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A NEW PROOF OF THE JOYAL-TIERNEY THEOREM

G. Bezhanishvili, L. Carai, and P. J. Morandi

Résumé. Nous donnons une preuve alternative du théorème bien connu de Joyal-Tierney dans la théorie des locales en utilisant la dualité de Priestley pour les cadres.
Abstract. We give an alternative proof of the well-known Joyal-Tierney Theorem in locale theory by utilizing Priestley duality for frames.
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1. Introduction and Preliminaries

A well-known result in locale theory, known as the Joyal-Tierney Theorem, states that a localic map $f\colon M \to L$ is open iff its left adjoint $f^*\colon L \to M$ is a complete Heyting homomorphism (see, e.g., [8, Prop. III.7.2]). In addition, if L is subfit, then f is open iff f^* is a complete lattice homomorphism (see, e.g., [8, Prop. V.1.8]). Our aim is to give another proof of this result utilizing the language of Priestley spaces.

Priestley duality [9, 10] establishes a dual equivalence between the categories of bounded distributive lattices and Priestley spaces. We recall that a *Priestley space* is a Stone space X equipped with a partial order $\leq$ such that $x \not\leq y$ implies the existence of a clopen upset U such that $x \in U$ and $y \notin U$. A *Priestley morphism* is a continuous order-preserving map.

Pultr and Sichler [12] showed how to restrict Priestley duality to the category of frames. We recall (see, e.g., [8, p. 10]) that a *frame* is a complete lattice L satisfying the infinite distributive law $a \wedge \bigvee S = \bigvee\{a \wedge s : s \in S\}$ for each $a \in L$ and $S \subseteq L$. A map $h\colon L \to M$ between frames is a *frame homomorphism* if h preserves finite meets and arbitrary joins. Let **Frm** be the category of frames and frame homomorphisms.

Definition 1.1.

1. A Priestley space X is a *localic space*, or simply an *L-space*, provided the closure of an open upset is a clopen upset.
2. A Priestley morphism $f\colon X \to Y$ between L-spaces is an *L-morphism* provided $\mathsf{cl} f^{-1}U = f^{-1}\mathsf{cl}U$ for each open upset U of Y.
3. Let **LPries** be the category of L-spaces and L-morphisms.

Proposition 1.2. [12, p. 198] **Frm** *is dually equivalent to* **LPries**.

Remark 1.3. Since frames are exactly complete Heyting algebras (see, e.g., [6, Prop. 1.5.4]), every L-space is an Esakia space, where we recall that a Priestley space X is an *Esakia space* provided ${\downarrow}U$ is clopen for each clopen $U \subseteq X$ (equivalently, the closure of an open upset is an upset).

Remark 1.4. The contravariant functors establishing Pultr-Sichler duality are the restrictions of the contravariant functors establishing Priestley duality. They are described as follows.

For an L-space X, let $\mathsf{ClopUp}(X)$ be the frame of clopen upsets of X. The functor $\mathsf{ClopUp}\colon$ **LPries** $\to$ **Frm** sends $X \in$ **LPries** to the frame $\mathsf{ClopUp}(X)$ and an **LPries**-morphism $f\colon X \to Y$ to the **Frm**-morphism $f^{-1}\colon \mathsf{ClopUp}(Y) \to \mathsf{ClopUp}(X)$.

For $L \in$ **Frm** let X_L be the set of prime filters of L ordered by inclusion and equipped with the topology whose basis is $\{\phi(a) \setminus \phi(b) : a, b \in L\}$, where $\phi\colon L \to \wp(X_L)$ is the *Stone map* $\phi(a) = \{x \in X_L : a \in x\}$. Then X_L is an L-space and the functor $\mathsf{pf}\colon$ **Frm** $\to$ **LPries** sends $L \in$ **Frm** to X_L and a **Frm**-morphism $h\colon L \to M$ to the **LPries**-morphism $h^{-1}\colon X_M \to X_L$.

Let L, M be frames. Every frame homomorphism $h\colon L \to M$ has a right adjoint $r = h_*\colon M \to L$, called a *localic map*. It is given by

$$r(b) = \bigvee\{a \in L : h(a) \leq b\}.$$

The following provides a characterization of localic maps:

Proposition 1.5. [8, Prop. II.2.3] *A map* $r\colon M \to L$ *between frames is a localic map iff*

(1) r *preserves all meets* (*so has a left adjoint* $h = r^*$);

(2) $r(a) = 1$ *implies* $a = 1$;

(3) $r(h(a) \to b) = a \to r(b)$.

Let **Loc** be the category of frames and localic maps. The following is obvious from Propositions 1.2 and 1.5:

Proposition 1.6. **Loc** *is dually isomorphic to* **Frm**, *and hence equivalent to* **LPries**.

