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DOUBLE GROUPOIDS AND POSTNIKOV INVARIANTS

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Résumé. Dans cet article, nous prouvons un théorème de classification pour les groupoïdes doubles (satisfaisant à une condition de remplissage supplémentaire, tout à fait naturelle) au moyen de troisièmes classes de cohomologie de groupoïdes. Dans une seconde étape, indépendante, nous montrons que la classe de cohomologie associée à un groupoïde double coïncide avec l'unique k-invariant non trivial de sa réalisation géométrique.

Abstract. In this paper, we prove a classification theorem for double groupoids (satisfying an extra, quite natural, filling condition) by means of third cohomology classes of groupoids. In a second, independent, step, we prove that the cohomology class associated to a double groupoid coincides with the unique non-trivial k-invariant of its geometric realization.

Keywords. Double groupoid, Cohomology of groupoids, Postnikov invariant, weak equivalence, homotopy type.

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Introduction and summary

Double groupoids (groupoid objects in the category of groupoids) go back to Ehresmann [14, 15, 16]. Roughly, they consist of objects, two kinds of morphisms between them, horizontal and vertical, and boxes whose boundaries are squares with morphisms as edges, usually depicted

$$\begin{tikzcd} d & b \arrow[l, "f"'] \\ c \arrow[u, "y"] & a \arrow[l, "g"] \arrow[u, "x"'] \arrow[ul, phantom, "\alpha"] \end{tikzcd},$$

together with horizontal and vertical composition of morphisms and boxes giving compatible groupoid structures and obeying middle four interchange on boxes. The double groupoids we encounter in practice, and certainly in this work, are small and satisfy a natural *filling condition*: Any filling problem

$$\begin{tikzcd} d & \cdot \arrow[l, dotted] \\ c \arrow[u, "y"] & a \arrow[l, "g"] \arrow[u, dotted] \arrow[ul, phantom, "\exists ?"] \end{tikzcd},$$

finds a solution in the double groupoid. This filling condition on double groupoids is often assumed in the case of double groupoids arising in different areas of mathematics, such as in weak Hopf algebra theory or in differential geometry (see, for instance, Andruskiewitsch and Natale [1] and Mackenzie [23]), and it is satisfied for those double groupoids that have emerged with an interest in algebraic topology, mainly thanks to the work of Brown, Higgins, Spencer, *et al.*, where the connection of double groupoids with crossed modules and a higher Seifert-van Kampen Theory has been established (see the surveys by Brown [3, 4, 5] and the references given there). Thus, the filling condition is easily proven for edge symmetric double groupoids (also called special double groupoids) with connections (see Brown and Higgins [6], Brown and Spencer [7], Brown, Hardie, Kamps and Porter [8] and Brown, Kamps and Porter [9]), for double groupoid objects in the category of groups (also termed cat^2-groups by Loday [22], see also Porter [25] and Bullejos, Cegarra and Duskin [10]), or, for example, for 2-groupoids (regarded as double groupoids where one of the side groupoids of morphisms is discrete (see for instance Moerdijk and Svensson [24] and Hardie, Kamps and Kieboom [20]).

Every (small) double groupoid $\mathcal{G}$ has a geometric realization, which is the topological space defined by first taking the double nerve $\mathrm{NN}\mathcal{G}$, which is a bisimplicial set, and then realizing geometrically the diagonal to obtain a space: $|\mathcal{G}| = |\triangle\, \mathrm{NN}\mathcal{G}|$. The usual definition of the homotopy invariants of a double groupoid $\mathcal{G}$ involves only its underlying topological space

$|\mathcal{G}|$ and does not take into account the algebraic structure. Our main goal in this paper is *to give a combinatorial definition of the* (unique) *Postnikov invariant of a double groupoid with the filling condition using only its algebraic structure*. Recall that a (2-dimensional) *Postnikov system* is a triple $(P, \mathcal{A}, \mathbf{k})$, where P is a groupoid, $\mathcal{A}$ is an abelian group valued functor on P, and $\mathbf{k} \in H^3(P, \mathcal{A})$ is a three-cohomology class of P with coefficients in $\mathcal{A}$. Our definitions and constructions here are suggested by previous work of the author and collaborators; particularly by the results in [11], where we address the homotopy types realized from double groupoids satisfying the filling condition. They are all the (not necessarily path-connected) homotopy 2-types, that is, the homotopy types of all CW-complexes whose homotopy groups at any base point vanish in degree 3 and higher.

After Section 1, where we briefly fix some notational conventions on double groupoids, in Sections 2 and 3, we review several needed definitions and results on the (algebraically defined) fundamental groupoid $\Pi\mathcal{G}$ and the homotopy groups $\pi_2(\mathcal{G}, a)$ of a double groupoid $\mathcal{G}$ satisfying the filling condition. Section 4 contains the new definition of the Postnikov invariant of such a double groupoid, which is the equivalence class of a Postnikov system $(\Pi\mathcal{G}, \pi_2\mathcal{G}, \mathbf{k}\mathcal{G})$ where $\mathbf{k}\mathcal{G} \in H^3(\Pi\mathcal{G}, \pi_2\mathcal{G})$ is a certain characteristic cohomology class of the fundamental groupoid of $\mathcal{G}$ with coefficients in the abelian group valued functor on $\Pi\mathcal{G}$ which assigns the homotopy group $\pi_2(\mathcal{G}, a)$ to each object a of $\mathcal{G}$. In Section 5, we mainly state and prove the expected classification result:

"The assignment $\mathcal{G} \mapsto (\Pi\mathcal{G}, \pi_2\mathcal{G}, \mathbf{k}\mathcal{G})$ induces a bijective correspondence between weak equivalence classes of double groupoids satisfying the filling condition and equivalence classes of Postnikov systems."

Finally, in Section 6 we prove

"The Postnikov invariant of a double groupoid $\mathcal{G}$ with the filling condition is equivalent to the Postnikov invariant of its geometric realization $|\mathcal{G}|$."

As a bonus, we find a new proof of the fact that the assignment $\mathcal{G} \mapsto |\mathcal{G}|$ induces a bijective correspondence between weak-equivalence classes of double groupoids satisfying the filling condition and homotopy 2-types.

1. Some conventions on double groupoids

The notion of double groupoid is well-known, we just specify in this preliminary section some basic terminology and notational conventions. We will work only with small double groupoids, so that in a double groupoid $\mathcal{G}$ we have a set of objects (usually denoted by $a, b, c, \ldots$), horizontal morphisms between them ($f, g, h, \ldots$), vertical morphisms between them ($x, y, z, \ldots$), both with composition written by juxtaposition, and boxes ($\alpha, \beta, \gamma, \ldots$), usually depicted as

$$\begin{tikzcd} d & b \arrow[l, "f"'] \\ c \arrow[u, "y"] & a \arrow[l, "g"] \arrow[u, "x"'] \arrow[ul, phantom, "\alpha"] \end{tikzcd} \tag{1}$$

where the horizontal morphisms f and g are, respectively, its vertical target and source and the vertical morphisms y and x are its respective horizontal target and source. The horizontal composition of boxes is denoted by the symbol $\circ_{\mathrm{h}}$:

$$\begin{tikzcd} \cdot & \cdot \arrow[l, "f'"'] & \cdot \arrow[l, "f'"'] \\ \cdot \arrow[u, "z"] & \cdot \arrow[l, "g'"] \arrow[u] \arrow[ul, phantom, "\alpha'"] & \cdot \arrow[l, "g"] \arrow[u, "x"'] \arrow[ul, phantom, "\alpha"] \end{tikzcd} \quad \mapsto \quad \begin{tikzcd} \cdot & \cdot \arrow[l, "f'f"'] \\ \cdot \arrow[u, "z"] & \cdot \arrow[l, "g'g"] \arrow[u, "x"'] \arrow[ul, phantom, "\alpha'\circ_{\mathrm{h}}\alpha"] \end{tikzcd}$$

and, similarly, the vertical composition of boxes is denoted by the symbol $\circ_{\mathrm{v}}$:

$$\begin{tikzcd} \cdot & \cdot \arrow[l, "f"'] \\ \cdot \arrow[u, "y"] & \cdot \arrow[l] \arrow[u, "x"'] \arrow[ul, phantom, "\alpha"] \\ \cdot \arrow[u, "y'"] & \cdot \arrow[l, "h"] \arrow[u, "x'"'] \arrow[ul, phantom, "\alpha'"] \end{tikzcd} \quad \mapsto \quad \begin{tikzcd} \cdot & \cdot \arrow[l, "f"'] \\ \cdot \arrow[u, "yy'"] & \cdot \arrow[l, "h"] \arrow[u, "xx'"'] \arrow[ul, phantom, "\alpha\circ_{\mathrm{v}}\alpha'"] \end{tikzcd}$$

Horizontal and vertical identities on objects and morphisms are respectively denoted by $\mathrm{I}^{\mathrm{h}}a$, $\mathrm{I}^{\mathrm{v}}a$, $\mathrm{I}^{\mathrm{h}}x$, $\mathrm{I}^{\mathrm{v}}f$, and $\mathrm{I}a := \mathrm{I}^{\mathrm{v}}\mathrm{I}^{\mathrm{h}}a = \mathrm{I}^{\mathrm{h}}\mathrm{I}^{\mathrm{v}}a$, depicted as

$$a = a \qquad \begin{array}{c} a \\ \| \\ a \end{array} \qquad \begin{tikzcd} \cdot \arrow[r, equal] & \cdot \\ \cdot \arrow[u, "x"] \arrow[r, equal] & \cdot \arrow[u, "x"'] \arrow[ul, phantom, "\mathrm{I}^{\mathrm{h}}x"] \end{tikzcd} \qquad \begin{tikzcd} \cdot & \cdot \arrow[l, "f"'] \\ \cdot \arrow[u, equal] & \cdot \arrow[l, "f"] \arrow[u, equal] \arrow[ul, phantom, "\mathrm{I}^{\mathrm{v}}f"] \end{tikzcd} \qquad \begin{tikzcd} a \arrow[r, equal] & a \\ a \arrow[u, equal] \arrow[r, equal] & a \arrow[u, equal] \arrow[ul, phantom, "\mathrm{I}a"] \end{tikzcd}$$

and horizontal and vertical inverses of boxes are respectively denoted by $\alpha^{-\mathrm{h}}$, $\alpha^{-\mathrm{v}}$, and $\alpha^{-\mathrm{hv}} := (\alpha^{-\mathrm{h}})^{-\mathrm{v}} = (\alpha^{-\mathrm{v}})^{-\mathrm{h}}$; that is,

$$
\begin{tikzcd}
\cdot & \cdot \arrow[l, "f^{-1}"'] \\
\cdot \arrow[u, "x"] & \cdot \arrow[l, "g^{-1}"] \arrow[u, "y"'] \arrow[ul, phantom, "\alpha^{-\mathrm{h}}"]
\end{tikzcd}
\qquad
\begin{tikzcd}
\cdot & \cdot \arrow[l, "g"'] \\
\cdot \arrow[u, "y^{-1}"] & \cdot \arrow[l, "f"] \arrow[u, "x^{-1}"'] \arrow[ul, phantom, "\alpha^{-\mathrm{v}}"]
\end{tikzcd}
\qquad
\begin{tikzcd}
\cdot & \cdot \arrow[l, "g^{-1}"'] \\
\cdot \arrow[u, "x^{-1}"] & \cdot \arrow[l, "f^{-1}"] \arrow[u, "y^{-1}"'] \arrow[ul, phantom, "\alpha^{-\mathrm{hv}}"]
\end{tikzcd}
$$

We will use several times the coherence theorem by Dawson and Paré [13, Theorem 1.2], which assures us that *if a compatible arrangement of boxes in a double groupoid is composable in two different ways, the resulting pasted boxes are equal*. Throughout the paper, an equality between pasting diagrams of boxes in a double groupoid means that the resulting pasted boxes are the same.

The double groupoids we are interested in satisfy the so-called filling condition: *Any filling problem*

$$
\begin{tikzcd}
\cdot & \cdot \arrow[l, dotted] \\
\cdot \arrow[u, "y"] & \cdot \arrow[l, "g"] \arrow[u, dotted] \arrow[ul, phantom, "\exists ?"]
\end{tikzcd},
$$

has a solution; that is, for any horizontal morphism g and any vertical morphism y such that the source of y coincides with the target of g, there is a box whose vertical source is g and whose horizontal target is y. This condition is more symmetric than it appears thanks to the following lemma by Andruskiewitsch and Natale [1, Lemma 1.12].

Lemma 1.1. *A double groupoid satisfies the filling condition if and only if any filling problem such as the one below has a solution.*

$$
\begin{tikzcd}
\cdot & \cdot \arrow[l, "f"'] \\
\cdot \arrow[u, dotted] & \cdot \arrow[l, dotted] \arrow[u, "x"'] \arrow[ul, phantom, "\exists ?"]
\end{tikzcd},
\qquad
\begin{tikzcd}
\cdot & \cdot \arrow[l, dotted] \\
\cdot \arrow[u, dotted] & \cdot \arrow[l, "g"] \arrow[u, "x"'] \arrow[ul, phantom, "\exists ?"]
\end{tikzcd},
\qquad
\begin{tikzcd}
\cdot & \cdot \arrow[l, "f"'] \\
\cdot \arrow[u, "y"] & \cdot \arrow[l, dotted] \arrow[u, dotted] \arrow[ul, phantom, "\exists ?"]
\end{tikzcd},
$$

Throughout the paper we make the assumption that the double groupoids we work with are small and satisfy the filling condition.

2. The fundamental groupoid $\Pi\mathcal{G}$

Let $\mathcal{G}$ be a double groupoid. If a_0, a_1 are objects of $\mathcal{G}$, we define a *path* in $\mathcal{G}$ from a_0 to a_1 to be a diagram (f, b, x) of the form

$$\begin{tikzcd} a_1 & b \arrow[l, "f"'] \\ & a_0 \arrow[u, "x"'] \end{tikzcd}$$

that is, where b is an object, f a horizontal morphism from b to a_1, and x a vertical morphism from a_0 to b. Throughout the paper, we identify paths in $\mathcal{G}$ of the form

$$\begin{tikzcd} a_1 & a_0 \arrow[l, "f"'] \arrow[d, equal] \\ & a_0 \end{tikzcd} \qquad \begin{tikzcd} a_1 \arrow[r, equal] & a_1 \\ & a_0 \arrow[u, "x"'] \end{tikzcd}$$

with the morphisms f and x respectively; that is, we write

$$f = (f, a_0, \mathrm{I}^{\mathrm{v}} a_0), \qquad x = (\mathrm{I}^{\mathrm{h}} a_1, a_1, x).$$

If (f, b, x) and (g, c, y) are two paths from a_0 to a_1, then we say that (f, b, x) is *homotopic* to (g, c, y), denoted by $(f, b, x) \simeq (g, c, y)$, if there is a box α in $\mathcal{G}$ of the form

$$\begin{tikzcd} b \arrow[d, equal] \arrow[dr, phantom, "\alpha"] & c \arrow[l, "f^{-1}g"'] \\ b \arrow[r, equal] & b \arrow[u, "yx^{-1}"'] \end{tikzcd} \tag{2}$$

that is, whose horizontal target and vertical source are identities, its horizontal source is yx^{-1}, and its its vertical target is $f^{-1}g$. We call such a box a *homotopy*, and we often write $\alpha : (f, b, x) \simeq (g, c, y)$ whenever we wish to display the homotopy.

Lemma 2.1. *Homotopy is an equivalence relation on the set of paths in $\mathcal{G}$ from a_0 to a_1.*

Proof. Reflexivity: For any path (f, b, x), clearly $\mathrm{I}b : (f, b, x) \simeq (f, b, x)$. *Symmetry*: If $\alpha : (f, b, x) \simeq (g, c, y)$ is a homotopy, then the pasted box of

$$\begin{tikzcd} c \arrow[d, equal] \arrow[dr, phantom, "\mathrm{I}^{\mathrm{v}}(g^{-1}f)"] & b \arrow[l, "g^{-1}f"'] \arrow[r, equal] \arrow[d, equal] \arrow[dr, phantom, "\alpha^{-\mathrm{v}}"] & b \\ c & b \arrow[l, "g^{-1}f"] & c \arrow[l, "f^{-1}g"] \arrow[u, "xy^{-1}"'] \end{tikzcd}$$

is a homotopy $(g, c, y) \simeq (f, b, x)$.
Transitivity: Assume that $\alpha : (f, b, x) \simeq (g, c, y)$ and $\beta : (g, c, y) \simeq (h, d, z)$. Then, we find a homotopy $\gamma : (f, b, x) \simeq (h, d, z)$ by pasting the diagram of boxes

$$\begin{tikzcd}
b \arrow[d, equal] \arrow[dr, phantom, "\mathrm{I}^{\mathrm{v}}(f^{-1}g)"] & c \arrow[l, "f^{-1}g"'] \arrow[d, equal] \arrow[dr, phantom, "\beta"] & d \arrow[l, "g^{-1}h"'] \\
b \arrow[d, equal] \arrow[drr, phantom, "\alpha"] & c \arrow[l, "f^{-1}g"] \arrow[r, equal] & c \arrow[u, "zy^{-1}"'] \\
b \arrow[rr, equal] & & c \arrow[u, "yx^{-1}"']
\end{tikzcd}$$

□

Let $[f, b, x]$ denote the homotopy class of a path (f, b, x) in $\mathcal{G}$.

We define the *fundamental groupoid* $\Pi\mathcal{G}$ of the double groupoid $\mathcal{G}$ to be a category having as objects all the objects of $\mathcal{G}$. An arrow in $\Pi\mathcal{G}$ from an object a_0 to an object a_1 is the homotopy class of a path in $\mathcal{G}$ from a_0 to a_1. Composition in $\Pi\mathcal{G}$ is as follows:

For each morphism in the fundamental groupoid $\rho \in \Pi\mathcal{G}(a_0, a_1)$, let us choose a representative path (f_ρ, b_ρ, x_ρ) of ρ,

$$\begin{tikzcd}
a_1 & b_\rho \arrow[l, "f_\rho"'] \\
& a_0, \arrow[u, "x_\rho"']
\end{tikzcd} \tag{3}$$

that is, such that $\rho = [f_\rho, b_\rho, x_\rho]$. If $a_2 \xleftarrow{\psi} a_1 \xleftarrow{\rho} a_0$ are any two composable morphisms in $\Pi\mathcal{G}$, by the filling condition on $\mathcal{G}$, we can select a box θ in $\mathcal{G}$ whose horizontal target is x_ψ and whose vertical source is f_ρ. Thus, we have a diagram in $\mathcal{G}$ of the form

$$\begin{tikzcd}
a_2 & b_\psi \arrow[l, "f_\psi"'] \arrow[dr, phantom, "\theta"] & b \arrow[l, "f"'] \\
& a_1 \arrow[u, "x_\psi"] & b_\rho \arrow[l, "f_\rho"] \arrow[u, "x"'] \\
& & a_0 \arrow[u, "x_\rho"']
\end{tikzcd} \tag{4}$$

and we define the composite $\psi\rho = [f_\psi f, b, x x_\rho] \in \Pi\mathcal{G}(a_0, a_2)$.

Lemma 2.2. *The composite $\psi\rho$ is well-defined, that is, it is independent of the choices of representative paths of ρ and ψ and of the choice of θ in* (4).

Proof. Suppose that $\alpha_\rho : (f_\rho, b_\rho, x_\rho) \simeq (g_\rho, c_\rho, y_\rho)$ and $\alpha_\psi : (f_\psi, b_\psi, x_\psi) \simeq (g_\psi, c_\psi, y_\psi)$ are homotopies and that we have selected boxes θ and θ' as in the diagrams below.

$$\begin{tikzcd}
a_2 & b_\psi \arrow[l, "f_\psi"'] & b \arrow[l, "f"'] \\
& a_1 \arrow[u, "x_\psi"] \arrow[ur, phantom, "\theta"] & b_\rho \arrow[l, "f_\rho"] \arrow[u, "x"'] \\
& & a_0 \arrow[u, "x_\rho"']
\end{tikzcd}
\qquad
\begin{tikzcd}
a_2 & c_\psi \arrow[l, "g_\psi"'] & c \arrow[l, "g"'] \\
& a_1 \arrow[u, "y_\psi"] \arrow[ur, phantom, "\theta'"] & c_\rho \arrow[l, "g_\rho"] \arrow[u, "y"'] \\
& & a_0 \arrow[u, "y_\rho"']
\end{tikzcd}$$

Then, we get a homotopy $\alpha : (f_\psi f, b, x x_\rho) \simeq (g_\psi g, c, y y_\rho)$ by pasting the diagram

$$\begin{tikzcd}
b & b_2 \arrow[l, "f^{-1}"'] & c_\psi \arrow[l, "f_\psi^{-1}g_\psi"'] & c \arrow[l, "g"'] \\
\arrow[r, phantom, "\theta^{-\mathrm{h}}"] & b_\psi \arrow[u, equal] \arrow[r, equal] \arrow[ur, phantom, "\alpha_\psi"] & b_\psi \arrow[u, "y_\psi x_\psi^{-1}"'] \arrow[r, phantom, "\theta'"] & {} \\
b_\rho \arrow[uu, "x"] & a_2 \arrow[l, "f_\rho^{-1}"] \arrow[u, "x_\psi"] \arrow[r, equal] \arrow[ur, phantom, "\mathrm{I}^{\mathrm{h}}x_\psi"] & a_2 \arrow[u, "x_\psi"'] & c_\rho \arrow[l, "g_\rho"] \arrow[uu, "y"'] \\
b_\rho \arrow[u, equal] \arrow[rrr, equal] \arrow[urrr, phantom, "\alpha_\rho"] & & & b_\rho \arrow[u, "y_\rho x_\rho^{-1}"'] \\
b \arrow[u, "x^{-1}"] \arrow[rrr, equal] \arrow[urrr, phantom, "\mathrm{I}^{\mathrm{h}}x^{-1}"] & & & b \arrow[u, "x^{-1}"']
\end{tikzcd}$$

□

For each object a of $\mathcal{G}$, let $id_a = [\mathrm{I}^{\mathrm{h}}a, a, \mathrm{I}^{\mathrm{v}}a] \in \Pi\mathcal{G}(a, a)$.

Theorem 2.3. *With these definitions, $\Pi\mathcal{G}$ is a groupoid.*

Proof. Identity: For every arrow $\rho = [f_\rho, b_\rho, x_\rho] \in \Pi\mathcal{G}(a_0, a_1)$, the diagrams in $\mathcal{G}$

$$\begin{tikzcd}
a_1 \arrow[r, equal] & a_1 & b_\rho \arrow[l, "f_\rho"'] \\
& a_1 \arrow[u, equal] \arrow[ur, phantom, "\mathrm{I}^{\mathrm{v}}f_\rho"] & b_\rho \arrow[l, "f_\rho"] \arrow[u, equal] \\
& & a_0 \arrow[u, "x_\rho"']
\end{tikzcd}
\qquad
\begin{tikzcd}
a_1 & b_\rho \arrow[l, "f_\rho"'] \arrow[r, equal] & b_\rho \\
& a_0 \arrow[u, "x_\rho"] \arrow[r, equal] \arrow[ur, phantom, "\mathrm{I}^{\mathrm{h}}x_\rho"] & a_0 \arrow[u, "x_\rho"'] \\
& & a_0 \arrow[u, equal]
\end{tikzcd}$$

show that $id_{a_1}\rho = \rho = \rho\, id_{a_0}$.

