, and running a computation on that. The work below presents a rather harmonious passage from the metric to topology, primarily without changing the objects of the category. This phenomenon may simplify the interaction between the metric input and the topological processing inherent to TDA.

Going back to the importance of the metric presentation of the point cloud, the current phrasing of TDA is limited to operate only on classical metric spaces. In this work we present an equivalency $\mathtt{TQ}\mathbf{Cat}_{\mathrm{cont}} \simeq \mathbf{Top}$, where the objects of the category on the left-hand side are generalised metric spaces, taking values in lattices more general than $[0, \infty]$. The approach we take singles out such suitable lattices that ensure the results are constructive. In other words, TDA remains applicable for data presented as an object in $\mathtt{TQ}\mathbf{Cat}_{\mathrm{cont}}$. This increases the domain of applicability of TDA and allows greater flexibility when modelling data. We also mention the importance and subtleties of developing algebraic topology constructively. For instance, it is vital for the basics of algebraic topology that a space admits the path joining property (see [21]). Classically, this is not an issue, but in order to lead to executable algorithms, the underlying mathematics must be constructive. For a metric space, rather than just a topological space, the path joining property can be inferred under suitable condition (again, see [21]). By exhibiting $\mathbf{Top}$ as equivalent to $\mathtt{TQ}\mathbf{Cat}_{\text{cont}}$, we open up the possibility of developing algebraic topology for generalized metric spaces, and in a constructive manner.

Already the case of the product of two spaces (alluded to above) demonstrates the potential of our approach. Suppose that X and Y are metric spaces, each thought of as the input for analysis. Applying TDA to $X \times Y$ then represents a case of multidimensional persistence; a well-known problem ([5]). A related scenario is that of multiparameter persistence, requiring sophisticated tools as developed in [14]. A metric in the classical sense for which $\mathcal{O}(X \times Y)$ is the product topology surely exists, e.g., the Euclidean metric or the $\inf$ metric. From a TDA perspective, the choice of which metric is used is paramount. The bar code that will be produced with either of the mentioned metrics does not record features as they occur in X and Y independently. Our approach offers an alternative: a metrisation of $X \times Y$ taking values in $[0, \infty] \otimes [0, \infty]$ as an approach to multiparameter analysis. As mentioned, since our results are constructive, existing TDA techniques are still applicable. Approaches to the foundations of TDA in general, and addressing multidimensionality in particular, that take a similar path to ours are, respectively, [9] and [8], emphasising constructive methods and topos theory.

1.3 Relation to other work

There is plenty of existing literature on quantale enriched categories, a survey of which is not intended. We point out here the references we are aware of that, at least tangentially, touch upon the issues we consider. We start off by mentioning [33], by the second named author. That work provides a comparison between Flagg's value quantales and their precursor concept, namely Kopperman's value semigroups. Some attempts were made there toward a construction of limits, but the background lattice theory was clunky and is much improved in this current work.

[30] discusses extension of functors in the context of quantale enrichment, clearly noting what happens when the quantale is constructively com-

pletely distributive. In particular, in that case, the Pompeiu-Hausdorff metric is obtained as such a functor extension. In this context we mention [1] and [26].

Quantale enrichment in a single quantale, namely $Q\mathbf{Cat}$, are studied in [28] as a rich source of concretely symmetric closed-monoidal topological categories. It is shown, conversely, that such a topological category induces a quantale. That article works toward characterising those categories equivalent to $Q\mathbf{Cat}$. The emphasis there is on a single quantale, and enriched functors as morphisms. In light of the information in the introduction above, it is interesting to extend the question and ask which categories occur, up to equivalence, as $\Gamma\mathbf{Cat}_{\mathrm{cont}}$ for a class Γ of quantales.

Categories enriched in quantales (and quantaloids, see [27]) are well studied in computer science. Here we mention [31], offering a topologically flavoured study, and [25], emphasising a dynamical interpretation. The latter notes that it is the abandonment of the commutativity of the quantale that results in dynamics. It is also primarily concerned with the categorical consequences of the complete distributivity of the quantale. Both aspects appear in our work, as we are careful to trace the role of commutativity, and the effects lattice properties have on the enrichment.

It is interesting that [2], when proving that $\mathbf{Top}^{\mathrm{op}}$ is a quasi-variety, uses complete distributivity, while we require complete distributivity when presenting $\mathbf{Top}$ as a category of enriched categories. Constructive complete distributivity features prominently in [22], which elaborates further on [2].

Finally, in [15] quantaloid enrichment is considered from a topological perspective close to ours. In particular, the authors associate with such an enrichment a closure operator and note simple conditions for the closure operator to land in topological spaces. Our work below addresses the closure operator alongside its interior operator twin, in the case of quantale enrichment. We expect that a similar story unfolds for quantaloid enrichment.

2. Topological quantales

For a metric space X, the closure operator is a monad on $\mathcal{P}(X)$, the interior operator is a monad on $\mathcal{P}(X)^{\mathrm{op}}$, and each monad determines the other via set complementation. The aim of this section is to identify a class of quantales Q for which this phenomenon holds for all small Q-categories X. We do so by examining what holds in general, and how lattice-theoretic properties of Q affect the situation. We start by furnishing such an X with closure and interior operators.

Definition 2.1. *Let Q be a quantale, X a small Q-category, and $S \subseteq X$ a set of objects. We write*

$$X(x,S) = \bigvee\{X(x,s) \mid s \in S\}$$

where $x \in X$ is an arbitrary object. The closure *of S is the set*

$$\mathsf{cl}(S) = \{x \in X \mid X(x,S) = \top\},$$

and its interior *is*

$$\mathsf{in}(S) = \{x \in S \mid \exists \varepsilon \lll \top\colon \quad \varepsilon \lll X(x,y) \implies y \in S\}.$$

For $r \in L$ and $x \in X$, the open ball *of radius r about x is*

$$\mathrm{B}_r(x) = \{y \in X \mid r \lll X(x,y)\},$$

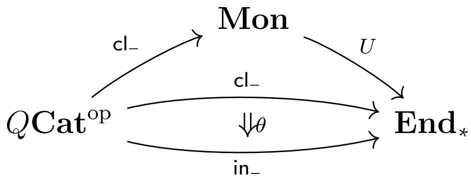
so that $x \in \mathsf{in}(S)$ is equivalently the existence of $\varepsilon \lll \top$ with $\mathrm{B}_\varepsilon(x) \subseteq S$, as usual.

Remark 2.2. When $Q = [0,\infty]$, these concepts attain the usual interpretations in a metric space. Unlike the definition of the closure operator, the interior operator requires justification. The inadequacy of naively using $\mathrm{B}_\varepsilon(x) = \{y \in X \mid r < X(x,y)\}$ instead is gleaned from Theorem 1.1 above — its validity depends on using $\lll$.