To define open localic maps, we recall the notion of a sublocale which generalizes that of a subspace. Let L be a frame. A subset S of L is a *sublocale* of L if S is closed under arbitrary meets and $x \to s \in S$ for each $x \in L$ and $s \in S$. Sublocales correspond to nuclei, where we recall (see, e.g., [8, Sec. III.5.3]) that a *nucleus* on L is a map $\nu\colon L \to L$ satisfying

1. $a \leq \nu a$;
2. $\nu\nu a \leq \nu a$;
3. $\nu(a \wedge b) = \nu a \wedge \nu b$.

We can go back and forth between nuclei and sublocales as follows. If ν is a nucleus on L, then $S_\nu := \nu[L]$ is a sublocale of L. Conversely, if S is a sublocale of L, then $\nu_S\colon L \to L$ is a nucleus on L, where ν_S is given by $\nu_S(a) = \bigwedge\{s \in S : a \leq s\}$. This correspondence is one-to-one (see, e.g., [8, Prop. III.5.3.2]).

If $a \in L$, then $\mathfrak{o}(a) := \{a \to x : x \in L\}$ is a sublocale of L, called an *open sublocale* of L, whose corresponding nucleus ν_a is given by $\nu_a(x) = a \to x$ (see, e.g., [8, pp. 33, 35]).

Definition 1.7. [8, p. 37] A localic map $r\colon M \to L$ is *open* if for each open sublocale S of M, the image $r[S]$ is an open sublocale of L.

2. The Joyal-Tierney Theorem

The Joyal-Tierney Theorem provides a characterization of open localic maps (see, e.g., [7, Prop. 7.3] or [8, pp. 37–38]):

Theorem 2.1 (Joyal-Tierney)**.** *Let $r\colon M \to L$ be a localic map between frames with left adjoint h. The following are equivalent*:

(1) *r is open.*

(2) *h is a complete Heyting homomorphism.*

(3) *h has a left adjoint $\ell = h^*$ satisfying the Frobenius condition*

$$\ell(a \wedge h(b)) = \ell(a) \wedge b$$

for each $a \in M$ and $b \in L$.

Our aim is to give an alternative proof of this result using Priestley duality for frames. For this we need to translate the algebraic conditions of Theorem 2.1 into geometric conditions about Priestley spaces. We will freely use the following well-known lemma. For parts (1) and (2) see [4, Lems. 11.21, 11.22]; for part (3) see [11, Prop. 2.6]; and part (4) is a consequence of Esakia's lemma (see [6, Lem. 3.3.12]).

Lemma 2.2.

(1) *For a Priestley space X, the set $\{U \setminus V : U, V \in \mathsf{ClopUp}(X)\}$ is a basis of open sets of X.*

(2) *Let X be a Priestley space. If F, G are disjoint closed subsets of X, with F an upset and G a downset, then there is a clopen upset U of X such that $F \subseteq U$ and $G \cap U = \varnothing$. In particular, every open upset is a union and every closed upset is an intersection of clopen upsets.*

(3) *If F is a closed subset of a Priestley space, then ${\uparrow}F$ and ${\downarrow}F$ are closed.*

(4) *Let $f\colon X \to Y$ be a continuous map between Priestley spaces. For each $x \in X$ we have*

$$f\left[\bigcap\{U \in \mathsf{ClopUp}(X) : x \in U\}\right] = \bigcap\{f[U] : x \in U \in \mathsf{ClopUp}(X)\}.$$

We recall (see, e.g., [4, p. 265]) that if $h\colon L \to M$ is a frame homomorphism and $f\colon X_M \to X_L$ is its Priestley dual, then

$$f^{-1}\phi(a) = \phi h(a). \tag{a}$$

We also recall that if $r\colon M \to L$ is a localic map and S is a sublocale of M, then $r[S]$ is a sublocale of L (see, e.g., [8, Prop. III.4.1]).

Lemma 2.3. *Let $r\colon M \to L$ be a localic map with left adjoint h. If S is a sublocale of M, then $\nu_{r[S]} = r\nu_S h$.*

Proof. Let $a \in L$. We have

$$\begin{aligned}\nu_{r[S]}(a) &= \bigwedge\{r(s) : s \in S,\, a \le r(s)\}\\ &= \bigwedge\{r(s) : s \in S,\, h(a) \le s\}\\ &= r\left(\bigwedge\{s \in S : h(a) \le s\}\right)\\ &= r\nu_S h(a).\end{aligned}$$

Therefore, $\nu_{r[S]} = r\nu_S h$. □

We thus see that a localic map $r\colon M \to L$, with left adjoint h, is open iff for each $a \in M$ there is $b \in L$ with $r\nu_a h = \nu_b$. We use this observation in the proof of the following lemma.