Associativity: if $a_3 \overset{\phi}{\leftarrow} a_2 \overset{\psi}{\leftarrow} a_1 \overset{\rho}{\leftarrow} a_0$ are any three composable morphisms in $\Pi\mathcal{G}$, we can choose boxes θ, θ' and θ'' as in the diagram

$$\begin{tikzcd}
a_3 & b_\phi \arrow[l, "f_\phi"'] & b' \arrow[l, "f'"'] & b'' \arrow[l, "f''"'] \\
 & a_2 \arrow[u, "x_\phi"] \arrow[ur, phantom, "\theta'"] & b_\psi \arrow[l, "f_\psi"] \arrow[u, "x'"'] \arrow[ur, phantom, "\theta''"] & b \arrow[l, "f"] \arrow[u, "x''"'] \\
 & & a_1 \arrow[u, "x_2"] \arrow[ur, phantom, "\theta"] & b_\rho \arrow[l, "f_\rho"] \arrow[u, "x"'] \\
 & & & a_0 \arrow[u, "x_\rho"']
\end{tikzcd}$$

whence,

$$(\phi\psi)\rho = [f_\phi f', b', x'x_\psi]\, \rho = [f_\phi f' f'', b'', x''x'x_\rho] = \phi\, [f_\psi f, b, x_\rho] = \phi(\psi\rho).$$

Inverse: For any morphism $\rho \in \Pi\mathcal{G}(a_0, a_1)$, we can select a box γ in $\mathcal{G}$ of the form

$$\begin{tikzcd}
a_0 & b \arrow[l, "f"'] \\
b_\rho \arrow[u, "x_\rho^{-1}"] \arrow[ur, phantom, "\gamma"] & a_1 \arrow[l, "f_\rho^{-1}"] \arrow[u, "x"']
\end{tikzcd}$$

and construct $\rho^{-1} = [f, b, x] \in \mathcal{G}(a_1, a_0)$. From the diagrams in $\mathcal{G}$

$$\begin{tikzcd}
a_0 & b \arrow[l, "f"'] & a_0 \arrow[l, "f^{-1}"'] \\
 & a_1 \arrow[u, "x"] \arrow[ur, phantom, "\gamma^{-\mathrm{h}}"] & b_\rho \arrow[l, "f_\rho"] \arrow[u, "x_\rho^{-1}"'] \\
 & & a_0 \arrow[u, "x_\rho"']
\end{tikzcd}
\qquad
\begin{tikzcd}
a_1 & b_\rho \arrow[l, "f_\rho"'] & a_1 \arrow[l, "f_\rho^{-1}"'] \\
 & a_0 \arrow[u, "x_\rho"] \arrow[ur, phantom, "\gamma^{-\mathrm{v}}"] & b \arrow[l, "f"] \arrow[u, "x^{-1}"'] \\
 & & a_1 \arrow[u, "x"']
\end{tikzcd}$$

it follows that $\rho^{-1}\rho = id_{a_0}$ and $\rho\rho^{-1} = id_{a_1}$. $\square$

Lemma 2.4. (i) *For any two composable horizontal morphisms*

$$a_2 \overset{g}{\leftarrow} a_1 \overset{f}{\leftarrow} a_0$$

and for any two composable vertical morphisms

$$\begin{tikzcd} a_2 \\ a_1 \arrow[u, "y"'] \\ a_0 \arrow[u, "x"'] \end{tikzcd}$$

the equalities $[g][f] = [gf]$ *and* $[y][x] = [yx]$ *hold in* $\Pi\mathcal{G}$.

(ii) *For any path* (f, b, x) *in* $\mathcal{G}$, $[f, b, x] = [f][x]$.

(iii) *The filling problem in* $\mathcal{G}$

$$\begin{tikzcd} a_1 & b \arrow[l, "f"'] \\ c \arrow[u, "y"] \arrow[ur, phantom, "\exists ?"] & a_0 \arrow[l, "g"] \arrow[u, "x"'] \end{tikzcd}$$

has a solution if and only if $[y][g] = [f][x]$ *in* $\Pi\mathcal{G}$.

Proof. (i) follows from the existence of the first two diagrams below and (ii) by the third one.

$$\begin{tikzcd} a_2 & a_1 \arrow[l, "g"'] \arrow[d, equal] \arrow[dr, phantom, "\mathrm{I}^{\mathrm{v}}f"] & a_0 \arrow[l, "f"'] \arrow[d, equal] \\ & a_1 & a_0 \arrow[l, "f"] \arrow[d, equal] \\ & & a_0 \end{tikzcd}
\quad
\begin{tikzcd} a_2 \arrow[r, equal] & a_2 \arrow[r, equal] & a_2 \\ & a_1 \arrow[u, "y"] \arrow[r, equal] \arrow[ur, phantom, "\mathrm{I}^{\mathrm{h}}y"] & a_1 \arrow[u, "y"'] \\ & & a_0 \arrow[u, "x"'] \end{tikzcd}
\quad
\begin{tikzcd} a_1 & b \arrow[l, "f"'] \arrow[r, equal] \arrow[d, equal] \arrow[dr, phantom, "\mathrm{I}b"] & b \arrow[d, equal] \\ & b \arrow[r, equal] & b \\ & & a_0 \arrow[u, "x"'] \end{tikzcd}$$

For (iii), suppose first θ is any solution to the given filling problem. Then, the diagram

$$\begin{tikzcd} a_1 \arrow[r, equal] & a_1 & b \arrow[l, "f"'] \\ & c \arrow[u, "y"] \arrow[ur, phantom, "\theta"] & a \arrow[l, "g"] \arrow[u, "x"'] \arrow[d, equal] \\ & & a_0 \end{tikzcd}$$

shows that $[y][g] = [f, b, x] \overset{(ii)}{=} [f][x]$. Conversely, assume that $[y][x] = [f][x] \overset{(ii)}{=} [f, b, x]$. By the filling condition on $\mathcal{G}$, we can select a box θ' of the form

$$\begin{tikzcd} a_1 & b' \arrow[l, "f'"'] \\ c \arrow[u, "y"] \arrow[ur, phantom, "\theta'"] & a_0 \arrow[l, "g"] \arrow[u, "x'"'] \end{tikzcd}$$

whence, by the already proven part, $[y][g] = [f'][x'] = [f', b', x']$. It follows that $[f, b, x] = [f', b', x']$, and therefore there is a homotopy $\alpha : (f', b', x') \simeq (f, b, x)$ which gives us the solution θ that we are seeking for the filling problem by pasting the diagram

$$\begin{tikzcd} a_1 & b' \arrow[l, "f'"'] & b \arrow[l, "f'^{-1}f"'] \\ d \arrow[u, equal] \arrow[ur, phantom, "\mathrm{I}^{\mathrm{v}}f'"] & b' \arrow[l, "f'"'] \arrow[u, equal] \arrow[r, equal] \arrow[ur, phantom, "\alpha"] & b' \arrow[u, "xx'^{-1}"'] \\ c \arrow[u, "y"] \arrow[ur, phantom, "\theta'"] & a_0 \arrow[l, "g"] \arrow[u, "x'"'] \arrow[r, equal] \arrow[ur, phantom, "\mathrm{I}^{\mathrm{h}}x'"] & a_0 \arrow[u, "x'"'] \end{tikzcd}$$

□

3. The functor $\pi_2\mathcal{G} : \Pi\mathcal{G} \to \mathsf{Ab}$

For each object a of $\mathcal{G}$, let $\pi_2(\mathcal{G}, a)$ denote the set of all boxes σ in $\mathcal{G}$ of the form

$$\begin{tikzcd} a \arrow[r, equal] & a \\ a \arrow[u, equal] \arrow[r, equal] \arrow[ur, phantom, "\sigma"] & a \arrow[u, equal] \end{tikzcd}$$

that is, whose horizontal source and target are both $\mathrm{I}^{\mathrm{v}}a$, the vertical identity of a, and whose vertical source and target are both $\mathrm{I}^{\mathrm{h}}a$, the horizontal identity of a. By the general Eckman-Hilton argument, the interchange law on $\mathcal{G}$ implies that both operations $\circ_{\mathrm{h}}$ and $\circ_{\mathrm{v}}$ on $\pi_2(\mathcal{G}, a)$ coincide and are commutative. Thus, $\pi_2(\mathcal{G}, a)$ is an abelian group with addition

$$\sigma + \tau := \sigma \circ_{\mathrm{h}} \tau = \sigma \circ_{\mathrm{v}} \tau,$$

zero $0 := \mathrm{I}a$, and opposites $-\sigma := \sigma^{-\mathrm{v}} = \sigma^{-\mathrm{h}}$.

The assignment $a \mapsto \pi_2(\mathcal{G}, a)$ is the function on objects of a functor $\pi_2\mathcal{G} : \Pi\mathcal{G} \to \mathsf{Ab}$, which acts on morphism as follows. There is an abelian group valued functor on the groupoid of horizontal morphisms which assigns to each horizontal morphism $f : a_0 \to a_1$ the homomorphism

$$f_* : \pi_2(\mathcal{G}, a_0) \to \pi_2(\mathcal{G}, a_1)$$

defined by $f_*\sigma = \mathrm{I}^{\mathrm{v}}f \circ_{\mathrm{h}} \sigma \circ_{\mathrm{h}} \mathrm{I}^{\mathrm{v}}f^{-1}$,

$$\begin{tikzcd} a_0 \arrow[r, equal] \arrow[d, equal] \arrow[dr, phantom, "\sigma"] & a_0 \arrow[d, equal] \\ a_0 \arrow[r, equal] & a_0 \end{tikzcd} \quad \overset{f_*}{\mapsto} \quad \begin{tikzcd} a_1 \arrow[d, equal] \arrow[dr, phantom, "\mathrm{I}^{\mathrm{v}}f"] & a_0 \arrow[l, "f"'] \arrow[r, equal] \arrow[d, equal] \arrow[dr, phantom, "\sigma"] & a_0 \arrow[d, equal] \arrow[dr, phantom, "\mathrm{I}^{\mathrm{v}}f^{-1}"] & a_1 \arrow[l, "f^{-1}"'] \arrow[d, equal] \\ a_1 & a_0 \arrow[l, "f"] \arrow[r, equal] & a_0 & a_1 \arrow[l, "f^{-1}"] \end{tikzcd}$$

and, similarly, there is an abelian group valued functor on the groupoid of vertical morphisms which assigns to each vertical morphism $x : a_0 \to a_1$ the homomorphism

$$x_* : \pi_2(\mathcal{G}, a_0) \to \pi_2(\mathcal{G}, a_1)$$

defined by $x_*\sigma = \mathrm{I}^{\mathrm{h}}x \circ_{\mathrm{v}} \sigma \circ_{\mathrm{v}} \mathrm{I}^{\mathrm{h}}x^{-1}$,

$$\begin{tikzcd} a_0 \arrow[r, equal] \arrow[d, equal] \arrow[dr, phantom, "\sigma"] & a_0 \arrow[d, equal] \\ a_0 \arrow[r, equal] & a_0 \end{tikzcd} \quad \overset{x_*}{\mapsto} \quad \begin{tikzcd} a_1 \arrow[r, equal] & a_1 \\ a_0 \arrow[u, "x"] \arrow[r, equal] \arrow[d, equal] \arrow[ur, phantom, "\mathrm{I}^{\mathrm{h}}x"] \arrow[dr, phantom, "\sigma"] & a_0 \arrow[u, "x"'] \arrow[d, equal] \\ a_0 \arrow[r, equal] & a_0 \\ a_1 \arrow[u, "x^{-1}"] \arrow[r, equal] \arrow[ur, phantom, "\mathrm{I}^{\mathrm{h}}x^{-1}"] & a_1 \arrow[u, "x^{-1}"'] \end{tikzcd}$$

Lemma 3.1. *If*

$$\begin{tikzcd} a_1 & b \arrow[l, "f"'] \\ c \arrow[u, "y"] \arrow[ur, phantom, "\theta"] & a_0 \arrow[l, "g"] \arrow[u, "x"'] \end{tikzcd}$$

is any box in $\mathcal{G}$, then the diagram below commutes.

$$\begin{tikzcd} \pi_2(\mathcal{G}, a_1) & \pi_2(\mathcal{G}, b) \arrow[l, "f_*"'] \\ \pi_2(\mathcal{G}, c) \arrow[u, "y_*"] & \pi_2(\mathcal{G}, a_0) \arrow[l, "g_*"] \arrow[u, "x_*"'] \end{tikzcd}$$

Proof. Let us consider, for any $\sigma \in \pi_2(\mathcal{G}, a_0)$, the following pasting diagram

$$\begin{tikzcd}
a_1 & b \arrow[l, "f"'] \arrow[r, equal] & b & a_1 \arrow[l, "f^{-1}"'] \\
c \arrow[u, "y"] \arrow[ur, phantom, "\theta"] & a_0 \arrow[l, "g"'] \arrow[u, "x"] \arrow[r, equal] \arrow[ur, phantom, "\mathrm{I}^{\mathrm{h}}x"] & a_0 \arrow[u, "x"] \arrow[ur, phantom, "\theta^{-\mathrm{h}}"] & c \arrow[l, "g^{-1}"'] \arrow[u, "y"'] \\
c \arrow[u, equal] \arrow[ur, phantom, "\mathrm{I}^{\mathrm{v}}g"] & a_0 \arrow[l, "g"'] \arrow[u, equal] \arrow[r, equal] \arrow[ur, phantom, "\sigma"] & a_0 \arrow[u, equal] \arrow[ur, phantom, "\mathrm{I}^{\mathrm{v}}g^{-1}"] & c \arrow[l, "g^{-1}"'] \arrow[u, equal] \\
a_1 \arrow[u, "y^{-1}"] \arrow[ur, phantom, "\theta^{-\mathrm{v}}"] & b \arrow[l, "f"] \arrow[u, "x^{-1}"] \arrow[r, equal] \arrow[ur, phantom, "\mathrm{I}^{\mathrm{h}}x^{-1}"] & b \arrow[u, "x^{-1}"'] \arrow[ur, phantom, "\theta^{-\mathrm{hv}}"] & a_1 \arrow[l, "f^{-1}"] \arrow[u, "y^{-1}"']
\end{tikzcd}$$

The two natural ways to paste this diagram yield, on the one hand, $f_* x_* \sigma$ and, on other hand, $y_* g_* \sigma$. Hence $f_* x_* \sigma = y_* g_* \sigma$. $\square$

For any morphism $\rho \in \Pi\mathcal{G}(a_0, a_1)$, we define the homomorphism

$$\rho_* := f_{\rho *} x_{\rho *} : \pi_2(\mathcal{G}, a_0) \to \pi_2(\mathcal{G}, a_1),$$

where (f_ρ, b_ρ, x_ρ) is a representative path of ρ.

Lemma 3.2. *The homomorphism $\rho_* : \pi_2(\mathcal{G}, a_0) \to \pi_2(\mathcal{G}, a_1)$ does not depend of the choice of representative path of ρ.*

Proof. If $(f_\rho, b_\rho, x_\rho) \simeq (g_\rho, c_\rho, y_\rho)$, there is a box in $\mathcal{G}$ as below.

$$\begin{tikzcd}
b_\rho & c \arrow[l, "f_\rho^{-1} g_\rho"'] \\
b_\rho \arrow[u, equal] \arrow[r, equal] \arrow[ur, phantom, "\alpha"] & b_\rho \arrow[u, "y_\rho x_\rho^{-1}"']
\end{tikzcd}$$

Then, by Lemma 3.1, $f_{\rho *}^{-1} g_{\rho *} y_{\rho *} x_{\rho *}^{-1} = id_{\pi_2(\mathcal{G}, b_\rho)}$ or, equivalently, $g_{\rho *} y_{\rho *} = f_{\rho *} x_{\rho *}$. $\square$

Theorem 3.3. *The assignments $a \mapsto \pi_2(\mathcal{G}, a)$, $\rho \mapsto \rho_*$, define a functor $\pi_2\mathcal{G} : \Pi\mathcal{G} \to \mathsf{Ab}$.*

Proof. That $(id_a)_* = id$, for any object a of $\mathcal{G}$, is clear. Let $a_2 \overset{\psi}{\leftarrow} a_1 \overset{\rho}{\leftarrow} a_0$ be two composable morphisms in $\Pi\mathcal{G}$. For any box θ as in (4), we have $\psi\rho = [f_\psi f, b, x x_\rho]$ Then, by Lemmas 3.2 and 3.1, $(\psi\rho)_* = f_{\psi *} f_* x_* x_{\rho *} = f_{\psi *} f_{\psi *} x_{\psi *} x_{\rho *} = \psi_* \phi_*$. $\square$

3.1 The action of $\pi_2\mathcal{G}$ on boxes of $\mathcal{G}$

For any box in $\mathcal{G}$

$$\begin{tikzcd} d & b \arrow[l, "f"'] \\ c \arrow[u, "y"] \arrow[ur, phantom, "\theta"] & a \arrow[l, "g"] \arrow[u, "x"'] \end{tikzcd}$$

and any $\sigma \in \pi_2(\mathcal{G}, d)$, we define the box $\sigma + \theta$ (with the same edges as θ) by

$$\begin{tikzcd} d & b \arrow[l, "f"'] \\ c \arrow[u, "y"] \arrow[ur, phantom, "\sigma+\theta"] & a \arrow[l, "g"] \arrow[u, "x"'] \end{tikzcd} := \begin{tikzcd} d \arrow[r, equal] \arrow[d, equal] \arrow[dr, phantom, "\sigma"] & d \arrow[d, equal] & b \arrow[l, "f"'] \\ d \arrow[r, equal] \arrow[dr, phantom, "\mathrm{I}^{\mathrm{h}}y"] & d \arrow[r, phantom, "\theta"] & {} \\ c \arrow[u, "y"] \arrow[r, equal] & c \arrow[u] & a \arrow[l, "g"] \arrow[uu, "x"'] \end{tikzcd} = \begin{tikzcd} d \arrow[r, equal] \arrow[d, equal] \arrow[dr, phantom, "\sigma"] & d \arrow[d, equal] \arrow[dr, phantom, "\mathrm{I}^{\mathrm{v}}f"] & b \arrow[l, "f"'] \arrow[d, equal] \\ d \arrow[r, equal] \arrow[dr, phantom, "\mathrm{I}^{\mathrm{h}}y"] & d \arrow[dr, phantom, "\theta"] & b \arrow[l] \\ c \arrow[u, "y"] \arrow[r, equal] & c \arrow[u] & a \arrow[l, "g"] \arrow[u, "x"'] \end{tikzcd} = \begin{tikzcd} d \arrow[r, equal] \arrow[d, equal] \arrow[dr, phantom, "\sigma"] & d \arrow[d, equal] \arrow[dr, phantom, "\mathrm{I}^{\mathrm{v}}f"] & b \arrow[l, "f"'] \arrow[d, equal] \\ d \arrow[r, equal] & d \arrow[d, phantom, "\theta"] & b \arrow[l] \\ c \arrow[u, "y"] & {} & a \arrow[ll, "g"] \arrow[u, "x"'] \end{tikzcd}$$

Clearly $0 + \theta = \theta$ and, for any $\tau, \sigma \in \pi_2(\mathcal{G}, d)$,

$$\tau + (\sigma + \theta) = \begin{tikzcd} d \arrow[r, equal] \arrow[d, equal] \arrow[dr, phantom, "\tau"] & d \arrow[r, equal] \arrow[d, equal] \arrow[dr, phantom, "\sigma"] & d \arrow[d, equal] \arrow[dr, phantom, "\mathrm{I}^{\mathrm{v}}f"] & b \arrow[l, "f"'] \arrow[d, equal] \\ d \arrow[r, equal] \arrow[dr, phantom, "\mathrm{I}^{\mathrm{h}}y"] & d \arrow[r, equal] \arrow[dr, phantom, "\mathrm{I}^{\mathrm{h}}y"] & d \arrow[dr, phantom, "\theta"] & b \arrow[l] \\ c \arrow[u, "y"] \arrow[r, equal] & c \arrow[u] \arrow[r, equal] & c \arrow[u] & a \arrow[l, "g"] \arrow[u, "x"'] \end{tikzcd} = \begin{tikzcd} d \arrow[r, equal] \arrow[d, equal] \arrow[dr, phantom, "\tau+\sigma"] & d \arrow[d, equal] \arrow[dr, phantom, "\mathrm{I}^{\mathrm{v}}f"] & b \arrow[l, "f"'] \arrow[d, equal] \\ d \arrow[r, equal] \arrow[dr, phantom, "\mathrm{I}^{\mathrm{h}}y"] & d \arrow[dr, phantom, "\theta"] & b \arrow[l] \\ c \arrow[u, "y"] \arrow[r, equal] & c \arrow[u] & a \arrow[l, "g"] \arrow[u, "x"'] \end{tikzcd} = (\tau + \sigma) + \theta. \tag{5}$$

Lemma 3.4. *For any $\sigma \in \pi_2(\mathcal{G}, d)$, any box θ as above, and any boxes*

$$\begin{tikzcd} c & a \arrow[l, "g"'] \\ \cdot \arrow[u] \arrow[ur, phantom, "\delta"] & \cdot \arrow[l] \arrow[u] \end{tikzcd} \quad \begin{tikzcd} b & \cdot \arrow[l] \\ a \arrow[u, "x"] \arrow[ur, phantom, "\gamma"] & \cdot \arrow[l] \arrow[u] \end{tikzcd} \quad \begin{tikzcd} \cdot & d \arrow[l, "h"'] \\ \cdot \arrow[u] \arrow[ur, phantom, "\alpha"] & c \arrow[l] \arrow[u, "y"'] \end{tikzcd} \quad \begin{tikzcd} \cdot & d \arrow[l] \\ d \arrow[u, "z"] \arrow[ur, phantom, "\beta"] & b \arrow[l, "f"] \arrow[u] \end{tikzcd}$$

the following equalities hold,

$$(\sigma + \theta) \circ_{\mathrm{v}} \delta = \sigma + (\theta \circ_{\mathrm{v}} \delta), \tag{6}$$

$$(\sigma + \theta) \circ_{\mathrm{h}} \gamma = \sigma + (\theta \circ_{\mathrm{h}} \gamma), \tag{7}$$

$$\alpha \circ_{\mathrm{h}} (\sigma + \theta) = h_*\sigma + (\alpha \circ_{\mathrm{h}} \theta), \tag{8}$$

$$\beta \circ_{\mathrm{v}} (\sigma + \theta) = z_*\sigma + (\beta \circ_{\mathrm{v}} \theta). \tag{9}$$

Moreover,

$$(\sigma + \theta)^{-\mathrm{h}} = -f_*^{-1}\sigma + \theta^{-\mathrm{h}}, \tag{10}$$

$$(\sigma + \theta)^{-\mathrm{v}} = -y_*^{-1}\sigma + \theta^{-\mathrm{v}}. \tag{11}$$

Proof. (6) (the proof of (7) is dual):

$$
(\sigma+\theta)\circ_{\mathrm{v}}\delta =
\begin{tikzcd}[sep=small]
d \arrow[r, equal] \arrow[d, equal] \arrow[r, phantom, shift right=3ex, "\sigma"] & d \arrow[d, equal] \arrow[r, phantom, shift right=3ex, "\mathrm{I}^{\mathrm{v}}f"] & b \arrow[l, "f"'] \arrow[d, equal] \\
d \arrow[r, equal] & d & b \arrow[l] \\
c \arrow[u, "y"] \arrow[rr, phantom, shift left=4ex, "\theta"] & & a \arrow[ll, "g"] \arrow[u, "x"'] \\
\cdot \arrow[u] \arrow[rr, phantom, shift left=4ex, "\delta"] & & \cdot \arrow[ll] \arrow[u]
\end{tikzcd}
= \sigma + (\theta \circ_{\mathrm{v}} \delta).
$$

(8) (the proof of (9) is dual):

$$
\alpha+(\sigma+\theta) =
\begin{tikzcd}[sep=small]
\cdot & d \arrow[l, "h"'] \arrow[r, equal] \arrow[d, equal] \arrow[r, phantom, shift right=3ex, "\sigma"] & d \arrow[d, equal] \arrow[r, phantom, shift right=3ex, "\mathrm{I}^{\mathrm{v}}f"] & b \arrow[l, "f"'] \arrow[d, equal] \\
\arrow[r, phantom, "\alpha"] & d \arrow[r, equal] & d & b \arrow[l] \\
\cdot \arrow[uu] & c \arrow[l] \arrow[u, "y"] \arrow[rr, phantom, shift left=4ex, "\theta"] & & a \arrow[ll, "g"] \arrow[u, "x"']
\end{tikzcd}
=
\begin{tikzcd}[sep=small]
\cdot \arrow[d, equal] \arrow[r, phantom, shift right=3ex, "\mathrm{I}^{\mathrm{v}}h"] & d \arrow[l, "h"'] \arrow[r, equal] \arrow[d, equal] \arrow[r, phantom, shift right=3ex, "\sigma"] & d \arrow[d, equal] \arrow[r, phantom, shift right=3ex, "\mathrm{I}^{\mathrm{v}}h^{-1}"] & \cdot \arrow[l, "h^{-1}"'] \arrow[d, equal] \arrow[r, phantom, shift right=3ex, "\mathrm{I}^{\mathrm{v}}(hf)"] & d \arrow[l, "hf"'] \arrow[d, equal] \\
\cdot \arrow[r, phantom, shift right=3ex, "\alpha"] & d \arrow[l] \arrow[r, equal] & d & \cdot \arrow[l] & d \arrow[l] \\
\cdot \arrow[u] & c \arrow[l] \arrow[u, "y"] \arrow[rrr, phantom, shift left=4ex, "\theta"] & & & a \arrow[lll, "g"] \arrow[u, "x"']
\end{tikzcd}
$$

$$
= h_*(\sigma) + \alpha \circ_{\mathrm{h}} \theta.
$$

(10) (the proof of (11) is dual):

$$
\begin{aligned}
(\sigma+\theta)\circ_{\mathrm{h}}(-f_*\sigma+\theta^{-\mathrm{h}}) &\overset{(7)}{=} \sigma + \big(\theta\circ_{\mathrm{h}}(-f_*\sigma+\theta^{-\mathrm{h}})\big)\\
&\overset{(9)}{=} \sigma + \big(-f_*f_*^{-1}\sigma + \theta\circ_{\mathrm{h}}\theta^{-\mathrm{h}}\big)\\
&\overset{(5)}{=} (\sigma-\sigma) + \mathrm{I}^{\mathrm{h}}y = 0 + \mathrm{I}^{\mathrm{h}}y = \mathrm{I}^{\mathrm{h}}y.
\end{aligned}
$$

□

Lemma 3.5. *For any two boxes with the same edges*

$$
\begin{tikzcd}[sep=small]
d \arrow[r, phantom, shift right=3ex, "\theta"] & b \arrow[l, "f"'] \\
c \arrow[u, "y"] & a \arrow[l, "g"] \arrow[u, "x"']
\end{tikzcd}
\qquad
\begin{tikzcd}[sep=small]
d \arrow[r, phantom, shift right=3ex, "\theta'"] & b \arrow[l, "f"'] \\
c \arrow[u, "y"] & a \arrow[l, "g"] \arrow[u, "x"']
\end{tikzcd}
$$

there is a unique $\sigma \in \pi_2(\mathcal{G}, d)$ *such that* $\sigma + \theta = \theta'$.

