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STONE DUALITY FOR TOPOLOGICAL CONVEXITY SPACES

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Résumé. Un espace de convexité est un ensemble X équipé d'une famille choisie de sous-ensembles (appelés les sous-ensembles convexes) fermée par intersections arbitraires et unions dirigées. On s'intéresse beaucoup aux espaces qui ont à la fois la structure d'espace de convexité et la structure d'espace topologique. Dans cet article, nous étudions la catégorie des espaces de convexité topologiques et étendons la dualité de Stone entre les coframes et les espaces topologiques á une adjonction entre la catégorie des espaces de convexité topologiques et la catégorie des treillis et des homomorphismes préservant le supremum. Cette adjonction peut etre factorisée à travers la catégorie des espaces de préconvexité (parfois appelés espaces de fermeture)
Abstract. A convexity space is a set X with a chosen family of subsets (called convex subsets) that is closed under arbitrary intersections and directed unions. There is a lot of interest in spaces that have both a convexity space and a topological space structure. In this paper, we study the category of topological convexity spaces and extend the Stone duality between coframes and topological spaces to an adjunction between topological convexity spaces and sup-lattices. We factor this adjunction through the category of preconvexity spaces (sometimes called closure spaces).
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1. Introduction

Stone duality is a contravariant equivalence of categories between categories of spaces and categories of lattices. The original Stone duality was between Stone spaces and Boolean algebras [15]. One of the most widely used extensions of Stone duality is between the categories of sober topological spaces and spatial coframes (or frames — since this is a 1-categorical duality, they are the same thing). This duality extends to an idempotent adjunction between topological spaces and coframes, given by the functors that send a topological space to its coframe of closed sets, and the functor that sends a coframe to its space of points.

In this paper, we develop an idempotent adjunction between topological convexity spaces and sup-lattices (the category whose objects are complete lattices, and morphisms are functions that preserve arbitrary suprema). Topological convexity spaces are sets equipped with both a chosen family of closed sets and a chosen family of convex sets. A canonical example is a metric space X with the usual metric topology, and convex sets being sets closed under the betweenness relation given by y is between x and z if $d(x, z) = d(x, y) + d(y, z)$. Many of the properties of metric spaces extend to topological convexity spaces. Homomorphisms of topological convexity spaces are continuous functions for which the inverse image of a convex set is convex.

Our approach to showing this adjunction goes via two equivalent intermediate categories. The first is the category of preconvexity spaces. A preconvexity space is a pair $(X, \mathcal{P})$ where $\mathcal{P}$ is a collection of subsets of X that is closed under arbitrary intersections and empty unions. We will refer to sets $P \in \mathcal{P}$ as *preconvex* subsets of X. A homomorphism of preconvexity spaces $f : (X, \mathcal{P}) \longrightarrow (X', \mathcal{P}')$ is a function $f : X \longrightarrow X'$ such that for any $P \in \mathcal{P}'$, we have $f^{-1}(P) \in \mathcal{P}$. This category of preconvexity spaces was also studied by [4], and shown to be closed under arbitrary limits and colimits.

The second intermediate category that is equivalent to the category of T_0-preconvexity spaces, is a full subcategory of Distributive Partial Sup lattices. This category was studied in [11]. Objects of this category are complete lattices with a chosen family of suprema which distribute over arbitrary infima. Morphisms are functions that preserve all infima and the chosen suprema.

The motivation for partial sup lattices was an adjunction between partial sup lattices and preconvexity spaces, which is shown in [11].

Before we begin presenting the extension of Stone duality to topological convexity spaces, Section 2 provides a review of the main ingredients needed. While these reviews do not contain substantial new results, they are presented with a different focus from much of the literature, so we hope that the reviews offer a new perspective on these well-studied subjects. We first recap the basics of topological convexity spaces. We then review Stone duality for topological spaces. We then review the category of distributive partial sup-lattices. This category was defined in [11], with the motivation of modelling various types of preconvexity spaces. However, the definition presented in this review is changed from the original definition in that paper to make it cleaner in a categorical sense.

2. Preliminaries

2.1 Topological Convexity Spaces

Definition 2.1. *A* topological convexity space *is a triple* $(X, \mathcal{F}, \mathcal{C})$*, where* X *is a set;* $\mathcal{F}$ *is a collection of subsets of* X *that is closed under finite unions and arbitrary intersections, i.e. the collection of closed sets for some topology on* X*; and* $\mathcal{C}$ *is a collection of subsets of* X *that is closed under directed unions and arbitrary intersections. Note that these include empty unions and intersections, so* X *and* $\varnothing$ *are in both* $\mathcal{F}$ *and* $\mathcal{C}$*. Sets in* $\mathcal{F}$ *will be called* closed *subsets of* X *and sets in* $\mathcal{C}$ *will be called* convex *subsets of* X*.*

The motivation here is that $(X, \mathcal{F})$ is a topological space, while $(X, \mathcal{C})$ is an abstract convexity space. Abstract convexity spaces are a generalisation of convex subsets of standard Euclidean spaces. Abstract convexity spaces were defined in [10], though in that paper, the definition did not require $\mathcal{C}$ to be closed under nonempty directed unions. Closure under directed unions was an additional property, called "domain finiteness". Later authors incorporated closure under directed unions into the definition of an abstract convexity space, and used the term *preconvexity space* for a set with a chosen collection of subsets that is closed under arbitrary intersections and contains the empty set [4].

While the definition of an abstract convexity space captures many of the important properties of convex sets in geometry, it also allows a large number of interesting examples far beyond the original examples from classical geometry, including many examples from combinatorics and algebra. The resulting category of convexity spaces has many natural closure properties [4].

The definition above does not include any interaction between the topological and convexity structures on X. While it will be convenient to deal with such general spaces, it is also useful to include compatibility axioms between the convexity and topological structures. The following axioms from [16] are often used to ensure suitable compatibility between topology and convexity structure.

(i) All convex sets are connected.

(ii) All polytopes (convex closures of finite sets) are compact.

(iii) The hull operation is uniformly continuous relative to a metric which generates the topology.

We will modify the third condition to not require the topology to come from a metric space, giving the weaker condition that the convex closure operation preserves compact sets.

Definition 2.2. *We will call a topological convexity space* compatible *if it satisfies the two conditions*

(i) All convex sets are connected.

(ii) The convex closure of a (topologically) closed compact set is (topologically) closed and compact.

We will call a topological convexity space precompatible *if it satisfies the two conditions*

(i) All convex sets are connected.

(ii') The convex closure of a finite set is (topologically) compact.

At this point, we will introduce some notation for describing topological convexity spaces. For any subset $A \subseteq X$, we will write $[A]$ for the intersection of all convex sets containing A. To simplify notation, when A is finite, we will write $[a_1, \ldots, a_n]$ instead of $[\{a_1, \ldots, a_n\}]$.

Examples 2.3.

1. If (X, d) is a metric space, then setting $\mathcal{F}$ to be the closed sets for the metric topology, i.e.

$$\mathcal{F} = \left\{ A \subseteq X \middle| (\forall x \in X) \left(\bigwedge_{y \in A} d(x, y) = 0 \Rightarrow x \in A \right) \right\}$$

and

$$\mathcal{C} = \{A \subseteq X | (\forall x, y, z \in X) ((x, z \in A \wedge d(x, z) = d(x, y) + d(y, z)) \Rightarrow y \in A)\}$$

we have that $(X, \mathcal{F}, \mathcal{C})$ is a topological convexity space. To ensure that convex sets are connected, we will often assume that geodesics exist — that is, for any $r < d(x, y)$, there is some $z \in [x, y]$ such that $d(x, z) = r$ and $d(y, z) = d(x, y) - r$, to ensure that convex sets are connected. We will usually also require that open balls are convex, and that the set $\{z \in X | d(x, y) = d(x, z) + d(z, x)\}$ is convex (and therefore the interval $[x, y]$). For common examples where these conditions hold, the convex closure of a compact set is compact, so that the space is compatible. However, it is not easy to prove compatibility of these spaces under simple conditions, or to find examples of metric spaces where this structure is not compatible.

2. Let L be a complete lattice. We define a topological convexity space structure by

$$\mathcal{F} = \left\{ \bigcap_{i \in I} F_i \middle| (\forall i \in I)(\exists x_1, \ldots, x_{n_i} \in X)(F_i = \downarrow\{x_1, \ldots, x_{n_i}\}) \right\} \cup \{\varnothing\}$$

and

$$\mathcal{C} = \{I \subseteq X | (\forall x_1, x_2 \in I) ((\forall y \leqslant x_1)(y \in I) \wedge (x_1 \vee x_2 \in I))\}$$

That is, $\mathcal{F}$ is the set of arbitrary intersections of finitely generated downsets, plus the emptyset (which are the closed sets for the weak topology [9]) and $\mathcal{C}$ is the set of (possibly empty) ideals of L. This topological convexity space is precompatible. To prove connectedness of convex sets, we want to show that an ideal cannot be covered by two disjoint weak-closed sets. Suppose U and V are disjoint weak-closed sets that cover I. Let $a \in I \cap U$ and $b \in I \cap V$. Then $a \vee b \in I$, and if $a \vee b \in U$, then $b \in U$ contradicting disjointness of U and V. Similarly if $a \vee b \in V$ then $a \in V$. This contradicts disjointness of U and V. The ideal generated by a finite set of elements in L is clearly principal, and therefore closed and compact. L is not in general compatible, since, for example, if L is the powerset of $\mathbb{N}$, then singletons in L are weak-closed, since for any set $X' \subseteq X$ containing two elements a and b, the downset $\downarrow\{\{a\}^c, \{b\}^c\}$ is finitely generated, and contains all singletons, but does not contain X'.

3. Let $n \in \mathbb{Z}^+$ be a positive integer. Let S_n be the group of permutations on n elements. Let $\mathcal{F}$ consist of all subsets of S_n, and for any partial order $\preceq$ on n, let

$$P_{\preceq} = \{\sigma \in S_n | (\forall i, j \in \{1, \ldots, n\})(i \preceq j \Rightarrow \sigma(i) \leqslant \sigma(j))\}$$

where $\leqslant$ is the usual total order on $\mathbb{Z}^+$. That is $P_{\preceq}$ is the set of permutations σ such that $\preceq$ is contained in $\sigma^{-1}(\leqslant)$. let

$$\mathcal{C} = \{P_{\preceq} | \preceq \text{ is a partial order on } \{1, \ldots, n\}\} \cup \{\varnothing\}$$

Since S_n is finite, to prove that $(S_n, \mathcal{F}, \mathcal{C})$ is a convexity space, we just need to show that $\mathcal{C}$ is closed under intersection. This is straightforward. Since partial orders are closed under intersection, the poset of partial orders on $\{1, \ldots, n\}$, with a top element adjoined, is a lattice. Thus the intersection $P_{\preceq} \cap P_{\sqsubseteq} = P_{\preceq \vee \sqsubseteq}$, so $\mathcal{C}$ is closed under intersection. This is a metric topology, with the metric given by $d(\sigma, \tau)$ is the Cayley distance from σ to τ, under the Coxeter generators. That is, $d(\sigma, \tau)$ is the length of the shortest word equal to $\tau\sigma^{-1}$ in the generators $\{\tau_i | i = 1, \ldots, n-1\}$, where

$$\tau_i(j) = \begin{cases} i+1 & \text{if } j = i \\ i & \text{if } j = i+1 \\ j & \text{otherwise} \end{cases}$$

is the transposition of i and $i + 1$.

4. If G is a topological group, or more generally a universal algebra equipped with a suitable topology, then we can define a topological convexity space by making subgroups (or more generally subalgebras) and the empty set convex, and keeping the closed sets from the topology.

Having defined the objects in the category of topological convexity spaces, we need to define the morphisms.

Definition 2.4. *A* homomorphism $f : (X, \mathcal{F}, \mathcal{C}) \longrightarrow (X', \mathcal{F}', \mathcal{C}')$ *between topological convexity spaces is a function* $f : X \longrightarrow X'$ *such that for every* $F \in \mathcal{F}'$*,* $f^{-1}(F) \in \mathcal{F}$ *and for every* $C \in \mathcal{C}'$*,* $f^{-1}(C) \in \mathcal{C}$*.*

The condition that $f^{-1}(F) \in \mathcal{F}$ is the condition that f is continuous as a function between topological spaces. The condition that $f^{-1}(C) \in \mathcal{C}$ is called *monotone* by [4], by analogy with the example of endofunctions of the real numbers. This was in the context of convexity spaces without topological structure. Dawson [4] uses the term $\mathcal{A}lign$ for the category of convexity spaces and monotone homomorphisms, and $\mathcal{C}onvex$ for the category of convexity spaces and functions whose forward image preserves convex sets. However, this terminology has not been widely used, and later authors have all considered the monotone homomorphisms as the natural homomorphisms of abstract convexity spaces. In the case of topological convexity spaces, the monotone condition is an even more natural choice because it aligns well with the continuity condition and leads to the Stone duality extension that we show in this paper.

Examples 2.5.

1. For the topological convexity space coming from a metric space, such that intervals are of the form $[x, y] = \{z \in X | d(x, z) + d(y, z) = d(x, y)\}$, a homomorphism is a function $f : X \longrightarrow Y$ such that whenever $d(x, z) = d(x, y) + d(y, z)$, we have $d(f(x), f(z)) = d(f(x), f(y)) + d(f(y), f(z))$. That is, f embeds geodesics from X into the geodesics in Y. To see that homomorphisms have this property, we have that $f^{-1}([f(x), f(z)])$ is convex, and contains x and z, so if $d(x, z) =$

$d(x,y)+d(y,z)$, then $f^{-1}([f(x),f(z)])$ must contain y. This means $f(y)\in[f(x),f(z)]=\{v|d(f(x),f(z))=d(f(x),v)+d(v,f(z))\}$. Conversely, if f has the given property, then for any convex $A\subseteq Y$, if $x,z\in f^{-1}(A)$, then for any y such that $d(x,z)=d(x,y)+d(y,z)$, we have $d(f(x),f(z))=d(f(x),f(y))+d(f(y),f(z))$, so by convexity, $f(y)\in A$, making $y\in f^{-1}(A)$, so $f^{-1}(A)$ is convex.

