CAHIERS DE TOPOLOGIE ET GEOMETRIE DIFFERENTIELLE CATEGORIQUES VOLUME LXIV-3 (2023)

PICARD GROUPOIDS AND Γ-CATEGORIES

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Résumé. Dans cet article, nous construisons une catégorie de modèles fermée monoïdale symétrique de groupoïdes de Picard commutatifs cohérents. Nous construisons une autre structure de catégorie de modèles sur la catégorie des (petites) catégories permutatives dont les objets fibrants sont des groupoïdes de Picard (permutatifs). Le résultat principal de cet article est que le foncteur nerf de Segal est un foncteur de Quillen droit d'une équivalence de Quillen entre les deux catégories de modèles susmentionnées. Sur la base de notre résultat principal, nous donnons une nouvelle preuve du résultat classique selon lequel les groupoïdes de Picard modélisent des monotypes d'homotopie stables.
Abstract. In this paper we construct a symmetric monoidal closed model category of coherently commutative Picard groupoids. We construct another model category structure on the category of (small) permutative categories whose fibrant objects are (permutative) Picard groupoids. The main result of this paper is that the Segal's nerve functor is a right Quillen functor of a Quillen equivalence between the two aforementioned model categories. Based on our main result, we give a new proof of the classical result that Picard groupoids model stable homotopy one-types.
Keywords. Your keywords come here.
Mathematics Subject Classification (2010). Your MSC numbers come here.

1. Introduction

Picard groupoids are interesting objects both in topology and algebra. A major reason for interest in topology is because they model stable homotopy 1-types which is a classical result appearing in various parts of the literature [JO12][Pat12][GK11][MOP$^+$20]. The category of Picard groupoids is the archetype example of a 2-Abelian category, see [Dup08]. A theory of 2-chain complexes of Picard groupoids was developed in [dRMMV05]. A simplicial cohomology with coefficients in Picard groupoids was introduced in the paper [CMM04].

This cohomology was used in [SV] to construct a TQFT called the *Dijkgraaf-Witten* theory. This (Picard) groupoidification of cohomology played a vital role in explaining a mysterious *integration theory* introduced in [FQ93], [Shab] which is pivotal in constructing the aforementioned TQFT functor.

A tensor product of Picard groupoids was defined in [Sch08]. However, a shortcoming of the category of Picard groupoids remains: unlike the category of abelian groups, it is not a symmetric monoidal closed category. In this paper we address this problem by proposing another model for Picard groupoids based on Γ- categories. A Γ- category is a functor from the (skeletal) category of finite based sets Γ^{op} into the category of all (small) categories $\mathbf{Cat}$. We denote the category of all Γ- categories and natural transformations between them by $\Gamma\mathbf{Cat}$. Along the lines of the construction of the stable Q-model category in [Sch99], we construct a symmetric monoidal closed model category $\Gamma\mathcal{P}ic$. The underlying category of $\Gamma\mathcal{P}ic$ is $\Gamma\mathbf{Cat}$ and we refer to it as the model category structure of *coherently commutative Picard groupoids*. A Γ- category X is called a *coherently commutative Picard groupoid* if it satisfies the Segal condition, see [Seg74] and moreover it has homotopy inverses. These Γ- categories are fibrant objects in our model category $\Gamma\mathcal{P}ic$. The main objective of this paper is to compare a (model) category of all (small) Picard groupoids with the model category of coherently commutative Picard groupoids $\Gamma\mathcal{P}ic$. We construct another model category structure on $\mathbf{Perm}$ whose fibrant objects are (permutative) Picard groupoids. This model category is denoted by $(\mathbf{Perm}, \mathbf{Pic})$ and called the model category of Picard groupoids. The main result of this paper, theorem 5.2, states that the following adjoint pair is a Quillen equivalence:

$$\mathcal{L} : \Gamma\mathcal{P}ic \rightleftarrows (\mathbf{Perm}, \mathbf{Pic}) : \mathcal{K}, \tag{1}$$

where $\mathcal{K}$ is the classical Segal's nerve functor which was originally defined in [Seg74] and a different description of it has recently appeared in[Shac].

A second aspect of our paper is about establishing a Quillen equivalence between a second pair of model category structures on the same two underlying categories, namely $\mathbf{Perm}$ and $\Gamma\mathbf{Cat}$. We first construct another cartesian closed (combinatorial) model category structure on $\mathbf{Cat}$, denoted by $(\mathbf{Cat}, \mathbf{Gpd})$, whose fibrant objects are groupoids. We then transfer this model category structure on the category of all permutative categories $\mathbf{Perm}$. Fibrant objects in this model category are permutative groupoids and it is denoted by $(\mathbf{Perm}, \mathbf{Gpd})$. We localize the model category of coherently commutative monoidal categories to get another symmetric monoidal closed model category $\Gamma\mathbf{Gpd}^{\otimes}$. The fibrant objects of this model category can be described as coherently commutative monoidal groupoids. These two model categories are instrumental in the construction of the model categories featuring in our main result. A second prominent result of this paper is that the following adjunction is a Quillen equivalence:

$$\mathcal{L} : \Gamma\mathbf{Gpd}^{\otimes} \rightleftarrows (\mathbf{Perm}, \mathbf{Gpd}) : \mathcal{K} \tag{2}$$

The Quillen equivalences (1) and (2) are proved in section 5. We first prove (2) and then use that to prove our main result (1).

In the last section of this paper we use our main result to prove a version of the *stable homotopy hypothesis* for Picard groupoids: the homotopy category of $(\mathbf{Perm}, \mathbf{Pic})$ is equivalent to a homotopy category of stable homotopy one-types. We recall that a stable homotopy one-type is a connective spectrum whose stable homotopy groups are trivial in all degrees greater than one.

We would like to mention that the Quillen equivalence (2) proved in theorem 5.1 can also be proved by an adaptation of some results of [dBM17] to (a suitable model category of) groupoids. This approach goes through the theory of dendroidal groupoids, more concretely, the approach appeals to the abstract relations between dendroidal groupoids and algebras over operads in groupoids and dendroidal groupoids and Γ-groupoids. In this paper we establish this Quillen equivalence without using any theory of dendroidal groupoids. Our approach is direct and is based on the results of [Shac].

Acknowledgments The author would like to thank Ieke Moerdijk for helping him understand the relation of his current and previous work with the theory of dendroidal categories. The author thanks André Joyal for several enlightening conversations regarding the paper. The author would also like to thank the anonymous referee for pointing out several inaccuracies in the paper and for making some helpful suggestions that have contributed to an improvement of the paper.

2. The Setup

In this section we will collect the machinery needed for the development of this paper. We begin with a review of permutative categories. We will also give a quick review of Γ- categories and collect some useful results about them. We will also construct a cartesian closed (simplicial) model category structure on the category of (small) categories $\mathbf{Cat}$ which will be used throughout this paper.

2.1 Review of Permutative categories

In this subsection we will briefly review the theory of permutative categories and monoidal and oplax functors between them. The definitions reviewed here and the notation specified here will be used throughout this paper.

Definition 2.1. A symmetric monoidal category C is called a *permutative* category or a *strict* symmetric monoidal category if its monoidal structure is strictly associative and unital.

Definition 2.2. An *oplax symmetric monoidal* functor F is a triple $(F, \lambda_F, \epsilon_F)$, where $F : C \to D$ is a functor between symmetric monoidal categories C and D,

$$\lambda_F : F \circ (- \underset{C}{\otimes} -) \Rightarrow (- \underset{D}{\otimes} -) \circ (F \times F)$$

is a natural transformation and $\epsilon_F : F(1_C) \to 1_D$ is a morphism in D, such that the following three conditions $OL.1$, $OL.2$ and $OL.3$ in [Shac, Defn. 2.4] are satisfied.

Notation 2.3. We will say that a functor $F : C \to D$ between two symmetric monoidal categories is *unital* or *normalized* if it preserves the unit of the symmetric monoidal structure *i.e.* $F(1_C) = 1_D$. In particular, we will say that an oplax symmetric monoidal functor is a unital (or normalized) oplax symmetric monoidal functor if the morphism ϵ_F is the identity.

Definition 2.4. An oplax symmetric monoidal functor $F = (F, \lambda_F, \epsilon_F)$ is called a *strong* symmetric monoidal functor (or just a symmetric monoidal functor) if λ_F is a natural isomorphism and ϵ_F is also an isomorphism.

Definition 2.5. An oplax symmetric monoidal functor $F = (F, \lambda_F, \epsilon_F)$ is called a *strict* symmetric monoidal functor if it is unital and λ_F is the identity natural transformation.

Definition 2.6. The *category of elements* of a $\mathbf{Cat}$ valued functor $F : C \to \mathbf{Cat}$, denoted by $\int^{c \in C} F(c)$ or *elF*, is a category whose objects are pairs (c, d), where $c \in C$ and $d \in F(c)$. A map from (c, d) to (a, b) in $\int^{c \in C} F(c)$ is a pair (f, α), where $f : c \to a$ is an arrow in C and $\alpha : F(f)(d) \to b$ is an arrow in $F(a)$.

Notation 2.7. Throughout this paper we will denote by $J : \mathbf{Cat} \to \mathbf{Gpd}$, a right adjoint of the inclusion functor $i : \mathbf{Gpd} \to \mathbf{Cat}$. For a category C, the groupoid $J(C)$ is obtained by discarding all non-invertible arrows of C.

2.2 Review of Γ- categories

In this subsection we will briefly review the theory of Γ- categories. We begin by introducing some notations which will be used throughout the paper.

Notation 2.8. We will denote by $\underline{n}$ the finite set $\{1, 2, \ldots, n\}$ and by n^+ the based set $\{0, 1, 2, \ldots, n\}$ whose basepoint is the element 0.

Notation 2.9. We will denote by $\mathcal{N}$ the skeletal category of finite unbased sets whose objects are $\underline{n}$ for all $n \geq 0$ and maps are functions of unbased sets. The category $\mathcal{N}$ is a (strict) symmetric monoidal category whose symmetric monoidal structure will be denoted by $+$. For two objects $\underline{k}, \underline{l} \in \mathcal{N}$ their *tensor product* is defined as follows:

$$\underline{k} + \underline{l} := \underline{k + l}.$$

Notation 2.10. We will denote by Γ^{op} the skeletal category of finite based sets whose objects are n^+ for all $n \geq 0$ and maps are functions of based sets.

Notation 2.11. Given a morphism $f : n^+ \to m^+$ in Γ^{op}, we denote by $\mathrm{Supp}(f)$ the largest subset of $\underline{n}$ whose image under f does not contain the basepoint of m^+. The set $\mathrm{Supp}(f)$ inherits an order from $\underline{n}$ and therefore could be regarded as an object of $\mathcal{N}$. We denote by $\mathrm{Supp}(f)^+$ the based set $\mathrm{Supp}(f) \sqcup \{0\}$ regarded as an object of Γ^{op} with order inherited from $\underline{n}$.

Definition 2.12. A map $f : n^+ \to m^+$ in Γ^{op} is called *inert* if its restriction to the set $\mathrm{Supp}(f)^+$ is a bijection.

Definition 2.13. A morphism f in Γ^{op} is called *active* if $f^{-1}(\{0\}) = \{0\}$ *i.e.* the pre-image of $\{0\}$ is the singleton set $\{0\}$.

Notation 2.14. A map $f : \underline{n} \to \underline{m}$ in the category $\mathcal{N}$ uniquely determines an active map in Γ^{op} which we will denote by $f^+ : n^+ \to m^+$. This map agrees with f on non-zero elements of n^+.

Remark 1. Each morphism in Γ^{op} can be factored into a composite of an inert map followed by an active map in Γ^{op}. The factorization is unique up to a unique isomorphism.

Definition 2.15. Each $n^+ \in \Gamma^{op}$ determines n *projection* maps $\delta^n_i : n^+ \to 1^+$ for $1 \leq i \leq n$ which are defined by $\delta^n_i(i) = 1$ and $\delta^n_i(j) = 0$ for $j \neq i$ and $j \in n^+$.

Definition 2.16. Each $n^+ \in \Gamma^{op}$ determines a *multiplication* map $m_n : n^+ \to 1^+$ which is the unique active map from n^+ to 1^+.