Proof. Uniqueness. For any $\sigma \in \pi_2(\mathcal{G}, d)$ and θ as above,

$$(\sigma + \theta) \circ_{\mathrm{h}} \theta^{-\mathrm{h}} \overset{(8)}{=} \sigma + (\theta \circ_{\mathrm{h}} \theta^{-\mathrm{h}}) = \sigma + \mathrm{I}^{\mathrm{h}} y = \sigma \circ_{\mathrm{v}} \mathrm{I}^{\mathrm{h}} y.$$

Hence, $\sigma + \theta$ determines σ as $\sigma = \big((\sigma + \theta) \circ_{\mathrm{h}} \theta^{-\mathrm{h}}\big) \circ_{\mathrm{v}} \mathrm{I}^{\mathrm{h}} y^{-1}$.

Existence. Taking

$$\sigma = \quad \begin{tikzcd} d & b \arrow[l, "f"'] & d \arrow[l, "f^{-1}"'] \\ c \arrow[u, "y"] \arrow[ur, phantom, "\theta'"] & a \arrow[l, "g"] \arrow[u, "x"'] \arrow[ur, phantom, "\theta^{-\mathrm{h}}"] & c \arrow[l, "g^{-1}"] \arrow[u, "y"'] \\ d \arrow[u, "y^{-1}"] \arrow[rr, equal] \arrow[urr, phantom, "\mathrm{I}^{\mathrm{h}} y^{-1}"] & & d \arrow[u, "y^{-1}"'] \end{tikzcd}$$

we have $\sigma + \theta = \big((\theta' \circ_{\mathrm{h}} \theta^{-\mathrm{h}}) \circ_{\mathrm{v}} \mathrm{I}^{\mathrm{h}} y^{-1} \circ_{\mathrm{v}} \mathrm{I}^{\mathrm{h}} y\big) \circ_{\mathrm{h}} \theta = \theta' \circ_{\mathrm{h}} \theta^{-\mathrm{h}} \circ_{\mathrm{h}} \theta = \theta'$ $\square$

4. The Postnikov invariant $[\Pi\mathcal{G}, \pi_2\mathcal{G}, \mathbf{k}\mathcal{G}]$

Let P be a groupoid. The category Ab^P of functors $\mathcal{A} : P \to \mathsf{Ab}$ is abelian and it has enough injectives and projective objects [19]. We can, thus, form the right derived functors of the functor $\varprojlim : \mathsf{Ab}^P \to \mathsf{Ab}$, which is given by

$$\varprojlim(\mathcal{A}) = \big\{(x_a) \in \prod_{a \in \mathrm{Ob} P} \mathcal{A}(a) \mid \rho_* x_a = x_b \text{ for every } \rho : a \to b \text{ in } P\big\},$$

where we write $\rho_* x$ for $\mathcal{A}(\rho)(x)$. The *cohomology groups of the groupoid P with coefficients in a functor $\mathcal{A} : P \to \mathsf{Ab}$* [26], denoted by $H^n(P, \mathcal{A})$, are defined by

$$H^n(P, \mathcal{A}) = (R^n \varprojlim)(A), \qquad n = 0, 1, \cdots .$$

To exhibit an explicit cochain complex that computes these cohomology groups, let $\mathrm{N}P$ be the nerve of P. That is, the simplicial set whose m-simplices are the composable sequences $\beta = (\beta m \overset{\beta_m}{\leftarrow} \cdots \overset{\beta_1}{\leftarrow} \beta 0)$ of m arrows in P (objects of P if $m = 0$). The face $d_i \beta$, for $0 < i < m$, is obtained from β by replacing the morphisms β_{i+1} and β_i by their composition $\beta_{i+1}\beta_i$, while $d_0\beta$ and $d_m\beta$ are obtained by leaving out $\beta 0$ and βm, respectively. The degeneracies $s_i\beta$ are obtained by inserting in β the identity morphism $id_{\beta i}$. This simplicial set $\mathrm{N}P$ is a Kan complex whose fundamental groupoid is P (and whose homotopy groups vanish in degree 2 and higher). Thus, every functor $\mathcal{A} : P \to \mathsf{Ab}$ defines a local coefficient system on $\mathrm{N}P$ and the cohomology groups $H^n(\mathrm{N}P, \mathcal{A})$ are defined [17, 18, 21]. By Illusie [21, Chap.VI, (3.4.2)] and Gabriel and Zisman [17, Appendix II, Prop. 3.3], there are natural isomorphisms

$$H^n(P, \mathcal{A}) \cong H^n(\mathrm{N}P, \mathcal{A}) \cong H^n C^\bullet(P, \mathcal{A}), \qquad n = 0, 1, \cdots .$$

where

$$C^\bullet(P, \mathcal{A}) : 0 \to C^0(P, \mathcal{A}) \to \cdots \to C^{m-1}(P, \mathcal{A}) \xrightarrow{\partial} C^m(P, \mathcal{A}) \to \cdots ,$$

denotes the *complex of normalized cochains of P with coefficients in $\mathcal{A}$.* Here, a normalized m-cochain $c \in C^m(P, \mathcal{A})$ is a function

$$c : \mathrm{N}P_m \to \bigsqcup_{a \in \mathrm{Ob}P} \mathcal{A}(a)$$

such that $c(\beta) \in \mathcal{A}(\beta m)$ and $c(\beta) = 0$ whenever some β_i is an identity. Each $C^m(P, \mathcal{A})$ is an abelian group with pointwise addition, and the coboundary $\partial : C^{m-1}(P, \mathcal{A}) \to C^m(P, \mathcal{A})$ is given by

$$\partial c(\beta) = \sum_{i=0}^{m-1} c(d_i\beta) + (-1)^m \beta_{m*} c(d_m\beta).$$

As usually, we write $Z^n(P, \mathcal{A})$ for the groups of n-cocycles of the complex $C^\bullet(P, \mathcal{A})$.

In this paper, we will only use the cohomology groups $H^3(P, \mathcal{A})$. For future reference let us specify that a normalized 3-cocycle $k \in Z^3(P, \mathcal{A})$ is a function assigning to each three composable morphisms in the groupoid $a_3 \overset{\phi}{\leftarrow} a_2 \overset{\psi}{\leftarrow} a_1 \overset{\rho}{\leftarrow} a_0$ an element $k(\phi, \psi, \rho) \in \mathcal{A}(a_3)$ such that, for any four composable morphisms $a_4 \overset{\delta}{\leftarrow} a_3 \overset{\phi}{\leftarrow} a_2 \overset{\psi}{\leftarrow} a_1 \overset{\rho}{\leftarrow} a_0$, the 3-cocycle condition

$$k(\delta, \phi, \psi) - k(\delta, \phi, \psi\rho) + k(\delta, \phi\psi, \rho) - k(\delta\phi, \psi, \rho) + \delta_* k(\phi, \psi, \rho) = 0.$$

holds, and $k(\phi, \psi, \rho) = 0$ if one of the morphisms ϕ, ψ or ρ is an identity.

A normalized 2-cochain $c \in C^2(P, \mathcal{A})$ is a function assigning to each pair of composable morphisms $a_2 \overset{\phi}{\leftarrow} a_1 \overset{\psi}{\leftarrow} a_0$ an element $c(\phi, \psi) \in \mathcal{A}(a_2)$, such that $c(\phi,\psi)=0$ whenever $\phi=id_{a_1}$ or $\psi=id_{a_0}$. The coboundary of such a 2-cochain is the 3-cocycle ∂c given by

$$\partial c(\phi,\psi,\rho)=c(\phi,\psi)-c(\phi,\psi\rho)+c(\phi\psi,\rho)-\phi_* c(\psi,\rho).$$

Two normalized 3-cocycles $k,k'\in Z^3(P,\mathcal{A})$ are cohomologous if and only if there is a normalized 2-cochain $c\in C^2(P,\mathcal{A})$ such that $k'=k+\partial c$.

Definition 4.1. *A (2-dimensional) Postnikov system $(P,\mathcal{A},\mathbf{k})$ consists of a groupoid P, an abelian group valued functor $\mathcal{A}:P\to\mathsf{Ab}$, and a cohomology class $\mathbf{k}\in H^3(P,\mathcal{A})$. Two such Postnikov systems $(P,\mathcal{A},\mathbf{k})$ and $(P',\mathcal{A}',\mathbf{k}')$ are equivalent if there exists an equivalence $\mathfrak{f}:P\xrightarrow{\sim}P'$ and a natural isomorphism $\mathfrak{F}:\mathcal{A}\cong\mathfrak{f}^*\mathcal{A}'$ such that $\mathfrak{f}^*(\mathbf{k}')=\mathfrak{F}_*(\mathbf{k})$, where*

$$\mathfrak{f}^*:H^3(P',\mathcal{A}')\cong H^3(P,\mathfrak{f}^*\mathcal{A}'),\qquad \mathfrak{F}_*:H^3(P,\mathcal{A})\cong H^3(P,\mathfrak{f}^*\mathcal{A}')$$

are the corresponding induced isomorphisms in cohomology.

Let $[P,\mathcal{A},\mathbf{k}]$ denote the equivalence class of a Postnikov system $(P,\mathcal{A},\mathbf{k})$.

Let $\mathcal{G}$ be a double groupoid. We associate to $\mathcal{G}$ a Postnikov system $(\Pi\mathcal{G},\pi_2\mathcal{G},\mathbf{k}\mathcal{G})$ as follows. For each morphism in the fundamental groupoid $\rho\in\Pi\mathcal{G}(a_0,a_1)$, let us choose a representative path (f_ρ,b_ρ,x_ρ) of ρ, as in (3). In particular, if $\rho=id_a$ for some object a of $\mathcal{G}$, we take $(\mathrm{I}^{\mathrm{h}}a,a,\mathrm{I}^{\mathrm{v}}a)$ as its representative path.

If $a_2\xleftarrow{\psi}a_1\xleftarrow{\rho}a_0$ are any two composable morphisms in $\Pi\mathcal{G}$, by Lemma 2.4, we have $[f_\psi][x_\psi][f_\rho][x_\rho]=\psi\rho=[f_{\psi\rho}][x_{\psi\rho}]$, whence

$$\begin{aligned}[x_\psi][f_\rho]&=[f_\psi^{-1}][f_{\psi\rho}][x_{\psi\rho}][x_\rho^{-1}]=[f_\psi^{-1}f_{\psi\rho}][x_{\psi\rho}x_\rho^{-1}]\\&=[f_\psi^{-1}f_{\psi\rho},b_{\psi\rho},x_{\psi\rho}x_\rho^{-1}],\end{aligned}$$

and therefore we can select a box $\theta_{\psi,\rho}$ in $\mathcal{G}$ as below.

$$\begin{tikzcd} b_\psi \arrow[d, phantom, ""] & b_{\psi\rho} \arrow[l, "f_\psi^{-1}f_{\psi\rho}"'] \\ a_1 \arrow[u, "x_\psi"] \arrow[ur, phantom, "\theta_{\psi,\rho}"] & b_\rho \arrow[l, "f_\rho"] \arrow[u, "x_{\psi\rho}x_\rho^{-1}"'] \end{tikzcd} \tag{12}$$

In particular, we choose

$$\theta_{id_{a_1},\rho}=\mathrm{I}^{\mathrm{v}}f_\rho,\qquad \theta_{\psi,id_{a_0}}=\mathrm{I}^{\mathrm{h}}x_\psi. \tag{13}$$

If $a_3 \overset{\phi}{\leftarrow} a_2 \overset{\psi}{\leftarrow} a_1 \overset{\rho}{\leftarrow} a_0$ are any three composable morphisms in $\Pi\mathcal{G}$, pasting in the diagrams

$$
\begin{tikzcd}
b_{\phi\psi} & b_\phi \arrow[l, "f_{\phi\psi}^{-1} f_\phi"'] & b_{\phi\psi\rho} \arrow[l, "f_\phi^{-1} f_{\phi\psi\rho}"'] \\
b_\phi \arrow[u, "x_{\phi\psi} x_\psi^{-1}"] \arrow[r, phantom, "\theta^{-\mathrm{h}}_{\phi,\psi}", near start, shift left=4] & a_2 \arrow[l, "f_\psi^{-1}"] \arrow[u] \arrow[r, phantom, "\theta_{\phi,\psi\rho}", shift left=4] & b_{\psi\rho} \arrow[l, "f_{\psi\rho}"] \arrow[u, "x_{\phi\psi\rho} x_{\psi\rho}^{-1}"']
\end{tikzcd}
\qquad
\begin{tikzcd}
b_{\phi\psi} & b_{\phi\psi\rho} \arrow[l, "f_{\phi\psi}^{-1} f_{\phi\psi\rho}"'] \\
a_1 \arrow[u, "x_{\phi\psi}"] \arrow[r, phantom, "\theta_{\phi\psi,\rho}", shift left=4] & b_\rho \arrow[l] \arrow[u, "x_{\phi\psi\rho} x_\rho^{-1}"'] \\
b_\psi \arrow[u, "x_\psi^{-1}"] \arrow[r, phantom, "\theta^{-\mathrm{v}}_{\psi,\rho}", shift left=4] & b_{\phi\psi} \arrow[l, "f_\psi^{-1} f_{\psi\rho}"] \arrow[u, "x_\rho x_{\psi\rho}^{-1}"']
\end{tikzcd}
$$

yields two boxes with the same edges, and therefore, by Lemma 3.5 and the isomorphism $f_{\phi\psi *} : \pi_2(\mathcal{G}, b_{\phi\psi}) \cong \pi_2(\mathcal{G}, a_3)$, there is a unique element $k^{\mathcal{G}}(\phi, \psi, \rho) \in \pi_2(\mathcal{G}, a_3)$ such that

$$
f_{\phi\psi *}^{-1} k^{\mathcal{G}}(\phi, \psi, \rho) + (\theta^{-\mathrm{h}}_{\phi,\psi} \circ_{\mathrm{h}} \theta_{\phi,\psi\rho}) = \theta_{\phi\psi,\rho} \circ_{\mathrm{v}} \theta^{-\mathrm{v}}_{\psi,\rho}. \tag{14}
$$

Note that composing horizontally with $\theta_{\phi,\psi}$ on the left in (14), by (8), gives

$$
f_{\phi *}^{-1} k^{\mathcal{G}}(\phi, \psi, \rho) + \theta_{\phi,\psi\rho} = \theta_{\phi,\psi} \circ_{\mathrm{h}} (\theta_{\phi\psi,\rho} \circ_{\mathrm{v}} \theta^{-\mathrm{v}}_{\psi,\rho}). \tag{15}
$$

More explicitly,

$$
k^{\mathcal{G}}(\phi, \psi, \rho) =
\begin{tikzcd}
a_3 \arrow[ddd, equal] & b_\phi \arrow[l, "f_\phi"'] & b_{\phi\psi} \arrow[l, "f_\phi^{-1} f_{\phi\psi}"'] & b_{\phi\psi\rho} \arrow[l, "f_{\phi\psi}^{-1} f_{\phi\psi\rho}"'] & a_3 \arrow[l, "f_{\phi\psi\rho}^{-1}"'] \arrow[ddd, equal] \\
& & a_1 \arrow[u, "x_{\phi\psi}"] \arrow[l, phantom, "\theta_{\phi,\psi}"] \arrow[ur, phantom, "\theta_{\phi\psi,\rho}"] & b_\rho \arrow[l, "f_\rho"'] \arrow[u, "x_{\phi\psi\rho} x_\rho^{-1}"'] \arrow[r, phantom, "\mathrm{I}^{\mathrm{v}} f_{\phi\psi\rho}^{-1}"] & \\
\arrow[r, phantom, "\mathrm{I}^{\mathrm{v}} f_\phi"] & a_2 \arrow[uu, "x_\phi"] & b_\psi \arrow[l, "f_\psi"] \arrow[u, "x_\psi^{-1}"] \arrow[ur, phantom, "\theta^{-\mathrm{v}}_{\psi,\rho}"] & b_{\psi\rho} \arrow[l, "f_\psi^{-1} f_{\psi\rho}"] \arrow[u, "x_\rho x_{\psi\rho}^{-1}"'] & \\
a_3 & b_\phi \arrow[l, "f_\phi"] \arrow[u, "x_\phi^{-1}"] \arrow[r, phantom, "\theta^{-\mathrm{v}}_{\phi,\psi\rho}", shift left=4] & & b_{\phi\psi\rho} \arrow[ll, "f_\phi^{-1} f_{\phi\psi\rho}"] \arrow[u, "x_{\psi\rho} x_{\phi\psi\rho}^{-1}"'] & a_3 \arrow[l, "f_{\phi\psi\rho}^{-1}"]
\end{tikzcd} \tag{16}
$$

In fact, composing vertically with $\theta^{-\mathrm{v}}_{\phi,\psi\rho}$ on the right in (15) yields

$$
f_{\phi *}^{-1} k^{\mathcal{G}}(\phi, \psi, \rho) + \mathrm{I}^{\mathrm{v}}(f_\phi^{-1} f_{\phi\psi\rho}) = \big(\theta_{\phi,\psi} \circ_{\mathrm{h}} (\theta_{\phi\psi,\rho} \circ_{\mathrm{v}} \theta^{-\mathrm{v}}_{\psi,\rho})\big) \circ_{\mathrm{v}} \theta^{-\mathrm{v}}_{\phi,\psi\rho},
$$

whence

$$\begin{aligned}
&\Gamma^{\mathrm{v}} f_\phi \circ_{\mathrm{h}} \Big(\big(\theta_{\phi,\psi} \circ_{\mathrm{h}} (\theta_{\phi\psi,\rho} \circ_{\mathrm{v}} \theta^{-\mathrm{v}}_{\psi,\rho})\big) \circ_{\mathrm{v}} \theta^{-\mathrm{v}}_{\phi,\psi\rho} \Big) \circ_{\mathrm{h}} \Gamma^{\mathrm{v}} f^{-1}_{\phi\psi\rho} \\
&\quad = \Gamma^{\mathrm{v}} f_\phi \circ_{\mathrm{h}} \big(f^{-1}_{\phi *} k(\phi,\psi,\rho) + \Gamma^{\mathrm{v}}(f^{-1}_\phi f_{\phi\psi\rho})\big) \circ_{\mathrm{h}} \Gamma^{\mathrm{v}} f^{-1}_{\phi\psi\rho} \\
&\quad \overset{(8)}{=} k^{\mathcal{G}}(\phi,\psi,\rho) + \Gamma^{\mathrm{v}}(f_\phi f^{-1}_\phi f_{\phi\psi\rho} f^{-1}_{\phi\psi\rho}) \\
&\quad = k^{\mathcal{G}}(\phi,\psi,\rho) + 0 = k^{\mathcal{G}}(\phi,\psi,\rho).
\end{aligned}$$

Lemma 4.2. *So defined, $k^{\mathcal{G}} \in Z^3(\Pi\mathcal{G}, \pi_2\mathcal{G})$, that is, $k^{\mathcal{G}}$ is normalized 3-cocycle of $\Pi\mathcal{G}$ with coefficients in $\pi_2\mathcal{G}$.*

Proof. That $k^{\mathcal{G}}$ is a normalized cochain, that is, $k^{\mathcal{G}}(\phi,\psi,\rho) = 0$ whenever one of the morphisms ϕ, ψ or ρ is an identity, follows from the selection in (13). For instance, if $\phi = id_{a_2}$, then $k^{\mathcal{G}}(id_{a_2},\psi,\rho) = 0$ since

$$\begin{aligned}
\theta^{-\mathrm{h}}_{id_{a_2},\psi} \circ_{\mathrm{h}} \theta_{id_{a_2},\psi\rho} &= \Gamma^{\mathrm{v}} f^{-1}_\psi \circ_{\mathrm{h}} \circ_{\mathrm{h}} \Gamma^{\mathrm{v}} f_{\psi\rho} = \Gamma^{\mathrm{v}}(f^{-1}_\psi f_{\psi\rho}) = \theta_{\psi,\rho} \circ_{\mathrm{v}} \theta^{-\mathrm{v}}_{\psi,\rho} \\
&= \theta_{id_{a_2}\psi,\rho} \circ_{\mathrm{v}} \theta^{-\mathrm{v}}_{\psi,\rho}.
\end{aligned}$$