Let $\mathbf{End}$ be the category of endofunctors; its objects are a category $\mathscr{C}$ together with a functor $F\colon \mathscr{C} \to \mathscr{C}$, with a typical morphism $(G,\theta)\colon (\mathscr{C},F) \to (\mathscr{C}',F')$ consisting of a functor $G\colon \mathscr{C} \to \mathscr{C}'$ and a natural transformation $\theta\colon F'G \Rightarrow GF$. Let $\mathbf{End}_*$ be the category of pointed endomorphisms, i.e., (F,η), where $\eta\colon \mathrm{Id}_\mathscr{C} \Rightarrow F$ is a natural transformation, and those morphisms (G,θ) that respect the points, in the sense that $G\theta = \theta\eta' G$. There is an evident forgetful functor $\mathbf{Mon} \to \mathbf{End}_*$ from the category of monads.

A consequence of the Axiom of Choice is that in any complete lattice L, if $x \lll \bigvee S$, then $x \lll s$ for some $s \in S$ (see Proposition 3.2 below). This plays an important role in the final part of the following result.

Theorem 2.3. *Let Q be an affine quantale. The assignments $X \mapsto (\mathcal{P}(X), \mathsf{cl}_X)$ and $X \mapsto (\mathcal{P}(X)^{\mathrm{op}}, \mathsf{in}_X)$ are the object parts of the functors*



to the category of pointed endofunctors, each of which acts on a Q-functor $f{:}\, X \to Y$ by sending f to the inverse image function $f^{\leftarrow}$. The functor $\mathsf{cl}_{\_}$ factorises through monads, and the two functors are related by the natural transformation θ carried by the set complementation functor $\neg{:}\, \mathcal{P}(X) \to \mathcal{P}(X)^{\mathrm{op}}$.

Proof. The claim that cl_X is a functor is that $S \subseteq S' \implies \mathsf{cl}_X(S) \subseteq \mathsf{cl}_X(S')$, which is clear. The claim that it is a pointed functor is that $S \subseteq \mathsf{cl}_X(S)$, which is just as clear. Similarly, and as trivially, in_X is a functor since $S \subseteq S' \implies \mathsf{in}_X(S) \subseteq \mathsf{in}_X(S')$, and it is pointed since $S \supseteq \mathsf{in}_X(S)$. The claim that cl_X is a monad is that $\mathsf{cl}_X^2(S) \subseteq \mathsf{cl}_X(S)$, so let $x \in X$ satisfy $X(x, \mathsf{cl}_X(S)) = \top$, and we must show that $X(x, S) = \top$. It suffices to show, for a given $y \in \mathsf{cl}_X(S)$, that $X(x, y) \le X(x, S)$. And indeed, using affineness,

$$X(x,y) = X(x,y) \cdot \top = X(x,y) \cdot X(y,S) \le X(x,S)$$

by the distributivity law in the quantale and the composition inequality in X. Finally, the existence of the natural transformation θ is the claim that $\mathsf{in}_X(\neg S) \subseteq \neg(\mathsf{cl}_X(S))$. To see its validity, assume to the contrary that $x \in \mathsf{in}_X(\neg S) \cap \mathsf{cl}_X(S)$, namely there exists $\varepsilon \lll \top$ with $\varepsilon \lll X(x,y) \implies y \notin S$, and $X(x,S) = \top$. But then $\varepsilon \lll X(x,S)$ and so, by Proposition 3.2 below, it must be that $\varepsilon \lll X(x,s)$ for some $s \in S$, a contradiction. □

Remark 2.4. A situation where $\mathsf{in}{:}\, \mathcal{P}(X)^{\mathrm{op}} \to \mathcal{P}(X)^{\mathrm{op}}$ fails to be a monad is given in Example 2.11.

Historically, Kuratowski favoured closed sets for the axiomatisation of topology while Sierpiński pioneered open sets. We allow this anecdote to dictate our choice of terminology.

Definition 2.5. *A quantale Q is* sierpiński *if $\varepsilon \lll t$ implies $\varepsilon \lll t \cdot \bigvee \Downarrow(\top)$, for all $t \in Q$.*

Proposition 2.6. *Let Q be a quantale and X a small Q-category. If Q is sierpiński, then $\mathsf{in}(\mathrm{B}_r(x)) = \mathrm{B}_r(x)$, for all $r \in Q$, and $\mathsf{in}\colon \mathcal{P}(X)^{\mathrm{op}} \to \mathcal{P}(X)^{\mathrm{op}}$ is a monad.*

Proof. Fix $x \in X$, $r \in Q$, and $y \in \mathrm{B}_r(x)$, i.e., $r \lll X(x,y)$. We require a $\delta \lll \top$ with $\mathrm{B}_\delta(y) \subseteq \mathrm{B}_r(x)$. Now, since $r \lll X(x,y) \cdot \bigvee \Downarrow(\top) = \bigvee\{X(x,y) \cdot \delta \mid \delta \lll T\}$, a $\delta \lll \top$ exists with $r \lll X(x,y) \cdot \delta$ (again by Proposition 3.2). If $\delta \lll X(y,z)$, then

$$r \lll X(x,y) \cdot \delta \le X(x,y) \cdot X(y,z) \le X(x,z)$$

and so $z \in \mathrm{B}_r(x)$, as required. The fact that in is now a monad, namely that $\mathsf{in}(S) \subseteq \mathsf{in}^2(S)$, follows at once. □

In the classical case $Q = [0,\infty]$, the monad cl_X is a kuratowski closure operator, namely its carrier functor $S \mapsto \mathsf{cl}(S)$ preserves finite unions. Similarly, the functor in_X preserves finite intersections. In other words, if $\mathbf{reEnd}_*$ denotes the full subcategory of $\mathbf{End}_*$ spanned by right exact endofunctors, then the functors cl_- and in_- factorise via the inclusion $\mathbf{reEnd}_* \to \mathbf{End}_*$. Neither claim holds generally.

Example 2.7. Consider the quantale $Q = \mathcal{P}(S)$ of all subsets of a set S. Viewed as a closed monoidal category with intersection as monoidal product, its self-enrichment structure yields the Q-category X with $\mathrm{ob}(X) = \mathcal{P}(S)$ and $X(x,y) = \{s \in S \mid s \in x \implies s \in y\} = \neg x \vee y$. For a collection $\mathcal{A} \subseteq X$ we have $X(x,\mathcal{A}) = \neg x \vee \bigvee \mathcal{A}$, and thus $\mathsf{cl}(\mathcal{A}) = \mathcal{P}(\bigvee \mathcal{A})$ — it need not preserve finite joins. Direct computation shows that $\varepsilon \lll \top$ if, and only if, ε is a sub-singleton. Noting that

$$\mathrm{B}_{\{s\}}(x) = \begin{cases} \mathcal{P}(X) & s \notin x \\ \{y \subseteq S \mid s \in y\} & s \in x \end{cases}$$

shows that $\mathsf{in}(\mathcal{A}) = \{a \in \mathcal{A} \mid \exists s \in a\colon \quad s \in y \implies y \in \mathcal{A}\}$ — it need not preserve finite meets.