2. If L and M are complete lattices with the weak topology and convex sets are ideals, then topological convexity space homomorphisms from L to M are exactly sup-homomorphisms. To see this, let $f: L\longrightarrow M$ be a sup-homomorphism. Let $I\subseteq M$ be an ideal. Since f is order-preserving, $f^{-1}(I)$ is clearly a downset, and for $a,b\in f^{-1}(I)$, $f(a\vee b)=f(a)\vee f(b)\in I$. Since inverse image preserves intersection, it is sufficient to show that the inverse image of a finitely-generated downset $F\subseteq M$ is weak-closed. Let $F=\downarrow\{m_1,\ldots,m_n\}$. For $i=1,\ldots,n$, let $l_i=f_*(m_i)$, where f_* is the order-theoretic right adjoint of f (which exists because f is a sup-homomorphism). We have $f(x)\leqslant m_i$, if and only if $x\leqslant l_i$. Thus, $f^{-1}(F)=\downarrow\{l_1,\ldots,l_n\}$. Conversely suppose $f: L\longrightarrow M$ is a topological convexity space homomorphism. Weak-closed ideals are easily seen to be principal ideals, since if I is an ideal, and $I\subseteq\downarrow\{x_1,\ldots,x_n\}$, then if there are elements $y_i\in I$ with $y_i\nleqslant x_i$, then $y_1\vee\cdots\vee y_n$ cannot be in $\downarrow\{x_1,\ldots,x_n\}$, which is a contradiction, so we must have $I\subseteq\downarrow x_i$ for some $i\in\{1,\ldots,n\}$. Thus the inverse image of a principal ideal is another principal ideal. In particular, $f^{-1}(\downarrow\bigvee\{f(a)|a\in A\}$ is a principal ideal containing A, so it contains $\bigvee A$, and thus $f(\bigvee A)\leqslant\bigvee\{f(a)|a\in A\}$ as required.

For the partial order convexity on S_n from Example 2.3.3, describing the topological convexity space morphisms is more challenging. We start by looking at half-spaces (convex sets with convex complements). Half-spaces of S_n are of the form $C_{ij}=P_{\preceq}$, where $\preceq$ is the partial order where the only non-trivial comparison is $i\preceq j$. That is, $C_{ij}=\{\sigma\in S_n|\sigma(i)\leqslant\sigma(j)\}$. We first consider automorphisms:

Lemma 2.6. *If i, j, k and l are distinct, then the only half-spaces that contain $C_{ij}\cap C_{kl}$ are C_{ij} and C_{kl}.*

Proof. For any half-space $C_{st}\notin\{C_{ij},C_{kl}\}$, we need to find some $\sigma\in C_{ij}\cap C_{kl}$ with $\sigma\notin C_{st}$. Suppose $s=j$ and $t\neq i$, then we can find a permutation

σ such that $\sigma(i) < \sigma(j) < \sigma(t) < \sigma(k) < \sigma(l)$. This σ is in $\mathcal{C}_{ij} \cap C_{kl}$, but not in C_{st} as required. Similar permutations work for all combinations. $\square$

Lemma 2.7. *An automorphism $f : (S_n, P(S_n), \mathcal{C}) \longrightarrow (S_n, P(S_n), \mathcal{C})$ is of the form $f(\sigma) = \theta\sigma\tau$ for some $\tau \in S_n$ and some $\theta \in \{e, \rho\}$ where e is the identity permutation and ρ is the permutation which reverses the order of all elements.*

Proof. It is easy to see that for $\tau \in S_n$, f_τ given by $f_\tau(\sigma) = \sigma\tau$ is an automorphism of $(S_n, P(S_n), \mathcal{C})$. Now we consider the stabiliser of the identity element. Since $\{C_{i(i+1)} | i = 1, \ldots, (n-1)\}$ is the only set of $n-1$ half-spaces whose intersection contains only the identity permutation, any automorphism which fixes the identity permutation must fix this set. Furthermore, since $C_{(i-1)i} \cap C_{i(i+1)} \subseteq C_{(i-1)(i+1)}$, it follows that

$$f^{-1}(C_{(i-1)i}) \cap f^{-1}(C_{i(i+1)}) \subseteq f^{-1}(C_{(i-1)(i+1)})$$

Since f is an automorphism, $f^{-1}(C_{(i-1)(i+1)})$ cannot be either $f^{-1}(C_{(i-1)i})$ or $f^{-1}(C_{i(i+1)})$. By Lemma 2.6, it follows that $f^{-1}(C_{(i-1)i})$ and $f^{-1}(C_{i(i+1)})$ are adjacent half-spaces. Since the set of half-spaces

$$\{C_{i(i+1)} | i = 1, \ldots, (n-1)\}$$

is permuted by f^{-1}, the only possible permutations are the identity and the reversal $C_{i(i+1)} \mapsto C_{(n-i)(n+1-i)}$. This reversal sends a permutation σ to $\rho\sigma\rho$.

We want to show that these are the only elements in the stabiliser of the identity. By applying $\rho\sigma\rho$ if necessary, we can change an element in the stabiliser of e to one such that f^{-1} fixes every $C_{i(i+1)}$. Now $C_{i(i+2)}$ is the unique half-space that contains $C_{i(i+1)} \cap C_{(i+1)(i+2)}$ that is not equal to either $C_{i(i+1)}$ or $C_{(i+1)(i+2)}$, so it is also fixed by f^{-1}. By induction, we can show that every C_{ij} is fixed by f^{-1}, and thus f is the identity. $\square$

Proposition 2.8. *$f\colon S_n \longrightarrow S_m$ is a surjective topological convexity space homomorphism, if and only if there is an injective function $g : m \longrightarrow n$, such that f is either given by*

1. *$f(\tau)(i) = |\{j \in \{1, \ldots, m\} | \tau(g(j)) \leqslant \tau(g(i))\}|$. That is, $f(\tau)$ is the automorphism part of the automorphism—order-preserving-inclusion factorisation of τg.*

$$\begin{array}{ccc} n & \xrightarrow{\tau} & n \\ {\scriptstyle g}\uparrow & & \uparrow{\scriptstyle i} \\ m & \xrightarrow[f(\tau)]{} & m \end{array}$$

or

2. $f(\tau)(i) = |\{j \in \{1, \ldots, m\} | \tau(g(j)) \geqslant \tau(g(i))\}|$. *That is, $f(\tau)$ is the automorphism part of the automorphism-order-preserving-inclusion factorisation of $\rho\tau g$, where ρ is the order-reversing permutation on n.*

$$\begin{array}{ccccc} n & \xrightarrow{\tau} & n & \xrightarrow{\rho} & n \\ {\scriptstyle g}\uparrow & & & & \uparrow{\scriptstyle i} \\ m & & \xrightarrow[f(\tau)]{} & & m \end{array}$$

Proof. Firstly, we show that for an injective function $g : m \longrightarrow n$, both the functions

$$\alpha_g(\sigma)(i) = |\{j \in \{1, \ldots, m\} | \sigma(g(j)) \leqslant \sigma(g(i))\}|$$

and

$$\delta_g(\sigma)(i) = |\{j \in \{1, \ldots, m\} | \sigma(g(j)) \geqslant \sigma(g(i))\}|$$

are surjective homomorphisms. We see that for any $i \neq j \in \{1, \ldots, m\}$,

$${\alpha_g}^{-1}(C_{ij}) = \{\sigma \in S_n | \alpha_g(\sigma)(i) < \alpha_g(\sigma)(j)\} = \{\sigma \in S_n | \sigma(g(i)) < \sigma(g(j))\} = C_{g(i)g(j)}$$

and

$${\delta_g}^{-1}(C_{ij}) = \{\sigma \in S_n | \delta_g(\sigma)(i) < \delta_g(\sigma)(j)\} = \{\sigma \in S_n | \sigma(g(i)) > \sigma(g(j))\} = C_{g(j)g(i)}$$

so α_g and δ_g are homomorphisms. For surjectivity, let $\phi \in S_m$. We need to show that $\phi = \alpha_g(\tau)$ for some $\tau \in S_n$. Given the injections $m \overset{g}{\rightarrowtail} n$ and $m \xrightarrow{\phi} m \xrightarrow{i} n$ for any injective order-preserving $m \overset{i}{\rightarrowtail} n$, $n \overset{g}{\leftarrowtail} m \overset{i\phi}{\rightarrowtail} n$ is a partial permutation of n, so it extends to a full permutation τ with $\alpha_g(\tau) = \phi$. Similarly, we have $\delta_g(\rho\tau) = \alpha_g(\tau) = \phi$, so α_g and δ_g are both surjective.

Conversely, let $f : S_n \longrightarrow S_m$ be a surjective homomorphism. Since $\{e\}$ is convex, where e is the identity homomorphism, $f^{-1}(\{e\})$ is convex. Furthermore, $f^{-1}(\{e\}) = \bigcap_{i<j} f^{-1}(C_{ij})$. Since f^{-1} preserves convex sets, for every $i \neq j \in \{1, \ldots, m\}$ $f^{-1}(C_{ij}) = C_{st}$ for some $s, t \in \{1, \ldots, n\}$. Furthermore, $f^{-1}(C_{ij} \cap C_{jk}) = C_{st} \cap C_{tu}$. Thus, we have $f^{-1}(\{e\}) = C_{i_1 i_2 \ldots i_m} = C_{i_1 i_2} \cap C_{i_2 i_3} \cdots \cap C_{i_{m-1} i_m}$. If $f^{-1}(C_{12}) = C_{i_1 i_2}$, then we can define $g(j) = i_j$, and we have that $f = \alpha_g$. If on the other hand $f^{-1}(C_{12}) = C_{i_{m-1} i_m}$, then we let $g(j) = i_{m+1-j}$ and we have $f = \delta_g$.

□

Describing general homomorphisms between these topological convexity spaces is more difficult, and outside the scope of this paper.

3. Preconvexity Spaces and the Adjunction with Topological Convexity Spaces

Definition 3.1. *A* preconvexity space *(sometimes called a* closure space*) is a pair $(X, \mathcal{P})$, where X is a set and $\mathcal{P}$ is a collection of subsets of X that is closed under arbitrary intersections and contains the empty set (since X is an empty intersection, we also have $X \in \mathcal{P}$).*

This was [10]'s original definition of a convexity space. However, later authors decided that closure under directed unions should be a required property for a convexity space, and [4] introduced the term preconvexity space for these spaces that do not require closure under directed unions.

Definition 3.2. *A* homomorphism $(X, \mathcal{P}) \xrightarrow{f} (X', \mathcal{P}')$ *of preconvexity spaces is a function $X \xrightarrow{f} X'$ such that for any preconvex set $P \in \mathcal{P}'$, the inverse image $f^{-1}(P) \in \mathcal{P}$.*

Examples 3.3.

If $(X, \mathcal{F}, \mathcal{C})$ is a topological convexity space, then $(X, \mathcal{F} \cap \mathcal{C})$ is a preconvexity space. The underlying function of any topological convexity space homomorphism $(X, \mathcal{F}, \mathcal{C}) \xrightarrow{f} (X', \mathcal{F}', \mathcal{C}')$ is a preconvexity homomorphism. Conversely, if $\mathcal{C}'$ consists of directed unions from $\mathcal{F}' \cap \mathcal{C}'$, and $\mathcal{F}'$

consists of intersections of finite unions from $\mathcal{F}' \cap \mathcal{C}'$, then any preconvexity homomorphism $(X, \mathcal{F} \cap \mathcal{C}) \xrightarrow{g} (X', \mathcal{F}' \cap \mathcal{C}')$ is a topological convexity homomorphism.

Example 3.3 gives a functor $\mathcal{C}onvex\mathcal{T}op \xrightarrow{CC} \mathcal{P}reconvex$ that sends every topological convexity space to the preconvexity space of closed convex sets. The action on morphisms simply reinterprets the topological convexity homomorphism as a preconvexity homomorphism.

This closed-convex functor has a right adjoint, IS, which sends the preconvexity space $(X, \mathcal{P})$ to $(X, \overline{\mathcal{P}}, \widetilde{\mathcal{P}})$ where $\widetilde{\mathcal{P}}$ is the closure of $\mathcal{P}$ under directed unions, and $\overline{\mathcal{P}}$ is the closure of $\mathcal{P}$ under finite unions and arbitrary intersections. We will show that this defines a topological convexity space and is a right adjoint.

Lemma 3.4. *For any preconvexity space $(X, \mathcal{P})$, the set $\widetilde{\mathcal{P}}$ is the collection $\{\bigcup \mathcal{D} | \mathcal{D} \subseteq \mathcal{P}$ directed$\}$.*

Proof. Let $\mathcal{Q} = \{\bigcup \mathcal{D} | \mathcal{D} \subseteq \mathcal{P}$ directed$\}$. We need to show that $\mathcal{Q}$ is closed under directed unions. Let $\mathcal{D} \subseteq \mathcal{Q}$ be directed. For each $D \in \mathcal{D}$, there is a directed $\mathcal{D}_D \subseteq \mathcal{P}$ such that $D = \bigcup \mathcal{D}_D$. Let $\widetilde{\mathcal{D}}$ be the closure of $\bigcup\{\mathcal{D}_D | D \in D\}$ under finite joins in $\mathcal{P}$ (which exist because $\mathcal{P}$ is closed under arbitrary intersections). By definition, $\widetilde{\mathcal{D}}$ is directed. We will show that $\bigcup \mathcal{D} = \bigcup \widetilde{\mathcal{D}}$. Suppose $x \in \bigcup \mathcal{D}$. Then there is some $D \in \mathcal{D}$ with $x \in D$, and since $D = \bigcup \mathcal{D}_D$, there is some $P \in \mathcal{D}_D \subseteq \widetilde{\mathcal{D}}$ with $x \in P$, so $x \in \bigcup \widetilde{\mathcal{D}}$. Conversely, if $x \in \bigcup \widetilde{\mathcal{D}}$, then there is some $P_1, \ldots, P_n \in \bigcup\{\mathcal{D}_D | D \in \mathcal{D}\}$ such that $x \in P_1 \vee \cdots \vee P_n$. Now let each $P_i \in \mathcal{D}_{D_i}$ for some $D_i \in \mathcal{D}$. This means that $P_i \subseteq D_i$. Since $\mathcal{D}$ is directed, there is an element of $\mathcal{D}$ that contains $D_1, \ldots, D_n$, and which must therefore contain $P_1 \vee \cdots \vee P_n$. □

Lemma 3.5. *For any preconvexity space $(X, \mathcal{P})$, the set $\widetilde{\mathcal{P}}$ is closed under directed unions and arbitrary intersections.*

Proof. By definition, $\widetilde{\mathcal{P}}$ is closed under directed unions, so we just need to show that it is closed under intersections. Let $\{P_i | i \in I\}$ be a family of elements of $\widetilde{\mathcal{P}}$. By definition, for every $i \in I$, there is a directed $\mathcal{D}_i \subseteq \mathcal{P}$ with $P_i = \bigcup \mathcal{D}_i$. W.l.o.g. assume every $\mathcal{D}_i$ is down-closed in $\mathcal{P}$. We will

show that
$$\bigcap_{i\in I} P_i = \bigcup_{\substack{f:I \longrightarrow \mathcal{P} \\ (\forall i\in I) f(i)\in\mathcal{D}_i}} \bigcap_{i\in I} f(i) \tag{1}$$

That is, the intersection of the family $\{P_i|i \in I\}$ is the union over all choice functions f, of the intersection of $\{f(i)|i \in I\}$. Every $f(i) \in \mathcal{P}$, so this intersection $\bigcap_{i\in I} f(i)$ is also in $\mathcal{P}$, and the set of choice functions is directed, since every $\mathcal{D}_i$ is directed and down-closed, so for choice functions $f, g : I \longrightarrow \mathcal{P}$ the join $(f \vee g)(i) = f(i) \vee g(i)$ is also a choice function. Equation (1) therefore shows that $\bigcap_{i\in I} P_i \in \widetilde{\mathcal{P}}$.