To prove that $k^{\mathcal{G}}$ is a 3-cocycle, suppose $a_4 \overset{\delta}{\leftarrow} a_3 \overset{\phi}{\leftarrow} a_2 \overset{\psi}{\leftarrow} a_1 \overset{\rho}{\leftarrow} a_0$ are morphisms in $\Pi\mathcal{G}$. By using first horizontal composition in the diagram below, we see, from (16), that the pasted boxes of the inner regions labeled with (A), (B) and (C) are

$$\begin{aligned}
(A) &= \Gamma^{\mathrm{v}} f^{-1}_\delta \circ_{\mathrm{h}} \ k^{\mathcal{G}}(\delta,\phi,\psi) \ \circ_{\mathrm{h}} \Gamma^{\mathrm{v}} f_{\delta\phi\psi\rho}, \\
(B) &= \Gamma^{\mathrm{v}} f^{-1}_\delta \circ_{\mathrm{h}} \ k^{\mathcal{G}}(\delta,\phi\psi,\rho) \ \circ_{\mathrm{h}} \Gamma^{\mathrm{v}} f_{\delta\phi\psi\rho}, \\
(C) &= \Gamma^{\mathrm{v}} f^{-1}_\delta \circ_{\mathrm{h}} \ \delta_* k^{\mathcal{G}}(\phi,\psi,\rho) \ \circ_{\mathrm{h}} \Gamma^{\mathrm{v}} f_{\delta\phi\psi\rho}.
\end{aligned}$$

$$
\begin{tikzcd}
 & b_\delta \arrow[ddr, phantom, "\theta_{\delta,\phi}"] & b_{\delta\phi} \arrow[l, "f_\delta^{-1}f_{\delta\phi}"'] \arrow[dr, phantom, "\theta_{\delta\phi,\psi}"] & b_{\delta\phi\psi} \arrow[l, "f_{\delta\phi}^{-1}f_{\delta\phi\psi}"'] \arrow[ddr, phantom, "\mathrm{I}^{\mathrm{v}}(f^{-1}_{\delta\phi\psi}f_{\delta\phi\psi\rho})"] & b_{\delta\phi\psi\rho} \arrow[l, "f^{-1}_{\delta\phi\psi}f_{\delta\phi\psi\rho}"'] \arrow[ddd, equal] \arrow[ddrr, phantom, "\mathrm{I}b_{\delta\phi\psi\rho}"] & & b_{\delta\phi\psi\rho} \arrow[ll, equal] \arrow[ddd, equal] \\
 & & a_2 \arrow[u, "x_{\delta\phi}"] & b_\psi \arrow[l, "f_\psi"'] \arrow[u, "x_{\delta\phi\psi}x_\psi^{-1}"'] \arrow[dl, phantom, "\theta^{-\mathrm{v}}_{\phi,\psi}"] & & & \\
(A) & a_3 \arrow[uu, "x_\delta"] & b_\phi \arrow[l, "f_\phi"] \arrow[u, "x_\phi^{-1}"] \arrow[dr, phantom, "\theta^{-\mathrm{v}}_{\delta,\phi\psi}"] & b_{\phi\psi} \arrow[l, "f_\phi^{-1}f_{\phi\psi}"] \arrow[u, "x_\psi x_{\phi\psi}^{-1}"'] & & & \\
 & b_\delta \arrow[u, "x_\delta^{-1}"] \arrow[ddrr, phantom, "\theta_{\delta,\phi\psi}"] & & b_{\delta\phi\psi} \arrow[ll, "f_\delta^{-1}f_{\delta\phi\psi}"] \arrow[u, "x_{\phi\psi}x_{\delta\phi\psi}^{-1}"'] \arrow[dr, phantom, "\theta_{\delta\phi\psi,\rho}"] & b_{\delta\phi\psi\rho} \arrow[l, "f^{-1}_{\delta\phi\psi}f_{\delta\phi\psi\rho}"] \arrow[ddrr, phantom, "\mathrm{I}^{\mathrm{h}}(x_{\delta\phi\psi\rho}x^{-1}_{\phi\psi\rho})"] & & b_{\delta\phi\psi\rho} \arrow[ll, equal] \\
 & & & a_1 \arrow[u, "x_{\delta\phi\psi}"] & b_\rho \arrow[l, "f_\rho"] \arrow[u, "x_{\delta\phi\psi\rho}x_\rho^{-1}"'] \arrow[dl, phantom, "\theta^{-\mathrm{v}}_{\phi\psi,\rho}"] & & \\
(B) & a_3 \arrow[uu, "x_\delta"] \arrow[drrrr, phantom, "\mathrm{I}^{\mathrm{h}}x_\delta^{-1}"] & b_\phi \arrow[l, "f_\phi"'] & b_{\phi\psi} \arrow[l, "f_\phi^{-1}f_{\phi\psi}"] \arrow[u, "x_{\phi\psi}^{-1}"] & b_{\phi\psi\rho} \arrow[l, "f^{-1}_{\phi\psi}f_{\phi\psi\rho}"] \arrow[u, "x_\rho x^{-1}_{\phi\psi\rho}"'] & a_3 \arrow[l, "f^{-1}_{\phi\psi\rho}"] \arrow[dr, phantom, "\theta^{-\mathrm{v}}_{\delta,\phi\psi\rho}"] & b_{\phi\psi\rho} \arrow[l, "f_{\phi\psi\rho}"'] \arrow[uu, "x_{\delta\phi\psi\rho}x^{-1}_{\phi\psi\rho}"'] \\
 & b_\delta \arrow[u, "x_\delta^{-1}"] \arrow[drrrr, phantom, "\mathrm{I}^{\mathrm{h}}x_\delta"] & & & & b_\delta \arrow[llll, equal] \arrow[u, "x_\delta^{-1}"'] \arrow[dddddr, phantom, "\mathrm{I}^{\mathrm{v}}(f_\delta^{-1}f_{\delta\phi\psi\rho})"] & b_{\delta\phi\psi\rho} \arrow[l, "f_\delta^{-1}f_{\delta\phi\psi\rho}"] \arrow[u, "x_{\phi\psi\rho}x^{-1}_{\delta\phi\psi\rho}"'] \arrow[ddddd, equal] \\
 & a_3 \arrow[u, "x_\delta"] \arrow[ddd, equal] \arrow[dddr, phantom, "\mathrm{I}^{\mathrm{v}}f_\phi"] & b_\phi \arrow[l, "f_\phi"'] \arrow[ddr, phantom, "\theta_{\phi,\psi}"] & b_{\phi\psi} \arrow[l, "f_\phi^{-1}f_{\phi\psi}"'] \arrow[dr, phantom, "\theta_{\phi\psi,\rho}"] & b_{\phi\psi\rho} \arrow[l, "f^{-1}_{\phi\psi}f_{\phi\psi\rho}"'] \arrow[ddr, phantom, "\mathrm{I}^{\mathrm{v}}f^{-1}_{\phi\psi\rho}"] & a_3 \arrow[l, "f^{-1}_{\phi\psi\rho}"'] \arrow[u, "x_\delta"'] \arrow[ddd, equal] & \\
 & & & a_1 \arrow[u, "x_{\phi\psi}"] & b_\rho \arrow[l, "f_\rho"'] \arrow[u, "x_{\phi\psi\rho}x_\rho^{-1}"'] \arrow[dl, phantom, "\theta^{-\mathrm{v}}_{\psi,\rho}"] & & \\
(C) & & a_2 \arrow[uu, "x_\phi"] \arrow[drr, phantom, "\theta^{-\mathrm{v}}_{\phi,\psi\rho}"] & b_\psi \arrow[l, "f_\psi"] \arrow[u, "x_\psi^{-1}"] & b_{\psi\rho} \arrow[l, "f_\psi^{-1}f_{\psi\rho}"] \arrow[u, "x_\rho x^{-1}_{\psi\rho}"'] & & \\
 & a_3 \arrow[drrrr, phantom, "\mathrm{I}^{\mathrm{h}}x_\delta^{-1}"] & b_\phi \arrow[l, "f_\phi"'] \arrow[u, "x_\phi^{-1}"] & & b_{\phi\psi\rho} \arrow[ll, "f_\phi^{-1}f_{\phi\psi\rho}"] \arrow[u, "x_{\psi\rho}x^{-1}_{\phi\psi\rho}"'] & a_3 \arrow[l, "f^{-1}_{\phi\psi\rho}"] & \\
 & b_\delta \arrow[u, "x_\delta^{-1}"] & & & & b_\delta \arrow[llll, equal] \arrow[u, "x_\delta^{-1}"'] & b_{\delta\phi\psi\rho} \arrow[l, "f_\delta^{-1}f_{\delta\phi\psi\rho}"']
\end{tikzcd}
$$

Hence, using now vertical composition of inner boxes in it, we see that

$\mathrm{I}^{\mathrm{v}}f_\delta^{-1}\circ_{\mathrm{h}}\big(k^{\mathcal{G}}(\delta,\phi,\psi)+k^{\mathcal{G}}(\delta,\phi\psi,\rho)+\delta_*k^{\mathcal{G}}(\phi,\psi,\rho)\big)\circ_{\mathrm{h}}\mathrm{I}^{\mathrm{v}}f_{\delta\phi\psi\rho}=$

$$
\begin{tikzcd}
b_\delta & b_{\delta\phi} \arrow[l, "f_\delta^{-1}f_{\delta\phi}"'] & b_{\delta\phi\psi} \arrow[l, "f_{\delta\phi}^{-1}f_{\delta\phi\psi}"'] & b_{\delta\phi\psi\rho} \arrow[l, "f_{\delta\phi\psi}^{-1}f_{\delta\phi\psi\rho}"'] \arrow[rr, equal] & & b_{\delta\phi\psi\rho} \\
a_3 \arrow[u, "x_\delta"] \arrow[ur, phantom, "\theta_{\delta,\phi}"] & b_\phi \arrow[l, "f_\phi"] \arrow[u] \arrow[ur, phantom, "\theta_{\delta\phi,\psi}\circ_{\mathrm{v}}\theta^{-\mathrm{v}}_{\phi,\psi}"] & b_{\phi\psi} \arrow[l, "f_\phi^{-1}f_{\phi\psi}"] \arrow[u] \arrow[ur, phantom, "\theta_{\delta\phi\psi,\rho}\circ_{\mathrm{v}}\theta^{-\mathrm{v}}_{\phi\psi,\rho}"] & b_{\phi\psi\rho} \arrow[l, "f_{\phi\psi}^{-1}f_{\phi\psi\rho}"] \arrow[u] \arrow[urr, phantom, "\mathrm{I}^{\mathrm{h}}(x_{\delta\phi\psi\rho}x^{-1}_{\phi\psi\rho})"] & a_3 \arrow[l, "f^{-1}_{\phi\psi\rho}"'] & b_{\phi\psi\rho} \arrow[l, "f_{\phi\psi\rho}"'] \arrow[u, "x_{\delta\phi\psi\rho}x^{-1}_{\phi\psi\rho}"'] \\
b_\delta \arrow[u, "x_\delta^{-1}"] \arrow[rrrr, equal] \arrow[urrrr, phantom, "\mathrm{I}^{\mathrm{h}}x_\delta^{-1}"] & & & & b_\delta \arrow[u, "x_\delta^{-1}"] \arrow[ur, phantom, "\theta^{-\mathrm{v}}_{\delta,\phi\psi\rho}"] & b_{\delta\phi\psi\rho} \arrow[l, "f_\delta^{-1}f_{\delta\phi\psi\rho}"] \arrow[u, "x_{\phi\psi\rho}x^{-1}_{\delta\phi\psi\rho}"'] \arrow[ddddd, equal] \\
a_3 \arrow[u, "x_\delta"] \arrow[urrrr, phantom, "\mathrm{I}^{\mathrm{h}}x_\delta"] \arrow[ddd, equal] & b_\phi \arrow[l, "f_\phi"'] & b_{\phi\psi} \arrow[l, "f_\phi^{-1}f_{\phi\psi}"'] & b_{\phi\psi\rho} \arrow[l, "f_{\phi\psi}^{-1}f_{\phi\psi\rho}"'] & a_3 \arrow[l, "f^{-1}_{\phi\psi\rho}"'] \arrow[u, "x_\delta"'] \arrow[ddd, equal] \arrow[ddr, phantom, "\mathrm{I}^{\mathrm{v}}(f_\delta^{-1}f_{\delta\phi\psi\rho})"] & \\
& & a_1 \arrow[u, "x_{\phi\psi}"] \arrow[l, phantom, "\theta_{\phi,\psi}"] \arrow[ur, phantom, "\theta_{\phi\psi,\rho}"] & b_\rho \arrow[l, "f_\rho"'] \arrow[u, "x_{\phi\psi\rho}x_\rho^{-1}"'] \arrow[r, phantom, "\mathrm{I}^{\mathrm{v}}f^{-1}_{\phi\psi\rho}"] & & \\
& a_2 \arrow[uu, "x_\phi"] \arrow[l, phantom, "\mathrm{I}^{\mathrm{v}}f_\phi"] & b_\psi \arrow[l, "f_\psi"'] \arrow[u, "x_\psi^{-1}"] \arrow[ur, phantom, "\theta^{-\mathrm{v}}_{\psi,\rho}"] & b_{\psi\rho} \arrow[l, "f_\psi^{-1}f_{\psi\rho}"] \arrow[u, "x_\rho x^{-1}_{\psi\rho}"'] & & \\
a_3 & b_\phi \arrow[l, "f_\phi"] \arrow[u, "x_\phi^{-1}"] \arrow[urr, phantom, "\theta^{-\mathrm{v}}_{\phi,\psi\rho}"] & & b_{\phi\psi\rho} \arrow[ll, "f_\phi^{-1}f_{\phi\psi\rho}"] \arrow[u, "x_{\psi\rho}x^{-1}_{\phi\psi\rho}"'] & a_3 \arrow[l, "f^{-1}_{\phi\psi\rho}"] & \\
b_\delta \arrow[u, "x_\delta^{-1}"] \arrow[rrrr, equal] \arrow[urrrr, phantom, "\mathrm{I}^{\mathrm{h}}x_\delta^{-1}"] & & & & b_\delta \arrow[u, "x_\delta^{-1}"'] & b_{\delta\phi\psi\rho} \arrow[l, "f_\delta^{-1}f_{\delta\phi\psi\rho}"']
\end{tikzcd}
$$

$$= \begin{tikzcd}
b_\delta & b_{\delta\phi} \arrow[l, "f_\delta^{-1}f_{\delta\phi}"'] & b_{\delta\phi\psi} \arrow[l, "f_{\delta\phi}^{-1}f_{\delta\phi\psi}"'] & b_{\delta\phi\psi\rho} \arrow[l, "f_{\delta\phi\psi}^{-1}f_{\delta\phi\psi\rho}"'] & & b_{\delta\phi\psi\rho} \arrow[ll, equal] \\
a_3 \arrow[u, "x_\delta"] \arrow[ddd, equal] \arrow[dddr, phantom, "\mathrm{I}^{\mathrm{v}}f_\phi"] \arrow[ur, phantom, "\theta_{\delta,\phi}"] & b_\phi \arrow[l] \arrow[u] \arrow[ddr, phantom, "\theta_{\phi,\psi}"] \arrow[ur, phantom, "\theta_{\delta\phi,\psi}\circ_{\mathrm{v}}\theta^{-\mathrm{v}}_{\phi,\psi}"] & b_{\phi\psi} \arrow[l] \arrow[u] \arrow[dr, phantom, "\theta_{\phi\psi,\rho}"] \arrow[ur, phantom, "\theta_{\delta\phi\psi,\rho}\circ_{\mathrm{v}}\theta^{-\mathrm{v}}_{\phi\psi,\rho}"] & b_{\phi\psi\rho} \arrow[l] \arrow[u] \arrow[dddr, phantom, "\mathrm{I}^{\mathrm{v}}f^{-1}_{\phi\psi\rho}"] \arrow[urr, phantom, "\mathrm{I}^{\mathrm{h}}(x_{\delta\phi\psi\rho}x^{-1}_{\phi\psi\rho})"] & a_3 \arrow[l, "f^{-1}_{\phi\psi\rho}"'] \arrow[ddd, equal] \arrow[dddr, phantom, "\mathrm{I}^{\mathrm{v}}f_{\phi\psi\rho}"] & b_{\phi\psi\rho} \arrow[l, "f_{\phi\psi\rho}"'] \arrow[u, "x_{\delta\phi\psi\rho}x^{-1}_{\phi\psi\rho}"'] \arrow[ddd, equal] \\
 & & a_1 \arrow[u] \arrow[dr, phantom, "\theta^{-\mathrm{v}}_{\psi,\rho}"] & b_\rho \arrow[l] \arrow[u, "x_{\phi\psi\rho}x_\rho^{-1}"'] & & \\
 & a_2 \arrow[uu, "x_\phi"] \arrow[drr, phantom, "\theta^{-\mathrm{v}}_{\phi,\psi\rho}"] & b_\psi \arrow[l] \arrow[u] & b_{\psi\rho} \arrow[l] \arrow[u, "x_\rho x^{-1}_{\psi\rho}"'] & & \\
a_3 \arrow[drrrr, phantom, "\mathrm{I}^{\mathrm{h}}x_\delta^{-1}"] & b_\phi \arrow[l, "f_\phi"] \arrow[u, "x_\phi^{-1}"] & & b_{\phi\psi\rho} \arrow[ll, "f_\phi^{-1}f_{\phi\psi\rho}"] \arrow[u, "x_{\psi\rho}x^{-1}_{\phi\psi\rho}"'] & a_3 \arrow[l, "f^{-1}_{\phi\psi\rho}"] \arrow[dr, phantom, "\theta^{-\mathrm{v}}_{\delta,\phi\psi\rho}"] & b_{\phi\psi\rho} \arrow[l, "f_{\phi\psi\rho}"'] \\
b_\delta \arrow[u, "x_\delta^{-1}"] & & & & b_\delta \arrow[llll, equal] \arrow[u, "x_\delta^{-1}"] & b_{\delta\phi\psi\rho} \arrow[l, "f_\delta^{-1}f_{\delta\phi\psi\rho}"] \arrow[u, "x_{\phi\psi\rho}x^{-1}_{\delta\phi\psi\rho}"']
\end{tikzcd}$$

$$= \begin{tikzcd}
b_\delta & b_{\delta\phi} \arrow[l, "f_\delta^{-1}f_{\delta\phi}"'] & b_{\delta\phi\psi} \arrow[l, "f_{\delta\phi}^{-1}f_{\delta\phi\psi}"'] & b_{\delta\phi\psi\rho} \arrow[l, "f_{\delta\phi\psi}^{-1}f_{\delta\phi\psi\rho}"'] \\
 & & a_1 \arrow[u, "x_{\delta\phi}"] \arrow[ur, phantom, "\theta_{\delta\phi\psi,\rho}"] \arrow[dr, phantom, "\theta^{-\mathrm{v}}_{\psi,\rho}"] & b_\rho \arrow[l, "f_\rho"'] \arrow[u, "x_{\delta\phi\psi\rho}x_\rho^{-1}"'] \\
 & a_2 \arrow[uu, "x_{\delta\phi}"] \arrow[uur, phantom, "\theta_{\delta\phi,\psi}"] \arrow[drr, phantom, "\theta^{-\mathrm{v}}_{\phi,\psi\rho}"] & b_\psi \arrow[l, "f_\psi"] \arrow[u, "x_\psi^{-1}"] & b_{\psi\rho} \arrow[l, "f_\psi^{-1}f_{\phi\rho}"] \arrow[u, "x_\rho x^{-1}_{\psi\rho}"'] \\
a_3 \arrow[uuu, "x_\delta"] \arrow[uuur, phantom, "\theta_{\delta,\phi}"] \arrow[drrr, phantom, "\theta^{-\mathrm{v}}_{\delta,\phi\psi\rho}"] & b_\phi \arrow[l, "f_\phi"] \arrow[u, "x_\phi^{-1}"] & & b_{\phi\psi\rho} \arrow[ll, "f_\phi^{-1}f_{\phi\psi\phi}"] \arrow[u, "x_{\psi\rho}x^{-1}_{\phi\psi\rho}"'] \\
b_\delta \arrow[u, "x_\delta^{-1}"] & & & b_{\delta\phi\psi\rho} \arrow[lll, "f_\delta^{-1}f_{\delta\phi\psi\rho}"] \arrow[u, "x_{\phi\psi\rho}x^{-1}_{\delta\phi\psi\rho}"']
\end{tikzcd} \tag{17}$$

Now, we realize that the diagram (17) above is also obtained by using vertical composition of inner boxes in the following diagram

$$
\begin{tikzcd}
 & b_\delta \arrow[ddd, equal] \arrow[dddr, phantom, "\Gamma^{\mathrm{v}}(f_\delta^{-1}f_{\delta\phi})"] & b_{\delta\phi} \arrow[l, "f_\delta^{-1}f_{\delta\phi}"'] & b_{\delta\phi\psi} \arrow[l, "f_{\delta\phi}^{-1}f_{\delta\phi\psi}"'] \arrow[dr, phantom, "\theta_{\delta\phi\psi,\rho}"] & b_{\delta\phi\psi\rho} \arrow[l, "f_{\delta\phi\psi}^{-1}f_{\delta\phi\psi\rho}"'] \\
 & & \arrow[r, phantom, "\theta_{\delta\phi,\psi}"] & a_1 \arrow[u, "x_{\delta\phi\psi}"] \arrow[dr, phantom, "\theta^{-\mathrm{v}}_{\psi,\rho}"] & b_\rho \arrow[l, "f_\rho"'] \arrow[u, "x_{\delta\phi\psi\rho}x_\rho^{-1}"'] \\
(D) & & a_2 \arrow[uu, "x_{\delta\phi}"] & b_\psi \arrow[l, "f_\psi"'] \arrow[u, "x_\psi^{-1}"] \arrow[d, phantom, "\theta^{-\mathrm{v}}_{\delta\phi,\psi\rho}"] & b_{\psi\rho} \arrow[l, "f_\phi^{-1}f_{\psi\rho}"] \arrow[u, "x_\rho x_{\psi\rho}^{-1}"'] \\
 & b_\delta \arrow[ddr, phantom, "\theta_{\delta,\phi}"] & b_{\delta\phi} \arrow[l, "f_\delta^{-1}f_{\delta\phi}"'] \arrow[u, "x_{\delta\phi}^{-1}"] \arrow[drr, phantom, "\theta_{\delta\phi,\psi\rho}"] & & b_{\delta\phi\psi\rho} \arrow[ll, "f_{\delta\phi}^{-1}f_{\delta\phi\psi\rho}"'] \arrow[u, "x_{\psi\rho}x_{\delta\phi\psi\rho}^{-1}"'] \\
 & & a_2 \arrow[u, "x_{\delta\phi}"] \arrow[drr, phantom, "\theta^{-\mathrm{v}}_{\phi,\psi\rho}"] & & b_{\psi\rho} \arrow[ll, "f_{\phi\psi}"'] \arrow[u, "x_{\delta\phi\psi\rho}x_{\psi\rho}^{-1}"'] \\
(E) & a_3 \arrow[uu, "x_\delta"] \arrow[drrr, phantom, "\theta^{-\mathrm{v}}_{\delta,\phi\psi\rho}"] & b_\phi \arrow[l, "f_\phi"] \arrow[u, "x_\phi^{-1}"] & & b_{\phi\psi\rho} \arrow[ll, "f_\phi^{-1}f_{\phi\psi\rho}"] \arrow[u, "x_{\psi\rho}x_{\phi\psi\rho}^{-1}"'] \\
 & b_\delta \arrow[u, "x_\delta^{-1}"] & & & b_{\delta\phi\psi\rho} \arrow[lll, "f_\delta^{-1}f_{\delta\phi\psi\rho}"] \arrow[u, "x_{\phi\psi\rho}x_{\delta\phi\psi\rho}^{-1}"']
\end{tikzcd}
$$

where the pasted boxes of the inner regions labeled with (D) and (E) are easily recognized, by (16), to be

$$
\begin{aligned}
(D) &= \Gamma^{\mathrm{v}} f_\delta^{-1} \circ_{\mathrm{h}} \ k^{\mathcal{G}}(\delta\phi, \psi, \rho) \ \circ_{\mathrm{h}} \Gamma^{\mathrm{v}} f_{\delta\phi\psi\rho},\\
(E) &= \Gamma^{\mathrm{v}} f_\delta^{-1} \circ_{\mathrm{h}} \ k^{\mathcal{G}}(\delta, \phi, \psi\rho) \ \circ_{\mathrm{h}} \Gamma^{\mathrm{v}} f_{\delta\phi\psi\rho}.
\end{aligned}
$$

So, the resulting pasted box of the diagram (17) is also

$$\Gamma^{\mathrm{v}} f_\delta^{-1} \circ_{\mathrm{h}} (k^{\mathcal{G}}(\delta\phi, \psi, \rho) + k^{\mathcal{G}}(\delta, \phi, \psi\rho)) \circ_{\mathrm{h}} \Gamma^{\mathrm{v}} f_{\delta\phi\psi\rho}.$$

This proves the 3-cocycle condition, that is,

$$k^{\mathcal{G}}(\delta\phi, \psi, \rho) + k^{\mathcal{G}}(\delta, \phi, \psi\rho) = k^{\mathcal{G}}(\delta, \phi, \psi) + k^{\mathcal{G}}(\delta, \phi\psi, \rho) + \delta_* k^{\mathcal{G}}(\phi, \psi, \rho).$$

□

Next, we observe the effect of different choices of (f_ρ, b_ρ, x_ρ) and $\theta_{\psi,\rho}$ in the construction of the 3-cocycle $k^{\mathcal{G}}$.

Lemma 4.3. (i) *If the choice of the boxes* $\theta_{\psi,\rho}$ *in* (12) *is changed, then* $k^{\mathcal{G}}$ *is changed to a cohomologous cocycle. By suitably changing the boxes* $\theta_{\psi,\rho}$*,* $k^{\mathcal{G}}$ *may by changed to any cohomologous cocycle.*

(ii) *If the choice of the representative paths* (f_ρ, b_ρ, x_ρ) *in* (3) *is changed, then a suitable new selection of the boxes* $\theta_{\psi,\rho}$ *leaves the cocycle* $k^{\mathcal{G}}$ *unaltered.*

Proof. (i) Let, for each two composable morphisms $a_2 \overset{\psi}{\leftarrow} a_1 \overset{\rho}{\leftarrow} a_0$ in $\Pi\mathcal{G}$,

$$\begin{tikzcd} b_\psi & b_{\psi\rho} \arrow[l, "f_\psi^{-1} f_{\psi\rho}"'] \\ a_1 \arrow[u, "x_\psi"] \arrow[ur, phantom, "\theta'_{\psi,\rho}"] & b_\rho \arrow[l, "f_\rho"] \arrow[u, "x_{\psi\rho} x_\rho^{-1}"'] \end{tikzcd}$$

be any other selection of boxes in (3), and let $k'^{\mathcal{G}} \in Z^3(\Pi\mathcal{G}, \pi_2)$ be the corresponding 3-cocycle.