Definition 2.8. *Let L be a complete lattice. If $\downarrow\!\!(\top)$ is closed under finite joins, then L is* kuratowski. *If $\Downarrow\!(\top)$ is closed under finite joins, then L is* sierpiński.

We say that a quantale Q is kuratowski if its underlying lattice is. We say that Q is *entirely sierpiński* if both it and its underlying lattice are sierpiński.

Proposition 2.9. *Let Q be an affine quantale and X a small Q-category. If Q is kuratowski, then* cl *is a kuratowski closure operator.*

Proof. We only need to verify preservation of finite unions. For binary unions it suffices to show that $\mathsf{cl}(S \cup S') \subseteq \mathsf{cl}(S) \cup \mathsf{cl}(S')$, which follows at once since $X(x, S \cup S') = X(x, S) \cup X(x, S')$. It remains to see that $\mathsf{cl}(\varnothing) = \varnothing$, and indeed, if $x \in \mathsf{cl}(\varnothing)$, then $X(x, \varnothing) = \top$, but the former is $\bot$, forcing Q to collapse. But then $\downarrow\!\!(\top)$ is not closed under the empty join. □

In agreement with our historical convention, we call the set-theoretic dual of a kuratowski closure operator, namely a comonad $\mathsf{in}\colon \mathcal{P}(X) \to \mathcal{P}(X)$ that preserves finite meets, a *sierpiński interior operator*. Recall from [18] the theory of free monads on (pointed) functors. It is clear that in admits a free monad; its value on S is $\mathsf{in}^{\alpha}(S)$, where α is a sufficiently large ordinal ensuring the stabilisation of the decreasing chain $\{\mathsf{in}^{\beta}(S)\}_{\beta}$, where $\mathsf{in}^{\beta+1} = \mathsf{in}(\mathsf{in}^{\beta}(S))$ and, for a limit ordinal γ, $\mathsf{in}^{\gamma}(S) = \bigcap\{\mathsf{in}^{\beta} \mid \beta < \gamma\}$.

Proposition 2.10. *Let Q be an affine quantale and X a small Q-category. If the underlying lattice of Q is sierpiński, then* in *preserves finite meets, and the free monad on it is a sierpiński interior operator. If Q is entirely sierpiński, then* in *is already a sierpiński interior operator.*

Proof. Assuming the underlying lattice is sierpiński, the equality $\mathsf{in}(S \cap S') = \mathsf{in}(S) \cap \mathsf{in}(S')$ follows at once since if $x \in \mathsf{in}(S) \cap \mathsf{in}(S')$, witnessed by $\varepsilon, \varepsilon' \lll \top$, respectively, then $\varepsilon \vee \varepsilon' \lll \top$ witnesses that $x \in \mathsf{in}(S \cap S')$. In order to show that $\mathsf{in}(X) = X$, note that the only obstruction to that equality is if Q admits no $\varepsilon \lll \top$ at all, which can happen only if Q collapses. But then $\Downarrow\!(\top)$ is not closed under the empty join.

It is now clear that if Q is entirely sierpiński, then in is a sierpiński interior operator. If we only know that in is a pointed finite-union preserving functor but not necessarily a monad, it is clear that the finite-union preservation survives the free monad construction, thus yielding a sierpiński interior operator. □

In the classical case $Q = [0,\infty]$, it is well known that open and closed sets are dual concepts: S is open/closed if, and only if, $X \setminus S$ is closed/open. Stated differently, set complementation $\neg\colon \mathcal{P}(X) \to \mathcal{P}(X)^{\mathrm{op}}$, since it is a complete lattice isomorphism, induces an isomorphism $\neg\colon \mathsf{Monad}(\mathcal{P}(X)) \to \mathsf{coMonad}(\mathcal{P}(X))$, given by $(\neg F)S = \neg(F(\neg S))$, that restricts to an isomorphism between the kuratowski and sierpiński operators. Thus, when $Q = [0,\infty]$, the natural transformation $\theta\colon \mathsf{in}_- \to \mathsf{cl}_-$ (which is carried by $\neg$) is a natural isomorphism. Direct verification shows that in the case considered in Example 2.7, one has that $\mathsf{in}_X(\neg S) = \neg\mathsf{cl}_X(S)$, namely the component of θ is an isomorphism. We shall shortly see why that holds true for all small Q-categories X for $Q = \mathcal{P}(S)$. We first observe that the same phenomenon does not hold true for arbitrary quantales.

Example 2.11. Let Q be a complete boolean algebra, viewed as a quantale with operation given by $\wedge$ (since any complete boolean algebra is a frame, this is legitimate). Let X be Q as a Q-category, thus $X(x,y) = \neg x \vee y$, where $\neg$ is the boolean complement operator. Clearly then, for $\mathcal{A} \subseteq X$, $X(x,\mathcal{A}) = \neg x \vee \bigvee \mathcal{A}$, and so $\mathsf{cl}(\mathcal{A}) = \downarrow(\bigvee \mathcal{A})$. Computing the interior operator requires knowledge of the set $\Downarrow(\top)$. Let us consider two extremes: the atomic and atom-less cases. If Q is an atomic complete boolean algebra, then $Q \cong \mathcal{P}(S)$, $\Downarrow(\top)$ is the set of sub-atomic elements, and the situation reduces to that of Example 2.7. If Q is atom-less, then $\Downarrow(\top) = \{\bot\}$, and it follows that

$$\mathrm{B}_\bot(x) = \{y \in X \mid \bot \lll \neg x \vee y\} = \begin{cases} X & x < \top \\ \Uparrow(\bot) & x = \top \end{cases}$$

using the simple observation that in any lattice, $\bot \lll x$ holds precisely when $x \neq \bot$. Therefore,

$$\mathsf{in}(\mathcal{A}) = \begin{cases} X & \mathcal{A} = X \\ \{\top\} & \mathcal{A} = \Uparrow(\bot) \\ \varnothing & \text{otherwise} \end{cases}$$

and in particular, $\mathsf{in}^2(\Uparrow(\bot)) \subsetneq \mathsf{in}(\Uparrow(\bot))$. The interior operator is thus not a monad. Since the closure operator is always a monad, it is thus impossible that θ is a natural isomorphism in this case.

The final piece of this section is a lattice-theoretic property under which θ is necessarily a natural isomorphism. The following terminology is explained in Subsection 3.1.

Definition 2.12. *A complete lattice L is* CCD *at* $\top$ *if* $\top = \bigvee \Downarrow(\top)$.

We say that Q is CCD at $\top$ if its underlying lattice is. Obviously, any CCD lattice is CCD at $\top$, and so, certainly, $\mathcal{P}(S)$ is CCD at $\top$ (cf. Example 2.7). A non-atomic complete boolean algebra satisfies $\bigvee \Uparrow(\top) = \bot$, so it is far from being CCD at $\top$ (cf. Example 2.11). The following results clarify much of the mechanics in both examples.