2.3 The model category structure of groupoids on Cat

In this subsection we will construct another model category structure on the category of all small categories $\mathbf{Cat}$ wherein an object is fibrant if and only if it is a groupoid and which we will refer to as the *model category structure of groupoids*. We remark that the model structure constructed here is different from the two well known model structures on $\mathbf{Cat}$, namely the natural model structure wherein all categories are fibrant and the Thomason model category structure which is Quillen equivalent to the Kan model category of simplicial sets $(\mathbf{sSets}, \mathbf{Kan})$. We will show that the weak equivalences in this model structure are those functors which induce a weak homotopy equivalence on their nerve. The model category structure is obtained by a *left Bousfield localization* of the *natural* model category structure on $\mathbf{Cat}$ with respect to the singleton set $\{i : 0 \to I$, where I is the category $0 \to 1$ and $i(0) = 0$. We review the definition and an existence result of left Bousfield localizations of model categories in appendix A.

Proposition 2.17. A category C is local with respect to the singleton set $\{i : 0 \to I\}$ if and only if it is a groupoid.

Proof. Let $\mathcal{J}$ denote the groupoid $0 \cong 1$. This groupoid is equipped with an inclusion functor $\iota : I \hookrightarrow \mathcal{J}$. A category C is a groupoid if and only if $J([\iota, C]) : J([\mathcal{J}, C]) \to J([I, C])$ is an equivalence of categories.

Since each object of the natural model category $\mathbf{Cat}$ is both cofibrant and fibrant, for any pair of categories C and D, the homotopy function complex is given as follows:

$$Map^h_{\mathbf{Cat}}(C, D) = N(J([C, D])).$$

This implies that a category C is $\{i\}$-local if and only if the following functor is an equivalence of groupoids:

$$J([i, C]) : J([I, C]) \to J(C).$$

Now we consider the following commutative diagram:

$$\begin{tikzcd} J([I,C]) \arrow[r, "J([i,C])"] & J(C) \\ & J([\mathcal{J},C]) \arrow[ul, "J([\iota,C])"] \arrow[u, "J([j,C])"'] \end{tikzcd}$$

where j is the inclusion functor $0 \hookrightarrow \mathcal{J}$. In light of the observation that the functor $J([j, C])$ is an equivalence of groupoids, the result now follows from the above commutative diagram of groupoids. $\square$

Theorem 2.18. There is a combinatorial model category structure on the category of (small) categories $\mathbf{Cat}$ in which a functor $F : A \to B$ is

1. a cofibration if it is monic on objects.

2. a weak equivalence if the following functor

$$[i, F] : [B, Z] \to [A, Z]$$

is an equivalence of categories for each groupoid Z.

3. a fibration if it has the right lifting property with respect to functors which satisfy both (1) and (2).

Proof. We want to carry out a left Bousfield localization of the natural model category of (small) categories with respect to the singleton set $\{i : 0 \to I$. The existence of this localization follows from theorem A.2. (1) follows from the aforementioned theorem. (2) follows from proposition 2.17 and [Shac, Lemma E.4]. (3) follows from the fact that fibrations in any model category are completely determined by cofibrations and weak equivalences.

□

Notation 2.19. We will refer to the above model category structure as the model category structure of *groupoids* on $\mathbf{Cat}$ and denote the model category by $(\mathbf{Cat}, \mathbf{Gpd})$. We will refer to a fibration in this model category as a *path fibration* of categories and refer to a weak equivalence as a *groupoidal equivalence* of categories.

Remark 2. Every category is cofibrant in the model category of groupoids. A category is fibrant if and only if it is a groupoid.

Remark 3. A groupoidal equivalence between groupoids is an equivalence of categories.

Proposition 2.20. The nerve of a path fibration of categories between two groupoids is a Kan fibration of simplicial sets.

Proof. Let $p : C \to D$ be a path fibration of categories such that both C and D are groupoids. Since C and D are fibrant in $(\mathbf{Cat}, \mathbf{Gpd})$, which is a left Bousfield localization of the natural model category structure on $\mathbf{Cat}$, therefore p is an isofibration from [Sha20, Lem. 4.17].

The nerve functor takes an isofibrations to a pseudo-fibration *i.e.* a fibration in the Joyal model category on simplicial sets so $N(p) : N(C) \to N(D)$ is a pseudo-fibration. However both $N(C)$ and $N(D)$ are Kan complexes. Now it follows that $N(p)$ is a Kan fibration , by the same aforementioned result [Sha20, Lem. 4.17] because $(\mathbf{sSets}, \mathbf{Kan})$ is a left Bousfield localization of the Joyal model category of quasi-categories $(\mathbf{sSets}, \mathbf{Q})$. □

Next we are interested in providing a characterization of weak equivalences and fibrations in the model category of groupoids. We first recall the notion of a *homotopy reflection*:

Definition 2.21. A Quillen adjunction (F, G) is called a *homotopy reflection* if the right derived functor of G is fully-faithful.

Lemma 2.22. The adjunction $\tau_1 : \mathbf{sSets} \rightleftarrows \mathbf{Cat} : N$ is a Quillen adjunction between the model category of groupoids and the Kan model category of simplicial sets. Further the adjunction is also a homotopy reflection.

Proof. The first statement follows from the observation that the adjunction in context is a composite of the following two Quillen adjunctions:

$$\tau_1 : (\mathbf{sSets}, \mathbf{Kan}) \rightleftarrows \mathbf{Cat} : N$$

and

$$id : \mathbf{Cat} \rightleftarrows (\mathbf{Cat}, \mathbf{Gpd}) : id$$

where $\mathbf{Cat}$ denotes the natural model category of (small) categories.

The second statement follows from the observation that both of the aforementioned Quillen adjunctions are homotopy reflections and the fact that a composite of homotopy reflections is again a homotopy reflection.

□

The following corollary is an easy consequence of the above lemma:

Corollary 2.23. A functor $F : G \to H$ between groupoids is a groupoidal equivalence if and only if it's nerve, $N(F)$, is a homotopy equivalence of Kan complexes.

The inclusion functor $\mathbf{Gpd} \to \mathbf{Cat}$, where $\mathbf{Gpd}$ is the full subcategory whose objects are groupoids, has a left adjoint which we denote by $\Pi_1 : \mathbf{Cat} \to \mathbf{Gpd}$. The groupoid $\Pi_1(C)$ is obtained from the category C by formally inverting all arrows in C *i.e.* $\Pi_1(C) = C[Ar(C)^{-1}]$.

Remark 4. In the paper [JT08] a model category structure was constructed on the full subcategory of $\mathbf{Cat}$ whose objects are groupoids $\mathbf{Gpd}$. We will refer to this model category as the *natural model category of groupoids*. The functor Π_1 is a left Quillen functor of a Quillen adjunction

$$\Pi_1 : \mathbf{Cat} \rightleftharpoons \mathbf{Gpd} : i$$

where $\mathbf{Cat}$ is endowed with the model category structure of groupoids and $\mathbf{Gpd}$ is the natural model category of groupoids. This Quillen adjunction is a Quillen equivalence.

The following proposition will be used repeatedly in this paper:

Proposition 2.24. The free groupoid functor $\Pi_1 : \mathbf{Cat} \to \mathbf{Gpd}$ preserves products.

Proposition 2.25. A functor $F : C \to D$ is a groupoidal equivalence if and only if the induced functor $\Pi_1(F) : \Pi_1(C) \to \Pi_1(D)$ is an equivalence of categories.

Proof. The unit of the adjunction $\Pi_1 \dashv i$ gives the following commutative diagram:

$$\begin{tikzcd} C \arrow[r] \arrow[d, "F"'] & \Pi_1(C) \arrow[d, "\Pi_1(F)"] \\ D \arrow[r] & \Pi_1(D) \end{tikzcd}$$

where both vertical functors are inclusions. We will first prove that these two inclusion maps are both weak equivalences. Since Π_1 is a left adjoint to the inclusion functor i therefore the inclusion functor $\iota_C : C \to \Pi_1(C)$ induces the following bijection for each groupoid G:

$$\mathbf{Cat}(\Pi_1(C), G) \cong \mathbf{Cat}(C, G).$$

Consider the following chain of bijections:

$$\begin{aligned} \mathbf{Cat}(I, [\Pi_1(C), G]) &\cong \mathbf{Cat}(I \times \Pi_1(C), G) \cong \mathbf{Cat}(\Pi_1(C), [I, G]) \\ &\cong \mathbf{Cat}(C, [I, G]) \cong \mathbf{Cat}(I \times C, G) \cong \mathbf{Cat}(I, [C, G]). \end{aligned}$$

The above two bijections together imply that we have the following equivalence of functor categories:

$$[\iota_C, G] : [\Pi_1(C), G] \to [C, G].$$

Now Theorem 2.18 (2) implies that the two inclusion maps are weak equivalences in the model category structure of groupoids. Now the theorem follows from the two out of three property of weak equivalences in a model category. □

Finally we would like to show that the groupoidal model category structure on $\mathbf{Cat}$ is cartesian closed.

Proposition 2.26. The groupoidal model category structure on $\mathbf{Cat}$ is cartesian closed.

Proof. The proposition follows from an application of theorem A.3 to the cartesian closed natural model category $\mathbf{Cat}$ with respect to the singleton set of maps $\{i_0\}$. □

Proposition 2.27. The model category of groupoids is a simplicial model category.

Proof. The proposition follows by an application of [Bar07, lem. 3.6] to the Quillen adjunction (τ_1, N) from lemma 2.22. The simplicial Hom is defined by the composite bifunctor:

$$\mathbf{Cat}^{op} \times \mathbf{Cat} \overset{[-,-]}{\to} \mathbf{Cat} \overset{N}{\to} \mathbf{sSets}.$$

The cotensor is defined by the following bifunctor:

$$\mathbf{sSets}^{op} \times \mathbf{Cat} \to \mathbf{Cat}^{op} \overset{\tau_1 \times id}{\times} \mathbf{Cat} \overset{[-,-]}{\to} \mathbf{Cat}$$

The tensor product bifunctor is defined by the following composite:

$$\mathbf{Cat} \times \mathbf{sSets} \overset{id \times \tau_1}{\to} \mathbf{Cat} \times \mathbf{Cat} \overset{- \times -}{\to} \mathbf{Cat}.$$

□

3. Two model category structures on $\mathbf{Perm}$

We denote by $\mathbf{Perm}$ the category whose objects are *permutative* categories, namely symmetric monoidal categories which are strictly unital and strictly associative. The morphisms of this category are *strict* symmetric monoidal functors, namely those symmetric monoidal functors which preserve the symmetric monoidal structure strictly. A model category structure on $\mathbf{Perm}$ was described in [Shac, Thm. 3.1]. This model category structure was obtained by transferring the natural model category structure on $\mathbf{Cat}$ to $\mathbf{Perm}$ and therefore it is aptly called the *natural* model category structure of permutative categories. In this section we will describe two new model category structures on $\mathbf{Perm}$ which can be described as the model category of permutative groupoids and the model category of (permutative) Picard groupoids.

3.1 The model category structure of Permutative groupoids

In this subsection we will construct the desired model category structure of permutative groupoids on $\mathbf{Perm}$ namely a model category structure whose fibrant objects are groupoids equipped with a permutative or strict symmetric monoidal structure. Before doing so we recall the following adjunction and also a permutative groupoid structure inherited by the fundamental groupoid of a permutative category:

$$\mathcal{F} : \mathbf{Cat} \rightleftharpoons \mathbf{Perm} : i \tag{3}$$

where i is the forgetful functor and $\mathcal{F}$ is its left adjoint namely the *free permutative category* functor. The following lemma recalls the aforementioned permutative structure:

Lemma 3.1. The fundamental groupoid of a permutative category is a permutative groupoid.