By Lemma 3.5 and the isomorphism $f_{\psi *} : \pi_2(\mathcal{G}, b_\psi) \cong \pi_2(\mathcal{G}, a_2)$, we can write $\theta'_{\psi,\rho} = f_{\psi *}^{-1} c(\psi,\rho) + \theta_{\psi,\rho}$ for a unique element $c(\psi,\rho) \in \pi_2(\mathcal{G}, a_2)$, and a normalized 2-cochain $c \in C^2(\Pi\mathcal{G}, \pi_2\mathcal{G})$ becomes so defined. Then, for every composable morphisms $a_3 \overset{\phi}{\leftarrow} a_2 \overset{\psi}{\leftarrow} a_1 \overset{\rho}{\leftarrow} a_0$, we have

$$\begin{aligned}
& f_{\phi *}^{-1} k'^{\mathcal{G}}(\phi,\psi,\rho) + f_{\phi *}^{-1} c(\phi,\psi\rho) + \theta_{\phi,\psi\rho} \\
& \overset{(15)}{=} (f_{\phi *}^{-1} c(\phi,\psi) + \theta_{\phi,\psi}) \circ_{\mathrm{h}} \big((f_{\phi\psi *}^{-1} c(\phi\psi,\rho) + \theta_{\phi\psi,\rho}) \circ_{\mathrm{v}} (f_{\psi *}^{-1} c(\psi,\rho) + \theta_{\psi,\rho})^{-\mathrm{v}}\big) \\
& \overset{(11)}{=} (f_{\phi *}^{-1} c(\phi,\psi) + \theta_{\phi,\psi}) \circ_{\mathrm{h}} \big((f_{\phi\psi *}^{-1} c(\phi\psi,\rho) + \theta_{\phi\psi,\rho}) \circ_{\mathrm{v}} (-x_{\psi *}^{-1} f_{\psi *}^{-1} c(\psi,\rho) + \theta_{\psi,\rho}^{-\mathrm{v}})\big) \\
& \overset{(9)}{=} (f_{\phi *}^{-1} c(\phi,\psi) + \theta_{\phi,\psi}) \circ_{\mathrm{h}} (f_{\phi\psi *}^{-1} c(\phi\psi,\rho) - x_{\phi\psi *} x_{\psi *}^{-1} f_{\phi *}^{-1} c(\psi,\rho) + \theta_{\phi\psi,\rho} \circ_{\mathrm{v}} \theta_{\psi,\rho}^{-\mathrm{v}}) \\
& \overset{(8)}{=} f_{\phi *}^{-1} c(\phi,\psi) + f_{\phi *}^{-1} c(\phi\psi,\rho) - f_{\phi *}^{-1} f_{\phi\psi *} x_{\phi\psi *} x_{\psi *}^{-1} f_{\psi *}^{-1} c(\psi,\rho) \\
& \qquad + \theta_{\phi,\psi} \circ_{\mathrm{h}} (\theta_{\phi\psi,\rho} \circ_{\mathrm{v}} \theta_{\psi,\rho}^{-\mathrm{v}}) \\
& \overset{(15)}{=} f_{\phi *}^{-1} c(\phi,\psi) + f_{\phi *}^{-1} c(\phi\psi,\rho) - f_{\phi *}^{-1} f_{\phi\psi *} x_{\phi\psi *} x_{\psi *}^{-1} f_{\psi *}^{-1} c(\psi,\rho) \\
& \qquad + f_{\phi *}^{-1} k^{\mathcal{G}}(\phi,\psi,\rho) + \theta_{\phi,\psi\rho}
\end{aligned}$$

whence, by Lemma 3.5,

$$\begin{aligned}
& k'^{\mathcal{G}}(\phi,\psi,\rho) + c(\phi,\psi\rho) \\
& \qquad = c(\phi,\psi) + c(\phi\psi,\rho) - f_{\phi\psi *} x_{\phi\psi *} x_{\psi *}^{-1} f_{\psi *}^{-1} c(\psi,\rho) + k^{\mathcal{G}}(\phi,\psi,\rho).
\end{aligned}$$

As, by Theorem 3.3 and Lemma 2.4,

$$f_{\phi\psi *} x_{\phi\psi *} x_{\psi *}^{-1} = (\phi\psi)_* x_{\psi *}^{-1} = \phi_* \psi_* x_{\psi *}^{-1} = \phi_* f_{\psi *} x_{\psi *} x_{\psi *}^{-1} = \phi_* f_{\psi *},$$

we finally conclude that

$$\begin{aligned} {k'}^{\mathcal{G}}(\phi,\psi,\rho) &= c(\phi,\psi) - c(\phi,\psi\rho) + c(\phi\psi,\rho) - \phi_* c(\psi,\rho) + k^{\mathcal{G}}(\phi,\psi,\rho) \\ &= \partial c(\phi,\psi,\rho) + k^{\mathcal{G}}(\phi,\psi,\rho). \end{aligned}$$

Thus, ${k'}^{\mathcal{G}} = \partial c + k^{\mathcal{G}}$ and therefore $k^{\mathcal{G}}$ and ${k'}^{\mathcal{G}}$ are cohomologous.

Conversely, suppose $c \in C^2(\Pi\mathcal{G}, \pi_2\mathcal{G})$ is any normalized 2-cochain and $k = \partial c + k^{\mathcal{G}}$. Then $f_{\psi *}^{-1} c(\psi,\rho) + \theta_{\psi,\rho}$ is an allowable choice of $\theta'_{\psi,\rho}$, for each pair of composable morphisms (ψ,ρ) in $\Pi\mathcal{G}$, for which, by the already shown above, the corresponding 3-cocycle just becomes ${k'}^{\mathcal{G}} = \partial c + k^{\mathcal{G}} = k$.

(ii) Suppose we have choosen another representative path (g_ρ, c_ρ, y_ρ) of each morphism ρ in $\Pi\mathcal{G}$. Then, we can select homotopies $\alpha_\rho : (f_\rho, b_\rho, x_\rho) \simeq (g_\rho, c_\rho, y_\rho)$ and construct, for each two morphisms $a_2 \overset{\psi}{\leftarrow} a_1 \overset{\rho}{\leftarrow} a_0$, the box

$$\begin{tikzcd} c_\psi & c_{\psi\rho} \arrow[l, "g_\psi^{-1} g_{\psi\rho}"'] \arrow[dl, phantom, "\theta'_{\psi,\rho}"] \\ a_1 \arrow[u, "y_\psi"] & c_\rho \arrow[l, "g_\rho"] \arrow[u, "y_{\psi\rho} y_\rho^{-1}"'] \end{tikzcd} \quad = \quad \begin{tikzcd} c_\psi & b_\psi \arrow[l, "g_\psi^{-1} f_\psi"'] & b_{\psi\rho} \arrow[l, "f_\psi^{-1} f_{\psi\rho}"'] & c_{\psi\rho} \arrow[l, "f_{\psi\rho}^{-1} g_{\psi\rho}"'] \\ b_\psi \arrow[u, "y_\psi x_\psi^{-1}"] \arrow[r, equal] \arrow[ur, phantom, "\alpha_\psi^{-\mathrm{h}}"] & b_\psi \arrow[u, equal] & b_{\psi\rho} \arrow[u, equal] \arrow[r, equal] \arrow[ur, phantom, "\alpha_{\psi\rho}"] & b_{\psi\rho} \arrow[u, "y_{\psi\rho} x_{\psi\rho}^{-1}"'] \\ & a_1 \arrow[u, "x_\psi"] \arrow[ur, phantom, "\theta_{\psi,\rho}"] & b_\rho \arrow[u] \arrow[r, equal] \arrow[ur, phantom, "\mathrm{I}^{\mathrm{h}}(x_{\psi\rho} x_\rho^{-1})"] & b_\rho \arrow[u, "x_{\psi\rho} x_\rho^{-1}"'] \\ a_1 \arrow[uu, "x_\psi"] \arrow[r, equal] \arrow[uur, phantom, "\mathrm{I}^{\mathrm{h}} x_\psi"] & a_1 \arrow[u, equal] & b_\rho \arrow[l, "f_\rho"] \arrow[u, equal] & c_\rho \arrow[l, "f_\rho^{-1} g_\rho"] \arrow[u, "x_\rho y_\rho^{-1}"'] \arrow[ul, phantom, "\alpha_\rho^{-\mathrm{v}}"] \end{tikzcd}$$

which, by the already proven part (i), we can use to define the corresponding 3-cocycle ${k'}^{\mathcal{G}} \in Z^3(\Pi\mathcal{G}, \pi_2\mathcal{G})$ from the new selected representative paths. Then, for any three composable morphisms $a_3 \overset{\phi}{\leftarrow} a_2 \overset{\psi}{\leftarrow} a_1 \overset{\rho}{\leftarrow} a_0$,

$$\mathrm{I}^{\mathrm{v}} g_\phi^{-1} \circ_{\mathrm{h}} {k'}^{\mathcal{G}}(\phi,\psi,\rho) \circ_{\mathrm{h}} \mathrm{I}^{\mathrm{v}} g_{\phi\psi\rho} = \big(\theta'_{\phi,\psi} \circ_{\mathrm{h}} (\theta'_{\phi\psi,\rho} \circ_{\mathrm{v}} {\theta'}^{-\mathrm{v}}_{\psi,\rho})\big) \circ_{\mathrm{v}} {\theta'}^{-\mathrm{v}}_{\phi,\psi\rho} =$$

$$
\begin{tikzcd}
c_\phi & b_\phi \arrow[l, "g_\phi^{-1}f_\phi"'] & b_{\phi\psi} \arrow[l, "f_\phi^{-1}f_{\phi\psi}"'] & c_{\phi\psi} \arrow[l, "f_{\phi\psi}^{-1}g_{\phi\psi}"'] & b_{\phi\psi} \arrow[l, "g_{\phi\psi}^{-1}f_{\phi\psi}"'] & b_{\phi\psi\rho} \arrow[l, "f_{\phi\psi}^{-1}f_{\phi\psi\rho}"'] & c_{\phi\psi\rho} \arrow[l, "f_{\phi\psi\rho}^{-1}g_{\phi\psi\rho}"'] \\
b_\phi \arrow[u, "y_\phi x_\phi^{-1}"] \arrow[r, equal] \arrow[ur, phantom, "\alpha_\phi^{-\mathrm{h}}"] & b_\phi \arrow[u, equal] & b_{\phi\psi} \arrow[u, equal] \arrow[r, equal] \arrow[ur, phantom, "\alpha_{\phi\psi}"] & b_{\phi\psi} \arrow[u] \arrow[r, equal] \arrow[ur, phantom, "\alpha_{\phi\psi}^{-\mathrm{h}}"] & b_{\phi\psi} \arrow[u, equal] & b_{\phi\psi\rho} \arrow[u, equal] \arrow[r, equal] \arrow[ur, phantom, "\alpha_{\phi\psi\rho}"] & b_{\phi\psi\rho} \arrow[u, "y_{\phi\psi\rho}x_{\phi\psi\rho}^{-1}"'] \\
 & & & \arrow[r, phantom, "\mathrm{I}^{\mathrm{h}}"] & a_1 \arrow[u, "x_{\phi\psi}"] \arrow[ur, phantom, "\theta_{\phi\psi,\rho}"] & b_\rho \arrow[u] \arrow[r, equal] \arrow[ur, phantom, "\mathrm{I}^{\mathrm{h}}"] & b_\rho \arrow[u, "x_{\phi\psi\rho}x_\rho^{-1}"'] \\
 & \arrow[r, phantom, "\theta_{\phi,\psi}"] & \arrow[r, phantom, "\mathrm{I}^{\mathrm{h}}"] & a_1 \arrow[uu, "x_{\phi\psi}"] \arrow[r, equal] & a_1 \arrow[u, equal] & b_\rho \arrow[l, "f_\rho"] \arrow[u, equal] \arrow[ur, phantom, "\alpha_\rho^{-\mathrm{v}}"] & c_\rho \arrow[l] \arrow[u, "x_\rho y_\rho^{-1}"'] \\
\arrow[r, phantom, "\mathrm{I}^{\mathrm{h}}"] & & & \arrow[r, phantom, "\mathrm{I}^{\mathrm{h}}"] & a_1 \arrow[u, equal] & b_\rho \arrow[u, equal] \arrow[r, equal] \arrow[ur, phantom, "\alpha_\rho"] & b_\rho \arrow[u, "y_\rho x_\rho^{-1}"'] \\
 & a_2 \arrow[uuuu, "x_\phi"] & b_\psi \arrow[uuuu] \arrow[r, equal] & b_\psi \arrow[uu, "x_\psi^{-1}"] \arrow[r, equal] & b_\psi \arrow[u, "x_\psi^{-1}"] \arrow[ur, phantom, "\theta_{\phi,\psi}^{-\mathrm{v}}"] & b_{\psi\rho} \arrow[u] \arrow[r, equal] \arrow[ur, phantom, "\mathrm{I}^{\mathrm{h}}"] & b_{\psi\rho} \arrow[u, "x_\rho x_{\psi\rho}^{-1}"'] \\
a_2 \arrow[uuuuu, "x_\phi"] \arrow[r, equal] & a_2 \arrow[u, equal] & b_\psi \arrow[l, "f_\psi"] \arrow[u, equal] \arrow[ur, phantom, "\alpha_\psi^{-\mathrm{v}}"] & c_\psi \arrow[l, "f_\psi^{-1}g_\psi"] \arrow[u] \arrow[ur, phantom, "\alpha_\psi^{-\mathrm{vh}}"] & b_\psi \arrow[l, "g_\psi^{-1}f_\psi"] \arrow[u, equal] & b_{\psi\rho} \arrow[l, "f_\psi^{-1}f_{\psi\rho}"] \arrow[u, equal] \arrow[ur, phantom, "\alpha_{\psi\rho}^{-\mathrm{v}}"] & c_{\psi\rho} \arrow[l] \arrow[u, "x_{\psi\rho}y_{\psi\rho}^{-1}"'] \\
\arrow[r, phantom, "\mathrm{I}^{\mathrm{h}}"] & a_2 \arrow[u, equal] & & & & b_{\psi\rho} \arrow[u, equal] \arrow[r, equal] \arrow[ur, phantom, "\alpha_{\psi\rho}"] & b_{\psi\rho} \arrow[u, "y_{\psi\rho}x_{\psi\rho}^{-1}"'] \\
b_\phi \arrow[uu, "x_\phi^{-1}"] \arrow[r, equal] & b_\phi \arrow[u, "x_\phi^{-1}"'] & & \arrow[u, phantom, "\theta_{\phi,\psi\rho}^{-\mathrm{v}}"] & & b_{\phi\psi\rho} \arrow[u] \arrow[r, equal] \arrow[ur, phantom, "\mathrm{I}^{\mathrm{h}}"] & b_{\phi\psi\rho} \arrow[u, "x_{\psi\rho}x_{\phi\psi\rho}^{-1}"'] \\
c_\phi \arrow[u, "x_\phi y_\phi^{-1}"] \arrow[ur, phantom, "\alpha_\phi^{-\mathrm{hv}}"] & b_\phi \arrow[l, "g_\phi^{-1}f_\phi"] \arrow[u, equal] & & & & b_{\phi\psi\rho} \arrow[llll, "f_\phi^{-1}f_{\phi\psi\rho}"] \arrow[u, equal] \arrow[ur, phantom, "\alpha_{\phi\psi\rho}^{-\mathrm{v}}"] & c_{\phi\psi\rho} \arrow[l, "f_{\phi\psi\rho}^{-1}g_{\phi\psi\rho}"] \arrow[u, "x_{\phi\psi\rho}y_{\phi\psi\rho}^{-1}"']
\end{tikzcd}
$$

$$
= \quad
\begin{tikzcd}
c_\phi \arrow[dddd, equal] \arrow[ddd, phantom, "\mathrm{I}^{\mathrm{v}}(g_\phi^{-1}f_\phi)"] & b_\phi \arrow[l, "g_\phi^{-1}f_\phi"'] \arrow[ddr, phantom, "\theta_{\phi,\psi}"] & b_{\phi\psi} \arrow[l, "f_\phi^{-1}f_{\phi\psi}"'] \arrow[dr, phantom, "\theta_{\phi\psi,\rho}"] & b_{\phi\psi\rho} \arrow[l, "f_{\phi\psi}^{-1}f_{\phi\psi\rho}"'] \arrow[ddd, phantom, "\mathrm{I}^{\mathrm{v}}(f_{\phi\psi\rho}^{-1}g_{\phi\psi\rho})"] & c_{\phi\psi\rho} \arrow[l, "f_{\phi\psi\rho}^{-1}g_{\phi\psi\rho}"'] \arrow[dddd, equal] \\
 & & a_1 \arrow[u, "x_{\phi\psi}"] \arrow[dr, phantom, "\theta^{-\mathrm{v}}_{\psi,\rho}"] & b_\rho \arrow[l, "f_\rho"'] \arrow[u, "x_{\phi\psi\rho}x_\rho^{-1}"'] & \\
 & a_2 \arrow[uu, "x_\phi"] \arrow[drr, phantom, "\theta^{-\mathrm{v}}_{\phi,\psi\rho}"] & b_\psi \arrow[l, "f_\psi"] \arrow[u, "x_\psi^{-1}"] & b_{\psi\rho} \arrow[l, "f_\psi^{-1}f_{\psi\rho}"] \arrow[u, "x_\rho x_{\psi\rho}^{-1}"'] & \\
 & b_\phi \arrow[u, "x_\phi^{-1}"] & & & \\
c_\phi & b_\phi \arrow[l, "g_\phi^{-1}f_\phi"] & & b_{\phi\psi\rho} \arrow[ll, "f_\phi^{-1}f_{\phi\psi\rho}"] \arrow[uu, "x_{\psi\rho}x_{\phi\psi\rho}^{-1}"'] & c_{\phi\psi\rho} \arrow[l, "f_{\phi\psi\rho}^{-1}g_{\phi\psi\rho}"]
\end{tikzcd}
$$

$= \mathrm{I}^{\mathrm{v}}g_\phi^{-1} \circ_{\mathrm{h}} k'^{\mathcal{G}}(\phi,\psi,\rho) \circ_{\mathrm{h}} \mathrm{I}^{\mathrm{v}}g_{\phi\psi\rho}$.

Hence $k'^{\mathcal{G}}(\phi,\psi,\rho) = k^{\mathcal{G}}(\phi,\psi,\rho)$, and the 3-cocycle $k^{\mathcal{G}}$ is unchanged. $\square$

Lemmas 4.2 and 4.3 prove that each double groupoid $\mathcal{G}$ has a three-dimensional cohomology class $\mathbf{k}\mathcal{G} = [k^{\mathcal{G}}] \in H^3(\Pi\mathcal{G}, \pi_2\mathcal{G})$ associated with it. We refer to

$$[\Pi\mathcal{G}, \pi_2\mathcal{G}, \mathbf{k}\mathcal{G}]$$

as the *Postnikov invariant* of $\mathcal{G}$.

A *double functor* $F : \mathcal{G} \to \mathcal{G}'$ between double groupoids takes objects, horizontal and vertical morphisms, and boxes in $\mathcal{G}$ to objects, horizontal and vertical morphisms, and squares in $\mathcal{G}'$, respectively, in such a way that all the structure categories are preserved. Clearly, such a double functor induces a functor $\Pi F : \Pi\mathcal{G} \to \Pi\mathcal{G}'$,

$$
\left[\begin{tikzcd}[cramped] a_1 & b \arrow[l, "f"'] \\ & a_0 \arrow[u, "x"'] \end{tikzcd}\right]
\quad\mapsto\quad
\left[\begin{tikzcd}[cramped] Fa_1 & b \arrow[l, "Ff"'] \\ & Fa_0 \arrow[u, "Fx"'] \end{tikzcd}\right],
$$

and a natural transformation $\pi_2 F : \pi_2\mathcal{G} \to (\Pi F)^*\pi_2\mathcal{G}'$, which consists of the homomorphisms $\pi_2(F,a) : \pi_2(\mathcal{G},a) \to \pi_2(\mathcal{G}',Fa)$ given by

$$
\begin{tikzcd}[cramped] a \arrow[r, equal] \arrow[d, equal] \arrow[dr, phantom, "\sigma"] & a \arrow[d, equal] \\ a \arrow[r, equal] & a \end{tikzcd}
\quad\mapsto\quad
\begin{tikzcd}[cramped] Fa \arrow[r, equal] \arrow[d, equal] \arrow[dr, phantom, "F\sigma"] & Fa \arrow[d, equal] \\ Fa \arrow[r, equal] & Fa. \end{tikzcd}
$$

We say that the double functor F is a *weak equivalence*, and write

$$F : \mathcal{G} \xrightarrow{\sim} \mathcal{G}',$$

whenever ΠF is an equivalence of groupoids and $\pi_2 F$ is an isomorphism. If, for any double groupoid $\mathcal{G}$ we define

$$\pi_0\mathcal{G} = \pi_0(\Pi\mathcal{G}),$$

the set of iso-classes of objects of its fundamental groupoid, and, for each object a of $\mathcal{G}$,

$$\pi_1(\mathcal{G}, a) = \Pi\mathcal{G}(a, a),$$

the group of automorphisms of a in its fundamental groupoid, this notion of weak equivalence is similar to the usual topological notion. Indeed, one readily verifies that a double functor $F : \mathcal{G} \to \mathcal{G}'$ is a weak equivalence if and only if F induces an isomorphism of sets $\pi_0\mathcal{G} \cong \pi_0\mathcal{G}'$ and for every object a of $\mathcal{G}$ isomorphisms of groups $\pi_i(\mathcal{G}, a) \cong \pi_i(\mathcal{G}', Fa)$ for $i = 1, 2$ (cf. [11, 3.4]).

We define two double groupoid $\mathcal{G}$ and $\mathcal{G}'$ to be *weak equivalent* if there exists a zig-zag chain of weak equivalences

$$\mathcal{G} = \mathcal{G}_0 \xrightarrow{\sim} \mathcal{G}_1 \xleftarrow{\sim} \mathcal{G}_2 \xrightarrow{\sim} \cdots \xleftarrow{\sim} \mathcal{G}_k = \mathcal{G}'.$$

connecting $\mathcal{G}$ and $\mathcal{G}'$ (see Corollary 5.4).

Let $[\mathcal{G}]$ denote the weak equivalence class of a double groupoid $\mathcal{G}$.

Theorem 4.4. *The Postnikov invariant* $[\Pi\mathcal{G}, \pi_2\mathcal{G}, \mathbf{k}\mathcal{G}]$ *of a double groupoid* $\mathcal{G}$ *only depends on its weak equivalence class* $[\mathcal{G}]$.

Proof. Let $F : \mathcal{G} \xrightarrow{\sim} \mathcal{G}'$ be a weak equivalence between double groupoids. Suppose that the construction of $k^{\mathcal{G}} \in Z^3(\Pi\mathcal{G}, \pi_2\mathcal{G})$ has been made by means of representative paths (f_ρ, b_ρ, x_ρ) of the morphisms ρ in $\Pi\mathcal{G}$, as in (3), and boxes $\theta_{\psi,\rho}$ for each pair of composable morphisms (ψ, ρ), as in (12). Then, for the construction of $k^{\mathcal{G}'} \in Z^3(\Pi\mathcal{G}', \pi_2\mathcal{G}')$, we can choose $(Ff_\rho, Fb_\rho, Fx_\rho)$ as representative paths of the morphisms $\Pi F\rho$ in $\Pi\mathcal{G}'$ as well as the boxes $\theta_{\Pi F\psi, \Pi F\rho} = F\theta_{\psi,\rho}$. If we do this, it follows from (16) that, for any triplet (ϕ, ψ, ρ) of composable morphisms in $\Pi\mathcal{G}$,

$$k^{\mathcal{G}'}(\Pi F\phi, \Pi F\psi, \Pi F\rho) = Fk^{\mathcal{G}}(\phi, \psi, \rho).$$

This means that $(\Pi F)^*(k^{\mathcal{G}'}) = (\pi_2 F)_*(k^{\mathcal{G}})$, whence $(\Pi F)^*(\mathbf{k}\mathcal{G}) = (\pi_2 F)_*(\mathbf{k}\mathcal{G})$. Thus, $[\Pi\mathcal{G}, \pi_2\mathcal{G}, \mathbf{k}\mathcal{G}] = [\Pi\mathcal{G}', \pi_2\mathcal{G}', \mathbf{k}\mathcal{G}']$. □

5. The Classification Theorem

Theorem 5.1. *The mapping* $[\mathcal{G}] \mapsto [\Pi\mathcal{G}, \pi_2\mathcal{G}, \mathrm{k}\mathcal{G}]$ *establishes a bijective correspondence between weak equivalence classes of double groupoids and equivalence classes of Postnikov systems.*

Proof. This follows from the following construction of a double groupoid $\mathcal{G}^k$ associated to each normalized 3-cocycle $k \in Z^3(P, \mathcal{A})$, of any groupoid P with coefficients in a functor $\mathcal{A} : P \to \mathsf{Ab}$, and Proposition 5.3 bellow. □

Let P be a groupoid, $\mathcal{A} : P \to \mathsf{Ab}$ a functor and $k \in Z^3(P, \mathcal{A})$ a normalized 3-cocycle of P with coefficients in $\mathcal{A}$. We construct a double groupoid, denoted by $\mathcal{G}^k$, as follows.

• The objects of $\mathcal{G}^k$ are the arrows of P.