Proposition 2.13. *Let Q be an affine quantale. If Q is* CCD *at* $\top$*, then* $\theta\colon \mathsf{cl}_{-} \Rightarrow \mathsf{in}_{-}$ *is a natural isomorphism.*

Proof. Since $\mathsf{in}(\neg S) \subseteq \neg\mathsf{cl}(S)$ always holds, we only need to show the reverse inclusion, so assume $X(x,S) < \top$. Since $\bigvee \Downarrow(\top) = \top$, there exists some $\varepsilon \lll \top$ such that $X(x,S) \ngeq \varepsilon$. But then $\mathrm{B}_\varepsilon(x) \subseteq \neg S$, since $\varepsilon \lll X(x,y)$ together with $y \in S$ leads to the contradiction $X(x,S) \geq X(x,y) \geq \varepsilon$. □

Proposition 2.14. *If an affine quantale Q is* CCD *at* $\top$*, then Q is sierpiński.*

Proof. We need to show that if $\varepsilon \lll t$, then $\varepsilon \lll t \cdot \bigvee \Downarrow(\top)$. But $\bigvee \Downarrow(\top) = \top$, and $\top$ is the quantale unit. □

As a consequence, if Q is an affine quantale that is CCD at $\top$, then in, and not just cl, is guaranteed to be a monad. The next result is slightly less immediate.

Proposition 2.15. *If Q is an affine quantale that is* CCD *at* $\top$*, then Q is kuratowski if, and only if, Q is entirely sierpiński.*

Proof. See [7, Proposition 3]. □

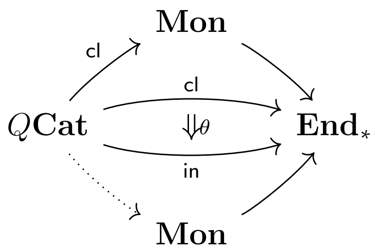
The above considerations highlight certain quantales as foundational in topology.

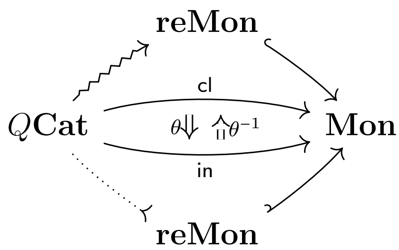
Definition 2.16. A topological quantale *is a commutative affine quantale Q that is* CCD *at* $\top$ *and kuratowski (and thus entirely sierpiński).*

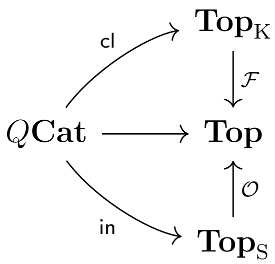
Remark 2.17. The commutativity of Q was not required in any of the result so far. The effect of commutativity is of importance when we come to consider coproducts of quantales below.

The following theorems embody the idea of viewing topological spaces as enriched categories.

Theorem 2.18. *Consider the diagrams*







where Q is an affine quantale. Regarding the functor $\mathsf{cl}\colon Q\mathbf{Cat} \to \mathbf{End}_*$ *factoring over* $\mathbf{Mon}$*, the functor* $\mathsf{in}\colon Q\mathbf{Cat} \to \mathbf{End}_*$*, and the natural transformation* $\mathsf{cl} \Rightarrow \mathsf{in}$ *from Theorem 2.3, we can specify that:*

1. *If Q is sierpiński, then* $\mathsf{in}\colon Q\mathbf{Cat} \to \mathbf{End}_*$ *factors over* $\mathbf{Mon}$.
2. *if Q is kuratowski, then* in *factors over* $\mathbf{reMon}$.
3. *if Q is entirely sierpiński, then* in *factors over* $\mathbf{reMon}$.
4. *if Q is* $\mathtt{CCD}$ *at* $\top$*, then* $\mathsf{cl} \Rightarrow \mathsf{in}$ *is a natural isomorphism.*
5. *if Q is kuratowski and* $\mathtt{CCD}$ *at* $\top$*, then all the above happens; in other words, both closure and interior operators are topological and specify the same topological space.*

Theorem 2.19. *Let* $\mathtt{TQ}$ *be the class of topological quantales and consider the category* $\mathtt{TQ}\mathbf{Cat}_{\mathrm{cont}}$ *whose objects are all small Q-categories where Q*

is a topological quantale, and morphisms the cauchy continuous functions $f: X \to Y$*, namely those satisfying the familiar* ε*–*δ *condition as described in the introduction. The unique functor* $\mathcal{TQ}\mathbf{Cat}_{\text{cont}} \to \mathbf{Top}$ *above, which we denote by* $X \mapsto \mathcal{O}(X)$*, is the functor that associates with a small* Q*-category its (unique!) topology, and acts as the identity on morphisms (which is valid since any* $\varepsilon - \delta$ *continuous function is also continuous with respect to the open ball topology). This functor takes the familiar form where* $U \subseteq X$ *is declared open precisely when*

$$\forall x \in U \quad \exists \varepsilon \lll \top: \quad \mathrm{B}_\varepsilon(x) \subseteq U.$$

These functors, for the various topological quantales Q*, patch up together to form the functor*

$$\mathcal{O}: \mathcal{TQ}\mathbf{Cat}_{\text{cont}} \to \mathbf{Top}$$

and this functor is an equivalence of categories.

Proof. The proof is essentially due to [12]. Obviously, $\mathcal{O}$ is faithful, and due to open balls being open sets, the standard textbook proof shows $\mathcal{O}$ is full. Flagg utilised the free frame construction $\Omega: \mathbf{Set} \to \mathbf{Frm}$, with frames viewed as quantales. In more detail, $\Omega(X)$ is the collection of all down closed collections of finite subsets of X, ordered by inclusion. Let us show that $\Omega(X)$ is kuratowski, so suppose $a, b < \top$, which means a misses a finite subset F_a, and b misses a finite subset F_b. But then if $a \vee b = \top$, then $F = F_a \cup F_b$ must be there, which would force it into either a or b. Since $F_a \subseteq F$, it cannot belong to a. b is similarly prohibited. To see that $\Omega(X)$ is $\mathtt{CCD}$ at $\top$ it suffices to note that $a \lll \top$ precisely when there exists a finite subset $F_a \subseteq X$ such that a consists only of subsets of F_a. The join of such elements a is thus the entire collection $\top$ of all finite subsets of X. In other words, Ω lands in topological quantales. Now, to show that $\mathcal{O}$ is surjective on objects, given a topology τ on X, let $X(x, y) \in \Omega(\tau)$ be the collection of all finite subsets of $\tau_{x \to y}$, where $\tau_{x \to y} = \{U \in \tau \mid x \in U \implies y \in U\}$. □

3. Coproducts of commutative quantales

The construction of limits in $\mathtt{TQ}\mathbf{Cat}_{\text{cont}}$ relies on coproducts of commutative quantales, and those rely on colimits of complete join lattices. We are particularly interested in stability properties of topological quantales under coproducts, and so proceed to introduce the relevant notions alongside a study of the totally below relation.