To prove Equation (1), first let $x \in \bigcap_{i\in I} P_i$. Since $(\forall i)(x \in P_i)$, and $P_i = \bigcup \mathcal{D}_i$, there is some $D_{i,x} \in \mathcal{D}_i$ with $x \in D_{i,x}$. Thus, we can take the choice function $f_x(i) = D_{i,x}$, and deduce $x \in \bigcap_{i\in I} f_x(i)$. Conversely, let

$$x \in \bigcup_{\substack{f:I \longrightarrow \mathcal{P} \\ (\forall i\in I) f(i)\in\mathcal{D}_i}} \bigcap_{i\in I} f(i)$$

There must be some choice function f with $x \in \bigcap_{i\in I} f(i)$. Since $f(i) \in \mathcal{D}_i$, it follows that $f(i) \subseteq P_i$, so $x \in P_i$ for every $i \in I$. Thus $x \in \bigcap_{i\in I} P_i$. □

Remark 3.6. The proof of Lemma 3.5 does not actually require the axiom of choice, because there are canonical choices for all choice functions needed — for each P_i, we need to choose a directed family $\mathcal{D}_i$ with $P_i = \bigcup \mathcal{D}_i$. We can let $\mathcal{D}_i = \{P \in \mathcal{P}|P \subseteq P_i\}$, and since every $\mathcal{D}_i$ is a downset, we can set $D_{i,x} = \overline{\{x\}}$ for every $i \in I$, where $\overline{\{x\}}$ is the convex-closed closure of $\{x\}$.

Lemma 3.7. *Every $F \in \overline{\mathcal{P}}$ is of the form $\bigcap \mathcal{F}$, where*

$$\mathcal{F} \subseteq \{P_1 \cup \cdots \cup P_n|P_1, \ldots, P_n \in \mathcal{P}\}$$

Proof. Let $\widehat{\mathcal{P}} = \{P_1 \cup \cdots \cup P_n|P_1, \ldots, P_n \in \mathcal{P}\}$ be the set of finite unions from $\mathcal{P}$. We need to show that the set $\left\{\bigcap \mathcal{F}|\mathcal{F} \subseteq \widehat{\mathcal{P}}\right\}$ is closed under finite unions. (By definition, it is closed under arbitrary intersections.) Let $F_1 = \bigcap \mathcal{F}_1$ and $F_2 = \bigcap \mathcal{F}_2$ for $\mathcal{F}_1, \mathcal{F}_2 \subseteq \widehat{\mathcal{P}}$. Let

$$\mathcal{F}_{12} = \{P_1 \cup P_2|P_1 \in \mathcal{F}_1, P_2 \in \mathcal{F}_2\}$$

We will show that $F_1 \cup F_2 = \bigcap \mathcal{F}_{12}$. Clearly, for every $P_1 \in \mathcal{F}_1$, and $P_2 \in \mathcal{F}_2$, we have $F_1 \subseteq P_1$ and $F_2 \subseteq P_2$, so $F_1 \cup F_2 \subseteq P_1 \cup P_2$. Conversely, suppose $x \notin F_1 \cup F_2$. Then there is some $P_1 \in \mathcal{F}_1$ and some $P_2 \in \mathcal{F}_2$ with $x \notin P_1$ and $x \notin P_2$. It follows that $x \notin P_1 \cup P_2 \in \mathcal{F}_{12}$, so $x \notin \bigcap \mathcal{F}_{12}$. □

Lemma 3.8.

1. *For a set X, the identity function on X is a preconvexity homomorphism $(X, \mathcal{P}) \longrightarrow (X, \mathcal{P}')$ if and only if $\mathcal{P}' \subseteq \mathcal{P}$.*

2. *For a set X, the identity function on X is a topological convexity homomorphism $(X, \mathcal{F}, \mathcal{C}) \longrightarrow (X, \mathcal{F}', \mathcal{C}')$ if and only if $\mathcal{F}' \subseteq \mathcal{F}$ and $\mathcal{C}' \subseteq \mathcal{C}$.*

Proof. This is immediate from the definition. □

Proposition 3.9. *The assignment IS that sends the preconvexity space $(X, \mathcal{P})$ to the topological convexity space $(X, \overline{\mathcal{P}}, \widetilde{\mathcal{P}})$ and the preconvexity homomorphism $(X, \mathcal{P}) \xrightarrow{f} (X', \mathcal{P}')$ to f considered as a topological convexity homomorphism, is a functor, and is right adjoint to the functor CC : $\mathcal{C}onvex\mathcal{T}op \longrightarrow \mathcal{P}reconvex$.*

Proof. Because the forgetful functor to **Set** sends IS to the identity functor, the functoriality of IS is automatic provided it is well-defined. That is, if any preconvexity homomorphism $(X, \mathcal{P}) \xrightarrow{f} (X', \mathcal{P}')$ is a topological convexity homomorphism from $(X, \overline{\mathcal{P}}, \widetilde{\mathcal{P}})$ to $(X', \overline{\mathcal{P}'}, \widetilde{\mathcal{P}'})$. For the adjunction, we need to demonstrate that for any topological convexity space $(X, \mathcal{F}, \mathcal{C})$ and any preconvexity space $(X', \mathcal{P}')$, a function $f : X \longrightarrow X'$ is a topological convexity space homomorphism $(X, \mathcal{F}, \mathcal{C}) \xrightarrow{f} (X', \overline{\mathcal{P}'}, \widetilde{\mathcal{P}'})$ if and only if it is a preconvexity homomorphism $(X, \mathcal{F} \cap \mathcal{C}) \xrightarrow{f} (X', \mathcal{P}')$. The "only if" part is obvious.

Suppose $(X, \mathcal{F} \cap \mathcal{C}) \xrightarrow{f} (X', \mathcal{P}')$ is a preconvexity homomorphism. Let $F \in \overline{\mathcal{P}'}$. We want to show that $f^{-1}(F) \in \mathcal{F}$. Now $F \in \overline{\mathcal{P}'}$ means $F = \bigcap \mathcal{U}$ where $\mathcal{U} \subseteq \widehat{\mathcal{P}'}$. Now if $P_1 \cup \cdots \cup P_n \in \widehat{\mathcal{P}'}$, then $f^{-1}(P_1 \cup \cdots \cup P_n) = f^{-1}(P_1) \cup \cdots \cup f^{-1}(P_n)$ is a finite union of sets from $\mathcal{F} \cap \mathcal{C}$, so since $\mathcal{F}$ is closed under finite unions, $f^{-1}(P_1 \cup \cdots \cup P_n) \in \mathcal{F}$. Therefore $f^{-1}(F) =$

$\bigcap\{f^{-1}U|U \in \mathcal{U}\}$ and $\{f^{-1}U|U \in \mathcal{U}\} \subseteq \mathcal{F}$, so as $\mathcal{F}$ is closed under arbitrary intersections, $f^{-1}(F) \in \mathcal{F}$. Similarly, let $C = \bigcup \mathcal{D}$, where $\mathcal{D} \subseteq \mathcal{P}'$ is a directed downset. For every $D \in \mathcal{D}$, we have $f^{-1}(D) \in \mathcal{C}$, and for any $D_1, D_2 \in \mathcal{D}$, there is some $D_{12} \in \mathcal{D}$ with $D_1 \subseteq D_{12}$ and $D_2 \subseteq D_{12}$. It follows that $f^{-1}(D_1) \subseteq f^{-1}(D_{12})$ and $f^{-1}(D_2) \subseteq f^{-1}(D_{12})$. Therefore, $\{f^{-1}(D)|D \in \mathcal{D}\}$ is directed. Now

$$f^{-1}(C) = f^{-1}(\bigcup \mathcal{D}) = \bigcup\{f^{-1}(D)|D \in \mathcal{D}\}$$

Since $\{f^{-1}(D)|D \in \mathcal{D}\} \subseteq \mathcal{C}$, and $\mathcal{C}$ is closed under directed unions, it follows that $f^{-1}(C) \in \mathcal{C}$. Thus f is a homomorphism of topological convexity spaces.

Well-definedness of the functor IS also follows from the adjunction, because $\mathcal{P} \subseteq \overline{\mathcal{P}} \cap \widetilde{\mathcal{P}}$, so the identity function on X is always a preconvexity homomorphism $(X, \overline{\mathcal{P}} \cap \widetilde{\mathcal{P}}) \xrightarrow{i} (X, \mathcal{P})$. Thus the composite

$$(X, \overline{\mathcal{P}} \cap \widetilde{\mathcal{P}}) \xrightarrow{i} (X, \mathcal{P}) \xrightarrow{f} (X', \mathcal{P}')$$

is a preconvexity homomorphism, so by the adjunction, it is a topological convexity space homomorphism $(X, \overline{\mathcal{P}}, \widetilde{\mathcal{P}}) \xrightarrow{f} (X', \overline{\mathcal{P}'}, \widetilde{\mathcal{P}'})$

□

Corollary 3.10. *The adjunction $CC \dashv IS$ is idempotent.*

Proof. The counit and unit of the adjunction are both the identity function viewed as a homomorphism in the relevant category. The triangle identities for the adjunction therefore give an isomorphism of spaces, showing that the adjunction is idempotent. □

For an idempotent adjunction, a natural question is what are the fixed points?

Proposition 3.11. *A topological convexity space $X = (X, \mathcal{F}, \mathcal{C})$ satisfies $IS \circ CC(X) = X$ if and only if X satisfies the conditions:*

1. *Every convex set is a directed union of closed convex sets.*
2. *For every $V \in \mathcal{F}$ and any $x \in X \backslash V$, there are sets $C_1, \ldots, C_n \in \mathcal{F} \cap \mathcal{C}$ such that $V \subseteq C_1 \cup \ldots \cup C_n$ and $x \notin C_1 \cup \ldots \cup C_n$.*

Proof. The counit of the adjunction is the identity function on the underlying sets. Thus $\overline{(\mathcal{F} \cap \mathcal{C})} \subseteq \mathcal{F}$ and $\widetilde{(\mathcal{F} \cap \mathcal{C})} \subseteq \mathcal{C}$. Let $A \in \mathcal{C}$ be convex in X. By Condition 1, A is a directed union of sets in $\mathcal{F} \cap \mathcal{C}$. By definition, this is in $\widetilde{(\mathcal{F} \cap \mathcal{C})}$.

Now let $V \in \mathcal{F}$. For any $W = C_1 \cup \cdots \cup C_n$ with $C_i \in \mathcal{F} \cap \mathcal{C}$, $W \in \overline{(\mathcal{F} \cap \mathcal{C})}$ by definition. Thus, by Condition 2, for every $x \in X \backslash V$, there is some $W \in \overline{(\mathcal{F} \cap \mathcal{C})}$ with $V \subseteq W$ and $x \notin W$. Now, clearly V is the intersection of all these W for all $x \notin V$. Since $\overline{(\mathcal{F} \cap \mathcal{C})}$ is closed under arbitrary intersections, this implies $V \in \overline{(\mathcal{F} \cap \mathcal{C})}$.

Conversely, if X is a fixed point of the adjunction, i.e. $IS \circ CC(X) = X$, then $\mathcal{C} = \widetilde{(F \cap C)}$, which is exactly Condition 1. Also $\mathcal{F} = \overline{(F \cap C)}$, meaning that for every $V \in \mathcal{F}$, we have $V = \bigcap \mathcal{U}$ where $\mathcal{U}$ is a family of finite unions of sets from $\mathcal{F} \cap \mathcal{C}$. Since $V = \bigcap \mathcal{U}$, for any $x \notin V$, there is some $U \in \mathcal{U}$ with $x \notin U$. By definition, $U = C_1 \cup \cdots \cup C_n$ for some $C_1, \ldots, C_n \in \mathcal{F} \cap \mathcal{C}$, which is Condition 2. □

We will call a topological convexity space *teetotal* if the conditions of Proposition 3.11 hold. The teetotal conditions are closely related to the compatible conditions from Definition 2.2. However, there are compatible spaces which are not teetotal.

Example 3.12. l^2 is the vector-space of square-summable sequences of real numbers, with the l^2 norm. Since l^2 is a metric space, it is easy to check that it is a compatible topological convexity space.

Let F be the unit sphere, which is a closed set, and let $x = 0$. In order for l^2 to be teetotal, we need to find a finite family of closed convex subsets $C_1, \ldots, C_n$ such that $F \subseteq C_1 \cup \cdots \cup C_n$ and $x \notin C_1 \cup \cdots \cup C_n$. For closed convex C_i and $x \notin C_i$, since C_i is closed, there is an open ball containing x disjoint from C_i. Let $d = \sup\{r \in \mathbb{R} | B(x, r) \cap C_i = \varnothing\}$ be the distance from x to C_i. Since $B(x, d)$ is the directed union of $\{B(x, r) | r < d\}$, it follows that $B(x, d) \cap C_i = \varnothing$.