• For any two arrows of P, there is a unique horizontal (resp. vertical) morphism in $\mathcal{G}^k$ between them whenever they have the same target (resp. source), whereas if they have different target (resp. source) then there are no horizontal (resp. vertical) morphisms between them. Compositions and identities are defined in the obvious manner. Thus, a path in $\mathcal{G}^k$

$$\begin{array}{ccc} \xi_1 & \longleftarrow & \eta \\ & & \uparrow \\ & & \xi_0 \end{array}$$

consists of three morphisms of P such that ξ_0 and η have the same source and η and ξ_1 have the same target. Notice that such a path writes uniquely as

$$\begin{array}{ccc} \xi_1 & \longleftarrow & \eta \\ & & \uparrow \\ & & \xi_0 \end{array} \quad = \quad \begin{array}{ccc} \phi\psi & \longleftarrow & \phi\psi\rho \\ & & \uparrow \\ & & \psi\rho \end{array}$$

with $\rho = \xi_1^{-1}\eta$, $\phi = \eta\xi_0^{-1}$ and $\psi = \xi_0\eta^{-1}\xi_1$ are three composable arrows $a_3 \overset{\phi}{\leftarrow} a_2 \overset{\psi}{\leftarrow} a_1 \overset{\rho}{\leftarrow} a_0$ in the groupoid P.

• A box $(\phi, \psi, \rho; u)$ in $\mathcal{G}^k$, with boundary as below

$$\begin{array}{ccc} \phi\psi & \longleftarrow & \phi\psi\rho \\ \uparrow & {\scriptstyle (\phi,\psi,\rho;u)} & \uparrow \\ \psi & \longleftarrow & \psi\rho, \end{array} \tag{18}$$

consists of three composable arrows in P, $a_3 \overset{\phi}{\leftarrow} a_2 \overset{\psi}{\leftarrow} a_1 \overset{\rho}{\leftarrow} a_0$, together with an element $u \in \mathcal{A}(a_3)$.

• For any four composable arrows in P, $a_4 \overset{\delta}{\leftarrow} a_3 \overset{\phi}{\leftarrow} a_2 \overset{\psi}{\leftarrow} a_1 \overset{\rho}{\leftarrow} a_0$, $u \in \mathcal{A}(a_3)$ and $v \in \mathcal{A}(a_4)$, the vertical composition of the boxes

$$\begin{tikzcd}
\delta\phi\psi & \delta\phi\psi\rho \arrow[l] \\
\phi\psi \arrow[u] & \phi\psi\rho, \arrow[l] \arrow[u] \arrow[ul, phantom, "{\scriptstyle (\delta,\phi\psi,\rho;v)}"] \\
\psi \arrow[u] & \psi\rho \arrow[l] \arrow[u] \arrow[ul, phantom, "{\scriptstyle (\phi,\psi,\rho;u)}"]
\end{tikzcd} \tag{19}$$

is given by

$$\begin{aligned}(\delta, \phi\psi, \rho; v)&\circ_{\mathrm{v}}(\phi, \psi, \rho; u) \\ &= \big(\delta\phi, \psi, \rho; v + \delta_* u + k(\delta, \phi, \psi) - k(\delta, \phi, \psi\rho)\big).\end{aligned} \tag{20}$$

• For any four composable arrows in P, $a_4 \overset{\phi}{\leftarrow} a_3 \overset{\psi}{\leftarrow} a_2 \overset{\rho}{\leftarrow} a_1 \overset{\lambda}{\leftarrow} a_0$, and $u, v \in \mathcal{A}(a_4)$, the horizontal composition of the boxes

$$\begin{tikzcd}
\phi\psi & \phi\psi\rho \arrow[l] & \phi\psi\rho\lambda \arrow[l] \\
\psi \arrow[u] & \psi\rho \arrow[l] \arrow[u] \arrow[ul, phantom, "{\scriptstyle (\phi,\psi,\rho;u)}"] & \psi\rho\lambda, \arrow[l] \arrow[u] \arrow[ul, phantom, "{\scriptstyle (\phi,\psi\rho,\lambda;v)}"]
\end{tikzcd} \tag{21}$$

is given by

$$(\phi, \psi, \rho; u) \circ_{\mathrm{h}} (\phi, \psi\rho, \lambda; v) = \big(\phi, \psi, \rho\lambda; u + v\big). \tag{22}$$

• The vertical and horizontal identity boxes are respectively defined by

$$\mathrm{I}^{\mathrm{v}}(\phi \leftarrow \phi\psi) = \begin{tikzcd}
\phi & \phi\psi \arrow[l] \\
\phi \arrow[u, equal] & \phi\psi \arrow[l] \arrow[u, equal] \arrow[ul, phantom, "{\scriptstyle (id_{a_2},\phi,\psi;0)}"]
\end{tikzcd} \qquad \mathrm{I}^{\mathrm{h}}\left(\begin{smallmatrix}\phi\psi \\ \uparrow \\ \psi\end{smallmatrix}\right) = \begin{tikzcd}
\phi\psi \arrow[r, equal] & \phi\psi \\
\psi \arrow[u] \arrow[r, equal] & \psi \arrow[u] \arrow[ul, phantom, "{\scriptstyle (\phi,\psi,id_{a_0};0)}"]
\end{tikzcd} \tag{23}$$

for any two composable arrows $a_2 \overset{\phi}{\leftarrow} a_1 \overset{\psi}{\leftarrow} a_0$ in P.

Lemma 5.2. *With these definitions, $\mathcal{G}^k$ is a double groupoid (satisfying the filling condition).*

Proof. We first observe that the vertical composition of boxes in $\mathcal{G}^k$ is associative thanks to the 3-cocycle condition of k. In fact, let

$$\begin{tikzcd}
\gamma\delta\phi\psi & \gamma\delta\phi\psi\rho \arrow[l] \\
\delta\phi\psi \arrow[u] & \delta\phi\psi\rho \arrow[l] \arrow[u] \arrow[ul, phantom, "{\scriptstyle (\gamma,\gamma\delta\phi\psi,\rho;w)}"] \\
\phi\psi \arrow[u] & \phi\psi\rho, \arrow[l] \arrow[u] \arrow[ul, phantom, "{\scriptstyle (\delta,\phi\psi,\rho;v)}"] \\
\psi \arrow[u] & \psi\rho \arrow[l] \arrow[u] \arrow[ul, phantom, "{\scriptstyle (\phi,\psi,\rho;u)}"]
\end{tikzcd}$$

be three vertically composable boxes, defined by five arrows

$$a_5 \xleftarrow{\gamma} a_4 \xleftarrow{\delta} a_3 \xleftarrow{\phi} a_2 \xleftarrow{\psi} a_1 \xleftarrow{\rho} a_0$$

of P and elements $u \in \mathcal{A}(a_3)$, $v \in \mathcal{A}(a_4)$ and $w \in \mathcal{A}(a_5)$. Then,

$$\begin{aligned}
&\big((\gamma, \gamma\delta\phi\psi, \rho; w) \circ_{\mathrm{v}} (\delta, \phi\psi, \rho; v)\big) \circ_{\mathrm{v}} (\phi, \psi, \rho; u) \\
&\qquad = \big(\gamma\delta, \phi\psi, \rho; w + \gamma_* v + k(\gamma, \delta; \phi\psi) - k(\gamma, \delta, \phi\psi\rho)\big) \circ_{\mathrm{v}} (\phi, \psi, \rho; u) \\
&\qquad = \big(\gamma\delta\phi, \psi, \rho; w + \gamma_* v + k(\gamma, \delta, \phi\psi) - k(\gamma, \delta; \phi\psi\rho) + \gamma_*\delta_* u \\
&\qquad\qquad\qquad\qquad\qquad\qquad + k(\gamma\delta, \phi, \psi) - k(\gamma\delta, \phi, \psi\rho)\big),
\end{aligned}$$

and, on the other hand,

$$\begin{aligned}
&(\gamma, \gamma\delta\phi\psi, \rho; w) \circ_{\mathrm{v}} \big((\delta, \phi\psi, \rho; v) \circ_{\mathrm{v}} (\phi, \psi, \rho; u)\big) \\
&\qquad = (\gamma, \gamma\delta\phi\psi, \rho; w) \circ_{\mathrm{v}} \big(\delta\phi, \psi, \rho; v + \delta_*(u) + k(\delta, \phi, \psi) - k(\delta, \phi, \psi\rho)\big) \\
&\qquad = \big(\gamma\delta\phi, \psi, \rho; w + \gamma_* v + \gamma_*\delta_* u + \gamma_* k(\delta, \phi, \psi) - \gamma_* k(\delta, \phi, \psi\rho) \\
&\qquad\qquad\qquad\qquad\qquad\qquad + k(\gamma, \delta\phi, \psi) - k(\gamma, \delta\phi, \psi\rho)\big).
\end{aligned}$$

Hence the result follows by comparison, using that the cocycle condition of k applied to the lists of arrows $a_5 \xleftarrow{\gamma} a_4 \xleftarrow{\delta} a_3 \xleftarrow{\phi} a_2 \xleftarrow{\psi\rho} a_0$ and $a_5 \xleftarrow{\gamma} a_4 \xleftarrow{\delta} a_3 \xleftarrow{\phi} a_2 \xleftarrow{\psi} a_1$ gives the equalities

$$\begin{aligned}
\gamma_* k(\delta, \phi, \psi\rho) + k(\gamma, \delta\phi, \psi\rho) &= k(\gamma\delta, \phi, \psi\rho) + k(\gamma, \delta, \phi\psi\rho) - k(\gamma, \delta, \phi), \\
\gamma_* k(\delta, \phi, \psi) + k(\gamma, \delta\phi, \psi) &= k(\gamma\delta, \phi, \psi) + k(\gamma, \delta, \phi\psi) - k(\gamma, \delta, \phi).
\end{aligned}$$

The associativity of the horizontal composition of boxes is easier. Let

$$\begin{array}{ccccccc}
\phi\psi & \longleftarrow & \phi\psi\rho & \longleftarrow & \phi\psi\rho\lambda & \longleftarrow & \phi\psi\rho\lambda\mu \\
\uparrow & {\scriptstyle(\phi,\psi,\rho;u)} & \uparrow & {\scriptstyle(\phi,\psi\rho,\lambda;v)} & \uparrow & {\scriptstyle(\phi,\psi\rho\lambda,\mu;w)} & \uparrow \\
\psi & \longleftarrow & \psi\rho & \longleftarrow & \psi\rho\lambda & \longleftarrow & \psi\rho\lambda\mu
\end{array}$$

be boxes, defined by arrows $a_5 \overset{\phi}{\leftarrow} a_4 \overset{\psi}{\leftarrow} a_3 \overset{\rho}{\leftarrow} a_2 \overset{\lambda}{\leftarrow} a_1 \overset{\mu}{\leftarrow} a_0$ of P and elements $u, v, w \in \mathcal{A}(a_5)$. Then,

$$\begin{aligned}
\big((\phi,\psi,\rho;u) \circ_{\mathrm{h}} (\phi,\psi\rho,\lambda;v)\big) &\circ_{\mathrm{h}} (\phi,\psi\rho\lambda,\mu;w) \\
&= (\phi,\psi,\rho\lambda;u+v) \circ_{\mathrm{h}} (\phi,\psi\rho\lambda,\mu;w) \\
&= (\phi,\psi,\rho\lambda\mu;u+v+w) = (\phi,\psi,\rho;u) \circ_{\mathrm{h}} (\phi,\psi\rho,\lambda\mu;v+w) \\
&= (\phi,\psi,\rho;u) \circ_{\mathrm{h}} \big((\phi,\psi\rho,\lambda;v)\big) \circ_{\mathrm{h}} (\phi,\psi\rho\lambda,\mu;w)\big).
\end{aligned}$$

For any box $(\phi,\psi,\rho;u)$ as in (18), its respective vertical and horizontal inverses

$$\begin{array}{ccc}
\psi & \longleftarrow & \psi\rho \\
\uparrow & {\scriptstyle(\phi,\psi,\rho;u)^{-\mathrm{v}}} & \uparrow \\
\phi\psi & \longleftarrow & \phi\psi\rho
\end{array}
\qquad
\begin{array}{ccc}
\phi\psi\rho & \longleftarrow & \phi\psi \\
\uparrow & {\scriptstyle(\phi,\psi,\rho;u)^{-\mathrm{h}}} & \uparrow \\
\psi\rho & \longleftarrow & \psi,
\end{array}$$

are given by

$$\begin{cases}(\phi,\psi,\rho;u)^{-\mathrm{v}} = (\phi^{-1},\phi\psi,\rho;k(\phi^{-1},\phi,\psi\rho) - k(\phi^{-1},\phi,\psi) - \phi_*^{-1}u), \\ (\phi,\psi,\rho;u)^{-\mathrm{h}} = (\phi,\psi,\rho^{-1};-u).\end{cases} \tag{24}$$

The only non-straightforward verification here is that

$$(\phi,\psi,\rho;u) \circ_{\mathrm{v}} (\phi,\psi,\rho;u)^{-\mathrm{v}} = \mathrm{I}^{\mathrm{v}}(\phi\psi \leftarrow \phi\psi\rho).$$

which is as follows

$$\begin{aligned}
(\phi,\psi,\rho;u) &\circ_{\mathrm{v}} (\phi^{-1},\phi\psi,\rho;k(\phi^{-1},\phi,\psi\rho) - k(\phi^{-1},\phi,\psi) - \phi_*^{-1}u) \\
&= \big(id_d,\psi\psi,\rho;u + \phi_*k(\phi^{-1},\phi,\psi\rho) - \phi_*k(\phi^{-1},\phi,\psi) - u \\
&\qquad\qquad + k(\phi^{-1},\phi,\phi\psi) - k(\phi^{-1},\phi,\phi\psi\rho)\big) \\
&= \big(id_{a_3},\psi\psi,\rho;\phi_*k(\phi^{-1},\phi,\psi\rho) - \phi_*k(\phi^{-1},\phi,\psi) + k(\phi^{-1},\phi,\phi\psi) \\
&\qquad\qquad - k(\phi^{-1},\phi,\phi\psi\rho)\big) \\
&\overset{(25)}{=} \big(id_{a_3},\phi\psi,\rho;k(\phi,\phi^{-1},\phi) - k(\phi,\phi^{-1},\phi)\big) = (id_{a_3},\phi\psi,\rho;0) \\
&= \mathrm{I}^{\mathrm{v}}(\phi\psi \leftarrow \phi\psi\rho),
\end{aligned}$$

where we have used the equality

$$\phi_* k(\phi^{-1}, \phi, \psi) - k(\phi, \phi^{-1}, \phi\psi) = k(\phi, \phi^{-1}, \phi) \tag{25}$$

which follows from the 3-cocycle and normalization conditions of k for the sequence of arrows $a_3 \xleftarrow{\phi} a_2 \xleftarrow{\phi^{-1}} a_3 \xleftarrow{\phi} a_2 \xleftarrow{\psi} a_1$ in P.

All other requirements are easily verified, except perhaps the interchange law which is proved as follows. Suppose given boxes

$$\begin{tikzcd}
\delta\phi\psi & \delta\phi\psi\rho \arrow[l] & \delta\phi\psi\rho\lambda \arrow[l] \\
\phi\psi \arrow[u] \arrow[r, phantom, "{\scriptstyle(\delta,\phi\psi,\rho;v)}"] & \phi\psi\rho, \arrow[l] \arrow[u] \arrow[r, phantom, "{\scriptstyle(\delta,\phi\psi\rho,\lambda;v')}"] & \phi\psi\rho\lambda \arrow[l] \arrow[u] \\
\psi \arrow[u] \arrow[r, phantom, "{\scriptstyle(\phi,\psi,\rho;u)}"] & \psi\rho \arrow[l] \arrow[u] \arrow[r, phantom, "{\scriptstyle(\phi,\psi\rho,\lambda;u')}"] & \psi\rho\lambda \arrow[l] \arrow[u]
\end{tikzcd}$$

defined by arrows of P, $a_5 \xleftarrow{\delta} a_4 \xleftarrow{\phi} a_3 \xleftarrow{\psi} a_2 \xleftarrow{\rho} a_1 \xleftarrow{\lambda} a_0$, and elements $v, v' \in \mathcal{A}(a_5)$ and $u, u' \in \mathcal{A}(a_4)$. Then,

$$\begin{aligned}
&\big((\delta, \phi\psi, \rho; v) \circ_{\mathrm{v}} (\phi, \psi, \rho; u)\big) \circ_{\mathrm{h}} \big((\delta, \phi\psi\rho, \lambda; v') \circ_{\mathrm{v}} (\phi, \psi\rho, \lambda; u')\big) \\
&\quad = \big(\delta\phi, \psi, \rho; v + \delta_* u + k(\delta, \phi, \psi) - k(\delta, \phi, \psi\rho)\big) \circ_{\mathrm{h}} \big(\delta\phi, \psi\rho, \lambda; v' \\
&\qquad\qquad + \delta_* u' + k(\delta, \phi, \psi\rho) - k(\delta, \phi, \psi\rho\lambda)\big) \\
&\quad = \big(\delta\phi, \psi, \rho\lambda; v + \delta_* u + v' + \delta_* u' + k(\delta, \phi, \psi) - k(\delta, \phi, \psi\rho\lambda)\big) \\
&\quad = (\delta, \phi\psi, \rho\lambda; v + v') \circ_{\mathrm{v}} (\phi, \psi, \rho\lambda; u + u') \\
&\quad = \big((\delta, \phi\psi, \rho; v) \circ_{\mathrm{h}} (\delta, \phi\psi\rho, \lambda; v')\big) \circ_{\mathrm{v}} \big((\phi, \psi, \rho; u) \circ_{\mathrm{h}} (\phi, \psi\rho, \lambda; u')\big).
\end{aligned}$$

$\square$

Proposition 5.3. (i) *Let $(P, \mathcal{A}, \mathbf{k})$ be a Postnikov system. For any representative 3-cocycle $k \in Z^3(P, \mathcal{A})$ of $\mathbf{k}$, the Postnikov invariant of the double groupoid $\mathcal{G}^k$ is equivalent to $(P, \mathcal{A}, \mathbf{k})$, that is,*

$$[\Pi\mathcal{G}^k, \pi_2\mathcal{G}^k, \mathbf{k}\mathcal{G}^k] = [P, \mathcal{A}, \mathbf{k}].$$

(ii) *Suppose $(P, \mathcal{A}, \mathbf{k})$ and $(P', \mathcal{A}', \mathbf{k}')$ are equivalent Postnikov systems. Then, for any representative 3-cocycles $k \in Z^3(P, \mathcal{A})$ and $k' \in Z^3(P', \mathcal{A}')$ of $\mathbf{k}$ and $\mathbf{k}'$ respectively, there is a weak equivalence $\mathcal{G}^k \stackrel{\sim}{\to} \mathcal{G}^{k'}$.*

(iii) *Let $\mathcal{G}$ be a double groupoid. For any* 3*-cocycle* $k \in Z^3(\Pi\mathcal{G}, \pi_2\mathcal{G})$ *representative of the cohomology class* $\mathbf{k}\mathcal{G}$*, there is a weak equivalence* $\mathcal{G}^k \xrightarrow{\sim} \mathcal{G}$.

Proof. Firstly notice that the homotopy relation between paths in $\mathcal{G}^k$ is trivial. In fact, suppose

$$\begin{array}{ccc} \xi_1 \longleftarrow \eta & & \xi_1 \longleftarrow \mu \\ \uparrow & \simeq & \uparrow \\ \xi_0 & & \xi_0 \end{array}$$

are two homotopic paths in $\mathcal{G}^k$. This means that there is a box in $\mathcal{G}^k$ of the form

$$\begin{tikzcd} \eta & \mu \arrow[l] \\ \eta \arrow[u, equal] \arrow[r, equal] & \eta \arrow[u] \arrow[ul, phantom, "(\phi,\psi,\rho;u)"] \end{tikzcd}$$

for some composable arrows $a_3 \overset{\phi}{\leftarrow} a_2 \overset{\psi}{\leftarrow} a_1 \overset{\rho}{\leftarrow} a_0$ in P and some $u \in \mathcal{A}(a_3)$. But then, we have the equalities $\psi = \eta = \psi\rho = \phi\psi$ and $\phi\psi\rho = \mu$ which imply $\eta = \mu$.

(i) There is a functor $\mathfrak{f}^k : P \to \Pi\mathcal{G}^k$ which carries each object a of P to the identity morphism id_a, regarded as an object of $\mathcal{G}^k$, and carries a morphism $\rho : a_0 \to a_1$ of P to the path

$$\mathfrak{f}^k\rho = \begin{array}{c} id_{a_1} \longleftarrow \rho \\ \uparrow \\ id_{a_0}. \end{array}$$

If $\psi : a_1 \to a_2$ is another morphism in P, the equality $\mathfrak{f}^k(\psi\rho) = \mathfrak{f}^k\psi\ \mathfrak{f}^k\rho$ follows from the diagram in $\mathcal{G}$

$$\begin{tikzcd} id_{a_2} & \psi \arrow[l] & \psi\rho \arrow[l] \\ & id_{a_1} \arrow[u] & \rho \arrow[l] \arrow[u] \arrow[ul, phantom, "(\psi,id_{a_1},\rho;0)"] \\ & & id_{a_0} \arrow[u] \end{tikzcd}$$

and, for any object a of P,

$$\mathfrak{f}^k id_a = \begin{array}{c} id_a = id_a \\ \| \\ id_a \end{array} = id_{\mathfrak{f}^k a}.$$

So, $\mathfrak{f}^k$ is actually a functor which is clearly fully faithful. Indeed, it is an equivalence of groupoids since any object $\rho : a_0 \to a_1$ is isomorphic in $\Pi\mathcal{G}^k$ to the object $\mathfrak{f}^k a_0 = id_{a_0}$ because of the path

$$\begin{tikzcd} id_{a_0} \arrow[r, equal] & id_{a_0} \\ & \rho. \arrow[u] \end{tikzcd}$$

Now, for any object a of P, the abelian group $\pi_2(\mathcal{G}^k, \mathfrak{f}a)$ just consists of all the boxes in $\mathcal{G}^k$ of the form

$$\begin{tikzcd} id_a \arrow[r, equal] \arrow[d, equal] \arrow[dr, phantom, "{\scriptstyle (id_a, id_a, id_a; u)}"] & id_a \arrow[d, equal] \\ id_a \arrow[r, equal] & id_a \end{tikzcd}$$

with $u \in \mathcal{A}(a)$. The mapping $\mathfrak{F}^k : \mathcal{A}(a) \to \pi_2(\mathcal{G}^k, \mathfrak{f}^k a)$,

$$u \mapsto (id_a, id_a, id_a; u),$$

is clearly an isomorphism of groups, for any object a of P, and thus we see that we are in presence of a natural isomorphism $\mathfrak{F}^k : \mathcal{A} \cong \mathfrak{f}^* \pi_2 \mathcal{G}^k$.