3.1 Constructive complete distributivity

Recall that $\downarrow(-): L \to \mathcal{D}(L)$, where L is a complete lattice and $\mathcal{D}(L)$ is the lattice of its down-closed subsets, has a left adjoint given by $\bigvee$, and that L is `CCD` precisely when $\bigvee$ has a left adjoint $\Downarrow(-)$. The definition of such a left adjoint dictates that

$$\Downarrow(x) = \{y \in L \mid y \lll x\}$$

in the sense of the totally below relation $y \lll x$, namely that for all $S \subseteq L$ with $x \leq \bigvee S$ there exists $s \in S$ with $y \leq s$.

Even if L is not `CCD`, the definition above still yields a functor $\Downarrow(-): L \to \mathcal{D}(L)$. Consider the functor $\bigsqcup: \mathcal{D}(L) \to L$ given by

$$\bigsqcup S = \bigvee\{x \in L \mid \Downarrow(x) \subseteq S\}.$$

Proposition 3.1. *The following conditions for a complete lattice L are equivalent:*

1. *For all $a \lll b$, if $b \leq \bigvee S$, then $a \lll s$ for some $s \in S$.*
2. *The functor $\Downarrow(-)$ is a left adjoint.*

Proof. Assuming the first condition, we show that $\Downarrow(-) \dashv \bigsqcup$. It suffices to demonstrate the unit and counit conditions, namely $x \leq \bigsqcup \Downarrow(x)$ and $\Downarrow(\bigsqcup S) \subseteq S$, of which the former is trivial. For the latter, suppose $x \lll \bigvee\{y \in L \mid \Downarrow(y) \subseteq S\}$, so, by the assumed condition, $x \lll y$ for some y with $\Downarrow(y) \subseteq S$, thus $x \in S$. For the converse, note that if $\Downarrow(-)$ is a left adjoint and $b \leq \bigvee S$, then $\Downarrow(b) \subseteq \Downarrow(\bigvee S) = \bigcup\{\Downarrow(s) \mid s \in S\}$. □

Proposition 3.2. *If the background set theory admits the Axiom of Choice, then, for all complete lattices L, the functor $\Downarrow(-)$ is a left adjoint.*

Proof. We demonstrate the first condition of Proposition 3.1. Proceeding by contradiction, suppose $a \lll b$, $b \leq \bigvee S$, and yet $a \lll s$ holds for not a single $s \in S$. Choose, for each $s \in S$, a set T_s with $s \leq \bigvee T_s$ and so that $a \leq t$ fails for all $t \in T_s$. The set $T = \bigcup\{T_s \mid s \in S\}$ contradicts $a \lll \bigvee S$. □

Remark 3.3. We proceed under the assumption that for all lattices L concerning us, the functor $\Downarrow(-)$ admits a right adjoint. In light of Proposition 3.2, this is automatic if the Axiom of Choice holds. Otherwise, the assumption we are making is that we restrict to those lattices that are sufficiently constructive to admit the required right adjoint.

For any complete lattice L let $\mathtt{CCD}(L) = \{x \in L \mid \bigvee \Downarrow(x) = x\}$, which we call the CCD core of L. It is easily seen that the CCD core of L is a complete join sublattice of it, but it need not itself be CCD. We obtain the following play on words.

Theorem 3.4. *Let L be a complete lattice. The following conditions are equivalent (and define what it means for L to be CCD):*

- $\bigvee$ *has a left adjoint*
- $\bigvee = \bigsqcup$
- $\mathit{CCD}(L) = L$.

3.2 Tensor product of complete join lattices

The category **CJLat** of complete join lattices is well known to support a symmetric closed monoidal structure ([17, 24]). The tensor product of complete lattices L_1, L_2 is a function $\beta\colon L_1 \times L_2 \to L_1 \otimes L_2$ that is universal among all functions $L_1 \times L_2 \to L$ that are join preserving in each variable. Much as in ring theory, the tensor product can be constructed as a quotient of a free lattice. Writing $x \otimes y$ for $\beta(x, y)$, and referring to such elements as elementary tensors, every element in $L_1 \otimes L_2$ is a join of elementary tensors, for all $x > \bot$ in L_1 and $y > \bot$ in L_2 we have that $x \otimes y \leq x' \otimes y'$ if, and only if, both $x \leq x'$ and $y \leq y'$, $\bigwedge_i x_i \otimes y_i = \bigwedge_i x_i \otimes \bigwedge_i y_i$, and $x \otimes \bigvee S = \bigvee\{x \otimes s \mid s \in S\}$. Thus meets are computed point-wisely in $L_1 \otimes L_2$. The join of arbitrary elementary tensors, however, does not admit such a simple formula.

The following result is [23, Lemma 37]:

Theorem 3.5. *If L_1 and L_2 are CCD, then so is their tensor product.*

We require a refinement of this result and an analysis of the totally below relation in the tensor product. It is convenient to use the fact ([17]) that

$$L_1 \otimes L_2 \cong \underline{\mathbf{CJLat}}(L_1, L_2^{\mathrm{op}})^{\mathrm{op}},$$

the opposite of the complete lattice of join preserving morphisms $L_1 \to L_2^{\mathrm{op}}$ (see [13] for a detailed description). In this model, the elementary tensors are given by

$$\beta(x,y)(a) = \begin{cases} \top & \text{if } a = \bot, \\ y & \text{if } \bot < a \leq x, \\ \bot & \text{if } a \nleq x, \end{cases}$$

and an arbitrary element $f \in L_1 \otimes L_2$ has the canonical presentation

$$f = \bigvee_{x \in L_1} \beta(x, f(x))$$

as a join of elementary tensors.

It is straightforward that if $a \otimes b \lll x \otimes y$, then both $a \lll x$ and $b \lll y$. The following result will assist in obtaining conditions for the converse implication.

Lemma 3.6. *Let S be a subset of the tensor product*

$$L_1 \otimes L_2 \cong \mathbf{CJLat}(L_1, L_2^{\mathrm{op}})^{\mathrm{op}}$$

of two complete lattices and write s for the point-wise join of S, i.e. $s(x) = \bigvee_{g \in S} g(x)$ computed in L_2 for all $x \in L_1$. Now define

$$f(x) = \bigwedge_{x' \lll x} s(x')$$

for all $x \in L_1$, again computed in L_2. The following properties hold.

1. *f is an upper bound of S in $L_1 \otimes L_2$.*
2. *If h is an upper bound of S, then $f(x) \leq h(x)$ for all $x \in \mathit{CCD}(L_1)$.*
3. *$(\bigvee S)(x) = f(x)$ for all $x \in \mathit{CCD}(L_1)$.*

Proof. Recall that $\Downarrow(-)$ is a left adjoint.