We first show that if C is a closed convex set that does not contain 0, then there is a unique $y \in C$ that minimises $\|y\|$. If there is no $y \in C$ that minimises $\|y\|$, then there must be a sequence $a_1, a_2, \ldots \in C$ such that $\|a_i\|$ is strictly decreasing and

$$\lim_{n \to \infty} \|a_n\| = \inf_{y \in C} \|y\|$$

Since $[a_1, \ldots, a_n]$ is compact for every n, there is a point $b_n \in [a_1, \ldots, a_n]$ that minimises $\|b\|$. In particular, this means that for any $i < n$ and any $0 < \epsilon < 1$, $\|b_n + \epsilon(b_i - b_n)\| \geqslant \|b_n\|$. Squaring both sides gives

$$2\epsilon\langle b_i, b_n\rangle - 2\epsilon\langle b_n, b_n\rangle + \epsilon^2\langle b_i - b_n, b_i - b_n\rangle > 0$$

Taking the limit as $\epsilon \to 0$ gives $\langle b_i, b_n\rangle > \langle b_n, b_n\rangle$. Thus

$$\begin{aligned}\|b_i - b_n\|^2 &= \|b_i\|^2 + \|b_n\|^2 - 2\langle b_i, b_n\rangle \\ &\leqslant \|b_i\|^2 - \|b_n\|^2\end{aligned}$$

Since $\|b_n\|^2$ is a decreasing sequence, bounded below by 0, it converges to some limit r. Thus $\|b_i - b_n\|^2 \leqslant \|b_i\|^2 - r$ for any $i < n$. Thus b_n is a Cauchy sequence, so it converges to some limit b_∞. Now since C is closed, $b_\infty \in C$, and

$$\|b_\infty\| = \lim_{n\to\infty}\|b_n\| = \inf_{y\in C}\|y\|$$

Thus b_∞ is a nearest point in C to 0. If y is another point with minimal norm, then $\frac{y+b_\infty}{2}$ must have smaller norm. Thus b_∞ is the unique point with smallest norm.

Now for any $y \in C$, since C is convex, we have that $\|b_\infty + \epsilon(y - b_\infty)\| > \|b_\infty\|$, and by the above argument, $\langle y, b_\infty\rangle \geqslant \langle b_\infty, b_\infty\rangle$. Thus $C \subseteq \{x \in l^2 | \langle x, b_\infty\rangle > \frac{1}{2}\|b_\infty\|^2\}$. That is, every closed convex set is contained in an open half-space that does not contain $x = 0$.

We can therefore find half-spaces $H_1, \ldots, H_n$ with $x \notin H_i$ and $C_i \subseteq H_i$. Thus, we may assume that $F \subseteq H_1 \cup \cdots \cup H_n$. Half-spaces that do not contain the origin are sets of the form $H_{w,a} = \{v \in l^2 | \langle v, w\rangle > a\}$ for some $w \in l^2$ and $a \in \mathbb{R}^+$. Given a finite family $H_1, \ldots, H_n = H_{w_1,a_1}, \ldots, H_{w_n,a_n}$, we can find a unit vector w that is orthogonal to all of $w_1, \ldots, w_n$. This means that $w \notin H_i$ for all i, and $w \in F$, contradicting the assumption that $F \subseteq H_1 \cup \cdots \cup H_n$. Therefore, l^2 does not satisfy the teetotal axioms.

The teetotal interior $IS \circ CC(l^2)$ has the same convex sets, but closed sets are intersections of finite unions of closed half-spaces. We can check that this is the product topology on l^2 as a real vector space.

Example 3.13. Let (X, d) be a metric space, where $X = \bigcup_{n\in\mathbb{N}}[n]^n$ is the set of finite lists with entries bounded by list length. The distance is given by $d(u, v) = l(u) + l(v) - l(u \cap v)$, where $l(u)$ is the length of the list u

and $u \cap v$ is the longest list which is an initial sublist of both u and v. The induced topology is clearly discrete. The complement of the empty list is not contained in a finite union of convex subsets that does not contain the empty list. In particular, a convex subset of X that does not contain $\varnothing$ must consist of lists that all start with the same first element. Since there are infinitely many possible first elements, a finite collection of convex sets that do not contain the empty list cannot cover $X \backslash \varnothing$.

The space (X, d) is a metric space and every closed ball is compact. However, it is not a fixed point of the adjunction between $\mathcal{C}onvex\mathcal{T}op$ and $\mathcal{P}reconvex$.

For a (pre)compatible topological space to be teetotal, an additional property is needed.

Proposition 3.14. *If $(X, \mathcal{F}, \mathcal{C})$ is a precompatible topological convexity space with the following properties:*

- *There is a basis of open sets that are convex, whose closure is convex and compact.*
- *$(X, \mathcal{F})$ is Hausdorff.*
- *If A is closed convex and $x \notin A$, then there is a closed convex set H such that H^c is convex, with $A \subseteq H$ and $x \notin H$. (This property, without the topological constraints, is often used in the literature, where it is called the Kakutani condition.)*

then $(X, \mathcal{F}, \mathcal{C})$ is fixed by the adjunction.

Proof. We need to show that for any closed $V \in \mathcal{F}$, and any $x \notin V$, there is a finite set of closed convex sets whose union covers V but does not contain x. Let U be an open subset of V^c, containing x such that U is convex and $\overline{U}$ is convex and compact. Let $A = \overline{U} \backslash U$. For any $a \in A$, by the Hausdorff property, we can find an open U_a that contains a, whose closure does not contain x. Since convex open sets with convex closure form a basis of open sets, we can find a convex open U'_a with convex closure that does not contain x. Since A is compact, it is covered by a finite subset $U'_{a_1} \cup \cdots \cup U'_{a_n}$. Now each $\overline{U'_{a_i}}$ is contained in a closed convex H_{a_i} which does not contain x, such that $H^c_{a_i}$ is also convex.

For any $y \in V$, since $[x, y]$ is connected (by compatibility), it cannot be the union $([x, y] \cap U) \cup \left([x, y] \cap \overline{U}^c\right)$, so $[x, y] \cap A \neq \varnothing$. Let $z \in [x, y] \cap A$. Since H_{a_i} cover A, we have $z \in H_{a_i}$ for some i. Now if $y \in {H_{a_i}}^c$, then since ${H_{a_i}}^c$ is convex and contains x, it follows that $z \in {H_{a_i}}^c$ contradicting $z \in H_{a_i}$. Thus, we must have $y \in H_{a_i}$. Since $y \in V$ is arbitrary, we have that $V \subseteq H_{a_1} \cup \cdots \cup H_{a_n}$ as required.

We also need to show that every convex set is a directed union of closed convex sets. Let $C \in \mathcal{C}$ be a convex set. Let $\mathcal{D} = \{[F] | F \subseteq C, F \text{ finite}\}$ be the collection of finitely generated convex subsets of C. Since finite sets are closed under binary unions, $\mathcal{D}$ is directed. Since the convex closure of any finite set is closed, it follows that C is a directed union of closed convex sets as required. □

For a metric space, these conditions can be simplified to give more natural conditions.

Lemma 3.15. *If X is a topological convexity space where intervals are closed, satisfying the Kakutani property that every pair of disjoint closed convex sets are separated by a closed half-space, then for any $x, s, t, p, q, r \in X$ with $s \in [x, p]$, $t \in [x, q]$ and $r \in [p, q]$, we have $[x, r] \cap [s, t] \neq \varnothing$.*

Proof. If $[x, r] \cap [s, t] = \varnothing$, then $[x, r]$ and $[s, t]$ are disjoint closed convex sets, so by the Kakutani propery, there is a closed half-space H such that $[x, r] \subseteq H$ and $[s, t] \subseteq H^c$. Now if $p \in H$, then since $x \in H$ and H is convex, we get $s \in H$, contradicting $[s, t] \subseteq H^c$. This is a contradiction, so we must have $p \in H^c$. A similar argument shows that $q \in H^c$. However, since H^c is convex, it follows that $r \in H^c$, contradicting $[x, r] \subseteq H$. This contradiction disproves $[x, r] \cap [s, t] = \varnothing$, so $[x, r] \cap [s, t] \neq \varnothing$ □

Lemma 3.16. *If (X, d) is a metric space, such that every open ball is convex, every pair of disjoint closed convex sets are separated by a closed half-space (a closed convex set with convex complement), and every interval $[a, b]$ is isomorphic (as a topological convexity space) to the real interval $[0, 1]$ then for any convex compact $A \subseteq X$ and any $x \in X$, we have*

$$[x, A] = \bigcup\{[x, y] | y \in A\}$$

Proof. We need to show that $\bigcup\{[x, y] | y \in A\}$ is closed under the betweenness relation. Let $s, t \in \bigcup\{[x, y] | y \in A\}$, and let $z \in [s, t]$. Let $s \in [x, p]$

and $t \in [x,q]$ for $p,q \in A$. We will show that $z \in [x,r]$ for some $r \in [p,q]$. Since $[s,t] \cong [0,1]$, we have that $[s,t] = [s,z] \cup [z,t]$. For $r \in [p,q]$, if $[x,r] \cap [s,z] \neq \varnothing$ and $[x,r] \cap [z,t] \neq \varnothing$, then clearly $z \in [x,r]$. Thus if $(\forall r \in [p,q])(z \notin [x,r])$, then

$$(\forall r \in [p,q])(([x,r] \cap [s,z] = \varnothing) \vee ([x,r] \cap [z,t] = \varnothing))$$

so

$$[p,q] = \{r \in [p,q] | [x,r] \cap [s,z] = \varnothing\} \cup \{r \in [p,q] | [x,r] \cap [z,t] = \varnothing\}$$

and this union is disjoint by Lemma 3.15. By connectedness of $[p,q]$, we just need to show that $\{r \in X | [x,r] \cap [s,z] = \varnothing\}$ and $\{r \in X | [x,r] \cap [z,t] = \varnothing\}$ are open to reach a contradiction, which would prove $z \in [x,r]$ for some $r \in [p,q]$. Let $U = \{r \in X | [x,r] \cap [s,z] = \varnothing\}$, and let $v \in U$. We want to show that there is some ϵ such that $B(v,\epsilon) \subseteq U$. Now $[s,z] \cap [x,v] = \varnothing$, which means $(\forall w \in [s,z])(d(x,w) + d(w,v) \neq d(x,v))$. Since $[s,z]$ is compact, the function $f(w) = d(x,w) + d(w,v) - d(x,v)$ is bounded away from zero on $[s,z]$. Let δ be a lower bound. Now if $v' \in B\left(v, \frac{\delta}{2}\right)$, then for any $w \in [s,z]$, we have

$$\begin{aligned} d(x,w) + d(w,v') &\geqslant d(x,w) + d(w,v) - d(v,v') \\ &> d(x,v) + \delta - \frac{\delta}{2} \\ &\geqslant d(x,v') - d(v',v) + \frac{\delta}{2} \\ &> d(x,v') \end{aligned}$$

Because the inequality is strict, we have $w \notin [x,v']$ for any $w \in [s,z]$, i.e. $v' \in U$. Thus $B\left(v, \frac{\delta}{2}\right) \subseteq U$, meaning U is open as required. □

Corollary 3.17. *If (X,d) is a metric space, such that every closed ball is compact, every open ball is convex, every pair of disjoint closed convex sets are separated by a closed half-space, and every interval $[a,b]$ is isomorphic to the real interval $[0,1]$ then the induced topological convexity space is fixed by the adjunction.*

Proof. We will show that the conditions of Proposition 3.14 hold in this case. The Hausdorff condition is always true for metric spaces.

The open balls form a basis for the topology, and are convex. By connectedness, if $d(x, y) = r$, then since $[x, y]$ is connected, there is a sequence $y_1, \ldots, y_n \to y$ in $[x, y]$, so $y_1, \ldots, y_n \in B(x, r)$. Thus, we have $\overline{B(x, r)} = \overline{B}(x, r) = \{y \in X | d(x, y) \leqslant r\}$. Thus, the closure of an open ball is compact. Also, $\overline{B}(x, r) = \bigcap_{R>r} B(x, R)$ is an intersection of convex sets, so closed balls are convex. Thus open balls are a basis of the topology with the required property.

Next, we need to show that the convex closure of a finite set is compact. We will do this inductively. By Lemma 3.16, we have that $[x_1, \ldots, x_n] = \bigcup\{[x_1, y] | y \in [x_2, \ldots, x_n]\}$. By the induction hypothesis $[x_2, \ldots, x_n]$ is compact. This means that $[x_2, \ldots, x_n] \subseteq B(x_1, r)$ for some $r \in \mathbb{R}^+$. It follows that $[x_1, \ldots, x_n] \subseteq B(x_1, r)$, since $B(x_1, r)$ is convex. Therefore, it is sufficient to prove that $[x_1, \ldots, x_n]$ is closed.

Let $z \notin [x_1, \ldots, x_n]$. We want to prove that there is some open ball about z that is disjoint from $[x_1, \ldots, x_n]$. For any $y \in [x_2, \ldots, x_n]$, we know $z \notin [x_1, y]$, so $d(x_1, z) + d(z, y) - d(x_1, y) > 0$. For $y \in [x_2, \ldots, x_n]$, let $f(y) = d(x_1, z) + d(z, y) - d(x_1, y)$. Then $f(y)$ is a continuous function $[x_2, \ldots, x_n] \to \mathbb{R}^+$. Since $[x_2, \ldots, x_n]$ is compact, f attains its lower bound, so in particular, there is some $\epsilon > 0$ such that $f(y) > \epsilon$ for all $y \in [x_2, \ldots, x_n]$. Now if $d(z, z') < \frac{\epsilon}{2}$, then for any $y \in [x_2, \ldots, x_n]$,

$$d(x_1, z') + d(z', y) > d(x_1, z) - \frac{\epsilon}{2} + d(z, y) - \frac{\epsilon}{2} > d(x_1, y)$$

so $z' \notin [x_1, y]$ because the inequality is strict and open balls are convex. It follows that $z' \notin [x_1, \ldots, x_n]$, so $[x_1, \ldots, x_n]$ is closed, as required. □

In the other direction, it is natural to ask which preconvexity spaces are fixed by the monad $CC \circ IS$. The functor $CC \circ IS$ sends a preconvexity space, $(X, \mathcal{P})$ to the space $(X, \overline{\mathcal{P}} \cap \widetilde{\mathcal{P}})$. We will call a preconvexity space $(X, \mathcal{P})$ *geometric* if $\overline{\mathcal{P}} \cap \widetilde{\mathcal{P}} = \mathcal{P}$.