To complete the proof, it is enough to prove that $\mathfrak{f}^{k*}(\mathbf{k}\mathcal{G}^k) = \mathfrak{F}^k_*(\mathbf{k})$. Indeed, we are going to prove that $\mathfrak{f}^{k*}(k^{\mathcal{G}^k}) = \mathfrak{F}^k_*(k)$ once we select, for each pair of composable arrows $a_2 \overset{\psi}{\leftarrow} a_1 \overset{\rho}{\leftarrow} a_0$ in P, the box

$$\theta_{\mathfrak{f}^k\psi, \mathfrak{f}^k\rho} = \begin{tikzcd} \psi & \psi\rho \arrow[l] \\ id_{a_1} \arrow[u] \arrow[ur, phantom, "{\scriptstyle (\psi, id_{a_1}, \rho; 0)}"] & \rho \arrow[l] \arrow[u] \end{tikzcd}$$

in the construction of the 3-cocycle $k^{\mathcal{G}^k}$. In fact, for any given composable arrows $a_3 \overset{\phi}{\leftarrow} a_2 \overset{\psi}{\leftarrow} a_1 \overset{\rho}{\leftarrow} a_0$ in P, by (16), the element

$$\mathfrak{f}^{k*}(k^{\mathcal{G}^k})(\phi, \psi, \rho) = k^{\mathcal{G}^k}(\mathfrak{f}^k\phi, \mathfrak{f}^k\psi, \mathfrak{f}^k\rho) \in \pi_2(\mathcal{G}^k, \mathfrak{f}^k a_3)$$

is the pasted box of the diagram

$$
\begin{tikzcd}
id_{a_3} \arrow[ddd, equal] \arrow[dddr, phantom, "\mathrm{I}^{\mathrm{v}}(id_{a_3}\leftarrow\phi)"] & \phi \arrow[l] & \phi\psi \arrow[l] \arrow[dr, phantom, "(\phi\psi{,}id_{a_2}{,}\rho;0)"] & \phi\psi\rho \arrow[l] \arrow[dddr, phantom, "\mathrm{I}^{\mathrm{v}}(\phi\psi\rho\leftarrow id_{a_3})"] & id_{a_3} \arrow[l] \arrow[ddd, equal] \\
 & & id_{a_1} \arrow[u] & \rho \arrow[l] \arrow[u] & \\
 & id_{a_2} \arrow[uu] \arrow[uur, phantom, "(\phi{,}id_{a_2}{,}\psi;0)"] & \psi \arrow[l] \arrow[u] \arrow[ur, phantom, "(\psi{,}id_{a_1}{,}\rho;0)^{-\mathrm{v}}"] & \psi\rho \arrow[l] \arrow[u] & \\
id_{a_3} & \phi \arrow[l] \arrow[u] \arrow[urr, phantom, "(\phi{,}id_{a_2}{,}\psi\rho;0)^{-\mathrm{v}}"] & & \phi\psi\rho \arrow[ll] \arrow[u] & id_{a_2} \arrow[l]
\end{tikzcd}
$$

Since, by (23) and (24),

$$
\begin{cases}
\mathrm{I}^{\mathrm{v}}(id_{a_3} \leftarrow \phi) = (id_{a_3}, id_{a_3}, \phi; 0),\\
\mathrm{I}^{\mathrm{v}}(\phi\psi\rho \leftarrow id_{a_3}) = (id_{a_3}, \phi\psi\rho, (\phi\psi\rho)^{-1}; 0),\\
(\psi, id_{a_1}, \rho; 0)^{-\mathrm{v}} = (\psi^{-1}, \psi, \rho; -k(\psi^{-1}, \psi, \rho)),\\
(\phi, id_{a_2}, \psi\rho; 0)^{-\mathrm{v}} = (\phi^{-1}, \phi, \psi\rho; -k(\phi^{-1}, \phi, \psi\rho)),
\end{cases}
$$

a direct computation, using (22) and (22), gives

$$
\begin{aligned}
&k^{\mathcal{G}^k}(\mathfrak{f}\phi, \mathfrak{f}\psi, \mathfrak{f}\rho)\\
&= \Big(id_{a_3}, id_{a_3}, id_{a_3}; -\phi_*\psi_* k(\psi^{-1}, \psi, \rho) + k(\phi\psi, \psi^{-1}, \psi\rho) - k(\phi\psi, \psi^{-1}, \psi)\\
&\qquad\qquad - \phi_* k(\phi^{-1}, \phi, \psi\rho) + k(\phi, \phi^{-1}, \phi\psi\rho) - k(\phi, \phi^{-1}, \phi)\Big)\\
&\overset{(25)}{=} \big(id_{a_3}, id_{a_3}, id_{a_3}; -\phi_*\psi_* k(\psi^{-1}, \psi, \rho) + k(\phi\psi, \psi^{-1}, \psi\rho) - k(\phi\psi, \psi^{-1}, \psi)\big).
\end{aligned}
$$

Moreover, since the cocycle condition of k on the sequence

$$
a_3 \overset{\phi\psi}{\leftarrow} a_1 \overset{\psi^{-1}}{\leftarrow} a_2 \overset{\psi}{\leftarrow} a_1 \overset{\rho}{\leftarrow} a_0
$$

yields

$$
\phi_*\psi_* k(\psi^{-1}, \psi, \rho) - k(\phi, \psi, \rho) - k(\phi\psi, \psi^{-1}, \psi\rho) + k(\phi\psi, \psi^{-1}, \psi) = 0
$$

we conclude that

$$
\begin{aligned}
k^{\mathcal{G}^k}(\mathfrak{f}^k\phi, \mathfrak{f}^k\psi, \mathfrak{f}^k\rho) &= \big(id_{a_3}, id_{a_3}, id_{a_3}; k(\phi, \psi, \rho)\big) = \mathfrak{F}^k k(\phi, \psi, \rho)\\
&= \mathfrak{F}^k_*(k)(\phi, \psi, \rho).
\end{aligned}
$$

(ii) By hypothesis, there is an equivalence $\mathfrak{f}: P \xrightarrow{\sim} P'$, a natural isomorphism $\mathfrak{F}: \mathcal{A} \cong \mathfrak{f}^*\mathcal{A}'$, and a normalized 2-cochain $c \in C^2(P, \mathfrak{f}^*\mathcal{A}')$ such that $\mathfrak{f}^*(k') = \mathfrak{F}_*(k) + \partial c$. A weak equivalence $F: \mathcal{G}^k \xrightarrow{\sim} \mathcal{G}^{k'}$ is then defined by the following assignments on objets, horizontal and vertical morphisms, and boxes

$$\rho \mapsto \mathfrak{f}\rho, \quad \big(\psi \leftarrow \psi\rho\big) \mapsto \big(\mathfrak{f}\psi \leftarrow \mathfrak{f}\psi\,\mathfrak{f}\rho\big), \quad \begin{pmatrix}\phi\psi\\ \uparrow\\ \psi\end{pmatrix} \mapsto \begin{matrix}\mathfrak{f}\phi\,\mathfrak{f}\psi\\ \uparrow\\ \mathfrak{f}\psi\end{matrix}$$

$$\begin{matrix}\phi\psi & \longleftarrow & \phi\psi\rho\\ \uparrow & {\scriptstyle(\phi,\psi,\rho;u)} & \uparrow\\ \psi & \longleftarrow & \psi\rho,\end{matrix} \quad\mapsto\quad \begin{matrix}\mathfrak{f}\phi\,\mathfrak{f}\psi & \longleftarrow & \mathfrak{f}\phi\,\mathfrak{f}\psi\,\mathfrak{f}\rho\\ \uparrow & {\scriptstyle\big(\mathfrak{f}\phi,\mathfrak{f}\psi,\mathfrak{f}\rho;\mathfrak{F}u+c(\phi,\psi)-c(\phi,\psi\rho)\big)} & \uparrow\\ \mathfrak{f}\psi & \longleftarrow & \mathfrak{f}\psi\,\mathfrak{f}\rho.\end{matrix}$$

So defined, one verifies easily that $F: \mathcal{G}^k \to \mathcal{G}^{k'}$ is actually a double functor. That F is a weak equivalence follows from the commutativity of the diagrams

$$\begin{tikzcd} P \arrow[r, "\mathfrak{f}^k"] \arrow[d, "\mathfrak{f}"'] & \Pi\mathcal{G}^k \arrow[d, "\Pi F"] \\ P' \arrow[r, "\mathfrak{f}^{k'}"] & \Pi\mathcal{G}^{k'} \end{tikzcd} \qquad \begin{tikzcd} \mathcal{A}(a) \arrow[r, "\mathfrak{F}^k"] \arrow[d, "\mathfrak{F}"'] & \pi_2(\mathcal{G}^k, \mathfrak{f}^k a) \arrow[d, "\pi_2 F"] \\ \mathcal{A}'(\mathfrak{f}a) \arrow[r, "\mathfrak{F}^{k'}"] & \pi_2(\mathcal{G}^{k'}, \mathfrak{f}^{k'} a) \end{tikzcd}$$

where $\mathfrak{f}$, $\mathfrak{f}^k$ and $\mathfrak{f}^{k'}$ are equivalences of groupoids and, for any object a of P, $\mathfrak{F}$, $\mathfrak{F}^k$ and $\mathfrak{F}^{k'}$ are isomorphisms of groups.

(iii) By Lemma 4.3 (i), we can assume that $k = k^{\mathcal{G}}$ for a certain selection of representative paths (f_ρ, b_ρ, x_ρ) of the morphisms ρ in $\Pi\mathcal{G}$ and the boxes $\theta_{\psi,\rho}$, as in (3) and (12). Then, a double functor $F: \mathcal{G}^k \xrightarrow{\sim} \mathcal{G}$ is defined by the following assignments on objets, horizontal and vertical morphisms, and boxes

$$F(\rho) = b_\rho, \quad F(\psi \leftarrow \psi\rho) = \big(b_\psi \xleftarrow{f_\psi^{-1} f_{\psi\rho}} b_{\psi\rho}\big)\ , \quad F\begin{pmatrix}\psi\rho\\ \uparrow\\ \rho\end{pmatrix} = \begin{matrix}b_{\psi\rho}\\ \uparrow{\scriptstyle x_{\psi\rho}x_\rho^{-1}}\\ b_\rho\end{matrix}\ ,$$

$$F\left(\begin{array}{ccc}\phi\psi & \longleftarrow & \phi\psi\rho \\ \uparrow & {\scriptstyle(\phi,\psi,\rho;u)} & \uparrow \\ \psi & \longleftarrow & \psi\rho\end{array}\right) = \begin{array}{ccc} b_{\phi\psi} & \xleftarrow{\ f_{\phi\psi}^{-1}f_{\phi\psi\rho}\ } & b_{\phi\psi\rho} \\ {\scriptstyle x_{\phi\psi}x_\psi^{-1}}\uparrow & {\scriptstyle f_{\phi\psi*}^{-1}(\sigma)+\theta_{\phi,\psi}^{-\mathrm{h}}\circ_{\mathrm{h}}\theta_{\phi,\psi\rho}} & \uparrow{\scriptstyle x_{\phi\psi\rho}x_{\psi\rho}^{-1}} \\ b_\psi & \xleftarrow[\ f_\psi^{-1}f_{\psi\rho}\ ]{} & b_{\psi\rho}\end{array}.$$

Many of the details to confirm that F, so defined, is a double functor are routine and easily verifiable, so are left to the reader. For instance, if ϕ, ψ, ρ are any three composable morphisms in $\mathcal{G}$,

$$F(\phi \leftarrow \phi\psi)\, F(\phi\psi \leftarrow \phi\psi\rho) = f_\phi^{-1}f_{\phi\psi}^{-1}f_{\phi\psi}f_{\phi\psi\rho}^{-1} = f_\phi^{-1}f_{\phi\psi\rho}^{-1} = F(\phi \leftarrow \phi\psi\rho)$$

and thus we see that F preserves horizontal composition of morphisms. The proof that F preserves composition of boxes is as follows. Suppose two vertically composable boxes in $\mathcal{G}^k$, as in (19). Then,

$$\begin{aligned}
&F(\delta,\phi\psi,\rho;\tau)\circ_{\mathrm{v}} F(\phi,\psi,\rho;\tau)\\
&= (f_{\delta\phi\psi*}^{-1}\tau + \theta_{\delta,\phi\psi}^{-\mathrm{h}}\circ_{\mathrm{h}}\theta_{\delta,\phi\psi\rho})\circ_{\mathrm{v}}(f_{\phi\psi*}^{-1}\sigma + \theta_{\phi,\psi}^{-\mathrm{h}}\circ_{\mathrm{h}}\theta_{\phi,\psi\rho})\\
&\overset{(14)}{=} \big(f_{\delta\phi\psi*}^{-1}(\tau - k(\delta,\phi\psi,\rho)) + \theta_{\delta\phi\psi,\rho}\circ_{\mathrm{v}}\theta_{\phi\psi,\rho}^{-\mathrm{v}}\big)\circ_{\mathrm{v}}\big(f_{\phi\psi*}^{-1}(\sigma - k(\phi,\psi,\rho))\\
&\qquad\qquad + \theta_{\phi\psi,\rho}\circ_{\mathrm{v}}\theta_{\psi,\rho}^{-\mathrm{v}}\big)\\
&\overset{(9)}{=} f_{\delta\phi\psi*}^{-1}(\tau - k(\delta,\phi\psi,\rho)) + x_{\delta\phi\psi*}x_{\phi\psi*}^{-1}f_{\phi\psi*}^{-1}(\sigma - k(\phi,\psi,\rho))\\
&\qquad\qquad + \theta_{\delta\phi\psi,\rho}\circ_{\mathrm{v}}\theta_{\phi\psi,\rho}^{-\mathrm{v}}\circ_{\mathrm{v}}\theta_{\phi\psi,\rho}\circ_{\mathrm{v}}\theta_{\psi,\rho}^{-\mathrm{v}}\\
&\overset{2.4}{=} f_{\delta\phi\psi*}^{-1}(\tau - k(\delta,\phi\psi,\rho)) + f_{\delta\phi\psi*}^{-1}f_{\delta*}x_{\delta*}(\sigma - k(\phi,\psi,\rho)) + \theta_{\delta\phi\psi,\rho}\circ_{\mathrm{v}}\theta_{\psi,\rho}^{-\mathrm{v}}\\
&\overset{2.4}{=} f_{\delta\phi\psi*}^{-1}(\tau - k(\delta,\phi\psi,\rho)) + f_{\delta\phi\psi*}^{-1}\delta_*(\sigma - k(\phi,\psi,\rho)) + \theta_{\delta\phi\psi,\rho}\circ_{\mathrm{v}}\theta_{\psi,\rho}^{-\mathrm{v}}\\
&= f_{\delta\phi\psi*}^{-1}(\tau + \delta_*\sigma) + f_{\delta\phi\psi*}^{-1}(-k(\delta,\phi\psi,\rho) - \delta_*k(\phi,\psi,\rho)) + \theta_{\delta\phi\psi,\rho}\circ_{\mathrm{v}}\theta_{\psi,\rho}^{-\mathrm{v}},
\end{aligned}$$

$$\begin{aligned}
&F\big((\delta,\phi\psi,\rho;\tau)\circ_{\mathrm{v}}(\phi,\psi,\rho;\tau)\big)\\
&\overset{(20)}{=} F(\delta\phi,\psi,\rho;\tau + \delta_*\sigma + k^{\mathcal{G}}(\delta,\phi,\psi) - k^{\mathcal{G}}(\delta,\phi,\psi\rho))\\
&= f_{\delta\phi\psi*}^{-1}(\tau + \delta_*\sigma + k^{\mathcal{G}}(\delta,\phi,\psi) - k^{\mathcal{G}}(\delta,\phi,\psi\rho)) + \theta_{\delta\phi,\psi}^{-\mathrm{h}}\circ_{\mathrm{h}}\theta_{\delta\phi,\psi\rho}\\
&\overset{(14)}{=} f_{\delta\phi\psi*}^{-1}(\tau + \delta_*\sigma) + f_{\delta\phi\psi*}^{-1}\big(k^{\mathcal{G}}(\delta,\phi,\psi) - k^{\mathcal{G}}(\delta,\phi,\psi\rho)) - k^{\mathcal{G}}(\delta\phi,\psi,\rho)\big)\\
&\qquad\qquad + \theta_{\delta\phi\psi,\rho}\circ_{\mathrm{v}}\theta_{\psi,\rho}^{-\mathrm{v}},
\end{aligned}$$

and the result follows from Lemma 3.5, thanks to the 3-cocycle condition of k. To prove that Γ preserves horizontal composition of boxes is easier. Suppose two horizontally composable boxes in $\mathcal{G}^k$, as in (21). Then,

$$\begin{aligned}
F(\phi,\psi,\rho;\sigma) \circ_{\mathrm{h}} F(\phi,\psi\rho,\lambda;\tau) \\
&= (f^{-1}_{\phi\psi *}\sigma + \theta^{-\mathrm{h}}_{\phi,\psi} \circ_{\mathrm{h}} \theta_{\phi,\psi\rho}) \circ_{\mathrm{h}} (f^{-1}_{\phi\psi\rho *}\tau + \theta^{-\mathrm{h}}_{\phi,\psi\rho} \circ_{\mathrm{h}} \theta_{\phi,\psi\rho\lambda}) \\
&\overset{(8)}{=} f^{-1}_{\phi\psi *}\sigma + f^{-1}_{\phi\psi *}\tau + \theta^{-\mathrm{h}}_{\phi,\psi} \circ_{\mathrm{h}} \theta_{\phi,\psi\rho} \circ_{\mathrm{h}} \theta^{-\mathrm{h}}_{\phi,\psi\rho} \circ_{\mathrm{h}} \theta_{\phi,\psi\rho\lambda} \\
&= f^{-1}_{\phi\psi *}(\sigma+\tau) + \theta^{-\mathrm{h}}_{\phi,\psi} \circ_{\mathrm{h}} \theta_{\phi,\psi\rho\lambda} \\
&= F(\phi,\psi,\rho\lambda;\sigma+\tau) = F\big((\phi,\psi,\rho;\sigma) \circ_{\mathrm{h}} (\phi,\psi\rho,\lambda;\tau)\big).
\end{aligned}$$

That F preserves identity boxes is also easily checked. For instance,

$$\begin{aligned}
F\mathrm{I}^{\mathrm{v}}(\phi \leftarrow \phi\psi) &= F(id,\phi,\psi;0) = \theta^{-\mathrm{h}}_{id,\phi} \circ_{\mathrm{h}} \theta_{id,\phi\psi} = \mathrm{I}^{\mathrm{v}} f^{-1}_{\phi}\, \mathrm{I}^{\mathrm{v}} f_{\phi\psi} \\
&= \mathrm{I}^{\mathrm{v}}(f^{-1}_{\phi} f_{\phi\psi}) = \mathrm{I}^{\mathrm{v}} F(\phi \leftarrow \phi\psi).
\end{aligned}$$

This double functor F is a weak equivalence. In fact, the induced functor on fundamental groupoids $\Pi F : \Pi\mathcal{G}^k \to \Pi\mathcal{G}$ is an equivalence since its composition with the equivalence $\mathfrak{f}^k : \Pi\mathcal{G} \simeq \Pi\mathcal{G}^k$ is the identity functor on $\Pi\mathcal{G}$: for any morphism $\rho \in \Pi\mathcal{G}(a,b)$,

$$\Pi F(\mathfrak{f}^k\rho) = \Pi F\begin{pmatrix} id_b \longleftarrow \rho \\ \hfill \uparrow \\ \hfill id_a \end{pmatrix} = \begin{bmatrix} b \xleftarrow{f_\rho} b_\rho \\ \hfill \uparrow{\scriptstyle x_\rho} \\ \hfill a \end{bmatrix} = [f_\rho, b_\rho, x_\rho] = \rho.$$

Furthermore, for any object a of $\mathcal{G}$, the induced map

$$\pi_2 F : \pi_2(\mathcal{G}^k, id_a) \to \pi_2(\mathcal{G}, a)$$

is the obvious isomorphism

$$\begin{array}{ccc} id_a & = & id_a \\ \| & {\scriptstyle (id_a,id_a,id_a;\sigma)} & \| \\ id_a & = & id_a \end{array} \quad \mapsto \quad \begin{array}{ccc} a & = & a \\ \| & \sigma & \| \\ a & = & a \end{array}.$$

□

Corollary 5.4. *Two double groupoids $\mathcal{G}$ and $\mathcal{G}'$ are weak equivalent if and only if there is a double groupoid $\mathcal{G}''$ with weak equivalences $\mathcal{G} \xleftarrow{\sim} \mathcal{G}'' \xrightarrow{\sim} \mathcal{G}'$.*

Proof. Suppose $\mathcal{G}$ and $\mathcal{G}'$ are weak equivalent. By Theorem 4.4, they have the same Postnikov invariant, that is, the Postnikov systems $(\Pi\mathcal{G}, \pi_2\mathcal{G}, \mathbf{k}\mathcal{G})$ and $(\Pi\mathcal{G}', \pi_2\mathcal{G}', \mathbf{k}\mathcal{G}')$ are equivalent. Then, by Proposition 5.3 (ii) and (iii), for any representative 3-cocycles of $\mathbf{k}\mathcal{G}$ and $\mathbf{k}\mathcal{G}'$, say k and k' respectively, there is a sequence of weak equivalences

$$\mathcal{G} \xleftarrow{\sim} \mathcal{G}^k \xrightarrow{\sim} \mathcal{G}^{k'} \xrightarrow{\sim} \mathcal{G}'.$$

□

6. Geometric realization

Theorem 6.1. *The Postnikov invariant of a double groupoid $\mathcal{G}$ agrees with the Postnikov invariant of its geometric realization $|\mathcal{G}|$.*

Proof. This follows from Proposition 6.2 below. □

For a groupoid P, let us recall from the beginning of Section 4 that $\mathrm{N}P$ denotes its nerve, that is, the simplicial set with m-simplices the composable sequences $\beta = (\beta m \overset{\beta_m}{\leftarrow} \cdots \overset{\beta_1}{\leftarrow} \beta 0)$ of m arrows in P. If $(P, \mathcal{A}, \mathbf{k})$ is any Postnikov system and we select any normalized 3-cocycle $k \in Z^3(P, \mathcal{A})$ representative of the cohomology class $\mathbf{k} \in H^3(P, \mathcal{A})$, then the equivalence class $[P, \mathcal{A}, \mathbf{k}]$ is justly realized as the unique Postnikov invariant of (the geometric realization of) the simplicial set homotopy colimit of the functor

$$K(\mathcal{A}, 2) : P \to \mathbf{Sset}, \qquad a \mapsto K(\mathcal{A}(a), 2),$$

twisted by the 3-cocycle k (see, for instance, Goerss and Jardine [18, Chapter VI, Lemma 5.8]). This simplicial set, which we denote by

$$\underset{P}{\mathrm{hocolim}}\, K(\mathcal{A}, 2; k), \tag{26}$$

has the same simplices as the ordinary homotopy colimit $\underset{P}{\mathrm{hocolim}}\, K(\mathcal{A}, 2)$, that is, its set of m-simplices is

$$\bigsqcup_{\beta \in \mathrm{N}P_m} K(\mathcal{A}(\beta m), 2)_m.$$

Its face and degeneracy maps are also the same as those of non-twisted homotopy colimit, except the last face maps which are here canonically affected by the cocycle k. This twisted homotopy colimit (26) becomes a Kan complex that is coskeletal in dimensions higher than three and whose 3-truncation can be described explicitly as below

$$\begin{tikzcd}
\bigsqcup\limits_{\beta\in \mathrm{N}P_3} \mathcal{A}(\beta 3)^3 \arrow[r, shift left=3, "d_0"] \arrow[r, shift left=1] \arrow[r, shift right=1] \arrow[r, shift right=3, "d_3"'] & \bigsqcup\limits_{\beta\in \mathrm{N}P_2} \mathcal{A}(\beta 2) \arrow[l, bend right=60, "s_0"'] \arrow[l, bend right=45] \arrow[l, bend right=30, "s_2"] \arrow[r, shift left=2, "d_0"] \arrow[r] \arrow[r, shift right=2, "d_2"'] & \mathrm{N}P_1 \arrow[l, bend right=50, "s_0"'] \arrow[l, bend right=35, "s_1"] \arrow[r, shift left=1, "d_0"] \arrow[r, shift right=1, "d_1"'] & \mathrm{N}P_0 \arrow[l, bend right=40, "s_0"']
\end{tikzcd}$$

where, for any $\beta \in \mathrm{N}P_2$ and $\sigma \in \mathcal{A}(\beta 2)$

$$d_i(\beta, \sigma) = d_i\beta, \quad 0 \leq i \leq 2,$$

for any $\beta \in \mathrm{N}P_3$ and $(\sigma_0, \sigma_1, \sigma_2) \in \mathcal{A}(\beta 3)^3$,

$$d_i(\beta, \sigma_0, \sigma_1 \sigma_2) = \begin{cases} (d_i\beta, \sigma_i) & \text{if} \quad 0 \leq i \leq 2, \\ \big(d_3\beta, \beta_{3*}^{-1}(k(\beta) + \sigma_2 - \sigma_1 + \sigma_0)\big) & \text{if} \quad i = 3, \end{cases}$$

for any $\beta \in \mathrm{N}P_1$,

$$s_i(\beta) = (s_i\beta, 0), \quad i = 0, 1,$$

and, for any $\beta \in \mathrm{N}P_2$ and $\sigma \in \mathcal{A}(\beta 2)$,

$$s_i(\beta, \sigma) = \begin{cases} (s_0\beta, \sigma, \sigma, 0) & \text{if} \quad i = 0, \\ (s_1\beta, 0, \sigma, \sigma) & \text{if} \quad i = 1, \\ (s_2\beta, 0, 0, \sigma) & \text{if} \quad i = 2. \end{cases}$$

Now, for a double groupoid $\mathcal{G}$, let $\mathrm{NN}\mathcal{G}$ denote its double nerve, that is, the bisimplicial set where a (p, q)-simplex is a subdivision of a box of $\mathcal{G}$ as a matrix of $p \times q$ horizontally and vertically composable boxes of the form

$$
\begin{array}{ccccccc}
a_{pq} & \xleftarrow{f_{pq}} & a_{p-1q} & \cdots & a_{1q} & \xleftarrow{f_{1q}} & a_{0q} \\
{\scriptstyle x_{pq}}\uparrow & \theta_{p,q} & \uparrow{\scriptstyle x_{p-1q}} & & {\scriptstyle x_{1q}}\uparrow & \theta_{1,q} & \uparrow{\scriptstyle x_{0q}} \\
a_{pq-1} & \xleftarrow[f_{pq-1}]{} & a_{p-1q-1} & \cdots & a_{1q} & \xleftarrow[f_{1q-1}]{} & a_{0q-1} \\
\uparrow & & \uparrow & & \uparrow & & \uparrow \\
\vdots & & \vdots & & \vdots & & \vdots \\
\uparrow & & \uparrow & & \uparrow & & \uparrow \\
a_{p1} & \xleftarrow{f_{p1}} & a_{p-11} & \cdots & a_{11} & \xleftarrow{f_{11}} & a_{01} \\
{\scriptstyle x_{p1}}\uparrow & \theta_{p,1} & \uparrow{\scriptstyle p-11} & & {\scriptstyle x_{11}}\uparrow & \theta_{1,1} & \uparrow{\scriptstyle x_{01}} \\
a_{p0} & \xleftarrow[f_{p0}]{} & a_{p-10} & \cdots & a_{10} & \xleftarrow[f_{10}]{} & a_{00}
\end{array}
$$

The bisimplicial face maps are the natural ones, induced by horizontal and vertical composition of boxes in $\mathcal{G}$, and the degeneracy ones by appropriate identity boxes. We picture $\mathrm{NN}\mathcal{G}$ so that the set of (p, q)-simplices is the set in the p-th row and q-th column. Thus, its p-th column, $\mathrm{NN}\mathcal{G}_{p\bullet}$, is the nerve of the "vertical" groupoid whose objects are strings $\cdot \overset{f_p}{\leftarrow} \cdots \overset{f_1}{\leftarrow} \cdot$ of p composable horizontal arrows in $\mathcal{G}$ and whose arrows are length p sequences of horizontally composable boxes

$$
\begin{array}{ccccccccccc}
\cdot & \xleftarrow{g_p} & \cdot & \cdots\cdots & \cdot & \longleftarrow & \cdot & \xleftarrow{g_1} & \cdot \\
\uparrow & \theta_p & \uparrow & \cdots & \uparrow & \theta_2 & \uparrow & \theta_1 & \uparrow \\
\cdot & \xleftarrow[f_p]{} & \cdot & \cdots\cdots & \cdot & \longleftarrow & \cdot & \xleftarrow[f_1]{} & \cdot
\end{array}
$$

Similarly, the q-th column, $\mathrm{NN}\mathcal{G}_{\bullet q}$, is the nerve of the "horizontal" groupoid whose objects are length q sequences of composable vertical morphisms in $\mathcal{G}$ and whose arrows are sequences of q vertically composable boxes. In particular, $\mathrm{NN}\mathcal{G}_{0\bullet}$ and $\mathrm{NN}\mathcal{G}_{\bullet 0}$ are, respectively, the nerves of the groupoids of vertical and horizontal morphisms of $\mathcal{G}$.