Proof. Let C be a permutative category and let $- \otimes - : C \times C \to C$ be bifunctor giving the permutative structure. From proposition 2.24, we have the isomorphism $\Pi_1(C \times C) \cong \Pi_1(C) \times \Pi_1(C)$. Since $\Pi_1(C)$ is a groupoid, the universal property of $\Pi_1(C \times C)$ and the above isomorphism imply that we have a dotted arrow in the following diagram which makes the diagram commutative:

$$\begin{tikzcd} C \times C \arrow[r, "- \otimes -"] \arrow[d] & C \arrow[d] \\ \Pi_1(C) \times \Pi_1(C) \arrow[r, dashed] & \Pi_1(C) \end{tikzcd}$$

This bifunctor, represented by the dotted arrow in the above diagram, provides a permutative structure on the groupoid $\Pi_1(C)$. The symmetry natural transformation of C is a functor

$$\gamma_C : C \times C \times J \to C$$

Once again by proposition 2.24 the free groupoid generated by $C \times C \times J$ is $\Pi_1(C) \times \Pi_1(C) \times J$. Again, the universal property of $\Pi_1(C \times C \times J)$ and the above isomorphism imply that we have a dotted arrow in the following diagram:

$$\begin{tikzcd} C \times C \times J \arrow[r, "\gamma_C"] \arrow[d] & C \arrow[d] \\ \Pi_1(C) \times \Pi_1(C) \times J \arrow[r, dashed, "\gamma_{\Pi_1(C)}"'] & \Pi_1(C) \end{tikzcd}$$

which is the symmetry natural isomorphism of $\Pi_1(C)$. $\square$

Remark 5. The functor Π_1 restricts to a functor on $\mathbf{Perm}$ such that the following diagram commutes:

$$\begin{tikzcd} \mathbf{Cat} \arrow[r, "\Pi_1"] & \mathbf{Gpd} \\ \mathbf{Perm} \arrow[u, "i"] \arrow[r, "\Pi_1"'] & \mathbf{PGpd} \arrow[u, "i"'] \end{tikzcd}$$

where $\mathbf{PGpd}$ denotes the category of permutative groupoids *i.e.*, the full subcategory of $\mathbf{Perm}$ having as objects those permutative categories whose underlying categories are groupoids.

We recall from [Shac] that Γ^1 is the representable Γ- category which is also the unit of the symmetric monoidal structure on the functor category $\Gamma\mathbf{Cat}$ which is tensored over $\mathbf{Cat}$. The inclusion map $i_0 : 0 \hookrightarrow I$ gives us the following map of Γ- categories by *tensoring* with Γ^1:

$$\Gamma^1 \otimes i_0 : \Gamma^1 \to \Gamma^1 \otimes I \tag{4}$$

We further recall from [Shac] the Quillen equivalence $\mathcal{L} : \Gamma\mathbf{Cat} \rightleftarrows \mathbf{Perm} : \mathcal{K}$. The image of the above map under the left Quillen functor $\mathcal{L}$ gives us the following strict symmetric monoidal functor which is the generator of the model structure to be constructed later in this subsection:

$$\mathcal{L}(\Gamma^1 \otimes i_0) : \mathcal{L}(\Gamma^1) \to \mathcal{L}(\Gamma^1 \otimes I) \tag{5}$$

Remark 6. The above strict symmetric monoidal functor $\mathcal{L}(\Gamma^1 \otimes i_0)$ has cofibrant domain and codomain.

Now we state the main theorem of this subsection:

Theorem 3.2. There is a model category structure on the category of all small permutative categories and strict symmetric monoidal functors $\mathbf{Perm}$ in which

1. A cofibration is a strict symmetric monoidal functor which is a cofibration in the natural model category structure on $\mathbf{Perm}$
2. A weak-equivalence is an $\{\mathcal{L}(\Gamma^1 \otimes i_0)\}$-local equivalence.
3. A fibration is a strict symmetric monoidal functor having the right lifting property with respect to all maps which are both cofibrations and weak equivalences.

Further, this model category structure is combinatorial and left-proper.

Proof. The desired model category structure is a left-Bousfield localization of the left-proper, combinatorial natural model category structure on $\mathbf{Perm}$ with respect to the singleton $\{\mathcal{L}(\Gamma^1 \otimes i_0)\}$. The existence follows from A.2. □

The following proposition characterizes fibrant objects of the above model category:

Proposition 3.3. A permutative category is fibrant in the above model category if and only if it's underlying category is a groupoid.

Proof. Let C be a permutative category. In light of 2.17, it is sufficient to show that C is fibrant in the above model category if and only if $J([i, C]) : J([I, C]) \to J(C)$ is an equivalence of groupoids. It follows from the results of [Shac, appendix D] that the two homotopy function complexes in context can be defined as follows:

$$Map^h(\mathcal{L}(\Gamma^1 \otimes I), C) = N(J([\mathcal{L}(\Gamma^1 \otimes I), C]_{\otimes}^{str}))$$

and

$$Map^h(\mathcal{L}(\Gamma^1), C) = N(J([\mathcal{L}(\Gamma^1), C]_{\otimes}^{str})).$$

Further, the simplicial map $Map^h(\mathcal{L}(\Gamma^1 \otimes i_0), C)$ is an equivalence of Kan complexes if and only if $J([\mathcal{L}(\Gamma^1 \otimes i_0), C]_{\otimes}^{str})$ is an equivalence of groupoids. Now the result can be deduced by the following commutative diagram in the category of groupoids:

$$\begin{tikzcd}
J([\mathcal{L}(\Gamma^1 \otimes I), C]_{\otimes}^{str}) \arrow[r, "\cong"] \arrow[dd, "J({[\mathcal{L}(\Gamma^1 \otimes i_0), C]_{\otimes}^{str}})"'] & J(\mathcal{M}ap_{\Gamma\mathbf{Cat}}(\Gamma^1 \otimes I, \mathcal{K}(C))) \arrow[d, "\cong"] & \\
 & J(\mathcal{M}ap_{\Gamma\mathbf{Cat}}(\Gamma^1, \mathbf{hom}_{\Gamma\mathbf{Cat}}(I, \mathcal{K}(C)))) \arrow[r, "\cong"] \arrow[d, "J(\mathcal{M}ap_{\Gamma\mathbf{Cat}}(\Gamma^1, \mathbf{hom}_{\Gamma\mathbf{Cat}}(i_0, \mathcal{K}(C))))"'] & J([I, C]) \arrow[d, "J({[i,C]})"] \\
J([\mathcal{L}(\Gamma^1), C]_{\otimes}^{str}) \arrow[r, "\cong"'] & J(\mathcal{M}ap_{\Gamma\mathbf{Cat}}(\Gamma^1, \mathcal{K}(C))) \arrow[r, "\cong"'] & J(C)
\end{tikzcd}$$

where $- \otimes -$, $\mathcal{M}ap_{\Gamma\mathbf{Cat}}(-, -)$ and $\mathbf{hom}_{\Gamma\mathbf{Cat}}(-, -)$ are the tensor product, categorical Hom and cotensor of $\Gamma\mathbf{Cat}$ over $\mathbf{Cat}$. See [Shac, Sec. 4] for details. $\square$

Notation 3.4. We will refer to the above model category as the model category of *permutative groupoids* and will be denoted by $(\mathbf{Perm}, \mathbf{Gpd})$.

The following proposition presents a characterization of weak-equivalences in $(\mathbf{Perm}, \mathbf{Gpd})$:

Proposition 3.5. A strict symmetric monoidal functor $F : C \to D$ is a weak-equivalence in $(\mathbf{Perm}, \mathbf{Gpd})$ if and only if its image $U(F)$, under the forgetful functor $U : \mathbf{Perm} \to \mathbf{Cat}$, is a groupoidal equivalence of (ordinary) categories.

Proof. The proof has two parts. In the first part we show that (Π_1, i) is a fibrant replacement functor on $(\mathbf{Perm}, \mathbf{Gpd})$. To show this it suffices to show that for each permutative category C, the map $i : C \to \Pi_1(C)$ is an acyclic cofibration in $(\mathbf{Perm}, \mathbf{Gpd})$. It is easy to see that i is a cofibration in $\mathbf{Perm}$ in light of the observation that $Ob(i)$ is the identity function and [Shac, Lem. 3.8]. In order to show that i is also a weak-equivalence we observe the following diagram:

$$\begin{tikzcd} C \arrow[r, "i_c"] \arrow[d, "i"'] & C^f \arrow[d, "i^f"] \\ \Pi_1(C) \arrow[r, equal] \arrow[ur, dashed] & \Pi_1(C) \end{tikzcd}$$

where $i = i^f \circ i_c$ is a factorization of i into an acyclic cofibration i_c and a fibration i^f in $(\mathbf{Perm}, \mathbf{Gpd})$. Since i^f is a fibration in $(\mathbf{Perm}, \mathbf{Gpd})$ therefore C^f is a permutative groupoid. Further, a fibration between permutative groupoids is a fibration in the natural model category $\mathbf{Perm}$ which is an isofibration. Thus i_f is a fibration in $(\mathbf{Cat}, \mathbf{Gpd})$. Since i is an acyclic cofibration in $(\mathbf{Cat}, \mathbf{Gpd})$ therefore we have a (dotted) lifting arrow which makes the whole diagram commutative. Since the two horizontal arrows in the above diagram are weak-equivalences in $(\mathbf{Perm}, \mathbf{Gpd})$, by the two out of six property of weak-equivalences in model categories we conclude that i is also a weak-equivalence in $(\mathbf{Perm}, \mathbf{Gpd})$.

In the second part of the proof we establish the desired result. Since (Π_1, i) is a fibrant replacement functor therefore F is a weak-equivalence in $(\mathbf{Perm}, \mathbf{Gpd})$ if and only if $\Pi_1(F)$ is one. However, $\Pi_1(F)$ is a weak-equivalence in $(\mathbf{Perm}, \mathbf{Gpd})$ if and only if it is a weak-equivalence in the natural model category $\mathbf{Perm}$. Thus $\Pi_1(F)$ is a weak-equivalence in $(\mathbf{Perm}, \mathbf{Gpd})$ if and only if $U(\Pi_1(F))$ is an equivalence of categories. Now the result follows from proposition 2.25.

□

The natural model category structure on $\mathbf{Perm}$ is a $\mathbf{Cat}$-model category structure [Shac, Thm. 3.1]. We recall that the *cotensor* of this enrichment is given by the bifunctor

$$[-,-] : \mathbf{Cat}^{op} \times \mathbf{Perm} \overset{id \times U}{\to} \mathbf{Cat} \times \mathbf{Cat} \overset{[-,-]}{\to} \mathbf{Perm} \tag{6}$$

where $[-,-]$ is the internal Hom of $\mathbf{Cat}$ but it takes values in $\mathbf{Perm}$ if the codomain category is permutative. The categorical Hom is the category of strict symmetric monoidal functors is given by the bifunctor

$$[-,-]^{str}_{\otimes} : \mathbf{Perm}^{op} \times \mathbf{Perm} \to \mathbf{Cat} \tag{7}$$

The *tensor product* of this enrichment is does not have a simple description but we will denote it as follows:

$$- \boxtimes - : \mathbf{Cat} \times \mathbf{Perm} \to \mathbf{Perm} \tag{8}$$

Proposition 3.6. The model category of permutative groupoids $(\mathbf{Perm}, \mathbf{Gpd})$ is a $(\mathbf{Cat}, \mathbf{Gpd})$-model category.

Proof. Let $i : U \to V$ be a functor which is monic on objects and $j : W \to X$ be a cofibration in $(\mathbf{Perm}, \mathbf{Gpd})$. We will show that the following map in $\mathbf{Perm}$ is a cofibration in $(\mathbf{Perm}, \mathbf{Gpd})$ which is acyclic whenever i or j is acyclic:

$$i \Box j : (V \boxtimes W) \underset{U \boxtimes W}{\sqcup} (U \boxtimes X) \to V \boxtimes X.$$

Since the cofibrations in the natural model category structure on $\mathbf{Perm}$ are the same as those in $(\mathbf{Perm}, \mathbf{Gpd})$ and the natural model category is a $\mathbf{Cat}$-model category, therefore $i \Box j$ is a cofibration in $(\mathbf{Perm}, \mathbf{Gpd})$.

Let us further assume that i is an acyclic cofibration in $(\mathbf{Cat}, \mathbf{Gpd})$. We will show that now $i \Box j$ is an acyclic cofibration. We recall a well-known fact that a map is an acyclic cofibration in a model category if and only if it has the left-lifting property with respect to all fibrations between fibrant objects. Let $p : A \to B$ be a fibration between permutative groupoids. By adjointness, the map $i \Box j$ has the left lifting property with respect to p if and only if there exists a (dotted) lifting arrow in the following diagram:

$$\begin{tikzcd}
U \arrow[r] \arrow[d, "i"'] & {[X, A]^{str}_{\otimes}} \arrow[d, "(j^*, p_*)"] \\
V \arrow[r] \arrow[ur, dashed, "L"] & {[X, B]^{str}_{\otimes} \underset{[W,B]^{str}_{\otimes}}{\times} [W, A]^{str}_{\otimes}}
\end{tikzcd}$$

Since the natural model category $\mathbf{Perm}$ is a $\mathbf{Cat}$-model category with categorical Hom given by $[-,-]^{str}_{\otimes}$, therefore the assumptions on j and p together imply that the map (j^*, p_*) is a fibration in the natural model structure on $\mathbf{Cat}$ namely an isofibration. However, it is an isofibration between groupoids, therefore it is a fibration in $(\mathbf{Cat}, \mathbf{Gpd})$. Hence there exists a(dotted) lifting arrow L which makes the whole diagram commutative. Thus, we have shown that $i \Box j$ is an acyclic cofibration when i is one.