1. Firstly, $f\colon L_1 \to L_2^{\mathrm{op}}$ belongs to $L_1 \otimes L_2$, namely it preserves joins, since

$$f(\bigvee \mathcal{A}) = \bigwedge_{x' \in \Downarrow(\bigvee \mathcal{A})} s(x') = \bigwedge_{a \in A} \bigwedge_{x' \in \Downarrow(a)} s(x') = \bigwedge_{a \in \mathcal{A}} f(a).$$

Now, for any $g \in S$ and $x' \lll x$, clearly, $g(x) \leq g(x') \leq s(x')$, showing that $g(x) \leq f(x)$ point-wisely, and thus f is an upper bound of S.

2. An upper bound h clearly satisfies $s(x') \leq h(x')$. Assuming that $x \in \mathtt{CCD}(L_1)$, we obtain that

$$f(x) \leq \bigwedge_{x' \lll x} h(x') = h(\bigvee_{x' \lll x} x') = h(x).$$

3. Immediate.

□

Proposition 3.7. *Let L_1 and L_2 be complete lattices, $a \lll x$ in L_1, and $b \lll y$ in L_2. If $x \in \mathit{CCD}(L_1)$, then $a \otimes b \lll x \otimes y$.*

Proof. Suppose that $x \otimes y \leq \bigvee S$ for some $S \subseteq L_1 \otimes L_2$. By Lemma 3.6 and the given conditions we have that

$$b \lll y \leq (\bigvee S)(x) = \bigwedge_{x' \lll x} \bigvee_{g \in S} g(x') \leq \bigvee_{g \in S} g(a),$$

and so $b \leq g_0(a)$ for some $g_0 \in S$. Therefore

$$a \otimes b = \beta(a, b) \leq g_0$$

as can be seen from the expression for $\beta(a, b)$, recalling that g_0 is antitone. □

We summarise as follows.

Theorem 3.8. *For complete lattices L_1 and L_2, if $x \in \mathit{CCD}(L_1)$ and $y \in \mathit{CCD}(L_2)$, then $x \otimes y \in \mathit{CCD}(L_1 \otimes L_2)$ and $t \lll x \otimes y$ if, and only if, $t \leq a \otimes b$ with $a \lll x$ and $b \lll y$. If L_1 and L_2 are* CCD *at $\top$ and are sierpiński, then $L_1 \otimes L_2$ is sierpiński.*

Proof. The characterisation of $t \lll x \otimes y$ follows from the observation that $x \otimes y = \bigvee\{a \otimes b \mid a \lll x, b \lll y\}$ as soon as $x \in \mathtt{CCD}(L_1)$ and $y \in \mathtt{CCD}(L_2)$, from which $x \otimes y \in \mathtt{CCD}(L_1 \otimes L_2)$ is immediate. With this property, the claim about the sierpiński property follows from $a \otimes b \vee a' \otimes b' \leq (a \vee a') \otimes (b \vee b')$. □

3.3 Coproducts of complete join lattices

Coproducts in $\mathbf{CJLat}$ ([17]) are particularly simple to describe, due to its strong self duality: if $f\colon L_1 \to L_2$ is join preserving, then it has a right adjoint $g\colon L_2 \to L_1$, which, when written as $f^{\mathrm{op}}\colon L_2^{\mathrm{op}} \to L_1^{\mathrm{op}}$, is join preserving, and thus yields an isomorphism $\mathbf{CJLat} \to \mathbf{CJLat}^{\mathrm{op}}$. The product L of lattices $\{L_k\}_{k\in I}$ is given by the usual product of the underlying sets, equipped with component-wise operations. The projections $\pi_k\colon L \to L_k$ preserve meets, and so admit left adjoints $\iota_k\colon L_k \to L$. It then holds that L with these morphisms is the coproduct in $\mathbf{CJLat}$, and $\pi_k \circ \iota_k = \mathrm{Id}_{L_k}$.

Proposition 3.9. *Let $\{L_k\}_{k\in I}$ be a collection of complete lattices. Then $a \lll x$ in the coproduct $\coprod L_k$ if, and only if, there exists $k_0 \in I$ and $\bar{a} \in L_{k_0}$ such that $\bar{a} \lll \pi_{k_0}(x)$ and $a = \iota_{k_0}(\bar{a})$.*

Proof. Clearly, $x = \bigvee\{\iota_k(\pi_k(x)) \mid k \in I\}$, so for $a \lll x$ the existence of k_0 follows from Proposition 3.2. □

3.4 Coproducts of commutative quantales

Since the category of commutative quantales is $\mathsf{cMon}(\mathbf{CJLat})$, it follows from general considerations that it is cocomplete (and complete). We require a concrete enough description of coproducts, sufficient to see that topological quantales admit coproducts.

It is a simple matter that, much as in the case of commutative rings, finite coproducts in $\mathbf{cQnt}$ are given by the tensor product. This follows again from general considerations of commutative monoid objects in a symmetric closed monoidal category. In a nutshell, the multiplication of a commutative quantale Q is a function $Q \times Q \to Q$, preserving joins in each variable, and thus corresponds to a morphism $\lfloor\cdot\rfloor\colon Q \otimes Q \to Q$ from the tensor product of the underlying lattice. If Q_1 and Q_2 are commutative quantales, then one obtains a binary operation on $Q_1 \otimes Q_2$, namely the one corresponding to

$$(Q_1 \otimes Q_2) \otimes (Q_1 \otimes Q_2) \to (Q_1 \otimes Q_1) \otimes (Q_2 \otimes Q_2) \xrightarrow{\lfloor\cdot\rfloor\otimes\lfloor\cdot\rfloor} Q_1 \otimes Q_2$$

utilising the canonical symmetry isomorphism $Q_2 \otimes Q_1 \to Q_1 \otimes Q_2$. It is easily seen that then $Q_1 \otimes Q_2$ is a commutative quantale, and that with the evident morphisms $Q_1 \to Q_1 \otimes Q_2 \leftarrow Q_2$ it is the coproduct in $\mathbf{cQnt}$.