Lemma 2.4. *Let $r\colon M \to L$ be a localic map, h the left adjoint of r, and $f\colon X_M \to X_L$ the Priestley dual of h. The following are equivalent*:

(1) *r is open.*

(2) *If U is a clopen upset of X_M, then $f[U]$ is a clopen upset of X_L.*

Proof. We start by showing that if $a \in M$ and $b, c \in L$, then

$$b \le (r\nu_a h)(c) \iff \phi(b) \cap f[\phi(a)] \subseteq \phi(c). \tag{b}$$

To see this,

$$\begin{aligned}b \le (r\nu_a h)(c) &\iff b \le r(a \to h(c)) \iff h(b) \le a \to h(c)\\ &\iff h(b) \wedge a \le h(c).\end{aligned}$$

Therefore, since $f[f^{-1}(B) \cap A] = B \cap f[A]$ for each A, B, by (a) we have

$$\begin{aligned} b \leq (r\nu_a h)(c) &\iff \phi h(b) \cap \phi(a) \subseteq \phi h(c) \\ &\iff f^{-1}\phi(b) \cap \phi(a) \subseteq f^{-1}\phi(c) \\ &\iff f[f^{-1}\phi(b) \cap \phi(a)] \subseteq \phi(c) \\ &\iff \phi(b) \cap f[\phi(a)] \subseteq \phi(c). \end{aligned}$$

(1)⇒(2). Let $U \in \mathsf{ClopUp}(X_M)$. Then $U = \phi(a)$ for some $a \in M$. By (1) and Lemma 2.3, there is $b \in L$ with $r\nu_a h = \nu_b$. Since $1 = \nu_b(b)$, we have $1 \leq (r\nu_a h)(b)$, so $\phi(1) \cap f[U] \subseteq \phi(b)$ by (b). Therefore, $f[U] \subseteq \phi(b)$. For the reverse inclusion, let $y \in \phi(b)$. If $y \notin f[U]$, then since $f[U]$ is closed in X_L, there is a clopen set containing y and missing $f[U]$. By Lemma 2.2(1), there are $c, d \in L$ with $y \in \phi(c) \setminus \phi(d)$ and $f[U] \cap (\phi(c) \setminus \phi(d)) = \varnothing$. Thus, $f[U] \cap \phi(c) \subseteq \phi(d)$, so $c \leq (r\nu_a h)(d) = \nu_b(d) = b \to d$ by (b). This gives $b \wedge c \leq d$, and hence $\phi(b) \cap \phi(c) \subseteq \phi(d)$, a contradiction since $y \in \phi(b) \cap \phi(c)$ but $y \notin \phi(d)$. Therefore, $y \in f[U]$, and so $\phi(b) \subseteq f[U]$. Consequently, $f[U] = \phi(b)$, and so $f[U] \in \mathsf{ClopUp}(X_L)$.

(2)⇒(1). Let $a \in M$ and set $U = \phi(a)$. Then $U \in \mathsf{ClopUp}(X_M)$, so $f[U] \in \mathsf{ClopUp}(X_L)$ by (2). Therefore, there is $b \in L$ with $\phi(b) = f[U]$. If $c, d \in L$, then by (b),

$$\begin{aligned} c \leq (r\nu_a h)(d) &\iff \phi(c) \cap f[U] \subseteq \phi(d) \\ &\iff \phi(c) \cap \phi(b) \subseteq \phi(d) \\ &\iff c \wedge b \leq d \\ &\iff c \leq b \to d \\ &\iff c \leq \nu_b(d). \end{aligned}$$

Thus, $r\nu_a h = \nu_b$, and hence r is open. □

We next give a dual characterization of when a frame homomorphism has a left adjoint. Let X be a Priestley space. Then we have two additional topologies on X, the topology of open upsets and the topology of open downsets. If cl_i and int_i are the corresponding closure and interior operators ($i = 1, 2$), then it is well known (see, e.g., [3, Lem. 6.5]) that for $A \subseteq X$ we have:

$$\begin{aligned} \mathsf{cl}_1 A = {\downarrow}\mathsf{cl} A \quad &\text{and} \quad \mathsf{int}_1(A) = X \setminus {\downarrow}(X \setminus \mathsf{int} A); \\ \mathsf{cl}_2 A = {\uparrow}\mathsf{cl} A \quad &\text{and} \quad \mathsf{int}_2(A) = X \setminus {\uparrow}(X \setminus \mathsf{int} A). \end{aligned}$$

Let L be a frame and let $a = \bigwedge S$ for $a \in L$ and $S \subseteq L$. Then

$$\phi(a) = \mathsf{int}_1 \bigcap \{\phi(s) : s \in S\} \tag{c}$$

(see, e.g., [2, Lem. 2.3]). This will be used in the following lemma.