A similar argument applied to j shows that if j is an acyclic cofibration in $(\mathbf{Perm}, \mathbf{Gpd})$, then so is $i \Box j$.

□

The $(\mathbf{Cat}, \mathbf{Gpd})$-model category structure described in the proposition above induces a simplicial model category structure on $(\mathbf{Perm}, \mathbf{Gpd})$:

Proposition 3.7. The model category of permutative groupoids $(\mathbf{Perm}, \mathbf{Gpd})$ is a simplicial model category.

Proof. The proof follows from [Bar07, lem. 3.6] and the $\mathbf{Cat}$-model category structure on the natural model category $\mathbf{Perm}$. However, we will describe the three bifunctors involved in this enrichment: The simplicial Hom bifunctor is defined to be the composite:

$$\mathbf{Perm}^{op} \times \mathbf{Perm} \overset{[-,-]^{str}_{\otimes}}{\to} \mathbf{Cat} \overset{N}{\to} \mathbf{sSets}. \tag{9}$$

This *cotensor* is defined as follows:

$$\mathbf{sSets}^{op} \times \mathbf{Perm} \overset{\tau_1^{op} \times id}{\to} \mathbf{Cat}^{op} \times \mathbf{Perm} \overset{[-,-]}{\to} \mathbf{Perm} \tag{10}$$

The *tensor product* bifunctor is defined by the following composite:

$$\mathbf{sSets} \times \mathbf{Perm} \overset{\tau_1 \times id}{\to} \mathbf{Cat} \times \mathbf{Perm} \overset{-\boxtimes-}{\to} \mathbf{Perm}. \tag{11}$$

$\square$

3.2 The model category of Picard groupoids

In this subsection we will construct yet another model category structure on $\mathbf{Perm}$ in which the fibrant objects are Picard groupoids. We obtain the desired model category by carrying out a left Bousfield localization of the model category constructed in the previous subsection, namely $(\mathbf{Perm}, \mathbf{Gpd})$. The model category we construct inherits an enrichment over $(\mathbf{Cat}, \mathbf{Gpd})$ and the Kan model category of simplicial sets from its parent model category.

Definition 3.8. A *Picard groupoid* G is a permutative groupoid such that one of the following two functors is an equivalences of categories:

$$G \times G \overset{(-\underset{G}{\otimes}-, p_1)}{\to} G \times G \quad \textit{and} \quad G \times G \overset{(-\underset{G}{\otimes}-, p_2)}{\to} G \times G, \tag{12}$$

where p_1 and p_2 are the two obvious projection maps.

Remark 7. If one of the two functors in the above definition is an equivalence of categories, then the permutative structure on the groupoid G in the above definition implies the other functor is also an equivalence.

Remark 8. A permutative groupoid is a Picard groupoid if and only if for each object $g \in Ob(G)$ there exists another object $g^{-1} \in Ob(G)$ and the following two isomorphisms in G:

$$g \underset{G}{\otimes} g^{-1} \cong 1_G, \qquad g^{-1} \underset{G}{\otimes} g \cong 1_G$$

We recall the construction of the permutative category $\mathcal{L}(1)$ from [Shac]. The permutative category $\mathcal{L}(1)$ is a groupoid whose object set consists of all finite sequences $(s_1, s_2, \dots, s_r)$, where either $s_i = 1$ or $s_i = 0$ for all $1 \le i \le r$. For an object $S = (s_1, s_2, \dots, s_r)$ in $\mathcal{L}(1)$ we denote by $\underline{S}$ the sum $\overset{r}{\underset{i=1}{+}} s_i$. A map $S = (s_1, s_2, \dots, s_r) \to T = (t_1, t_2, \dots, t_k)$ in $\mathcal{L}(1)$ is a bijection $f : \underline{S} \to \underline{T}$. The symmetric monoidal structure on $\mathcal{L}(1)$ is given by concatenation. It follows from [Shac, Lem. 3.8] that $\mathcal{L}(1)$ is cofibrant in the natural model category $\mathbf{Perm}$.

Proposition 3.9. For any permutative groupoid G, the evaluation map

$$ev_{(1)} : [\mathcal{L}(1), G]^{str}_{\otimes} \to G$$

is an equivalence of categories.

Proof. The free permutative category $\mathcal{F}(\underline{1})$, see (3), can be described as follow: The objects are finite sets $\underline{n}$ for all $n \ge 0$. A morphism is a bijection between finite sets. The permutative category $\mathcal{F}(1)$ is cofibrant in the natural model category $\mathbf{Perm}$. This category has the property that the evaluation functor on the object $\underline{1}$:

$$ev_{\underline{1}} : [\mathcal{F}(\underline{1}), C]^{str}_{\otimes} \to C$$

is an isomorphism for any permutative category C. This category is equipped with an inclusion functor

$$i : \mathcal{F}(\underline{1}) \to \mathcal{L}(1),$$

such that $i(\underline{1}) = (1)$, which is an equivalence of categories. Now the 2 out of 3 and the following commutative diagram prove the proposition:

$$\begin{tikzcd}
{[\mathcal{L}(1), G]^{str}_{\otimes}} \arrow[r, "ev_{(1)}"] \arrow[d, "{[i,G]^{str}_{\otimes}}"'] & G \\
{[\mathcal{F}(\underline{1}), G]^{str}_{\otimes}} \arrow[ur, "\cong"'] &
\end{tikzcd}$$

□

The maps of finite sets $m_2 : 2^+ \to 1^+$, $\delta^2_1 : 2^+ \to 1^+$ and $\delta^2_2 : 2^+ \to 1^+$ together induce the following two maps in $\mathbf{Perm}$

$$\mathcal{L}(1) \vee \mathcal{L}(1) \overset{(\mathcal{L}(m_2), \mathcal{L}(\delta^2_1))}{\to} \mathcal{L}(2) \quad \textit{and} \quad \mathcal{L}(1) \vee \mathcal{L}(1) \overset{(\mathcal{L}(\delta^2_1), \mathcal{L}(\delta^2_2))}{\to} \mathcal{L}(2) \tag{13}$$

Remark 9. For each $n \in \mathbb{N}$, the permutative groupoid $\mathcal{L}(n)$ [Shac, Defn. 5.4] is canonically isomorphic to the permutative groupoid $\mathcal{L}(\Gamma^n)$, where $\mathcal{L} : \Gamma\mathbf{Cat} \to \mathbf{Perm}$ is the left adjoint of the Segal's nerve functor, see [Shac, Sec. 5]. This implies that the map $(\mathcal{L}(m_2), \mathcal{L}(\delta^2_1))$ is isomorphic to $\mathcal{L}(\Gamma^{(m_2,\delta^2_1)})$ and $(\mathcal{L}(m_2), \mathcal{L}(\delta^2_2))$ is isomorphic to $\mathcal{L}(\Gamma^{(m_2,\delta^2_2)})$, where the maps $\Gamma^{(m_2,\delta^2_1)}$ and $\Gamma^{(m_2,\delta^2_2)}$ are defined in (17) and (18) respectively.

Remark 10. The above remark and the fact that $\mathcal{L}$ is a left Quillen functor together imply that the permutative groupoid $\mathcal{L}(n)$ is cofibrant in the natural model category $\mathbf{Perm}$. The symmetric monoidal structure on $\mathcal{L}(n)$ is concatenation.

By [Shac, Lemma 5.29] the strict symmetric monoidal functor $(\mathcal{L}(\delta^2_1), \mathcal{L}(\delta^2_2))$ is an acyclic cofibration in the natural model category structure on $\mathbf{Perm}$. This implies that for any permutative category C, we have the following equivalence of categories:

$$[(\mathcal{L}(\delta^2_1), \mathcal{L}(\delta^2_2)), C]^{str}_{\otimes} : [\mathcal{L}(2), C]^{str}_{\otimes} \to [\mathcal{L}(1), C]^{str}_{\otimes} \times [\mathcal{L}(1), C]^{str}_{\otimes}. \tag{14}$$

Lemma 3.10. A permutative groupoid G is a Picard groupoid if and only if it is a $\{(\mathcal{L}(m_2), \mathcal{L}(\delta^2_1))\}$-local object.

Proof. The permutative groupoid G is $\{(\mathcal{L}(m_2), \mathcal{L}(\delta^2_1))\}$-local if and only if we have the following weak homotopy equivalence of simplicial sets:

$$Map^h((\mathcal{L}(m_2), \mathcal{L}(\delta^2_1)), G) : Map^h(\mathcal{L}(2), G) \to Map^h(\mathcal{L}(1) \vee \mathcal{L}(1), G)$$

We recall that the function complex for a pair of permutative categories C and D in $(\mathbf{Perm}, \mathbf{Gpd})$, where C is cofibrant and D is a permutative groupoid is defined as follows:

$$\mathcal{M}ap^h_{(\mathbf{Perm},\mathbf{Gpd})}(C, D) := N([C, D]^{str}_{\otimes})$$

which implies that $Map^h((\mathcal{L}(m_2), \mathcal{L}(\delta^2_1)), G)$ is a homotopy equivalence if and only if the functor:

$$[(\mathcal{L}(m_2), \mathcal{L}(\delta^2_1)), G]^{str}_{\otimes} : [\mathcal{L}(2), G]^{str}_{\otimes} \to [\mathcal{L}(1) \vee \mathcal{L}(1), G]^{str}_{\otimes} \cong [\mathcal{L}(1), G]^{str}_{\otimes} \times [\mathcal{L}(1), G]^{str}_{\otimes}.$$

is an equivalence of categories. Thus we get the following (composite) weak equivalence in $(\mathbf{Perm}, \mathbf{Gpd})$:

$$[\mathcal{L}(2), G]^{str}_{\otimes} \overset{p}{\to} [\mathcal{L}(1), G]^{str}_{\otimes} \times [\mathcal{L}(1), G]^{str}_{\otimes} \overset{(ev_{(1)}, ev_{(1)})}{\to} G \times G \tag{15}$$

where $p = [(\mathcal{L}(m_2), \mathcal{L}(\delta^2_1)), G]^{str}_{\otimes}$. There is another composite map in $\mathbf{Perm}$ which is the following:

$$[\mathcal{L}(2), G]^{str}_{\otimes} \overset{q}{\to} [\mathcal{L}(1), G]^{str}_{\otimes} \times [\mathcal{L}(1), G]^{str}_{\otimes} \overset{(ev_{(1)}, ev_{(1)})}{\to} G \times G \overset{r}{\to} G \times G \tag{16}$$

where $q = [(\mathcal{L}(\delta_1^2), \mathcal{L}(\delta_2^2)), G]_{\otimes}^{str}$ and the map $r = (- \underset{G}{\otimes} -, p_2)$. We will now construct a natural isomorphism (in $\mathbf{Cat}$) $H : (ev_{(1)}, ev_{(1)}) \circ p \Rightarrow r \circ (ev_{(1)}, ev_{(1)}) \circ q$ between the above two functors. For each $F \in [\mathcal{L}(2), G]_{\otimes}^{str}$ let us denote $F((\underline{2}))$ by g_{12}. The isomorphism $p_{12} : (\underline{2}) \cong (\{1\}, \{2\})$ in $\mathcal{L}(2)$ gives an isomorphism $F(p_{12}) : g_{12} \cong g_1 \otimes g_2$, where $g_1 = F((\{1\}))$ and $g_2 = F((\{2\}))$. We observe that $r \circ (ev_{(1)}, ev_{(1)}) \circ q(F) = (g_1 \otimes g_2, g_1)$ and $(ev_{(1)}, ev_{(1)}) \circ p(F) = (g_{12}, g_1)$. We define $H(F) := F(p_{12})$. Let $\sigma : F \Rightarrow G$ be a (monoidal) natural transformation and denoting $G((\underline{2}))$ by g'_{12}, $G((1))$ by g'_1 and $G((2))$ by g'_2 we get an isomorphism $G(p_{12}) : g'_{12} \cong g'_1 \otimes g'_2$. The following diagram commutes:

$$\begin{tikzcd} g_{12} \arrow[r, "H(F)"] \arrow[d, "\sigma((\underline{2}))"'] & g_1 \otimes g_2 \arrow[d, "\sigma((\{1\},\{2\}))"] \\ g'_{12} \arrow[r, "H(G)"'] & g'_1 \otimes g'_2 \end{tikzcd}$$

because σ is a natural isomorphism. Hence we have constructed the desired natural isomorphism H. The construction of H implies that the strict symmetric monoidal functor $r \circ (ev_{(1)}, ev_{(1)}) \circ q$ is a groupoidal equivalence if and only if $(ev_{(1)}, ev_{(1)}) \circ p$ is one. We know that the functors q and $(ev_{(1)}, ev_{(1)})$ are both equivalence of categories. Let us assume that G is a aforementioned local object then $(ev_{(1)}, ev_{(1)}) \circ p$ is a groupoidal equivalence and, by the above argument, so is the composite functor $r \circ (ev_{(1)}, ev_{(1)}) \circ q$. By two out of three property of weak equivalences this implies that r is a weak equivalence which implies that G is a Picard groupoid. Conversely, let us assume that G is a Picard groupoid in which case r is a groupoidal equivalence which means that both $r \circ (ev_{(1)}, ev_{(1)}) \circ q$ and $(ev_{(1)}, ev_{(1)}) \circ p$ are groupoidal equivalences. Again by the two out of three property, p is a groupoidal equivalence which implies that G is local.