Proposition 3.18. *If X is finite, then any preconvexity space $(X, \mathcal{P})$ is geometric.*

Proof. If X is finite, then $\widetilde{\mathcal{P}} = \mathcal{P}$, so $\widetilde{\mathcal{P}} \cap \overline{\mathcal{P}} = \mathcal{P}$ as required. □

A natural question is whether this extends to topologically discrete spaces. In fact, there are preconvexity spaces where all sets are in both $\overline{\mathcal{P}}$ and $\widetilde{\mathcal{P}}$, but not in $\mathcal{P}$.

Example 3.19. Let $X = \mathbb{N}$. Let $\mathcal{P}$ consist of all subsets of $\mathbb{N}$ whose complement is infinite or empty. Clearly every subset of $\mathbb{N}$ is a finite union from $\mathcal{P}$, and also a directed union from $\mathcal{P}$ (as $\mathcal{P}$ contains all finite sets). Thus $(X, \mathcal{P})$ is a non-geometric example where all sets are closed and all sets are convex.

Proposition 3.20. *Every T_0 preconvexity space (meaning for any $x \neq y$, there is a preconvex set containing exactly one of x and y) embeds in a geometric preconvexity space.*

Proof. For a T_0 preconvexity space $(X, \mathcal{P})$, let $(Y, \mathcal{Q})$ be given by $Y = \mathcal{P}$ and $\mathcal{Q} = \big\{\{S \in \mathcal{P} | S \subseteq R\} | R \in \mathcal{P}\big\}$. Now the inclusion $X \xrightarrow{i} Y$ given by $i(x) = \bigcap\{P \in \mathcal{P} | x \in P\}$, is an embedding of preconvexity spaces, meaning that for $A \subseteq X$, we have $A \in \mathcal{P}$ if and only if $A = i^{-1}(B)$ for some $B \in \mathcal{Q}$. Clearly if $A \in \mathcal{P}$, then $\{S \in \mathcal{P} | S \subseteq A\} \in \mathcal{Q}$. Now it is easy to see that $a \in i^{-1}\big(\{S \in \mathcal{P} | S \subseteq A\}\big)$ if and only if $i(a) \subseteq A$, if and only if $a \in A$. Thus $A = i^{-1}\big(\{S \in \mathcal{P} | S \subseteq A\}\big)$. Conversely, let $\mathcal{R} \in \mathcal{Q}$. By definition, there is some $P \in \mathcal{P}$ such that $\mathcal{R} = \{S \in \mathcal{P} | S \subseteq P\}$. It is easy to see that $i^{-1}(\mathcal{R}) = P$.

We need to show that $(Y, \mathcal{Q})$ is geometric. Y is a complete lattice, ordered by set-inclusion, and $\mathcal{Q}$ is the set of principal downsets of Y. This means that $\widetilde{\mathcal{Q}}$ is the set of ideals in Y, and $\overline{\mathcal{Q}}$ is the set of closed sets of the weak topology. From Examples 2.5.2, we know that the intersection of these is $\mathcal{Q}$. □

This leads to the natural question is what subspaces of a geometric preconvexity space are geometric.

Proposition 3.21. *If $(X, \mathcal{P})$ is a geometric preconvexity space and $A \in \mathcal{P}$, then the restriction $(A, \mathcal{P}|_A)$ is a geometric preconvexity space.*

Proof. Since $\mathcal{P}$ is closed under intersection, $\mathcal{P}|_A \subseteq \mathcal{P}$. Now let $C \subseteq A$ be both a directed union of sets from $\mathcal{P}|_A$ and an intersection of finite unions of sets from $\mathcal{P}|_A$. Since $\mathcal{P}|_A \subseteq \mathcal{P}$, C is both a directed union of sets from $\mathcal{P}$ and an intersection of finite unions of sets from $\mathcal{P}$. Since $(X, \mathcal{P})$ is geometric, it follows that $C \in \mathcal{P}$, and since $C \subseteq A$, we have $C \in \mathcal{P}|_A$ as required. □

On the other hand, closed or convex subspaces of geometric preconvexity spaces are not necessarily geometric.

Example 3.22. Let $X = \mathbb{R}^2 \backslash \{(0,0)\}$, and let

$$Y = \{(x, y) \in [0,1]^2 | (|2x - 1| - 1)(|2y - 1| - 1) = 0\}$$

be the unit square with one corner at the origin. It is straightforward to check that X and Y, with the preconvexities coming from closed convex subsets of $\mathbb{R}^2$, are geometric. However, $X \cap Y$ is a closed subspace of X, and a convex subspace of Y, but the subset $\{(x, y) \in X \cap Y | x > 0$ or $y = 1\}$ is both closed and convex, but is not closed convex, so $X \cap Y$ is not geometric.

4. Stone Duality

4.1 Stone Duality for Topological Spaces

In this section, we review Stone duality for topological spaces. While a lot of what we review is well-known, some parts are written from an unusual perspective, and are not as well-known as they might be.

Given a topological space, the collection of closed sets form a coframe. (Many authors refer to the frame of open sets, but for our purposes the closed sets are more natural, and since we are not considering 2-categorical aspects, it does not matter since $\mathcal{C}\mathit{oframe} = \mathcal{F}\mathit{rame}^{\text{co}}$.) Furthermore, the inverse image of a continuous function between topological spaces is by definition a coframe homomorphism between the coframes of closed spaces. This induces a functor $C : \mathcal{T}\mathit{op} \longrightarrow \mathcal{C}\mathit{oframe}^{\text{op}}$. Not every coframe arises as closed sets of a topological space. Coframes that do arise in this way are called *spatial* and are said to "have enough points".

In some cases, there can be many topological spaces that have the same coframe of closed sets. If multiple points have the same closure, then there is no way to separate them by looking at the coframe of closed sets. Therefore, we restrict our attention to T_0 spaces, where the function from X to $C(X)$ sending a point to its closure is injective. The functor $T_0\text{-}\mathcal{T}\mathit{op} \xrightarrow{C} \mathcal{C}\mathit{oframe}^{\text{op}}$ is faithful.

We can recover a T_0 topological space from its lattice of closed sets and from the subset $S \subset C(X)$ consisting of the closures of singletons. For a coframe L, the elements which could arise as closures of singletons for a topological space corresponding to L are non-zero elements that cannot be written as a join of two strictly smaller elements (called *join-irreducible* elements). These are called the "points" of L since they correspond to coframe homomorphisms $f : L \longrightarrow 2$, where the 2-element coframe, 2, is the terminal object in $\mathcal{C}oframe^{\mathrm{op}}$. If we let $P_L \subseteq L$ be the set of points, a topological space X corresponds to a coframe $L = C(X)$ with a chosen subset $S \subseteq P_L$ such that for every $x \in L$, we have $x = \bigvee(S \cap \downarrow x)$ (that is, S is join-dense in L). Continuous functions $X \xrightarrow{g} Y$ correspond to coframe homomorphisms $C(Y) \xrightarrow{C(g)} C(X)$ whose left adjoint $C(X) \xrightarrow{C(g)^*} C(Y)$ (in the category of order-preserving maps) sends S_X to S_Y. We can express this left adjoint condition topologically as: for every $s \in S_X$, $C(g)^{-1}(\downarrow s)$ is a principal downset in $C(Y)$, and the top element is in S_Y, where $S_X \subseteq P_{C(X)}$ and $S_Y \subseteq P_{C(Y)}$ are the chosen sets of points that correspond to elements of X and Y respectively.

More formally, let $\mathcal{S}patial\mathcal{C}oframe_*$ be the category of pointed spatial coframes. Objects are pairs (L, S) where L is a coframe and $S \subseteq P_L$ is a join dense set of points of L (meaning $(\forall a \in L)(a = \bigvee(S \cap \downarrow a))$). These pairs are introduced in [6, 7], where they are called prime-based complete lattices. Morphisms $(M, T) \xrightarrow{g} (L, S)$ are coframe homomorphisms $M \xrightarrow{g} L$ whose left adjoint $L \xrightarrow{g^*} M$ as order-preserving homomorphisms restricts to a function $S \xrightarrow{g^*_S} T$.

Proposition 4.1 ([6]). *The category of T_0 topological spaces and continuous functions is equivalent to the category $\mathcal{S}patial\mathcal{C}oframe_*^{\mathrm{op}}$.*

Proof. The functor $C : T_0\text{-}\mathcal{T}op \longrightarrow \mathcal{S}patial\mathcal{C}oframe_*^{\mathrm{op}}$ sends a topological space X to the pair $\left(C(X), \left\{\overline{\{x\}}\middle| x \in X\right\}\right)$, where $C(X)$ is the coframe of closed subsets of X. It sends a continuous function $f : X \longrightarrow Y$ to $f^{-1} : C(Y) \longrightarrow C(X)$. We need to show that this is a homomorphism in $\mathcal{S}patial\mathcal{C}oframe_*$. It is clearly a coframe homomorphism, so we need to show that for any $x \in X$,

$$\bigwedge \left\{t \in \left\{\overline{\{y\}}\middle| y \in Y\right\}\middle|\overline{\{x\}} \leqslant f^{-1}(t)\right\} \in \{\overline{\{y\}}|y \in Y\}$$

We will show that $\bigwedge \left\{t \in \left\{\overline{\{y\}}\middle| y \in Y\right\}\middle|\overline{\{x\}} \leqslant f^{-1}(t)\right\} = \overline{\{f(x)\}}$. We need to show that $\overline{\{x\}} \subseteq f^{-1}\left(\overline{\{f(x)\}}\right)$, and if $\overline{\{x\}} \subseteq f^{-1}(A)$ for any closed

$A \subseteq Y$, then $\overline{\{f(x)\}} \subseteq A$. Clearly, $x \in f^{-1}\left(\overline{\{f(x)\}}\right)$, so $f^{-1}\left(\overline{\{f(x)\}}\right)$ is a closed set containing x, so $\overline{\{x\}} \leqslant f^{-1}\left(\overline{\{f(x)\}}\right)$. On the other hand, suppose $\overline{\{x\}} \leqslant f^{-1}(A)$. Then $x \in f^{-1}(A)$, so $f(x) \in A$, so $\overline{\{f(x)\}} \leqslant A$. Thus f^{-1} is a morphism in $\mathcal{S}patial\mathcal{C}oframe_*^{\mathrm{op}}$.

In the opposite direction, the functor $P : \mathcal{S}patial\mathcal{C}oframe_*^{\mathrm{op}} \longrightarrow T_0\text{-}\mathcal{T}op$ sends the pair (L, S) to the topological space with elements S and closed sets $\{S \cap \downarrow a | a \in L\}$. For the morphism $(L, S) \xrightarrow{f} (M, T)$, we define $T \xrightarrow{f^*} S$ as the restriction of the left adjoint of f to T. By definition of $\mathcal{S}patial\mathcal{C}oframe_*$, this is a well-defined function. For any $F \in L$, we have $f^*(t) \in \downarrow F$ if and only if $t \leqslant f(F)$ by definition, so $(f^*)^{-1}(\downarrow F \cap S) = T \cap \downarrow f(F)$ is a closed subset of $P(M, T)$. Thus f^* is continuous.

Finally, we need to show that the two functors defined above form an equivalence. For a topological space X, we see that PCX has the same elements as X and closed sets of PCX are of the form $\downarrow F \cap \left\{\overline{\{x\}}\middle| x \in X\right\}$ for $F \in C(X)$. It is clear that $\overline{\{x\}} \leqslant F$ if and only if $x \in F$, so closed sets of PCX are exactly closed sets of X, so $PCX \cong X$.

For a coframe L with a chosen subset $S \subseteq L$, we want to show that $CP(L, S) \cong (L, S)$. By definition, elements of $CP(L, S)$ are $\{\downarrow a \cap S | a \in L\}$. Since $(\forall a \in L)(a = \bigvee(\downarrow a \cap S))$, it follows that the coframe of $CP(L, S)$ is isomorphic to L. The chosen elements are $\left\{\overline{\{s\}}\middle| s \in S\right\}$, where $\overline{\{s\}}$ is the closure of $\{s\}$ in $P(L, S)$. Closed sets of $P(L, S)$ are of the form $\downarrow a \cap S$ for $a \in L$, so in particular $\overline{\{s\}} = \downarrow s \cap S$. This clearly induces an isomorphism $(L, S) \cong CP(L, S)$. □

An alternative approach due to [17] is take the embedding of the coframe of closed sets into the completely distributive lattice of arbitrary unions of closed sets. That is, for (L, S) a pointed spatial coframe, we have the coframe inclusion $L \rightarrowtail DS$, where DS is the completely distributive lattice of down-closed subsets of S (where S is viewed as a sub-poset of L). In topological terms, S is the collection of points of the space, with the specialisation order. Downsets of S correspond to arbitrary unions of closed sets, and the inclusion of L into DS is the obvious inclusion. In lattice theoretic terms, the inclusion $L \rightarrowtail DS$ sends $x \in L$ to $\{s \in S | s \leqslant x\}$. For a homomorphism $f : (M, T) \longrightarrow (L, S)$, the condition that f^* restricts to a function $S \xrightarrow{f^*} T$ means that the inverse image function $Df^* : DT \longrightarrow DS$ is a complete lattice homomorphism. Furthermore, the diagram

$$\begin{array}{ccc} M & \xrightarrow{f} & L \\ \downarrow & & \downarrow \\ DT & \xrightarrow[Df^*]{} & DS \end{array}$$

commutes, since Df^* sends $T \cap \downarrow x$ to

$$\{s \in S | f^*(s) \in T \cap \downarrow x\} = \{s \in S | f^*(s) \leqslant x\} = \{s \in S | s \leqslant f(x)\} = S \cap \downarrow f(x)$$

The condition that $S \subseteq L$ means that all totally compact elements of DS (elements $x \in DS$ such that for any $A \subseteq DS$, if $\bigvee A \geqslant x$, then there is some $a \in A$ such that $a \geqslant x$) are in L, so every element of DS is a join of elements in L. We will refer to such lattice inclusions as dense. Thus the category of T_0 topological spaces is equivalent to the category of dense inclusions of spatial coframes into totally compactly generated completely distributive lattices.