Theorem 3.10. *The category* $\mathbf{tQnt}$ *of topological quantales, as a full subcategory of* $\mathbf{cQnt}$*, is closed under finite coproducts, and* $\varepsilon \lll \top$ *holds in* $Q_1 \otimes \cdots \otimes Q_n$ *if, and only if, there exist* $\varepsilon_k \in Q_k$ *with* $\varepsilon_k \lll \top$ *such that* $\varepsilon \leq \varepsilon_1 \otimes \cdots \otimes \varepsilon_n$*.*

Proof. The empty coproduct is the quantale $\mathcal{B} = \{\bot < \top\}$ of boolean truth values, and it is clearly topological. The characterisation of $\varepsilon \lll \top$ in this case simply says that all $\varepsilon \in \mathcal{B}$ satisfy $\varepsilon \lll \top$. Suppose that Q_1 and Q_2 are topological quantales. It is clear that $Q = Q_1 \otimes Q_2$ is affine and the rest of the claims follow from the fact that the underlying lattice of Q is the tensor product in $\mathbf{CJLat}$, together with Theorem 3.8. □

This leaves the case of infinite coproducts. As in any category, directed colimits together with finite coproducts suffice to construct all coproducts, as follows. For a set I consider the poset $\mathrm{Fin}(I)$ of all finite subsets of I, under inclusion. For a small collection $\{Q_k\}_{k \in I}$ of commutative quantales indexed by I their coproduct is the colimit

$$\coprod_{k \in I} Q_k = \operatorname*{colim}_{\mathrm{Fin}(I)} (S \mapsto \bigotimes_{k \in S} Q_k).$$

Directed colimits are (again) particularly simple in $\mathbf{cQnt}$, namely they are created by the functor $\mathbf{cQnt} \to \mathbf{CJLat}$ (see [16, C1.1, Lemma 1.1.8], and the discussion surrounding it).

We continue to use the elementary tensor notation $x_1 \otimes \cdots \otimes x_n$ to stand for $\iota_1(x_1) \vee \cdots \vee \iota_n(x_n)$, where ι_k is the canonical injection into the (possibly infinite) coproduct.

Theorem 3.11. *The category* $\mathbf{tQnt}$ *of topological quantales, as a full subcategory of* $\mathbf{cQnt}$*, is closed under infinite coproducts, and* $\varepsilon \lll \top$ *holds in* $\bigotimes_{k \in I} Q_k$ *if, and only if, there exist finitely many* $\varepsilon_k \in Q_k$ *with* $\varepsilon \lll \top$ *such that* $\varepsilon \leq \varepsilon_1 \otimes \cdots \otimes \varepsilon_n$*.*

Proof. Combine Proposition 3.9 with Theorem 3.8 and the form of the infinite coproduct of commutative quantales. □

To conclude this section we note that coproducts of quantales occur naturally in applications. For instance, recall the quantales $[0, 1]$ with multiplication and $[0, \infty]^{\mathrm{op}}$ with addition. Their coproduct is usually denoted by

Δ, the quantale of distance distribution functions. Small Δ-categories are also known as probabilistic metric spaces. Since the constituent quantales are topological, so is Δ. This quantale was in circulation long before it was realised that it is simply the coproduct of two very naturally occurring quantales. Further, it is clear that $[0,1] \cong [0,\infty]^{\mathrm{op}}$ as quantales, and so, up to isomorphism, Δ is simply the coproduct of $[0,\infty]^{\mathrm{op}}$ with itself. This can be iterated to any cardinality, yielding a transfinite ladder of topological quantales. See [6] for more details.

4. Limits of spaces

In this section we construct limits in $\mathtt{TQ}\mathbf{Cat}_{\mathrm{cont}}$. The novelty, of course, is not in the completeness of the category but in the techniques used. The interest in these techniques is their very existence. Not all formalisms are created equal; while it is fairly straightforward to define the product topology in terms of open sets, doing so in terms of closed sets is not readily achieved. It is thus not a priori clear that an enriched-categorically flavoured construction of products exists.

It suffices to construct all small products and all equalisers. For a Q-category X and a subset $A \subseteq X$ of its objects, the full subcategory on A is the Q-category with $A(x,y) = X(x,y)$.

Theorem 4.1. *The equaliser of $f, g\colon X \to Y$ in $\mathtt{TQ}\mathbf{Cat}_{\mathrm{cont}}$ is the full subcategory of X on $E = \{x \in X \mid f(x) = g(x)\}$.*

Proof. Straightforward. □

We now turn to products, so fix a family $\{X_k\}_{k\in I}$ of objects in $\mathtt{TQ}\mathbf{Cat}_{\mathrm{cont}}$, indexed by a set I. Each X_k is a small Q_k-category where Q_k is a topological quantale. Let $Q = \coprod_{k\in I} Q_k$ be the coproduct in the category of $\mathbf{cQnt}$, equipped with the canonical injections $\iota_k\colon Q_k \to Q$. Let X be the Q-category with

$$\mathrm{ob}(X) = \prod_{k\in I} \mathrm{ob}(X_k) \quad \text{and} \quad X(x,y) = \bigvee_{k\in I} \iota_k(X_k(\pi_k(x), \pi_k(y)))$$

with the join computed in Q. It is easily seen to be a small Q-category, by extending the fact that, for elementary tensors in $Q_1 \otimes Q_2$, $(a\otimes b)\cdot(c\otimes d) = (a\cdot c)\otimes(b\cdot d)$, to the coproduct of quantales.

In the next proof, we use the following observation. A function $f: X \to Y$ is continuous at x precisely when for all $\varepsilon \lll \top$ there exists $\delta \lll \top$ such that $X(x,y) \leq \delta \implies Y(fx, fy) \leq \varepsilon$. The equivalency with the definition of continuity as given above follows from the fact that $\Downarrow(-)$ is a left adjoint.

Theorem 4.2. *With the evident projection functions $X \to X_k$, the Q-category X is the product of $\{X_k\}_{k\in I}$ in $\mathcal{TQ}\mathbf{Cat}_{\mathrm{cont}}$.*

Proof. The quantale Q is topological, and it is clear that the projection functions are continuous. It remains to establish the universal property, so assume continuous functions $f_k: Y \to X_k$ from some small R-category Y are given, where R is some topological quantale. The function $g: Y \to X$ we seek is dictated to be the unique one satisfying $\pi_k \circ g = f_k$, so we only need to show that g is continuous. For that, let $y \in Y$ and $\varepsilon \lll \top$ be given, where ε is chosen in Q. By Theorem 3.11, $\varepsilon \leq \iota_{k_1}(\varepsilon_1) \vee \cdots \vee \iota_{k_n}(\varepsilon_n)$, where $\varepsilon_{k_i} \lll \top$ holds in Q_{k_i}. By the continuity of f_{k_i}, there exists $\delta_{k_i} \lll \top$ in R such that $\delta_{k_i} \leq Y(y,y') \implies \varepsilon_{k_i} \leq X_{k_i}(f_{k_i}(y), f_{k_i}(y'))$. Let $\delta = \delta_{k_1} \vee \cdots \vee \delta_{k_n}$, which satisfies $\delta \lll \top$ since R is sierpiński. We claim that $\delta \leq Y(y,y') \implies \varepsilon \leq X(g(y), g(y'))$, namely that g is continuous at y. Assume $\delta \leq Y(y,y')$, and fix k_i. Then certainly $\delta_{k_i} \leq Y(y,y')$, and thus $\varepsilon_{k_i} \leq X_{k_i}(f_{k_i}(y), f_{k_i}(y')) = X_{k_i}(\pi_{k_i}(g(y)), \pi_{k_i}(g(y')))$. Upon applying ι_{k_i} we obtain that $\iota_{k_i}(\varepsilon_{k_i}) \leq X(g(y), g(y'))$, and as this holds for $k_1, \ldots, k_n$, it follows that $\iota_{k_1}(\varepsilon_{k_1}) \vee \cdots \vee \iota_{k_n}(\varepsilon(k_n)) \leq X(g(y), g(y'))$. By the choice of ε this inequality completes the proof. □