□

Theorem 3.11. There is a combinatorial model category structure on the category of (small) permutative categories $\mathbf{Perm}$ in which a functor $F : A \to B$ is

1. a cofibration if it is a cofibration in the natural model category structure on $\mathbf{Perm}$.

2. a weak equivalence if the following functor

   $$\mathcal{M}ap^h_{(\mathbf{Perm},\mathbf{Gpd})}(F, P) : \mathcal{M}ap^h_{(\mathbf{Perm},\mathbf{Gpd})}(B, P) \to \mathcal{M}ap^h_{(\mathbf{Perm},\mathbf{Gpd})}(A, P)$$

   is a homotopy equivalence of simplicial sets for each Picard groupoid P.

3. a fibration if it has the right lifting property with respect to the set of maps which are both cofibrations and weak equivalences.

A permutative category is a fibrant objects of this model category if and only if it ia a Picard groupoid.

Proof. We will prove this theorem by localizing the model category of permutative groupoids $(\mathbf{Perm}, \mathbf{Gpd})$ with respect to the map

$$\mathcal{L}(1) \vee \mathcal{L}(1) \overset{(\mathcal{L}(m_2), \mathcal{L}(\delta_1^2))}{\to} \mathcal{L}(2).$$

The existence of this left Bousfield localization follows from theorem A.2. A left Bousfield localization preserves cofibrations therefore the cofibrations in the new model category are the same as those in $(\mathbf{Perm}, \mathbf{Gpd})$. Lemma 3.10 above tells us that a permutative groupoid is a $\{(\mathcal{L}(m_2), \mathcal{L}(\delta_1^2))\}$-local object if and only if it is a Picard groupoid.

$\square$

Notation 3.12. We will refer to the above model category as the model category of *Picard groupoids*. We denote this model category by $(\mathbf{Perm}, \mathbf{Pic})$.

Adaptations of arguments used in the proof of propositions 3.6 and 3.7, to the model category $(\mathbf{Perm}, \mathbf{Pic})$ prove the following two analogous propositions:

Proposition 3.13. The bifunctors (8), (6) and (7) equip the model category of Picard groupoids $(\mathbf{Perm}, \mathbf{Pic})$ with a $(\mathbf{Cat}, \mathbf{Gpd})$-model category structure.

and

Proposition 3.14. The bifunctors (11), (10) and (9) equip the model category of Picard groupoids $(\mathbf{Perm}, \mathbf{Pic})$ with a simplicial model category structure.

4. The model category structures

A Γ- category is a functor from Γ^{op} to $\mathbf{Cat}$. The category of functors from Γ^{op} to $\mathbf{Cat}$ and natural transformations between them $[\Gamma^{op}, \mathbf{Cat}]$ will be denoted by $\Gamma\mathbf{Cat}$. The main objective of this section is to construct two new symmetric monoidal closed model category structures on $\Gamma\mathbf{Cat}$. Some notations used in this section have been defined in [Shac, Sec. 4]. We recall the following definition:

Definition 4.1. A Q-cofibration is a cofibration in the strict (or projective) model category structure on $\Gamma\mathbf{Cat}$.

4.1 Coherently commutative monoidal groupoids

In the paper [Shac, Sec. 4] a symmetric monoidal closed model category structure was constructed on $\Gamma\mathbf{Cat}$ whose fibrant objects are coherently commutative monoidal categories, see definition [Shac, Defn. 4.15]. These objects should be understood as categories equipped with a multiplication which is associative, unital and commutative only up to higher coherence data. In this subsection we want to construct another symmetric monoidal closed model category structure on $\Gamma\mathbf{Cat}$ whose fibrant objects are groupoids equipped with a multiplication which is associative, unital and commutative only up to higher coherence data. In other words, the underlying category of a fibrant object in the desired model category is a fibrant object in the groupoidal model category $(\mathbf{Cat}, \mathbf{Gpd})$. We will construct the desired model category as a left Bousfield localization of the model category of coherently commutative monoidal categories with respect to the map $\Gamma^1 \otimes i_0 : \Gamma^1 \to \Gamma^1 \otimes I$, see (4).

Definition 4.2. We will refer to a $\{\Gamma^1 \otimes i_0\}$-local equivalence as an equivalence of coherently commutative monoidal groupoids.

Definition 4.3. We will refer to a fibrant $\{\Gamma^1 \otimes i_0\}$-local object as a coherently commutative monoidal groupoid.

Proposition 4.4. A Γ- category X is a coherently commutative monoidal groupoid if and only if the following two conditions are satisfied:

1. For each $k^+ \in Ob(\Gamma^{op})$, $X(k^+)$ is a groupoid.
2. For each $k^+, l^+ \in Ob(\Gamma^{op})$

$$(X(\delta^{k+l}_k), X(\delta^{k+l}_l)) : X((k+l)^+) \to X(k^+) \times X(l^+)$$

is a groupoidal equivalence.

Proof. The model category of coherently commutative monoidal categories is a $\mathbf{Cat}$-model category and its categorical Hom $\mathcal{M}ap_{\Gamma\mathbf{Cat}}(-,-)$ is defined in [Shac, Sec. 4]. Now it follows from [Shac, Appendix D] that for any cofibrant C and fibrant X in the model category of coherently commutative monoidal categories

$$\mathcal{M}ap^h_{\Gamma\mathbf{Cat}}(C, X) = N\left(J\left(\mathcal{M}ap_{\Gamma\mathbf{Cat}}(C, X)\right)\right).$$

We begin by observing that conditions (1) and (2) are satisfied by X if and only if X is a coherently commutative monoidal category and $X(1^+)$ is a groupoid. Now it is sufficient to show that X is a coherently commutative monoidal groupoid if and only if $X(1^+)$ is a groupoid and X is a coherently commutative monoidal category. The Γ- category X is a coherently commutative monoidal groupoid if and only if it is a coherently commutative monoidal category and the following simplicial map is a weak homotopy equivalence:

$$N\left(J\left(\mathcal{M}ap_{\Gamma\mathbf{Cat}}(\Gamma^1 \otimes i_0, X)\right)\right) : N\left(J\left(\mathcal{M}ap_{\Gamma\mathbf{Cat}}(\Gamma^1 \otimes I, X)\right)\right) \to N\left(J\left(\mathcal{M}ap_{\Gamma\mathbf{Cat}}(\Gamma^1, X)\right)\right).$$

This simplicial map of Kan complexes is a homotopy equivalence if and only if the following functor between groupoids is a groupoidal equivalence:

$$\left(J\left(\mathcal{M}ap_{\Gamma\mathbf{Cat}}(\Gamma^1 \otimes i_0, X)\right)\right) : \left(J\left(\mathcal{M}ap_{\Gamma\mathbf{Cat}}(\Gamma^1 \otimes I, X)\right)\right) \to \left(J\left(\mathcal{M}ap_{\Gamma\mathbf{Cat}}(\Gamma^1, X)\right)\right).$$

By adjointness, the above functor is a groupoidal equivalence if and only if the following functor is a groupoidal equivalence:

$$\left(J\left(\mathcal{M}ap_{\Gamma\mathbf{Cat}}(\Gamma^1, \mathbf{hom}_{\Gamma\mathbf{Cat}}(i_0, X))\right)\right) : \left(J\left(\mathcal{M}ap_{\Gamma\mathbf{Cat}}(\Gamma^1, \mathbf{hom}_{\Gamma\mathbf{Cat}}(I, X))\right)\right) \to \left(J\left(\mathcal{M}ap_{\Gamma\mathbf{Cat}}(\Gamma^1, X)\right)\right) \cong J\left(X(1^+)\right).$$

Unwinding definition, the functor $\left(J\left(\mathcal{M}ap_{\Gamma\mathbf{Cat}}(\Gamma^1, \mathbf{hom}_{\Gamma\mathbf{Cat}}(i_0, X))\right)\right)$ is isomorphic to the following functor:

$$J[i_0; X(1^+)] : J[I; X(1^+)] \to J(X(1^+)).$$

This implies that X is a coherently commutative monoidal groupoid if and only if it is a coherently commutative monoidal category and $X(1^+)$ is a groupoid *i.e.* a i_0-local object in $\mathbf{Cat}$.

□

A left-Bousfield localization with respect to the map $\{\Gamma^1 \otimes i_0\}$ gives us the following model category.

Theorem 4.5. There is a left proper, combinatorial model category structure on the category of Γ- categories, $\Gamma\mathbf{Cat}$, in which

1. The class of cofibrations is the same as the class of Q-cofibrations of Γ- categories.

2. The weak equivalences are equivalences of coherently commutative monoidal groupoids.

An object is fibrant in this model category if and only if it is a coherently commutative monoidal groupoid.

Proof. The model category structure follows from an application of A.2 to the model category of coherently commutative monoidal categories with respect to the singleton set $\{\Gamma^1 \otimes i_0\}$. The characterization of fibrant objects also follows from the same theorem.

□

Notation 4.6. We will refer to the above model category as the model category of coherently commutative monoidal categories and denote it by $\Gamma\mathbf{Gpd}^{\otimes}$

The following proposition will be useful in proving the main result of this subsection:

Proposition 4.7. The model category $\Gamma\mathbf{Gpd}^{\otimes}$ is a $(\mathbf{Cat}, \mathbf{Gpd})$-model category.

The rest of this subsection is devoted to showing that the model category $\Gamma\mathbf{Gpd}^{\otimes}$ is a symmetric monoidal closed model category under the Day convolution. In order to do so we will need the following result:

Lemma 4.8. For each Q-cofibrant Γ- category W, the mapping object $\underline{\mathcal{M}ap}_{\Gamma\mathbf{Cat}}(W, A)$ is a coherently commutative monoidal groupoid if A is one.

Proof. Since A is also a coherently commutative monoidal category *i.e.* a fibrant object in the model category of coherently commutative monoidal categories, the symmetric monoidal closed structure on the aforementioned model category, [Shac, Thm. 4.27], implies that $\underline{\mathcal{M}ap}_{\Gamma\mathbf{Cat}}(W, A)$ is a coherently commutative monoidal category. Now, in light of proposition 4.4, it is sufficient to show that $\underline{\mathcal{M}ap}_{\Gamma\mathbf{Cat}}(W, A)(k^+)$ is a groupoid, for all $k^+ \in \Gamma^{op}$. Since W is cofibrant, therefore we have the following equality:

$$\underline{\mathcal{M}ap}_{\Gamma\mathbf{Cat}}(W, A)(k^+) = \mathcal{M}ap_{\Gamma\mathbf{Cat}}(W * \Gamma^k, A).$$

We recall that Γ^k is a Q-cofibrant Γ- category. Since W is Q-cofibrant by assumption therefore $W * \Gamma^k$ is also Q-cofibrant by [Shac, Thm. 4.27]. The result now follows from the above observation that the domain Γ- category $W * \Gamma^k$ is Q-cofibrant and the model category $\Gamma\mathbf{Gpd}^{\otimes}$ is a $(\mathbf{Cat}, \mathbf{Gpd})$-model category which together imply that the category $\mathcal{M}ap_{\Gamma\mathbf{Cat}}(W * \Gamma^k, A)$ is a groupoid.