The collections L^{op} of open subsets of the topological space, and DS of arbitrary unions of closed sets, generate the open sets of a larger topology, called the *Skula* topology [14]. Putting these three lattices together gives a structure called the Skula biframe. A *biframe* [2], consists of a frame L_0 with two chosen subframes L_1 and L_2, such that $L_1 \cup L_2$ generates L_0. The biframe (L_0, L_1, L_2) is strictly zero-dimensional if every element of L_1 is complemented in L_0, and the complement is in L_2. Every zero-dimensional biframe is determined by the complement inclusion $(L_1)^{\mathrm{op}} \rightarrowtail L_2$, so the functor that sends a topological space to the Skula biframe is one half of an equivalence between the category of T_0 topological spaces and the category of strictly zero-dimensional biframes [13].

For all of these representations of T_0 topological spaces, the fibres of the forgetful functor

$$T_0\text{-}\mathcal{T}op \xrightarrow{C} \mathcal{S}patial\mathcal{C}oframe^{\mathrm{op}}$$

correspond to additional structure on the coframe, and are partially ordered by inclusion of this additional structure. Every fibre has a top element, which gives an adjoint to the forgetful functor C, sending a spatial coframe to the top element of the fibre over it. (In fact, this adjoint extends to all coframes, because spatial coframes are reflective in all coframes). These top elements of the fibres are exactly the sober spaces.

Not all fibres have bottom elements. However, a large number of the fibres of the forgetful functor $T_0\text{-}\mathcal{T}op \xrightarrow{C} \mathcal{S}patial\mathcal{C}oframe^{\mathrm{op}}$ do have bottom elements and are actually complete Boolean algebras. This is probably easiest to see from the representation as coframes with a chosen set of elements which are closures of points of the topological space. If S_0 is the smallest such set of closed sets that can arise as closures of points, and S_1 is the largest set, then any set between S_0 and S_1 is a valid set of points, making the poset of possible sets of points isomorphic to the Boolean algebra $P(S_1 \backslash S_0)$. The topological spaces that can occur as the bottom elements of fibres are spaces where the closure of every point cannot be expressed as a union of closed sets not containing that point. That is, for every $x \in X$, $\overline{\{x\}} \backslash \{x\}$ is closed. Spaces with this property are called T_D spaces [1].

Clearly, all T_1 spaces are T_D because in a T_1 space $\overline{\{x\}} \backslash \{x\} = \varnothing$ is closed. However, even if we restrict to T_1 spaces and atomic spatial coframes, the assignment of an atomic spatial coframe to the bottom element in the corresponding fibre is not functorial, since the adjoint to a coframe homomorphism between T_D spaces does not necessarily preserve join-indecomposable elements, or even atoms. This is why the focus of attention in most of the literature has been on sober spaces, rather than T_D spaces. In order to model the morphisms between T_D spaces, we need to restrict to coframe homomorphisms whose adjoint preserves join-indecomposable elements. While most of the topological spaces of interest are T_D, many of the fibres of the forgetful functor $T_0\text{-}\mathcal{T}op \xrightarrow{C} \mathcal{S}patial\mathcal{C}oframe^{\mathrm{op}}$ contain only a singleton T_0 topological space, which is therefore both sober and T_D. (Several equivalent characterisations of when this occurs are given in [8].) Thus many important topological spaces are sober.

4.2 Stone Duality for Preconvexity Spaces

There is in many ways, a very similar picture for the category of preconvexity spaces. Instead of the coframe of closed sets, the structure that defines the preconvexity spaces is the complete lattice of preconvex sets $\mathcal{P}$. Because the inverse image function for a preconvexity space homomorphism preserves preconvex sets, it induces an inf-homomorphism between the lattices of preconvex sets. Thus, we have a functor $\mathcal{P}reconvex \xrightarrow{P} \mathcal{I}nf^{\mathrm{op}}$, where $\mathcal{I}nf$ is the category of complete lattices with infimum-preserving homomorphisms between them, sending every preconvexity space to its lattice of preconvex sets, and every homomorphism to the inverse image function. This has many of the nice properties of the Stone duality functor $\mathcal{T}op \xrightarrow{F} \mathcal{C}oframe^{\mathrm{op}}$.

As in the topology case, there is an equivalent category of sup-lattices with a set of chosen elements. Let $\mathcal{T}CGP artial\mathcal{S}up$ be the category whose objects are pairs (L, S), where L is a complete lattice and $S \subseteq L\backslash\{\bot\}$ is sup-dense, i.e. $(\forall x \in L)(x = \bigvee(S \cap \downarrow x))$. Morphisms $(L, S) \xrightarrow{f} (M, T)$ in $\mathcal{T}CGP artial\mathcal{S}up$ are sup-homomorphisms $L \xrightarrow{f} M$ with the property that $(\forall x \in S)(f(x) \in T)$. These pairs are called *based complete lattices* in [6].

Proposition 4.2 ([6]). *The categories $\mathcal{T}CGP artial\mathcal{S}up$ and T_0-$\mathcal{P}reconvex$ are equivalent.*

Proof. There is a functor T_0-$\mathcal{P}reconvex \xrightarrow{F} \mathcal{T}CGP artial\mathcal{S}up$ given by $F(X, \mathcal{P}) = \left(\mathcal{P}, \left\{\overline{\{x\}}\middle| x \in X\right\}\right)$ on objects and $F(f) \dashv f^{-1}$ on morphisms, where the adjoint is as a partial order homomorphism and exists because f^{-1} is an inf-homomorphism. To show this is well-defined, since $F(f)$ is a left adjoint, it is a sup-homomorphism, and can be given explicitly by $F(f)(A) = \bigcap\{B \in \mathcal{P}' | A \subseteq f^{-1}(B)\}$. In particular, if $A = \overline{\{x\}}$, then

$$F(f)(A) = \bigcap\{B \in \mathcal{P}' | x \in f^{-1}(B)\} = \bigcap\{B \in \mathcal{P}' | f(x) \in B\} = \overline{\{f(x)\}}$$

To complete the proof that F is well-defined, we need to show that $\left\{\overline{\{x\}}\middle| x \in X\right\}$ is sup-dense in $\mathcal{P}$. For any $P \in \mathcal{P}$, and any $x \in P$, we have $\overline{\{x\}} \subseteq P$. Thus $P = \bigcup\left\{\overline{\{x\}}\middle| x \in P\right\}$ as required. Thus F is well-defined, and functoriality is obvious.

In the other direction, we define $G : \mathcal{T}CGP artial\mathcal{S}up \longrightarrow \mathcal{P}reconvex$ by $G(L, S) = (S, \{S \cap \downarrow x | x \in L\})$ on objects and $G(f)(s) = f(s)$ on morphisms. To show well-definedness, for $(L, S) \xrightarrow{f} (M, T)$ a morphism of $\mathcal{T}CGP artial\mathcal{S}up$, we need to show that $G(f)$ is a preconvexity homomorphism. That is, for any $s \in S$, we have $f(s) \in T$, and for any $m \in M$,

$G(f)^{-1}(T \cap \downarrow m) = S \cap \downarrow x$ for some $x \in L$. The first condition is by definition of a homomorphism. Since f is a sup-homomorphism, it has a right adjoint f_* given by $f_*(m) = \bigwedge\{x \in L | f(x) \geqslant m\}$. Now

$$G(f)^{-1}(T \cap \downarrow m) = \{s \in S | f(s) \leqslant m\} = \{s \in S | s \leqslant f_*(m)\} = S \cap \downarrow f_*(m)$$

which gives the required homomorphism property. Finally, we want to show that F and G form an equivalence of categories. For a preconvexity space $(X, \mathcal{P})$, we have that

$$GF(X,\mathcal{P}) = G\left(\mathcal{P}, \left\{\overline{\{x\}} | x \in X\right\}\right) = \left(\left\{\overline{\{x\}} | x \in X\right\}, \left\{\left\{\overline{\{x\}} | x \in X\right\} \cap \downarrow P \middle| P \in \mathcal{P}\right\}\right)$$

It is obvious that the function sending x to $\overline{\{x\}}$ is a natural isomorphism of preconvexity spaces. In the other direction, for $(L, S) \in \mathrm{ob}(\mathcal{T}CGPartial\mathcal{S}up)$, we have

$$FG(L,S) = F(S, \{S \cap \downarrow x | x \in L\}) = \left(\{S \cap \downarrow x | x \in L\}, \left\{\overline{\{s\}} \middle| s \in S\right\}\right)$$

For $s \in S$, $\overline{\{s\}} = S \cap \downarrow s$, so the function $L \xrightarrow{i} \{S \cap \downarrow x | x \in L\}$ given by $i(x) = S \cap \downarrow x$ is easily seen to be an isomorphism in $\mathcal{T}CGPartial\mathcal{S}up$. Thus we have shown the equivalence of categories. □

Under this equivalence (and the adjoint isomorphism $\mathcal{S}up \cong \mathcal{I}nf^{\mathrm{op}}$), the functor T_0-$\mathcal{P}reconvex \xrightarrow{P} \mathcal{I}nf^{\mathrm{op}}$ corresponds to the forgetful functor $\mathcal{T}CGPartial\mathcal{S}up \xrightarrow{U} \mathcal{S}up$ sending (L, S) to L. As in the topological case, it is easy to see that the fibres of the functor U are partial orders. Each fibre clearly has a top element setting $S = L$. This induces a right adjoint to U. Furthermore, we can show that this right adjoint extends to all preconvexity spaces.

Proposition 4.3. *The preconvex set lattice functor $\mathcal{P}reconvex \xrightarrow{U} \mathcal{S}up$ has a right adjoint $\mathcal{S}up \xrightarrow{P} \mathcal{P}reconvex$.*

Proof. The right adjoint P is defined by $P(L) = (L, \{\varnothing\} \cup \{\downarrow x | x \in L\})$. That is, it sends the complete lattice L to L with the preconvexity where only principal downsets are preconvex. From the equivalence, between T_0-$\mathcal{P}reconvex$ and $\mathcal{T}CGPartial\mathcal{S}up$, this P sends L to the pair (L, L), which is clearly the top element of the fibre of the forgetful functor, U, when restricted to T_0 preconvexity spaces. To show that P is right adjoint to U, we need to show the hom-sets $\mathcal{S}up(UX, L)$ and $\mathcal{P}reconvex(X, PL)$ are naturally isomorphic. For $f \in \mathcal{S}up(UX, L)$, the corresponding element of $\mathcal{P}reconvex(X, PL)$ is $\hat{f}$ given by $\hat{f}(x) = f(\overline{\{x\}})$. It is easy to see that $\hat{f}$ is a preconvexity homormophism, since preconvex sets in PL are principal downsets of L, and

$$(\hat{f})^{-1}(\downarrow(y)) = \{x \in X | f(x) \leqslant y\} = \{x \in X | \overline{\{x\}} \subseteq f_*(y)\} = f_*(y)$$

is preconvex. For $f \in \mathcal{P}reconvex(X, PL)$, the corresponding element of $\mathcal{S}up(UX, L)$ is $\tilde{f}$ given by $\tilde{f}(A) = \bigvee_{x \in A} f(x)$. Since f is a preconvexity homomorphism, we have that $f^{-1}(\downarrow(y))$ is preconvex for any $y \in L$. For $\mathcal{A} \subseteq UX$, we want to show that $\tilde{f}(\bigvee \mathcal{A}) \leqslant \bigvee_{A \in \mathcal{A}} \tilde{f}(A)$. If y is an upper bound for $\{\tilde{f}(A) | A \in \mathcal{A}\}$, then since f is a preconvexity homomorphism, $f^{-1}(\downarrow y)$ is preconvex, and for any $A \in \mathcal{A}$, we have $\tilde{f}(A) \leqslant y$, so $A \subseteq f^{-1}(\downarrow y)$. Thus $f^{-1}(\downarrow y)$ is an upper bound for $\mathcal{A}$, in the lattice of preconvex subsets of X, so it contains $\bigvee \mathcal{A}$, as required. □

Bottom elements of the fibres are of the form (L, S) where S is the smallest subset of L satisfying $(\forall x \in L)(x = \bigvee S \cap \downarrow x)$. For any $x \in L$, if we can find a downset $D \subseteq L$ with $\bigvee D = x$ and $x \notin D$, then clearly if $(L, S) \in \operatorname{ob} \mathcal{TCGP}artial\mathcal{S}up$, then $(L, (S \backslash \{x\}) \cup D) \in \operatorname{ob} \mathcal{TCGP}artial\mathcal{S}up$, so if there is a minimum set S, then we cannot have $x \in S$. Conversely, if the only downset whose supremum is x is the principal downset $\downarrow x$, then for any (L, S) in $\mathcal{TCGP}artial\mathcal{S}up$, we must have $x \in S$. Thus, if there is a smallest element of the fibre above L, it must be given by (L, S), where

$$S = \left\{ x \in L \middle| (\forall D \subseteq L) \left(\left(\bigvee D = x \right) \Rightarrow x \in D) \right) \right\}$$

This is similar to the total compactness condition on elements of a sup-lattice, but an element x is called totally compact if it satisfies

$$(\forall D \subseteq L) \left(\left(\bigvee D \geqslant x \right) \Rightarrow (\exists y \in D)(x \leqslant y) \right)$$

which is a stronger condition. This condition is that x is totally compact in the sub-lattice $\downarrow x$.

As in the topological case, when bottom elements of the fibre exist, they are usually the spaces of greatest interest. For example, spaces where every singleton set is preconvex are always the bottom elements of the corresponding fibre. However, the fibres of the forgetful functor are very rarely singletons, so the top elements of the fibres are not of as much interest as in the topological case.

It is also worth noting that we have the chain of adjunctions

$$\mathcal{C}onvex\mathcal{T}op \underset{IS}{\overset{CC}{\underset{\longleftarrow}{\overset{\longrightarrow}{\bot}}}} \mathcal{P}reconvex \underset{P}{\overset{U}{\underset{\longleftarrow}{\overset{\longrightarrow}{\bot}}}} \mathcal{S}up$$

which gives an adjunction between the category of topological convexity spaces and the category of sup-lattices. This adjunction sends a topological convexity space $(X,\mathcal{F},\mathcal{C})$ to the lattice of sets $\mathcal{F}\cap\mathcal{C}$ ordered by set inclusion, and a topological convexity space homomorphism to the left adjoint of its inverse image. The right adjoint sends a sup-lattice L to the topological convexity space $(L,\mathcal{S},\mathcal{I})$, where $\mathcal{S}$ is the set of weak-closed subsets of L, namely intersections of finitely-generated downsets in L, and $\mathcal{I}$ is the set of ideals in L.