To conclude, let us speculate on the applicability of this last construction in data analysis. Any data analysis endevour starts with recording the data, very often as a point-cloud data structure, i.e., a metric space or, in our terminology, a small $[0,\infty]^{\mathrm{op}}$-category. Often, the data does not naturally appear in metric form and some manipulation, including simplification or arbitrary choice, is required in order to obtain a metric space. Higher dimensional data is often encoded in terms of some metric on $\mathbb{R}^n$, again possibly skewing the data. Having more quantales at hand provides more flexibility. For instance, suppose data is collected coordinate wise as ordinary metric spaces X_i, but the data analysis requires patching the coordinates together. The last theorem provides a canonical metrisation for the entire space of coordinates. It is expected to introduce less bias or distortion into the data, while ensuring the topologicity of the scenario.

References

[1] Akhvlediani, A., Clementino, M.M., Tholen, W.: On the categorical meaning of hausdorff and gromov distances, i. Topology and its Applications **157**(8), 1275–1295 (2010)

[2] Barr, M., Pedicchio, M.C.: $\mathbf{Top}^{\mathrm{op}}$ is a quasi-variety. Cahiers de topologie et géométrie différentielle catégoriques **36**(1), 3–10 (1995)

[3] Carlsson, G.: Topology and data. Bulletin of the American Mathematical Society **46**(2), 255–308 (2009)

[4] Carlsson, G., Mémoli, F.: Classifying clustering schemes. Foundations of Computational Mathematics **13**(2), 221–252 (2013)

[5] Carlsson, G., Zomorodian, A.: The theory of multidimensional persistence. Discrete & Computational Geometry **42**(1), 71–93 (2009)

[6] Cook, D.S., Weiss, I.: Diagrams of quantales and lipschitz norms. Fuzzy Sets and Systems (2021)

[7] Cook, D.S., Weiss, I.: The topology of a quantale valued metric space. Fuzzy Sets and Systems **406**, 42–57 (2021)

[8] Costa, J.P., Johansson, M.V., Škraba, P.: Variable sets over an algebra of lifetimes: a contribution of lattice theory to the study of computational topology. arXiv preprint arXiv:1409.8613 (2014)

[9] Costa, J.P., Škraba, P., Vejdemo-Johansson, M.: Aspects of an internal logic for persistence. arXiv preprint arXiv:1409.3762 (2014)

[10] Eklund, P., Gutiérrez García, J., Höhle, U., Kortelainen, J.: Semigroups in complete lattices: quantales, modules and related topics, vol. 54. Springer (2018)

[11] Fawcett, B., Wood, R.J.: Constructive complete distributivity. i. In: Mathematical Proceedings of the Cambridge Philosophical Society, vol. 107, pp. 81–89. Cambridge University Press (1990)

[12] Flagg, R.C.: Quantales and continuity spaces. Algebra Universalis **37**(3), 257–276 (1997)

[13] Gutiérrez García, J., Höhle, U., Kubiak, T.: Tensor products of complete lattices and their application in constructing quantales. Fuzzy Sets and Systems **313**, 43–60 (2017)

[14] Harrington, H.A., Otter, N., Schenck, H., Tillmann, U.: Stratifying multiparameter persistent homology. SIAM J. Appl. Algebra Geom. **3**(3), 439–471 (2019). DOI 10.1137/18M1224350

[15] Hofmann, D., Stubbe, I.: Topology from enrichment: the curious case of partial metrics. Cah. Topol. Géom. Différ. Catég. **59**(4), 307–353 (2018)

[16] Johnstone, P.T., et al.: Sketches of an Elephant: A Topos Theory Compendium: Volume 2, vol. 2. Oxford University Press (2002)

[17] Joyal, A., Tierney, M.: An extension of the Galois theory of Grothendieck, vol. 309. American Mathematical Soc. (1984)

[18] Kelly, G.M.: A unified treatment of transfinite constructions for free algebras, free monoids, colimits, associated sheaves, and so on. Bulletin of the Australian Mathematical Society **22**(1), 1–83 (1980)

[19] Kelly, G.M., Kelly, M.: Basic concepts of enriched category theory, vol. 64. CUP Archive (1982)

[20] Lawvere, F.W.: Metric spaces, generalized logic and closed categories. Reprint. Repr. Theory Appl. Categ. **2002**(1), 1–37 (2002)

[21] Palmgren, E.: From intuitionistic to point-free topology: on the foundation of homotopy theory. In: Logicism, Intuitionism, and Formalism, pp. 237–253. Springer (2009)

[22] Pedicchio, M.C., Wood, R.: Groupoidal completely distributive lattices. Journal of Pure and Applied Algebra **143**(1-3), 339–350 (1999)

[23] Rosebrugh, R., Wood, R.J.: Constructive complete distributivity iv. Applied categorical structures **2**(2), 119–144 (1994)

[24] Shmuely, Z.: The structure of galois connections. Pacific Journal of Mathematics **54**(2), 209–225 (1974)

[25] Stubbe, I.: Towards dynamic domains: totally continuous cocomplete q-categories. Theoretical Computer Science **373**(1-2), 142–160 (2007)

[26] Stubbe, I.: "hausdorff distance" via conical cocompletion. Cahiers de topologie et géométrie différentielle catégoriques **51**(1), 51–76 (2010)

[27] Stubbe, I.: An introduction to quantaloid-enriched categories. Fuzzy Sets and Systems **256**, 95–116 (2014)

[28] Tholen, W.: Met-like categories amongst concrete topological categories. Applied Categorical Structures **26**(5), 1095–1111 (2018)

[29] Vejdemo-Johansson, M.: Sketches of a platypus: a survey of persistent homology and its algebraic foundations. In: Algebraic topology: applications and new directions. Stanford symposium on algebraic topology: applications and new directions, Stanford University, Stanford, CA, USA, July 23–27, 2012. Proceedings, pp. 295–319. Providence, RI: American Mathematical Society (AMS) (2014)

[30] Velebil, J., Kurz, A., Balan, A.: Extending set functors to generalised metric spaces. Logical Methods in Computer Science **15** (2019)

[31] Wagner, K.R.: Liminf convergence in ω-categories. Theoretical Computer Science **184**(1-2), 61–104 (1997)

[32] Weiss, I.: A note on the metrizability of spaces. Algebra universalis **73**(2), 179–182 (2015)

[33] Weiss, I.: Value semigroups, value quantales, and positivity domains. Journal of Pure and Applied Algebra **223**(2), 844–866 (2019)

School of Mathematics and Physics
University Portsmouth
Lion Gate Building
PO1 3HF
England
ittay.weiss@port.ac.uk