□

The following theorem is the main result of this section:

Theorem 4.9. The model category of coherently commutative monoidal groupoids $\Gamma\mathbf{Gpd}^{\otimes}$ is a symmetric monoidal closed model category under the Day convolution product.

Proof. The generating cofibrations in the model category of coherently commutative monoidal categories are maps between Q-cofibrant objects. For a Q-cofibrant object W and a coherently commutative monoidal groupoid A, the mapping object $\underline{\mathcal{M}ap}_{\Gamma\mathbf{Cat}}(W, A)$ is a coherently commutative monoidal groupoid by lemma 4.8. The model category of coherently commutative monoidal categories is symmetric monoidal closed under the Day convolution product by [Shac, Thm. 4.27]. Now the result follows from Theorem A.3.

□

4.2 Coherently commutative Picard groupoids

In this subsection we will introduce a notion of a *coherently commutative* Picard groupoid. We will go on to construct another model category structure on $\Gamma\mathbf{Cat}$ whose fibrant objects are the aforementioned objects. A prominent result of this section is that this new model category is symmetric monoidal closed under the Day convolution product thereby giving us a tensor product of Picard groupoids.

The mode of construction of this new model category will be localization. The following two pairs of maps, see definitions 2.16 and 2.15, of based sets:

$$m_2 : 2^+ \to 1^+ \quad \textit{and} \quad \delta^2_1 : 2^+ \to 1^+$$

and

$$m_2 : 2^+ \to 1^+ \quad \textit{and} \quad \delta^2_2 : 2^+ \to 1^+$$

induce two maps of Γ- categories

$$\Gamma^{(m_2, \delta^2_1)} : \Gamma^1 \vee \Gamma^1 \to \Gamma^2 \tag{17}$$

and

$$\Gamma^{(m_2, \delta^2_2)} : \Gamma^1 \vee \Gamma^1 \to \Gamma^2 \tag{18}$$

Remark 11. We recall that for each $k \geq 0$, the representatble Γ- category Γ^k is Q-cofibrant. Further the coproduct of two Q-cofibrant Γ- categories is again Q-cofibrant. This implies that the above two maps are between Q-cofibrant Γ- categories.

Notation 4.10. We denote the set $\{\Gamma^{(m_2, \delta^2_1)}\}$ by $\mathcal{P}_\infty$.

Definition 4.11. A *coherently commutative Picard groupoid* is a coherently commutative monoidal groupoid which is also a $\mathcal{P}_\infty$-local object.

Unravelling the above definition gives us the following characterization of a coherently commutative Picard groupoid:

Proposition 4.12. A Γ- category X is a coherently commutative Picard groupoid if and only if it satisfies the following three conditions:

1. For each $k^+ \in Ob(\Gamma^{op})$, $X(k^+)$ is a groupoid.
2. For each $k^+, l^+ \in Ob(\Gamma^{op})$
$$(X(\delta^{k+l}_k), X(\delta^{k+l}_l)) : X((k+l)^+) \to X(k^+) \times X(l^+)$$
is a groupoidal equivalence.
3. One of the following two maps, and hence both maps, are groupoidal equivalences:
$$(X(m_2), X(\delta^2_1)) : X(2^+) \to X(1^+) \times X(1^+) \textit{ and } (X(m_2), X(\delta^2_2)) : X(2^+) \to X(1^+) \times X(1^+)$$

Definition 4.13. A stable equivalence of Γ- categories is a $\mathcal{P}_\infty$-local equivalence.

An application of theorem A.2 to the model category $\Gamma\mathbf{Gpd}^{\otimes}$ with respect to the set $\mathcal{P}_\infty$ gives us the following model category:

Theorem 4.14. There is a left proper, combinatorial model category structure on the category of Γ- categories, $\Gamma\mathbf{Cat}$, in which

1. The class of cofibrations is the same as the class of Q-cofibrations of Γ- categories.
2. The weak equivalences are stable equivalences of Γ- categories.

An object is fibrant in this model category if and only if it is a coherently commutative Picard groupoid.

Notation 4.15. We denote the above model category by $\Gamma\mathcal{P}ic$.

The following lemma will be useful in the proof of the main result of this section:

Lemma 4.16. For each Q-cofibrant Γ- category W, the mapping object $\underline{\mathcal{M}ap}_{\Gamma\mathbf{Cat}}(W, A)$ is a coherently commutative Picard groupoid if A is one.

Proof. If A is a coherently commutative Picard groupoid then it is also a fibrant object in the model category of coherently commutative monoidal groupoids, the symmetric monoidal closed structure on the aforementioned model category, 4.9, implies that $\underline{\mathcal{M}ap}_{\Gamma\mathbf{Cat}}(W, A)$ is a coherently commutative monoidal groupoid because W is Q-cofibrant by assumption. Thus we have verified (1) and (2) in proposition 4.12. In order to verify (3) in the same proposition we need to show that the following functor is a groupoidal equivalence:

$$\begin{aligned}\mathcal{M}ap_{\Gamma\mathbf{Cat}}(W * \Gamma^{(m_2,\delta_1^2)}, A) : \mathcal{M}ap_{\Gamma\mathbf{Cat}}(W * \Gamma^2, A) \to \\ \mathcal{M}ap_{\Gamma\mathbf{Cat}}(W * \Gamma^1, A) \times \mathcal{M}ap_{\Gamma\mathbf{Cat}}(W * \Gamma^1, A)\end{aligned}$$

By adjointness, the morphism of Γ- categories $\mathcal{M}ap_{\Gamma\mathbf{Cat}}(W * \Gamma^{(m_2,\delta_1^2)}, A)$ is a groupoidal equivalence if and only if its adjunct map

$$\begin{aligned}\mathcal{M}ap_{\Gamma\mathbf{Cat}}(W, \underline{\mathcal{M}ap}_{\Gamma\mathbf{Cat}}(\Gamma^{(m_2,\delta_1^2)}, A)) : \mathcal{M}ap_{\Gamma\mathbf{Cat}}(W, \underline{\mathcal{M}ap}_{\Gamma\mathbf{Cat}}(\Gamma^2, A)) \to \\ \mathcal{M}ap_{\Gamma\mathbf{Cat}}(W, \underline{\mathcal{M}ap}_{\Gamma\mathbf{Cat}}(\Gamma^1, A)) \times \mathcal{M}ap_{\Gamma\mathbf{Cat}}(W, \underline{\mathcal{M}ap}_{\Gamma\mathbf{Cat}}(\Gamma^1, A))\end{aligned}$$

is one. Since W is Q-cofibrant, it is sufficient to show that the morphism

$$\underline{\mathcal{M}ap}_{\Gamma\mathbf{Cat}}(\Gamma^{(m_2,\delta_1^2)}, A) : \underline{\mathcal{M}ap}_{\Gamma\mathbf{Cat}}(\Gamma^2, A) \to \underline{\mathcal{M}ap}_{\Gamma\mathbf{Cat}}(\Gamma^1, A) \times \underline{\mathcal{M}ap}_{\Gamma\mathbf{Cat}}(\Gamma^1, A)$$

is a strict equivalence of Γ- groupoids. Since the Γ- categories $\underline{\mathcal{M}ap}_{\Gamma\mathbf{Cat}}(\Gamma^2, A)$ and $\underline{\mathcal{M}ap}_{\Gamma\mathbf{Cat}}(\Gamma^1, A)$ are both coherently commutative monoidal groupoids therefore the morphism $\underline{\mathcal{M}ap}_{\Gamma\mathbf{Cat}}(\Gamma^{(m_2,\delta_1^2)}, A)$ will be a strict equivalence of Γ- groupoids if and only if $(\underline{\mathcal{M}ap}_{\Gamma\mathbf{Cat}}(\Gamma^{(m_2,\delta_1^2)}, A))(1^+)$ is a groupoidal equivalence. The following commutative diagram :

$$\begin{tikzcd} \underline{\mathcal{M}ap}_{\Gamma\mathbf{Cat}}(\Gamma^2, A)(1^+) \arrow[r, "U"] \arrow[d, "\cong"'] & \underline{\mathcal{M}ap}_{\Gamma\mathbf{Cat}}(\Gamma^1, A)(1^+) \times \underline{\mathcal{M}ap}_{\Gamma\mathbf{Cat}}(\Gamma^1, A)(1^+) \arrow[d, "\cong"] \\ A(2^+) \arrow[r, "A((m_2,\delta_1^2))"'] & A(1^+) \times A(1^+) \end{tikzcd}$$

where $U = (\underline{\mathcal{M}ap}_{\Gamma\mathbf{Cat}}(\Gamma^{(m_2,\delta_1^2)}, A))(1^+)$, implies that this map is a groupoidal equivalence because A is a coherently commutative Picard groupoid by assumption. $\square$

Theorem 4.17. The model category of coherently commutative Picard groupoids $\Gamma\mathcal{P}ic$ is a symmetric monoidal closed model category under the Day convolution product.

Proof. The generating cofibrations of $\Gamma\mathbf{Gpd}^{\otimes}$ are maps between Q-cofibrant objects. For a Q-cofibrant object W and a coherently commutative Picard groupoid A, the mapping object $\underline{\mathcal{M}ap}_{\Gamma\mathbf{Cat}}(W, A)$ is a coherently commutative Picard groupoid by lemma 4.16. Now an application of theorem A.3 to the model category $\Gamma\mathbf{Gpd}^{\otimes}$ with the set of morphisms $\mathcal{S} = \mathcal{P}_{\infty}$, see (4.10), proves the theorem. □

5. The Quillen equivalences

In this section we prove that the following two adjoint pairs are Quillen equivalences :

$$\mathcal{L} : \Gamma\mathcal{P}ic \rightleftarrows (\mathbf{Perm}, \mathbf{Pic}) : \mathcal{K}$$

and

$$\mathcal{L} : \Gamma\mathbf{Gpd}^{\otimes} \rightleftarrows (\mathbf{Perm}, \mathbf{Gpd}) : \mathcal{K}$$

where $\mathcal{K}$ is the classical Segal's nerve functor, see [Seg74], [Man10],[EM06] [Shac]. We begin with a proof of the later result:

Theorem 5.1. The Quillen pair $(\mathcal{L}, \mathcal{K})$ is a Quillen equivalence between the model category of coherently commutative permutative groupoids $\Gamma\mathbf{Gpd}^{\otimes}$ and the model category of permutative groupoids $(\mathbf{Perm}, \mathbf{Gpd})$.

Proof. We recall that the model category of coherently commutative monoidal groupoids $\Gamma\mathbf{Gpd}^{\otimes}$ is a left Bousfield localization of the model category of coherently commutative monoidal categories [Shac, Thm. 4.20] with respect to a single map $\Gamma^1 \otimes i_0$, see (4). By remark 11, this is a map between Q-cofibrant Γ- categories. We further recall that the model category of permutative groupoids $(\mathbf{Perm}, \mathbf{Gpd})$ is a left Bousfield localization of the natural model category $\mathbf{Perm}$ with respect to the image of $\Gamma^1 \otimes i_0$ under the left adjoint $\mathcal{L}$. Moreover, the adjoint pair $(\mathcal{L}, \mathcal{K})$ is a Quillen equivalence between the model category of coherently commutative monoidal categories and the natural model category $\mathbf{Perm}$ [Shac, Cor. 6.19]. Now the result follows from [Hir02, Thm. 3.3.20.]. □

Now we prove the main result of this paper:

Theorem 5.2. The Quillen pair $(\mathcal{L}, \mathcal{K})$ is a Quillen equivalence between the model category of coherently commutative Picard groupoids $\Gamma\mathcal{P}ic$ and the model category of permutative groupoids $(\mathbf{Perm}, \mathbf{Pic})$.

Proof. We recall that the model category of coherently commutative Picard groupoids $\Gamma\mathcal{P}ic$ is a left Bousfield localization of the model category of coherently commutative monoidal groupoids $\Gamma\mathbf{Gpd}^{\otimes}$ with respect to a single map $\Gamma^{(m_2, \delta_1^2)}$, see (17).