Theorem 4.4. *There is an adjunction between the category of topological convexity spaces and the category of sup-lattices. The left adjoint sends a topological convexity space $(X,\mathcal{F},\mathcal{C})$ to the lattice $\mathcal{F}\cap\mathcal{C}$ of closed convex sets, ordered by inclusion, and a topological convexity space homomorphism $X \xrightarrow{f} Y$ to the adjoint of its inverse image function. The right adjoint sends a sup-lattice L to the topological convexity space $(L,\mathcal{S},\mathcal{I})$ from Example 2.3(2), and a sup-homomorphism $L \xrightarrow{f} K$ to f viewed as a topological convexity space homomorphism.*

Proof. It is straightforward to check that these functors are the composites of the adjunctions

$$\mathcal{C}onvex\mathcal{T}op \underset{IS}{\overset{CC}{\underset{\longleftarrow}{\overset{\longrightarrow}{\bot}}}} \mathcal{P}reconvex \underset{P}{\overset{U}{\underset{\longleftarrow}{\overset{\longrightarrow}{\bot}}}} \mathcal{S}up$$

shown in Proposition 3.9 and Proposition 4.3. □

Remark 4.5. In the abstract, we described the relation between topological convexity spaces and inf-lattices as an extension to the Stone duality between topological spaces and coframes. Any topological space is a topological convexity space with the discrete convexity, where all sets are convex. Similarly, the category of coframes is a subcategory of the category of inf-lattices. The following diagram commutes:

$$\begin{array}{ccc} \mathcal{T}op & \xrightarrow{D} & \mathcal{C}onvex\mathcal{T}op \\ {\scriptstyle C}\downarrow & & \downarrow{\scriptstyle UCC} \\ \mathcal{C}oframe^{\mathrm{op}} & \longrightarrow & \mathcal{I}nf^{\mathrm{op}} \end{array}$$

However, the adjoint ISP to UCC does not restrict to an adjoint to the closed set coframe functor, C, because $ISP(L)$ is not in general a topological space, even if L is a coframe. For $ISP(L)$ to be a topological space, all subsets of L would need to be ideals, which is impossible for non-trivial lattices. Thus only the forgetful functor is truly an extension, and the duality is not an extension.

4.3 Distributive Partial-Sup Lattices

The equivalence T_0-$\mathcal{P}reconvex \cong \mathcal{T}CGPartialSup$ is based on previous work [11]. We present this work in a more abstract framework here. The idea is that for a preconvexity space $(X, \mathcal{P})$, the sets in $\mathcal{P}$ are partially ordered by inclusion. This partial order has an infimum operation given by intersection, but union of sets only gives a partial supremum operation because a union of preconvex sets is not necessarily preconvex. (Because of the existence of arbitrary intersections, there is a supremum operation given by union followed by the induced closure operation, but this supremum is not related to the structure of the preconvexity space. Unions of preconvex sets better reflect the structure of the preconvexity space. We therefore add a partial operation to the structure to describe these unions where possible.) For a preconvexity space, the operations are union and intersection, so we have a distributivity law between the partial join operation and the infimum operation. This can be neatly expressed by saying that the partial join structure is actually a partial join structure in the category $\mathcal{I}nf$. We define a partial join structure as a partial algebra for the downset monad. The downset monad exists in the category of partial orders, and also in the category of inf-lattices.

We begin by recalling the following definitions:

Definition 4.6 ([12]). *A* KZ-doctrine *on a 2-category $\mathcal{C}$ is a monad (T, η, μ) on $\mathcal{C}$ with a modification $T\eta \overset{\lambda}{\Longrightarrow} \eta_T$ such that $\lambda\eta$, $\mu\lambda$ and $\mu T\mu\lambda_T$ are all identity 2-cells.*

Definition 4.7 ([3]). *A 2-functor $\mathcal{C} \xrightarrow{F} \mathcal{D}$ is* sinister *if for every morphism f in $\mathcal{C}$, Ff has a right adjoint in $\mathcal{D}$.*

In particular, if F is sinister, then it gives rise to a functor from the category of partial maps in $\mathcal{C}$, to $\mathcal{D}$, sending the partial map $X \overset{a}{\leftarrowtail} A \xrightarrow{f} Y$ to the map $FX \xrightarrow{(Fa)_*} FA \xrightarrow{Ff} FY$.

Definition 4.8. *A* lax partial algebra *for a sinister KZ-doctrine in an order-enriched category is a partial map $TX \overset{\theta}{\rightharpoonup} X$ such that*

$$\begin{array}{ccc} X & \xrightarrow{\eta} & TX \\ & {\scriptstyle 1_X} \searrow & \downarrow{\scriptstyle \theta} \\ & & X \end{array}$$

commutes and there is a 2-cell

$$\begin{array}{ccc} TTX & \xrightarrow{T\theta} & TX \\ {\scriptstyle \mu_X}\downarrow & \Leftarrow & \downarrow{\scriptstyle \theta} \\ TX & \underset{\theta}{\rightharpoonup} & X \end{array}$$

A homomorphism *of lax partial algebras from (X, θ) to (Y, τ) is a morphism $X \xrightarrow{f} Y$, together with a 2-cell*

$$\begin{array}{ccc} TX & \xrightarrow{Tf} & TY \\ {\scriptstyle \theta}\downarrow & \Rightarrow & \downarrow{\scriptstyle \tau} \\ X & \underset{f}{\longrightarrow} & Y \end{array}$$

Remark 4.9. It is possible to define lax partial algebras for KZ monads in general 2-categories. However, this requires more careful consideration of coherence conditions, so to focus on the particular case of distributive partial sup-lattices, we have restricted attention to $\mathcal{O}rd$-enriched categories, where $\mathcal{O}rd$ is the category of partially-ordered sets and order-preserving functions between them.

Definition 4.10. *A* partial sup-lattice *is a lax partial algebra for the sinister KZ doctrine* $(D, \downarrow, \bigcup)$ *in* $\mathcal{O}rd$, *where* D *is the downset functor,* $\downarrow_X$ *is the function sending an element* $x \in X$ *to the principal downset it generates, and* $\bigcup_X : DDX \longrightarrow DX$ *sends a collection of downsets to its union.*

A distributive partial sup-lattice *is a lax partial algebra for the sinister KZ doctrine* $(D, \downarrow, \bigcup)$ *in* $\mathcal{I}nf$, *where* D *is the downset functor,* $\downarrow_X$ *is the function sending an element* $x \in X$ *to the principal downset it generates, and* $\bigcup_X : DDX \longrightarrow DX$ *sends a collection of downsets to its union.*

The definition given in [11] is

Definition 4.11 ([11])**.** *A* partial sup lattice *is a pair* (L, J) *where* L *is a complete lattice,* J *is a collection of downsets of* L *with the following properties:*

- J *contains all principal downsets.*
- J *is closed under arbitrary intersections.*
- *If* $A \in J$ *has supremum* x, *then any downset* B *with* $A \subseteq B \subseteq \downarrow x$ *has* $B \in J$.
- *If* $\mathcal{A} \subseteq J$ *is down-closed,* $Y \in J$ *has* $\bigvee Y = x$ *and for any* $a \in Y$, *there is some* $A \in \mathcal{A}$ *with* $\bigvee A \geqslant a$, *then there is some* $B \subseteq \bigcup \mathcal{A}$ *with* $B \in J$ *and* $\bigvee B \geqslant x$.

A partial sup-lattice, (L, J), *is* distributive *if for any* $\mathcal{D} \subseteq J$, *we have* $\bigwedge \{\bigvee D | D \in \mathcal{D}\} = \bigvee \bigcap \mathcal{D}$.

An inf-homomorphism $L \xrightarrow{f} M$ *is a* partial sup-lattice homomorphism $(L, J) \xrightarrow{f} (M, K)$ *if for any* $A \in J$, *we have* $\downarrow\{f(a) | a \in A\} \in K$, *and* $\bigvee \downarrow\{f(a) | a \in A\} = f(\bigvee A)$.

Proposition 4.12. *Definitions 4.10 and 4.11 give equivalent definitions of distributive partial sup lattices.*

Proof. We need to show that if $DL \xrightarrow{\theta} L$ is a lax partial algebra for the downset monad in $\mathcal{I}nf$, then there is some $J \subseteq DL$ satisfying the conditions of Definition 4.11. We will show that setting J as the domain of the partial algebra morphism $DL \xrightarrow{\theta} L$ works. We will let j denote the inclusion

$J \rightarrowtail DL$, and write j^{-1} for the inverse image map $DDL \xrightarrow{j^{-1}} DJ$ that is right adjoint to Dj.

From the unit condition

$$\begin{array}{ccccc} L & \xrightarrow{\downarrow} & DL & \xleftarrow{j} & J \\ & \searrow^{1_L} & & & \downarrow{\scriptstyle\theta} \\ & & & & L \end{array}$$

we have that all principal downsets must be contained in J. This allows us to show that θ is the join whenever it is defined. For $A \in J$, if $x = \bigvee A$, then $A \leqslant \downarrow x$ in J, and for any $a \in A$, we have $\downarrow a \leqslant A$ in J. Since θ is order-preserving, this gives $a = \theta(\downarrow a) \leqslant \theta(A) \leqslant \theta(\downarrow x) = x$, so $\theta(A)$ is an upper bound of A, and is below $x = \bigvee A$. Thus $\theta(A) = \bigvee A$. Since the inclusion $J \overset{j}{\rightarrowtail} DL$ is an inf-homomorphism, we get that J is closed under arbitrary intersections. Suppose $A \in J$ has supremum x, and $B \in DL$ satisfies $A \subseteq B \subseteq \downarrow x$. We want to show that $B \in J$.

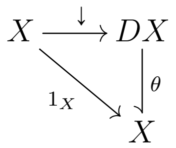
The lax partial algebra condition gives

$$\begin{array}{ccccc} DDL & \xrightarrow{j^{-1}} & DJ & \xrightarrow{D\theta} & DL \\ & & & & \uparrow{\scriptstyle j} \\ \bigcup\downarrow & & \geqslant & & J \\ & & & & \downarrow{\scriptstyle\theta} \\ DL & \xleftarrow{j} & J & \xrightarrow{\theta} & L \end{array}$$

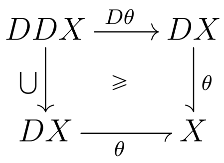
In particular, since $A \in J \cap \downarrow B$, we have $D\theta(J \cap \downarrow B) = \downarrow x$, and since $\theta(\downarrow x) = x$ is defined, we have that the upper composite partial morphism is defined on $J \cap \downarrow B$. For the lower composite, we have $\bigcup(J \cap \downarrow B) = B$, so for the lower composite to be defined, we must have $B \in J$.

Finally if $\mathcal{A} \in DJ$, $Y \in J$ has $\bigvee Y = x$ and for any $a \in Y$, there is some $A_a \in \mathcal{A}$ with $\bigvee A_a \geqslant a$, then clearly $A_a \cap \downarrow a \in J$, and since $J \xrightarrow{\theta} L$ is an inf-homomorphism, $\bigvee(A_a \cap \downarrow a) = \theta(A_a \cap \downarrow a) = a$. Thus, setting $\mathcal{B} = \downarrow\{A_a \cap \downarrow a | a \in Y\}$ gives $D\theta(\mathcal{B}) = Y$, so the upper composite is defined for $\mathcal{B}$, and is equal to x. Thus, the lower composite gives $B = \bigcup \mathcal{B} \in J$ with $\theta(B) = x$, which proves the last condition.

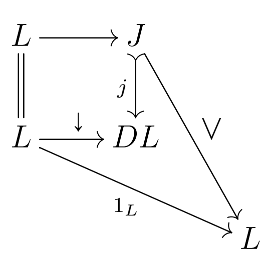
Conversely, suppose that (L, J) is a distributive partial sup lattice as in Definition 4.11. We want to show that $DL \xleftarrow{j} J \xrightarrow{\bigvee} L$ is a lax partial algebra for the downset KZ monad. That is, we want to show that

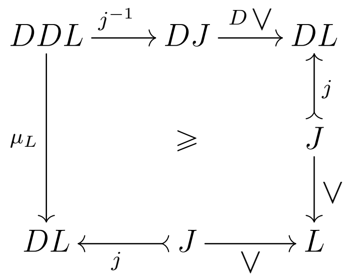


commutes, and



We expand the partial morphisms to get the following diagrams





The first diagram commutes because J contains all principal downsets. For the second diagram, if the upper-right composite of the diagram is defined for $\mathcal{A}$, we have $j^{-1}(\mathcal{A}) = \mathcal{A} \cap J$, and $Y = {\downarrow}\{\bigvee A | A \in \mathcal{A} \cap J\} \in J$. By definition, for every $a \in Y$, there is some $A_a \in \mathcal{A} \cap J$ such that $\bigvee A_a \geqslant a$. Now, by the fourth condition in Definition 4.11, there is some $B \subseteq \bigcup \mathcal{A}$, with $B \in J$ and $\bigvee B \geqslant \bigvee Y$. For $x \in \bigcup \mathcal{A}$, we have ${\downarrow}(x) \in \mathcal{A} \cap J$, so $x \in Y$, and therefore $\bigvee Y = \bigvee (\bigcup \mathcal{A})$, so $\bigvee B = \bigvee Y = \bigvee (\bigcup \mathcal{A})$. Now by the third condition of Definition 4.11, it follows that $\bigcup \mathcal{A} \in J$, so the lower-left composite is defined for $\mathcal{A}$, giving the required inequality of partial maps. □

Proposition 4.13. *The definition of distributive partial sup-lattice homomorphisms given in Definition 4.11 is equivalent to a lax partial algebra homomorphism between lax partial algebras.*

Proof. Because θ is the restriction of the supremum operation, the lax partial algebra homomorphism condition is exactly that J factors through the pullback

$$\begin{array}{ccc} K^* & \longrightarrow & K \\ \downarrow & & \downarrow \\ DL & \xrightarrow[Df]{} & DM \end{array}$$

and for any $A \in J$, $f\left(\bigvee A\right) \leqslant \bigvee \downarrow\{f(a)|a \in A\}$.

The pullback is given by $K^* = \{A \in DL|Df(A) \in K\}$. Thus the inclusion is equivalent to the condition for any $A \in J$, $\downarrow\{f(a)|a \in A\} \in K$.