The geometric realization $|\mathcal{G}|$ of the double groupoid $\mathcal{G}$ is, by definition, the geometric realization of the simplicial set *diagonal* of its double nerve, that is, $|\mathcal{G}| = |\triangle\, \mathrm{NN}\mathcal{G}|$. By Cegarra-Remedios [12, Therem 1.1] or Zisman [27], $|\mathcal{G}|$ can be also realized, up to homotopy equivalence, as the geometric realization of the Artin-Mazur *total* simplicial set [2, Section III] (aka *codiagonal* or $\overline{W}$) of the double nerve, $\nabla\,\mathrm{NN}\mathcal{G}$. A direct analysis of this simplicial set tell us that it is a Kan complex in which any simplex of dimension higher than two is determined by any three of its faces. In particular, it is coskeletal in dimensions higher than 3, so that it is completely determined by its 3-truncation, which is explicitly described as follows. Its vertices are the objects a of $\mathcal{G}$. The 1-simplices ξ_1 are the paths of $\mathcal{G}$

$$\xi_1: \quad \begin{tikzcd} a_{11} & a_{01} \arrow[l, "f_{11}"'] \\ & a_{00} \arrow[u, "x_{01}"'] \end{tikzcd}$$

whose faces are $d_0\xi_1 = a_{11}$ and $d_1\xi_1 = a_{00}$. The 2-simplices ξ_2 are the diagrams in $\mathcal{G}$

$$\xi_2: \quad \begin{tikzcd} a_{22} & a_{12} \arrow[l, "f_{22}"'] & a_{02} \arrow[l, "f_{12}"'] \\ & a_{11} \arrow[u, "x_{12}"] \arrow[ur, phantom, "\theta_{12}"] & a_{01} \arrow[l, "f_{11}"] \arrow[u, "x_{02}"'] \\ & & a_{00} \arrow[u, "x_{01}"'] \end{tikzcd}$$

with faces

$$d_0\xi_2 = \begin{tikzcd} a_{22} & a_{12} \arrow[l, "f_{22}"'] \\ & a_{11}, \arrow[u, "x_{12}"'] \end{tikzcd} \quad d_1\xi_2 = \begin{tikzcd} a_{22} & a_{02} \arrow[l, "f_{22}f_{12}"'] \\ & a_{00}, \arrow[u, "x_{02}x_{01}"'] \end{tikzcd} \quad d_2\xi_2 = \begin{tikzcd} a_{11} & a_{01} \arrow[l, "f_{11}"'] \\ & a_{00}, \arrow[u, "x_{01}"'] \end{tikzcd}$$

and its 3-simplices ξ_3 are the diagrams in $\mathcal{G}$

$$\xi_3: \quad \begin{tikzcd} a_{33} & a_{23} \arrow[l, "f_{33}"'] & a_{13} \arrow[l, "f_{23}"'] & a_{03} \arrow[l, "f_{13}"'] \\ & a_{22} \arrow[u, "x_{23}"] \arrow[ur, phantom, "\theta_{23}"] & a_{12} \arrow[l, "f_{22}"] \arrow[u, "x_{13}"'] \arrow[ur, phantom, "\theta_{13}"] & a_{02} \arrow[l, "f_{12}"] \arrow[u, "x_{03}"'] \\ & & a_{11} \arrow[u, "x_{12}"] \arrow[ur, phantom, "\theta_{12}"] & a_{01} \arrow[l, "f_{11}"] \arrow[u, "x_{02}"'] \\ & & & a_{00} \arrow[u, "x_{01}"'] \end{tikzcd}$$

with faces

$$d_0\xi_3 = \begin{tikzcd} a_{33} & a_{23} \arrow[l, "f_{33}"'] & a_{13} \arrow[l, "f_{23}"'] \\ & a_{22} \arrow[u, "x_{23}"] \arrow[ur, phantom, "\theta_{23}"] & a_{12} \arrow[l, "f_{22}"] \arrow[u, "x_{13}"'] \\ & & a_{11}, \arrow[u, "x_{12}"'] \end{tikzcd} \qquad d_1\xi_3 = \begin{tikzcd} a_{33} & a_{23} \arrow[l, "f_{33}"'] & a_{03} \arrow[l, "f_{23}f_{13}"'] \\ & a_{22} \arrow[u, "x_{23}"] \arrow[ur, phantom, "\theta_{23}\circ_{\mathrm{h}}\theta_{13}"] & a_{02} \arrow[l, "f_{22}f_{12}"] \arrow[u, "x_{03}"'] \\ & & a_{00}, \arrow[u, "x_{02}x_{01}"'] \end{tikzcd}$$

$$d_2\xi_3 = \begin{tikzcd} a_{33} & a_{13} \arrow[l, "f_{33}f_{23}"'] & a_{03} \arrow[l, "f_{13}"'] \\ & a_{11} \arrow[u, "x_{13}x_{12}"] \arrow[ur, phantom, "\theta_{13}\circ_{\mathrm{v}}\theta_{12}"] & a_{01} \arrow[l, "f_{11}"] \arrow[u, "x_{03}x_{02}"'] \\ & & a_{00}, \arrow[u, "x_{01}"'] \end{tikzcd} \qquad d_3\xi_3 = \begin{tikzcd} a_{22} & a_{12} \arrow[l, "f_{22}"'] & a_{02} \arrow[l, "f_{12}"'] \\ & a_{11} \arrow[u, "x_{12}"] \arrow[ur, phantom, "\theta_{12}"] & a_{01} \arrow[l, "f_{11}"] \arrow[u, "x_{02}"'] \\ & & a_{00}. \arrow[u, "x_{01}"'] \end{tikzcd}$$

Degeneracies are defined by

$$s_0 a = \begin{tikzcd} a \arrow[r, equal] & a \\ & a \arrow[u, equal] \end{tikzcd} \quad s_0\xi_1 = \begin{tikzcd} a_{11} & a_{01} \arrow[l, "f_{11}"'] \arrow[r, equal] & a_{01} \\ & a_{11} \arrow[u] \arrow[r, equal] \arrow[ur, phantom, "\mathrm{I}^{\mathrm{h}}x_{01}"] & a_{01} \arrow[u, "x_{01}"'] \\ & & a_{00} \arrow[u, equal] \end{tikzcd} \quad s_1\xi_1 = \begin{tikzcd} a_{11} \arrow[r, equal] & a_{11} & a_{01} \arrow[l, "f_{11}"'] \\ & a_{11} \arrow[u, equal] \arrow[ur, phantom, "\mathrm{I}^{\mathrm{v}}f_{11}"] & a_{01} \arrow[l] \arrow[u, equal] \\ & & a_{00} \arrow[u, "x_{01}"'] \end{tikzcd}$$

$$s_0\xi_2 = \begin{tikzcd} a_{22} & a_{12} \arrow[l, "f_{22}"'] & a_{02} \arrow[l, "f_{12}"'] \arrow[r, equal] & a_{02} \\ & a_{11} \arrow[u, "x_{12}"] \arrow[ur, phantom, "\theta_{12}"] & a_{01} \arrow[l, "f_{11}"] \arrow[u] \arrow[r, equal] \arrow[ur, phantom, "\mathrm{I}^{\mathrm{h}}x_{02}"] & a_{01} \arrow[u, "x_{02}"'] \\ & & a_{00} \arrow[u] \arrow[r, equal] \arrow[ur, phantom, "\mathrm{I}^{\mathrm{h}}x_{01}"] & a_{00} \arrow[u, "x_{01}"'] \\ & & & a_{00} \arrow[u, equal] \end{tikzcd} \qquad s_1\xi_2 = \begin{tikzcd} a_{22} & a_{12} \arrow[l, "f_{22}"'] \arrow[r, equal] & a_{12} & a_{02} \arrow[l, "f_{12}"'] \\ & a_{11} \arrow[u, "x_{12}"] \arrow[r, equal] \arrow[ur, phantom, "\mathrm{I}^{\mathrm{h}}x_{12}"] & a_{11} \arrow[u] \arrow[ur, phantom, "\theta_{12}"] & a_{01} \arrow[l] \arrow[u, "x_{02}"'] \\ & & a_{11} \arrow[u, equal] \arrow[ur, phantom, "\mathrm{I}^{\mathrm{v}}f_{11}"] & a_{01} \arrow[l, "f_{11}"] \arrow[u, equal] \\ & & & a_{00} \arrow[u, "x_{01}"'] \end{tikzcd}$$

$$s_2\xi_2 = \begin{tikzcd} a_{22} \arrow[r, equal] & a_{22} & a_{12} \arrow[l, "f_{22}"'] & a_{02} \arrow[l, "f_{12}"'] \\ & a_{22} \arrow[u, equal] \arrow[ur, phantom, "\mathrm{I}^{\mathrm{v}}f_{22}"] & a_{12} \arrow[l] \arrow[u, equal] \arrow[ur, phantom, "\mathrm{I}^{\mathrm{v}}f_{12}"] & a_{02} \arrow[l] \arrow[u, equal] \\ & & a_{11} \arrow[u, "x_{12}"] \arrow[ur, phantom, "\theta_{12}"] & a_{01} \arrow[l, "f_{11}"] \arrow[u, "x_{02}"'] \\ & & & a_{00} \arrow[u, "x_{01}"'] \end{tikzcd}$$

Proposition 6.2. *Let $\mathcal{G}$ be a double groupoid. For any normalized 3-cocycle $k \in Z^3(\Pi\mathcal{G}, \pi_2\mathcal{G})$ representing the cohomology class $\mathbf{k}\mathcal{G}$, there is a weak equivalence of simplicial sets*

$$\Gamma : \operatorname*{hocolim}_{\Pi\mathcal{G}} K(\pi_2\mathcal{G}, 2; k) \xrightarrow{\sim} \nabla\, \mathrm{NN}\mathcal{G}.$$

Proof. By Lemma 4.3 (i), we can assume that $k = k^{\mathcal{G}}$ for a certain selection of representative paths (f_ρ, b_ρ, x_ρ) of the morphisms ρ in $\Pi\mathcal{G}$ and the boxes $\theta_{\psi,\rho}$, as in (3) and (12). The claimed simplicial map Γ, which is completely defined by its 3-truncation

$$\begin{tikzcd}
\coprod\limits_{\beta\in \mathrm{N}\Pi\mathcal{G}_3} \pi_2(\mathcal{G},\beta 3)^3 \arrow[r, shift left=3, "d_0"] \arrow[r, shift left=1] \arrow[r, shift right=1] \arrow[r, shift right=3, "d_3"'] \arrow[d, "\Gamma_3"'] & \coprod\limits_{\beta\in \mathrm{N}\Pi\mathcal{G}_2} \pi_2(\mathcal{G},\beta 2) \arrow[l, bend right=40, "s_0"'] \arrow[l, bend right=30] \arrow[l, bend right=20, "s_2"] \arrow[r, shift left=2, "d_0"] \arrow[r] \arrow[r, shift right=2, "d_2"'] \arrow[d, "\Gamma_2"] & \mathrm{N}\Pi\mathcal{G}_1 \arrow[l, bend right=40, "s_0"'] \arrow[l, bend right=25, "s_1"] \arrow[r, shift left=1, "d_0"] \arrow[r, shift right=1, "d_1"'] \arrow[d, "\Gamma_1"] & \mathrm{N}\Pi\mathcal{G}_0 \arrow[l, bend right=30, "s_0"'] \arrow[d, equal, "\Gamma_0"] \\
\nabla\mathrm{NN}\mathcal{G}_3 \arrow[r, shift left=3, "d_0"] \arrow[r, shift left=1] \arrow[r, shift right=1] \arrow[r, shift right=3, "d_3"'] & \nabla\mathrm{NN}\mathcal{G}_2 \arrow[l, bend right=40, "s_0"'] \arrow[l, bend right=30] \arrow[l, bend right=20, "s_2"] \arrow[r, shift left=2, "d_0"] \arrow[r] \arrow[r, shift right=2, "d_2"'] & \nabla\mathrm{NN}\mathcal{G}_1 \arrow[l, bend right=40, "s_0"'] \arrow[l, bend right=25, "s_1"] \arrow[r, shift left=1, "d_0"] \arrow[r, shift right=1, "d_1"'] & \nabla\mathrm{NN}\mathcal{G}_0, \arrow[l, bend right=30, "s_0"']
\end{tikzcd}$$

is given as follows: Γ_0 is the identity map on the objects of the double groupoid $\mathcal{G}$. For any morphism $\rho \in \Pi\mathcal{G}(a_0, a_1)$,

$$\Gamma_1(\rho) = \begin{tikzcd}
a_1 & b_\rho \arrow[l, "f_\rho"'] \\
 & a_0, \arrow[u, "x_\rho"']
\end{tikzcd}$$

If $a_2 \overset{\psi}{\leftarrow} a_1 \overset{\rho}{\leftarrow} a_0$ are any two morphisms in $\Pi\mathcal{G}$ and $\sigma \in \pi_2(\mathcal{G}, a_2)$,

$$\Gamma_2(\psi,\rho;\sigma) = \begin{tikzcd}
a_2 & b_\psi \arrow[l, "f_\psi"'] & b_{\psi\rho} \arrow[l, "f_\psi^{-1}f_{\psi\rho}"'] \\
 & a_1 \arrow[u, "x_\psi"] \arrow[ur, phantom, "f^{-1}_{\psi *}(\sigma)+\theta_{\psi,\rho}"] & b_\rho \arrow[l, "f_\rho"] \arrow[u, "x_{\psi\rho}x_\rho^{-1}"'] \\
 & & a_0, \arrow[u, "x_\rho"']
\end{tikzcd}$$

and, for any $a_3 \overset{\phi}{\leftarrow} a_2 \overset{\psi}{\leftarrow} a_1 \overset{\rho}{\leftarrow} a_0$ in $\Pi\mathcal{G}$ and $(\sigma_0, \sigma_1, \sigma_2) \in \pi_2(\mathcal{G}, a_3)^3$,

$$\Gamma_3(\phi, \psi, \rho; \sigma_0, \sigma_1, \sigma_2) = \begin{tikzcd}
a_3 & b_\phi \arrow[l, "f_\phi"'] & b_{\phi\psi} \arrow[l, "f_\phi^{-1}f_{\phi\psi}"'] & b_{\phi\psi\rho} \arrow[l, "f_{\phi\psi}^{-1}f_{\phi\psi\rho}"'] \\
 & a_2 \arrow[u, "x_\phi"] \arrow[ur, phantom, "f^{-1}_{\phi*}(\sigma_0)+\theta_{\phi,\psi}"] & b_\psi \arrow[l, "f_\psi"] \arrow[u] \arrow[ur, phantom, "{\begin{array}{c} f^{-1}_{\phi\psi*}(\sigma_1-\sigma_0) \\ + \\ \theta^{-\mathrm{h}}_{\phi,\psi}\circ_{\mathrm{h}}\theta_{\phi,\psi\rho} \end{array}}"] & b_{\psi\rho} \arrow[l] \arrow[u, "x_{\phi\psi\rho}x^{-1}_{\psi\rho}"'] \\
 & & a_1 \arrow[u, "x_\psi"] \arrow[ur, phantom, "{\begin{array}{c} f^{-1}_{\psi*}\phi^{-1}_*(\sigma_0-\sigma_1+\sigma_2) \\ + \\ f^{-1}_{\psi*}\phi^{-1}_*k(\phi,\psi,\rho) \\ + \\ \theta_{\psi,\rho} \end{array}}"] & b_\rho \arrow[l, "f_\rho"] \arrow[u, "x_{\psi\rho}x^{-1}_\rho"'] \\
 & & & a_0. \arrow[u, "x_\rho"']
\end{tikzcd}$$

So defined, all the simplicial identities to verify that Γ is actually a simplicial map are easily checked, except perhaps that $d_i\Gamma_3 = \Gamma_2 d_i$ for $i = 1, 2$. For $i = 1$, the required equality $d_1\Gamma_3(\phi, \psi, \rho, \sigma_0, \sigma_1, \sigma_2) = \Gamma_2(\phi, \psi\rho, \sigma_1)$ follows from the equalities

$$\begin{aligned}
&\big(f^{-1}_{\phi*}(\sigma_0)+\theta_{\phi,\psi}\big) \circ_{\mathrm{h}} \big(f^{-1}_{\phi\psi*}(\sigma_1 - \sigma_0) + \theta^{-\mathrm{h}}_{\phi,\psi} \circ_{\mathrm{h}} \theta_{\phi,\psi\rho}\big) \\
&\qquad \overset{(7)(8)}{=} f^{-1}_{\phi*}(\sigma_0) + f^{-1}_{\phi*}(\sigma_1 - \sigma_0) + \theta_{\phi,\psi} \circ_{\mathrm{h}} \theta^{-\mathrm{h}}_{\phi,\psi} \circ_{\mathrm{h}} \theta_{\phi,\psi\rho} \\
&\qquad = f^{-1}_{\phi*}(\sigma_1) + \theta_{\phi,\psi\rho}.
\end{aligned}$$

The case $i = 2$ is somewhat more complicated. In this case, the required equality

$$d_2\Gamma_3(\phi, \psi, \rho, \sigma_0, \sigma_1, \sigma_2) = \Gamma_2(\phi\psi, \rho, \sigma_2)$$

follows from the equalities

$$
\begin{aligned}
&\big(f^{-1}_{\phi\psi*}(\sigma_1-\sigma_0)+\theta^{-\mathrm{h}}_{\phi,\psi}\circ_{\mathrm{h}}\theta_{\phi,\psi\rho}\big)\circ_{\mathrm{v}}\big(f^{-1}_{\psi*}\phi^{-1}_{*}(\sigma_0-\sigma_1+\sigma_2+k(\phi,\psi,\rho)+\theta_{\psi,\rho}\big)\\
&\qquad\overset{(9)}{=} f^{-1}_{\phi\psi*}(\sigma_1-\sigma_0)+x_{\phi\psi*}x^{-1}_{\psi*}f^{-1}_{\psi_*}\phi^{-1}_{*}(\sigma_0-\sigma_1+\sigma_2+k(\phi,\psi,\rho))\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad+(\theta^{-\mathrm{h}}_{\phi,\psi}\circ_{\mathrm{h}}\theta_{\phi,\psi\rho})\circ_{\mathrm{v}}\theta_{\psi,\rho}\\
&\qquad\overset{2.4(ii)}{=} f^{-1}_{\phi\psi*}(\sigma_1-\sigma_0)+x_{\phi\psi*}\psi^{-1}_{*}\phi^{-1}_{*}(\sigma_0-\sigma_1+\sigma_2+k(\phi,\psi,\rho))\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad+(\theta^{-\mathrm{h}}_{\phi,\psi}\circ_{\mathrm{h}}\theta_{\phi,\psi\rho})\circ_{\mathrm{v}}\theta_{\psi,\rho}\\
&\qquad\overset{3.3}{=} f^{-1}_{\phi\psi*}(\sigma_1-\sigma_0)+x_{\phi\psi*}(\phi\psi)^{-1}_{*}(\sigma_0-\sigma_1+\sigma_2+k(\phi,\psi,\rho))\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad+(\theta^{-\mathrm{h}}_{\phi,\psi}\circ_{\mathrm{h}}\theta_{\phi,\psi\rho})\circ_{\mathrm{v}}\theta_{\psi,\rho}\\
&\qquad\overset{2.4(ii)}{=} f^{-1}_{\phi\psi*}(\sigma_1-\sigma_0)+x_{\phi\psi*}x^{-1}_{\phi\psi*}f^{-1}_{\phi\psi*}(\sigma_0-\sigma_1+\sigma_2+k(\phi,\psi,\rho))\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad+(\theta^{-\mathrm{h}}_{\phi,\psi}\circ_{\mathrm{h}}\theta_{\phi,\psi\rho})\circ_{\mathrm{v}}\theta_{\psi,\rho}\\
&\qquad= f^{-1}_{\phi\psi*}\sigma_2+f^{-1}_{\phi\psi*}k(\phi,\psi,\rho)+(\theta^{-\mathrm{h}}_{\phi,\psi}\circ_{\mathrm{h}}\theta_{\phi,\psi\rho})\circ_{\mathrm{v}}\theta_{\psi,\rho}\\
&\qquad\overset{(14)}{=} f^{-1}_{\phi\psi*}\sigma_2+\theta_{\phi\psi,\rho}.
\end{aligned}
$$

That Γ induces an isomorphism on the fundamental groupoids follows from the observation that homotopies $(f,b,x)\simeq(g,c,y)$ in $\mathcal{G}$ between two paths from an object a_0 to an object a_1, as in (2), are in bijection with homotopies $(f,b,x)\simeq(g,c,y)$ in the simplicial set $\nabla\,\mathrm{NN}\mathcal{G}$, by the mapping

$$
\begin{tikzcd}
b \arrow[d, equal] \arrow[dr, phantom, "\alpha"] & c \arrow[l, "f^{-1}g"'] \\
b \arrow[r, equal] & b \arrow[u, "yx^{-1}"']
\end{tikzcd}
\quad\mapsto\quad
\begin{tikzcd}
a_1 \arrow[r, equal] & a_1 \arrow[d, equal] \arrow[dr, phantom, "\Gamma^{\mathrm{v}}f\circ_{\mathrm{h}}\alpha"] & c \arrow[l, "g"'] \\
 & a_1 & b \arrow[l, "f"] \arrow[u, "yx^{-1}"'] \\
 & & a_0 \arrow[u, "x"']
\end{tikzcd}
$$

Furthermore, for any object a of $\mathcal{G}$, the induced homomorphism by Γ on the second homotopy groups with base a,

$$\pi_2\big(\operatorname*{hocolim}_{\Pi\mathcal{G}} K(\pi_2\mathcal{G},2;k),a\big)\to\pi_2(\nabla\mathrm{NN}\mathcal{G},a),$$

is explicitly given by

$$(id_a, id_a; \sigma) \overset{\Gamma_2}{\longmapsto} \begin{array}{ccccc} a & = & a & = & a \\ & & \| & \sigma & \| \\ & & a & = & a \\ & & & & \| \\ & & & & a \end{array}$$

and clearly is an isomorphism.

Since the homotopy groups of $\underset{\Pi\mathcal{G}}{\mathrm{hocolim}}\, K(\pi_2\mathcal{G}, 2; k)$ and of $\nabla\mathrm{NN}\mathcal{G}$ vanish in degree 3 and higher, Γ is actually a weak homotopy equivalence. $\square$

As a consequence of Theorems 5.1 and 6.1, we get a new proof of the following fact (cf. [12, Theorem 13] for a more general result).

Corollary 6.3. *The mapping $\mathcal{G} \mapsto |\mathcal{G}|$ induces a bijective correspondence between weak equivalence classes of double groupoids and weak homotopy classes of* 2*-types.*

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