We observe that this is a map between Q-cofibrant Γ- categories. We further recall that the model category of Picard groupoids $(\mathbf{Perm}, \mathbf{Pic})$ is a left Bousfield localization of the model category $(\mathbf{Perm}, \mathbf{Gpd})$ with respect to a single map $(\mathcal{L}(m_2), \mathcal{L}(\delta_1^2))$ which is isomorphic to the image of $\Gamma^{(m_2, \delta_1^2)}$ under the left adjoint $\mathcal{L}$, see remark (9). Further, the adjoint pair $(\mathcal{L}, \mathcal{K})$ is a Quillen equivalence between the model category of coherently commutative monoidal groupoids $\Gamma\mathbf{Gpd}^{\otimes}$ and the model category $(\mathbf{Perm}, \mathbf{Gpd})$ by theorem 5.1. Now the result follows from [Hir02, Thm. 3.3.20.]. $\square$

In light of the natural weak-equivalence [Shac, Cor. 6.19] between the Segal's nerve functor $\mathcal{K}$ and the thickened Segal's nerve functor $\overline{\mathcal{K}}$, constructed in [Shac, Sec. 6], the following two theorems follow from the above two Quillen equivalences:

Theorem 5.3. The Quillen pair $(\overline{\mathcal{L}}, \overline{\mathcal{K}})$ is a Quillen equivalence between the model category of coherently commutative permutative groupoids $\Gamma\mathbf{Gpd}^{\otimes}$ and the model category of permutative groupoids $(\mathbf{Perm}, \mathbf{Gpd})$.

Theorem 5.4. The Quillen pair $(\overline{\mathcal{L}}, \overline{\mathcal{K}})$ is a Quillen equivalence between the model category of coherently commutative Picard groupoids and the model category of permutative groupoids $(\mathbf{Perm}, \mathbf{Pic})$.

6. Stable homotopy hypothesis for Picard groupoids

In this section we give a new proof of the classical result that Picard groupoids model *stable homotopy one-types*. This result has been referred to in the literature as the *stable homotopy hypothesis* for Picard groupoids. The main objective of this section is to show that the homotopy category of our model category $\Gamma\mathcal{P}ic$ is equivalent to a (suitably defined) homotopy category of stable homotopy one-types. We use the language of *relative categories* in this section, see [BK12]. We define two relative categories for the objects in context: a relative category $(\mathbf{Pic}, Str)$ of Picard groupoids whose homotopy category is equivalent to that of $(\mathbf{Perm}, \mathbf{Pic})$ and another relative category of stable homotopy one-types $(\Gamma\mathcal{S}_{\bullet}^{f}[1], Str)$. We prove a stronger result, namely we establish a homotopy equivalence of the two aforementioned relative categories which implies that their homotopy categories are equivalent. Our proof of the homotopy equivalence is based on the main result of this paper, namely theorem 5.4. A short time before the first version of this paper was released, a different proof of another version of the aforementioned homotopy equivalence was given in the paper [MOP^{+}20]. This proof is based the stable homotopy hypothesis proved in the same paper. In this section we will be dealing with the model category of pointed spaces $(\mathbf{sSets}_\bullet, \mathbf{Kan})$ and we recall that a map in this model category is a weak equivalence if and only if its underlying (unpointed) simplicial map is a weak homotopy equivalence.

Definition 6.1. A stable homotopy one type is a functor $X : \Gamma^{op} \to \mathbf{sSets}_\bullet$ such that the following conditions are satisfied:

1. For each $n^+ \in \Gamma^{op}$, the (pointed) simplicial set $X(n^+)$ is a Kan complex.
2. All homotopy groups of pointed simplicial set $X(1^+)$ vanish in degree greater than one *i.e.*, $\pi_n(X(1^+)) = *$ for $n \geq 2$.
3. For each pair of objects $k^+, l^+ \in \Gamma^{op}$, the following simplicial map is a weak homotopy equivalence:
$$(X(\delta^{k+l}_k), X(\delta^{k+l}_l)) : X((k+l)^+) \to X(k^+) \times X(l^+)$$
4. One of the following two maps, and hence both maps:
$(X(m_2), X(\delta^2_1)) : X(2^+) \to X(1^+) \times X(1^+)$ *and* $(X(m_2), X(\delta^2_2)) : X(2^+) \to X(1^+) \times X(1^+)$
are homotopy equivalences of pointed simplicial sets.

Remark 12. Each stable homotopy one type is a fibrant object in the stable Q-model category constructed in [Sch99].

Remark 13. Each stable homotopy one-type determines a *connective spectrum* with at most two non-trivial homotopy groups in degree zero or one, see [BF78].

Remark 14. The adjoint pair of functors (τ, N) induce an adjunction
$$[\Gamma^{op}, \tau] : \Gamma\mathbf{Cat} \rightleftharpoons \Gamma\mathcal{S} : [\Gamma^{op}, N]$$
This adjunction is a Quillen pair with respect to the strict (or projective) model category structure on the two functor categories, see [Lur09, Remark A.2.8.6]. Since the counit of (τ, N) is the identity, therefore the counit of the induced adjunction is also identity.

We recall from [Shaa] the adjoint pair $((-)^{nor}, U)$ which determines a Quillen equivalence between the JQ-model category of Γ-spaces [Sha20, Notation 4.11] and the JQ-model category of normalized Γ-spaces [Sha20, Notation C.19]. We recall from [Sha20] that a normalized Γ-space is a functor from $X : \Gamma^{op} \to \mathbf{sSets}_\bullet$ such that $X(0^+) = *$. It is easy to see that each coherently commutative monoidal Picard groupoid X determines a Γ-space upon composition with the nerve functor, we denote this Γ-space by $N(X)$. Applying the left adjoint gives us a normalized Γ-space $(N(X))^{nor}$. This leads us to the following proposition:

Proposition 6.2. For each coherently commutative Picard groupoid X, the normalized Γ-space $(N(X))^{nor}$ is a stable homotopy one-type.

Proof. The nerve functor preserves products, maps groupoids to Kan complexes and also maps groupoidal equivalences between two groupoids to homotopy equivalences of simplicial sets therefore $N(X)$ is a coherently commutative monoidal quasi-category in which $N(X)(k^+)$ is a Kan complex for each $k^+ \in \Gamma^{op}$. It follows that the following simplicial maps:

$$N(X(m_2), X(\delta_1^2)) : N(X(2^+)) \to N(X(1^+)) \times N(X(1^+))$$

and

$$N(X(m_2), X(\delta_2^2)) : N(X(2^+)) \to N(X(1^+)) \times N(X(1^+))$$

are both homotopy equivalences of Kan complexes. It follows from [Shaa, Prop. 6.6] that the unit simplicial map $\eta_{N(X)} : N(X) \to U((N(X))^{nor})$ is a strict JQ-equivalence of Γ-spaces. This implies that the normalized Γ-space $(N(X))^{nor}$ is a stable homotopy one-type.

□

We recall that a *relative category* $C = (C, W)$ consists of a pair of categories (C, W) which have the same set of objects and the set arrows of W is a subset of arrows of C and the maps of W are called *weak-equivalences* of C. A morphism of relative categories $F : (C, W) \to (D, X)$ is a functor $F : C \to D$ that preserves weak-equivalences *i.e.* $F(W) \subseteq X$. A morphism of relative categories is called a *functor of relative categories*.

Definition 6.3. A *strict homotopy* between two functors of relative categories $F : (C, W) \to (D, X)$ and $G : (C, W) \to (D, X)$ is a natural transformation $H : F \Rightarrow G$ such that for each object $c \in C$, the map $H(c)$ lies in X, *i.e.*, it is a weak-equivalence in D.

More generally, we will say that there exists a *homotopy* between F and G if they can be joined by a finite zig-zag of strict homotopies.

Based on the notion of homotopy, we define another notion of homotopy equivalence:

Definition 6.4. A functor of relative categories $F : (C, W) \to (D, X)$ is called a *strict homotopy equivalence* if there exists another functor of relative categories $F^{-1} : (D, X) \to (C, W)$ and two strict homotopies $\eta : id \Rightarrow F^{-1} \circ F$ and $\epsilon : F \circ F^{-1} \Rightarrow id$.

F will be called a *homotopy equivalence* if η and ϵ are just homotopies, namely, zig-zags of strict homotopies.

Remark 15. A homotopy equivalence induces an equivalence on the homotopy categories of its domain and codomain relative categories.

Next we will construct three relative categories:

Definition 6.5. We denote by $(\mathbf{Pic}, \mathit{Str})$ the relative category in which $\mathbf{Pic}$ is the category whose objects are permutative Picard groupoids and arrows are strict symmetric monoidal functors. The morphisms of *Str* are those strict symmetric monoidal functors whose underlying functors are equivalences of categories.

Remark 16. The homotopy category of the relative category $(\mathbf{Pic}, \mathit{Str})$ is equivalent to the homotopy category of the model category $(\mathbf{Perm}, \mathbf{Pic})$.

Definition 6.6. We denote by $(\Gamma\mathcal{P}ic^f, \mathit{Str})$ the relative category in which $\Gamma\mathcal{P}ic^f$ is the full subcategory of $\Gamma\mathbf{Cat}$ whose objects are coherently commutative Picard groupoids. The morphisms of *Str* are strict equivalences of Γ- categories.

Remark 17. The homotopy category of the relative category $(\Gamma\mathcal{P}ic^f, \mathit{Str})$ is equivalent to the homotopy category of the model category of coherently commutative Picard groupoids $\Gamma\mathcal{P}ic$.

Definition 6.7. We denote by $(\Gamma\mathcal{S}_\bullet^f[1], \mathit{Str})$ the relative category in which $\Gamma\mathcal{S}_\bullet^f[1]$ is the full subcategory of $\Gamma\mathcal{S}_\bullet$ (the category of normalized Γ-spaces, see [Shaa]) whose objects are stable homotopy one types, see definition (6.1). The morphisms of *Str* are strict JQ-equivalences of normalized Γ-spaces, see [Shaa].

Remark 18. The homotopy category of the relative category $(\Gamma\mathcal{S}_\bullet^f[1], \mathit{Str})$ is equivalent to the full subcategory of the homotopy category of the stable Q-model category, constructed in [Sch99], whose objects are normalized Γ-spaces having at most two non-zero stable homotopy groups only in degree zero or one.

We recall the classical result that the homotopy theory of one-types *i.e.*, Kan complexes (fibrant simplicial sets) whose homotopy groups are trivial in degrees 2 and above is equivalent to the homotopy theory of groupoids. This result can be expressed by the following (strict) homotopy equivalence:

$$\tau_1 : (\mathbf{sSets}^1, \mathit{WH}) \rightleftharpoons (\mathbf{Gpd}, \mathit{Eq.}) : N \tag{19}$$

where $\mathbf{sSets}^1$ denotes the full subcategory of $\mathbf{sSets}$ whose objects are one-types and the maps in *WH* are homotopy equivalences of simplicial sets. The functors in *Eq* are equivalences of categories.

Notation 6.8. We denote by $N^{nor}(-)$ the composite functor

$$\Gamma\mathbf{Cat} \overset{N}{\to} \Gamma\mathcal{S} \overset{(-)^{nor}}{\to} \Gamma\mathcal{S}_\bullet$$

where N denotes the functor $[\Gamma^{op}, N] : \Gamma\mathbf{Cat} \to \Gamma\mathcal{S}$.

Proposition 6.2 above implies that the functor $N^{nor}(-)$ restricts to:

$$N^{nor}(-) : \Gamma\mathcal{P}ic^f \to \Gamma\mathcal{S}_\bullet^f[1]. \tag{20}$$

Lemma 6.9. The functor $N^{nor}(-)$ is a homotopy equivalence of relative categories.

Proof. We begin by observing that the following composite functor:

$$\Gamma\mathcal{S}_\bullet \overset{U}{\to} \Gamma\mathcal{S} \overset{\tau_1}{\to} \Gamma\mathbf{Cat},$$

restricts to a functor

$$\tau^{un}(-) : \Gamma\mathcal{S}_\bullet^f[1] \to \Gamma\mathcal{P}ic^f.$$

This follows by an argument similar to the one in the proof of Proposition 6.2 based on the fact that U and τ_1 preserve strict JQ-equivalences as well as products. We claim that this functor $\tau^{un}(-)$ is a homotopy inverse of $N^{nor}(-)$. We observe that the functor $N^{nor}(-)$ is a functor of relative categories because $N = [\Gamma^{op}, N]$ is a right Quillen functor and therefore preserves weak-equivalences (strict equivalences) between fibrant objects. The functor $(-)^{nor}$ preserves strict equivalences by [Shaa, Prop. 6.2]. Similarly, the functor $\tau^{un}(-)$ preserves strict equivalences because both U and $\tau_1 = [\Gamma^{op}, \tau_1]$ do so.