Since f is order-preserving, for $a \in A$, we have that $f(a) \leqslant f\left(\bigvee A\right)$, so $f\left(\bigvee A\right)$ is an upper bound for $\downarrow\{f(a)|a \in A\}$, and thus $\bigvee \downarrow\{f(a)|a \in A\} \leqslant f\left(\bigvee A\right)$. Thus the second condition that $\bigvee \downarrow\{f(a)|a \in A\} \geqslant f\left(\bigvee A\right)$ is equivalent to $\bigvee \downarrow\{f(a)|a \in A\} = f\left(\bigvee A\right)$ as required.

□

Definition 4.14. *An element a of a partial sup-lattice (L, J) is* totally compact *if for any downset $D \in J$, $\bigvee D \geqslant a \Rightarrow a \in D$. (Note that $\varnothing \in J$, so $\bot$ is not totally compact.) A partial sup-lattice (L, J) is* totally compactly generated *if for any $x \in L$, there is some $C \subseteq L$ such that every $c \in C$ is totally compact, and such that $\downarrow C \in J$ and $\bigvee C = x$.*

Proposition 4.15. *The full subcategory of totally compactly generated distributive partial sup-lattices and partial sup-lattice homomorphisms is equivalent to the category $\mathcal{TCGP}artial\mathcal{S}up^{\mathrm{op}}$ defined at the start of Section 4.2.*

Proof. Given a totally compactly generated distributive partial sup-lattice (L, J), let $K \subseteq L$ be the set of totally compact elements of (L, J). Then (L, K) is an element of $\mathcal{TCGP}artial\mathcal{S}up$. Conversely, for the object $(L, S) \in \mathrm{ob}\,\mathcal{TCGP}artial\mathcal{S}up$, let $J = \{D \in DL|S \cap \downarrow\left(\bigvee D\right) \subseteq D\}$ be the set of downsets of L that contain all totally compact elements below their supremum. It is clear that performing these two constructions gives an isomorphic structure. To show an equivalence of categories, we need to show that $L \xrightarrow{f} M$ is a distributive partial sup-lattice homomorphism if and only if it is a morphism in $\mathcal{TCGP}artial\mathcal{S}up^{\mathrm{op}}$. Since distributive partial sup-lattice homomorphisms preserve infima, they have left adjoints. If f is a partial sup-lattice homomorphism, and f^* is its left adjoint, then f^* is a sup-homomorphism, and for any totally compact $a \in M$, if $B \in J$ has $\bigvee B \geqslant f^*(a)$, then the adjunction gives $f\left(\bigvee B\right) \geqslant a$. Since f is a partial sup-homomorphism, we have $f\left(\bigvee B\right) = \bigvee\{f(b)|b \in B\} \geqslant a$. As a is totally compact, we must have $a \leqslant f(b)$ for some $b \in B$. By the adjunction, this gives $f^*(a) \leqslant b$. Thus we have shown that if $B \in J$ has $\bigvee B \geqslant f^*(a)$, then $f^*(a) \in B$. That is, $f^*(a)$ is totally compact.

Conversely, if g is a sup-homomorphism between totally compactly generated distributive partial sup-lattices, that preserves totally compact elements, then its right adjoint is a partial sup-homomorphism, since if $B \in J$ has $\bigvee B = x$, then if $a \leqslant g_*(x)$ is totally compact, then $g(a) \leqslant x$ is also totally compact, so $g(a) \in B$. It follows that $a \in \downarrow\{g_*(b) | b \in B\}$, so $\bigvee \downarrow\{g_*(b)|b \in B\} = g_*(x)$ as required. □

5. Final Remarks and Future Work

We have extended the left adjoint functor from Stone duality, sending a topological space to its coframe of closed sets to a functor sending a topological convexity spaces to its sup-lattice of closed convex sets. As in the topological Stone duality, this functor has a right adjoint. This right adjoint is not an extension of the topological case.

In many ways, the theory is nicer in this situation than in the topological case. For example, there are no non-spatial sup-lattices: every sup-lattice arises as the closed convex sets of a topological convexity space. However, in some ways this nicer theory makes the results less useful, because in topology, the non-spatial locales fill some problematic gaps in the category of topological spaces. With every sup-lattice arising as the closed convex sets of a topological convexity space, there are no new spaces to be added, so we are not filling the gaps.

Another significant difference between this and Stone duality for topological spaces is that for the topological Stone duality, many interesting topological spaces are in the singleton fibres of the functor, meaning that the closed set functor is full and faithful for these spaces, so we can study the categorical structure of large classes of interesting topological spaces using the category of coframes. For topological convexity spaces, there are no singleton fibres, and the top elements of fibres (on which the functor is full and faithful) are not very interesting topological convexity spaces. The most interesting topological convexity spaces are the bottom elements of their fibres, and when we restrict the functor to these spaces, it is faithful, but not full, meaning that from a categorical perspective, $\mathcal{C}onvex\mathcal{T}op$ and $\mathcal{S}up$ are not so closely related.

The adjunction between topological convexity spaces and sup-lattices factors through the category of preconvexity spaces, or the equivalent category of totally compactly-generated distributive partial sup-lattices. The adjunction between topological convexity spaces and preconvexity spaces is potentially more interesting, with most interesting topological convexity spaces being fixed-points of the induced comonad on $\mathcal{ConvexTop}$. We have characterised which topological convexity spaces are fixed by this comonad in Proposition 3.11, and given some important examples in Proposition 3.14. In the opposite direction, for the question of which preconvexity spaces are fixed by the induced monad on $\mathcal{Preconvex}$, we have only been able to show this for a few special cases.

5.1 Future Work

The study of topological convexity spaces is an extremely promising area of research, including some classical geometric examples and also some very interesting combinatorial examples. The adjunctions from this paper are likely to prove extremely valuable in the study of topological convexity spaces. In this section, we discuss a number of important problems about topological convexity spaces that may be addressed using these adjunctions.

5.1.1 Restricting this to a Duality

In topology, it is often convenient to restrict Stone duality to an isomorphism of categories between sober topological spaces and spatial locales. Sober topological spaces can be described in a number of topologically natural ways. Similarly, spatial locales can be easily described. It is easy to describe the topological convexity spaces from this adjunction, as they come directly from lattices. However, for the intermediate adjunction between topological convexity spaces and preconvexity spaces, the conditions for fixed points are less clear. The characterising conditions in Proposition 3.11 are not particularly natural, while the natural and commonly used conditions in Proposition 3.14 exclude a number of interesting combinatorial examples. A result between these two that includes the interesting combinatorial examples but also consists of natural, easy-to-understand conditions would be extremely valuable. In the other direction, describing the geometric preconvexity spaces is more challenging, and could lead to a lot of fruitful research.

5.1.2 Euclidean Spaces

The motivating examples for topological convexity spaces are real vector spaces, particularly finite-dimensional ones. The author has nearly completed a characterisation of these spaces within the category of topological convexity spaces, which will be presented in another paper.

5.1.3 Convexity Manifolds

In differential geometry, a manifold is a space which has a local differential structure. That is, the space is covered by a family of open sets, each of which has a local differential structure. There are examples of spaces with a cover by open subsets with a local convexity structure. The motivating example here is real projective space. We cannot assign a global convexity structure to projective space, but if we remove a line from the projective plane, then the remaining space is isomorphic to the Euclidean plane, and so has a canonical convexity space. Furthermore, these convexity spaces have a certain compatibility condition — given a subset C of the intersection which is convex in both convexity spaces, the convex subsets of C are the same in both spaces. This gives us the outline for a definition of convexity manifolds. Further work is needed to identify the Euclidean projective spaces within the category of convexity manifolds, and to determine what geometric structure is retained at this level of generality.

5.1.4 Metrics and Measures

There are connections between metrics and measures. For example, on the real line, any metric that induces the usual convexity space structure corresponds to a monotone function $\mathbb{R} \xrightarrow{d} \mathbb{R}$ with 0 as a fixed point. Such a function naturally induces a measure on the Lebesgue sets of $\mathbb{R}$. Conversely, for every measure on the Lebesgue sets of $\mathbb{R}$, we obtain a monotone endofunction of $\mathbb{R}$ by integrating. Thus, for the real numbers, there is a bijective correspondence between metrics that induce the usual topological convexity structure and measures on $\mathbb{R}$. This property is specific to $\mathbb{R}$, and does not generalise to other spaces like $\mathbb{R}^2$.

There is a more general connection between topological convexity spaces, sigma algebras, measures and metrics.

Example 5.1. Let $(X, \mathcal{B})$ be a Σ-algebra. There is a topological convexity space $(\mathcal{B}, \mathcal{F}, \mathcal{I})$ where $\mathcal{I}$ is the set of intervals in the lattice $\mathcal{B}$ and $\mathcal{F}$ is the set of collections of measurable sets closed under limits of characteristic functions. That is, for $B_1, B_2, \ldots \in \mathcal{B}$, say B is the *limit* of $B_1, B_2, \ldots$ if for any $x \in B$, there is some $k \in \mathbb{Z}^+$ such that $x \notin B_i \Rightarrow i < k$, and for any $x \notin B$, there is some $k \in \mathbb{Z}^+$ such that $x \in B_i \Rightarrow i < k$. $\mathcal{F}$ is the collection of subsets $F \subseteq \mathcal{B}$ such that for any $B_1, B_2, \ldots \in F$, if B is the limit of $B_1, B_2, \ldots$, then $B \in F$.

Proposition 5.2. *If μ is a finite measure on $(X, \mathcal{B})$ such that no non-empty set has measure zero, then $d : \mathcal{B} \times \mathcal{B} \longrightarrow \mathbb{R}$ given by $d(A, B) = \mu(A \triangle B)$, is a metric and induces the topological convexity space structure $(\mathcal{B}, \mathcal{F}, \mathcal{I})$ or a finer structure. Furthermore, all metrics inducing this topological convexity space structure on $\mathcal{B}$ are of this form.*

Proof. We have $d(A, A) = \mu(\varnothing) = 0$ and $d(A, B) = d(B, A)$, so we need to prove the triangle inequality. That is, for $A, B, C \in \mathcal{B}$, we have $d(A, C) \leqslant d(A, B) + d(B, C)$. This is clear because $A \triangle C \subseteq A \triangle B \cup B \triangle C$. Thus d is a metric. To prove that it induces this topological convexity structure, we note that $d(A, C) = d(A, B) + d(B, C)$ if and only if $A \triangle C = A \triangle B \amalg B \triangle C$. This only happens if $A \cap C \subseteq B \subseteq A \cup C$, which means that convex sets must be intervals. Finally, we need to show the topology from the metric is finer than $\mathcal{F}$. That is, if B is the limit of $B_1, B_2, \ldots$, then $d(B_i, B) \to 0$. By definition, $\bigcap_{i=1}^{\infty} B_i \triangle B = \varnothing$. Thus, we need to show that for a sequence $A_i = B_i \triangle B$ of measurable sets with empty intersection, $\mu(A_i) \to 0$. Let $C_i = \bigcup_{j \geqslant i} A_j$. Since $A_i \to \varnothing$, we get $\bigcap_{i=1}^{\infty} C_i = \varnothing$. Since the C_i are nested, we have $\lim_{i \to \infty} \mu(C_i) = \mu\left(\bigcap_{i=1}^{\infty} C_i\right) = 0$.

To show that every metric is of that form, let $d : \mathcal{B} \times \mathcal{B} \longrightarrow \mathbb{R}$ be a metric on $\mathcal{B}$ whose induced topology and convexity are finer than $\mathcal{F}$ and $\mathcal{I}$ respectively. We want to show that there is a finite measure μ on $(X, \mathcal{B})$ such that $d(A, B) = \mu(A \triangle B)$. By the convexity, whenever $A = B \amalg C$ is a disjoint union, we have $d(A, B) + d(B, \varnothing) = d(\varnothing, A) = d(\varnothing, C) + d(C, A)$

and $d(B, \varnothing) + d(\varnothing, C) = d(B, C) = d(B, A) + d(A, C)$. It follows that

$$2d(A, B) + d(B, \varnothing) + d(A, C) = d(A, C) + d(B, \varnothing) + 2d(\varnothing, C)$$

so $d(A, B) = d(C, \varnothing)$. For general B, we have $A \cap B$ is between A and B, so

$$d(A, B) = d(A, A \cap B) + d(A \cap B, B) = d(\varnothing, A \backslash B) + d(\varnothing, B \backslash A) = d(\varnothing, A \triangle B)$$

Thus, if we define $\mu(B) = d(\varnothing, B)$, then d is defined by $d(A, B) = \mu(A \triangle B)$. We need to show that μ is a measure on $(X, \mathcal{B})$. That is, that if A and B are disjoint, we have $\mu(A \cup B) = \mu(A) + \mu(B)$ and if $B_1 \subseteq B_2 \subseteq \cdots$, then $\mu(\bigcup_{i=1}^{\infty} B_i) = \lim_{i=1}^{\infty} \mu(B_i)$. We have already shown that the first of these comes from the convexity. The second comes from the topology. Consider the sequence $A_i = \left(\bigcup_{j=1}^{\infty} B_j\right) \backslash B_i$. By the convexity, we have $\mu(A_i) = \mu\left(\bigcup_{j=1}^{\infty} B_j\right) - \mu(B_i)$, so it is sufficient to show that $\mu(A_i) \to 0$, when $(A_i)_{i=1}^{\infty}$ is a decreasing sequence with empty intersection. If $(A_i)_{i=1}^{\infty}$ is a decreasing sequence with empty intersection, then for any $x \in X$, we have $(\exists k \in \mathbb{Z}^+)(x \notin A_k)$. Thus $\varnothing$ is a limit of $(A_i)_{i=1}^{\infty}$. Thus we have $\mu(A_i) \to \mu(\varnothing) = 0$ as required.

□

5.1.5 Sheaves

A lot of information about topological spaces can be obtained by studying their categories of sheaves. A natural question is whether a similar category of sheaves can be constructed for a topological convexity space. Part of the difficulty here is that the usual construction of the sheaf category is described in terms of open sets. However, for topological convexity spaces, closed sets are more fundamental, so it is necessary to redefine sheaves in terms of closed sets. This is conceptually strange. One interpretation of sheaves is as sets with truth values given by open sets. In this interpretation, closed sets correspond to the truth values of negated statements, such as inequality. [5] argues that inequality is a more fundamental concept for studying lattices of equivalence relations as a form of logical statement, so a definition of sheaves in terms of closed sets could be linked to this work.

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