Next , we will construct a (strict) homotopy $\beta^c : id \Rightarrow \tau^{un}(-) \circ N^{nor}(-)$ with the identity (relative) functor on $(\Gamma\mathcal{P}ic^f, Str)$. For each $X \in Ob(\Gamma\mathcal{P}ic^f)$, the unit of the Quillen equivalence $((-)^{nor}, U)$ provides a strict equivalence of Γ-spaces $\eta_X : N(X) \to U(N(X)^{nor})$. Applying the left Quillen functor τ_1, we get a weak equivalence in $(\Gamma\mathcal{P}ic^f, Str)$, namely, $\tau_1(\eta_X) : X = \tau_1(N(X)) \to \tau_1(U(N(X)^{nor}))$. We define $\beta^c_X = \tau_1(\eta_X)$. One can easily check that this defines a natural transformation β^c. Now we define a (strict) homotopy $\beta^u : id \Rightarrow N^{nor}(-) \circ \tau^{un}(-)$. Let Y be a stable homotopy one type. The unit map of the Quillen adjunction (τ_1, N) gives a map $\eta_Y : U(Y) \to N(\tau_1(U(Y)))$. Since Y is a stable homotopy one type therefore this map is a weak homotopy equivalence by (19). Now applying the functor $(-)^{nor}$, we get a weak homotopy equivalence

$$(\eta_Y)^{nor} : Y = (U(Y))^{nor} \to N(\tau(U(Y)))^{nor}.$$

Now we define $\beta^u_Y = (\epsilon_Y)^{nor}$. One can easily check that this defines a natural transformation. Thus we have established a (strict) homotopy equivalence. $\square$

It follows from theorem [Shac, 6.17] that the left adjoint functor $\overline{\mathcal{L}}$ restricts to a functor of relative categories

$$\overline{\mathcal{L}} : (\Gamma\mathcal{P}ic^f, Str) \to (\mathbf{Pic}, Str).$$

Further, it follows from [Shac, lem. 6.14] that the right Quillen functor $\overline{\mathcal{K}}$ restricts to a functor of relative categories:

$$\overline{\mathcal{K}} : (\mathbf{Pic}, Str) \to (\Gamma\mathcal{P}ic^f, Str).$$

This leads us to the final lemma of this section:

Lemma 6.10. The pair of functors of relative categories $(\overline{\mathcal{L}}, \overline{\mathcal{K}})$ determines a (strict) homotopy equivalence between the relative categories $(\Gamma\mathcal{P}ic^f, Str)$ and $(\mathbf{Pic}, Str)$.

The proof follows from the two observations above, namely $\overline{\mathcal{K}}$ and $\overline{\mathcal{L}}$ are functors of relative categories and theorem 5.4. Now the previous two lemms give us the main result of this section:

Theorem 6.11. The composite functor of relative categories $N^{nor}(-) \circ \overline{\mathcal{K}}$ is a homotopy equivalence between the relative categories $(\mathbf{Pic}, Str)$ and $(\Gamma\mathcal{S}^f_\bullet[1], Str)$.

A. Localization in model categories

In this appendix we review the definition and a fundamental existence theorem of a *left Bousfield localization* of a model category. The original result of this section is theorem A.3 which formulates a condition on a symmetric monoidal closed model category so that a left Bousfield localization preserves the symmetric monoidal closed structure. A thorough exposition on homotopy function complexes in model categories can be found in [Hir02], [DK80].

Definition A.1. Let $\mathcal{M}$ be a model category and let $\mathcal{S}$ be a class of maps in $\mathcal{M}$. The left Bousfield localization of $\mathcal{M}$ with respect to $\mathcal{S}$ is a model category structure $L_{\mathcal{S}}\mathcal{M}$ on the underlying category of $\mathcal{M}$ such that

1. The class of cofibrations of $L_{\mathcal{S}}\mathcal{M}$ is the same as the class of cofibrations of $\mathcal{M}$.
2. A map $f : A \to B$ is a weak equivalence in $L_{\mathcal{S}}\mathcal{M}$ if it is an $\mathcal{S}$-local equivalence, namely, for every fibrant $\mathcal{S}$-local object X, the induced map on homotopy function complexes

$$f^* : Map^h_{\mathcal{M}}(B, X) \to Map^h_{\mathcal{M}}(A, X)$$

is a weak homotopy equivalence of simplicial sets. Recall that an object X is called fibrant $\mathcal{S}$-local if X is fibrant in $\mathcal{M}$ and for every element $g : K \to L$ of the set $\mathcal{S}$, the induced map on homotopy function complexes

$$g^* : Map^h_{\mathcal{M}}(L, X) \to Map^h_{\mathcal{M}}(K, X)$$

is a weak homotopy equivalence of simplicial sets.

We recall the following theorem which will be the main tool in the construction of the desired model category. This theorem first appeared in an unpublished work [Smi] but a proof was later provided by Barwick in [Bar07].

Theorem A.2. [Bar07, Theorem 2.11] If $\mathcal{M}$ is a left proper, combinatorial model category and $\mathcal{S}$ is a small set of homotopy classes of morphisms of $\mathcal{M}$, the left Bousfield localization $L_{\mathcal{S}}\mathcal{M}$ of $\mathcal{M}$ along any set representing $\mathcal{S}$ exists and satisfies the following conditions.

1. The model category $L_{\mathcal{S}}\mathcal{M}$ is left proper and combinatorial.
2. As a category, $L_{\mathcal{S}}\mathcal{M}$ is simply $\mathcal{M}$.
3. The cofibrations of $L_{\mathcal{S}}\mathcal{M}$ are exactly those of $\mathcal{M}$.
4. The fibrant objects of $L_{\mathcal{S}}\mathcal{M}$ are the fibrant $\mathcal{S}$-local objects Z of $\mathcal{M}$.
5. The weak equivalences of $L_{\mathcal{S}}\mathcal{M}$ are the $\mathcal{S}$-local equivalences.

The next theorem provides a condition for a left Bousfield localization to preserves the symmetric monoidal model category structure:

Theorem A.3. Let $\mathcal{M}_{\mathcal{O}}$ be a combinatorial model category such that the generating cofibrations in $\mathcal{M}_{\mathcal{O}}$ are maps between cofibrant objects. Let the underlying category of $\mathcal{M}_{\mathcal{O}}$, denoted by $\mathcal{M}$, have a symmetric monoidal closed structure which endows on $\mathcal{M}_{\mathcal{O}}$ a symmetric monoidal closed model category structure. Let us denote by $\mathcal{M}_{\mathcal{S}}$ the model category, which is a left Bousfield localization of $\mathcal{M}_{\mathcal{O}}$, with respect to a set of maps $\mathcal{S}$ in $\mathcal{M}$. If the internal mapping object $\mathcal{M}ap_{\mathcal{M}}(X, Y)$ is an $\mathcal{S}$-local object whenever X is cofibrant in $\mathcal{M}_{\mathcal{O}}$ and Y is an $\mathcal{S}$-local object, then the model category $\mathcal{M}_{\mathcal{S}}$ is also symmetric monoidal closed.

Proof. Let $i : U \to V$ be a cofibration in $\mathcal{M}_{\mathcal{S}}$ and $j : Y \to Z$ be another cofibration in $\mathcal{M}_{\mathcal{S}}$. We will prove the theorem by showing that the following *pushout product* morphism

$$i \square j : U * Z \coprod_{U * Y} V * Y \to V * Z$$

is a cofibration in $\mathcal{M}_{\mathcal{S}}$ which is also an $\mathcal{S}$-local equivalence whenever either i or j is one. We first deal with the case of i being a generating cofibration in $\mathcal{M}_{\mathcal{O}}$. The assumption of a symmetric monoidal closed model category structure on $\mathcal{M}_{\mathcal{O}}$ implies that $i\Box j$ is a cofibration in $\mathcal{M}_{\mathcal{O}}$ and we recall that the cofibrations in $\mathcal{M}_{\mathcal{S}}$ are exactly cofibration in $\mathcal{M}_{\mathcal{O}}$. Thus $i\Box j$ is a cofibration in $\mathcal{M}_{\mathcal{S}}$. Let us assume that j is an acyclic cofibration *i.e.* j is a cofibration in $\mathcal{M}_{\mathcal{S}}$ and also an $\mathcal{S}$-local equivalence. We recall that the fibrant objects of $\mathcal{M}_{\mathcal{S}}$ are exactly $\mathcal{S}$-local objects and fibrations in $\mathcal{M}_{\mathcal{S}}$ between $\mathcal{S}$-local objects are fibrations in $\mathcal{M}_{\mathcal{O}}$. According to [Shac, Proposition 4.22] the cofibration $i\Box j$ is an $\mathcal{S}$-local equivalence if and only if it has the left lifting property with respect to all fibrations in $\mathcal{M}_{\mathcal{S}}$ between $\mathcal{S}$-local objects. Let $p : W \to X$ be a fibration in $\mathcal{M}_{\mathcal{S}}$ between two $\mathcal{S}$-local objects. A (dotted) lifting arrow would exists in the following diagram

$$\begin{tikzcd} U * Z \coprod_{U*Y} V * Y \arrow[r] \arrow[d] & W \arrow[d, "p"] \\ V * Z \arrow[r] \arrow[ur, dotted] & Y \end{tikzcd}$$

if and only if a (dotted) lifting arrow exists in the following adjoint commutative diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d, "j"'] & \mathcal{M}ap_{\mathcal{M}}(V,W) \arrow[d, "(i^*,p^*)"] \\ Y \arrow[r] \arrow[ur, dotted] & \mathcal{M}ap_{\mathcal{M}}(U,X) \underset{\mathcal{M}ap_{\mathcal{M}}(U,Y)}{\times} \mathcal{M}ap_{\mathcal{M}}(V,Y) \end{tikzcd}$$

The map (i^*, p^*) is a fibration in $\mathcal{M}_{\mathcal{O}}$ by [Hov99, lem. 4.2.2] and the assumption that $\mathcal{M}_{\mathcal{O}}$ is a symmetric monoidal closed model category with internal Hom denoted by $\mathcal{M}ap_{\mathcal{M}}(-,-)$. Further the assumption of cofibrancy on both V and U and the assumption on internal mapping objects together imply that (j^*, p^*) is a fibration in $\mathcal{M}_{\mathcal{O}}$ between $\mathcal{S}$-local objects and therefore a fibration in the model category $\mathcal{M}_{\mathcal{S}}$. Since j is an acyclic cofibration in $\mathcal{M}_{\mathcal{S}}$ by assumption, therefore the (dotted) lifting arrow exists in the above diagram. Thus, we have shown that if i is a generating cofibration in $\mathcal{M}_{\mathcal{O}}$ and j is a cofibration in $\mathcal{M}_{\mathcal{O}}$ which is also an $\mathcal{S}$-local equivalence then $i\Box j$ is an acyclic cofibration in the model category $\mathcal{M}_{\mathcal{S}}$. Now we deal with the general case of i being an arbitrary cofibration in $\mathcal{M}_{\mathcal{O}}$. Consider the following set:

$$\mathcal{I} = \{i : U \to V |\ i\Box j \ \textit{ is an acyclic cofibration in } \ \mathcal{M}_{\mathcal{S}}\}$$

We have proved above that the set $\mathcal{I}$ contains all generating cofibration in $\mathcal{M}_{\mathcal{S}}$. We observe that the set $\mathcal{I}$ is closed under pushouts, transfinite compositions and retracts. Thus, I contains all cofibration in $\mathcal{M}_{\mathcal{O}}$. Thus, we have proved that $i\Box j$ is a cofibration which is acyclic if j is acyclic. The same argument as above when applied to the second argument of the Box product (*i.e.*, in the variable j) shows that $i\Box j$ is an acyclic cofibration whenever i is an acyclic cofibration in $\mathcal{M}_{\mathcal{S}}$.

$\square$

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