Lemma 2.5. *Let* $h\colon L \to M$ *be a frame homomorphism and* $f\colon X_M \to X_L$ *its Priestley dual. The following are equivalent*:

(1) h *has a left adjoint.*

(2) h *preserves all meets.*

(3) $f^{-1}\mathsf{int}_1 F = \mathsf{int}_1 f^{-1} F$ *for each closed upset* $F \subseteq X_L$.

(4) $\uparrow f[U]$ *is clopen for each clopen upset* $U \subseteq X_M$.

Proof. (1)⇔(2). This is well known (see, e.g., [4, Prop. 7.34]).

(2)⇒(3). Let F be a closed upset of X_L. By Lemma 2.2(2), we may write $F = \bigcap\{\phi(s) : s \in S\}$ for some $S \subseteq L$. By (a),

$$\begin{aligned} f^{-1}(F) = f^{-1}\left(\bigcap\{\phi(s) : s \in S\}\right) &= \bigcap\{f^{-1}\phi(s) : s \in S\} \\ &= \bigcap\{\phi h(s) : s \in S\}, \end{aligned}$$

so

$$\mathsf{int}_1 f^{-1}(F) = \mathsf{int}_1 \bigcap\{\phi h(s) : s \in S\} = \phi\left(\bigwedge h[S]\right).$$

On the other hand, by (c) we have

$$\mathsf{int}_1 F = \mathsf{int}_1 \bigcap\{\phi(s) : s \in S\} = \phi\left(\bigwedge S\right).$$

Therefore, using (a) again yields

$$f^{-1}(\mathsf{int}_1 F) = f^{-1}\phi\left(\bigwedge S\right) = \phi h\left(\bigwedge S\right).$$

Thus, by (2) we have

$$\mathsf{int}_1 f^{-1}(F) = \phi\left(\bigwedge h[S]\right) = \phi h\left(\bigwedge S\right) = f^{-1}(\mathsf{int}_1 F).$$

(3)⇒(4). Let $U \in \mathsf{ClopUp}(X_M)$ and set $F = {\uparrow} f[U]$. By Lemma 2.2(3), F is a closed upset of Y. By (3),

$$U \subseteq \mathsf{int}_1 f^{-1}(f[U]) \subseteq \mathsf{int}_1 f^{-1}F = f^{-1}\mathsf{int}_1 F,$$

so $f[U] \subseteq \mathsf{int}_1 F$, and hence ${\uparrow} f[U] \subseteq \mathsf{int}_1 F = \mathsf{int}_1 {\uparrow} f[U]$. Thus, ${\uparrow} f[U]$ is clopen.

(4)⇒(1). Let $a \in M$. By (4), ${\uparrow} f[\phi(a)] \in \mathsf{ClopUp}(X_L)$. Therefore, there is a unique $b \in L$ such that $\phi(b) = {\uparrow} f[\phi(a)]$. Letting $\ell(a) = b$ defines a function $\ell\colon M \to L$ such that

$$\phi\ell(a) = {\uparrow} f[\phi(a)]. \tag{d}$$

To see that ℓ is left adjoint to h, let $c \in L$. Since $\phi(c)$ is an upset, by (a) we have

$$\begin{aligned} \ell(a) \le c &\iff \phi\ell(a) \subseteq \phi(c) \iff {\uparrow} f[\phi(a)] \subseteq \phi(c) \iff f[\phi(a)] \subseteq \phi(c) \\ &\iff \phi(a) \subseteq f^{-1}\phi(c) \iff \phi(a) \subseteq \phi h(c) \iff a \le h(c). \end{aligned}$$

□

We recall (see, e.g., [6, p. 9]) that a map $f\colon X \to Y$ between posets is a *bounded morphism* or a *p-morphism* if ${\downarrow} f^{-1}(y) = f^{-1}({\downarrow} y)$ for each $y \in Y$. Let $h\colon L \to M$ be a frame homomorphism between frames and $f\colon X_M \to X_L$ its Priestley dual. Then f is an L-morphism. It follows from Esakia duality for Heyting algebras [5, 6] that h preserves $\to$ iff f is a p-morphism. This together with Lemma 2.5 yields:

Lemma 2.6. *Let $h\colon L \to M$ be a frame homomorphism and $f\colon X_M \to X_L$ its dual L-morphism. Then h is a complete Heyting homomorphism iff f is a p-morphism and ${\uparrow} f[U]$ is clopen for each clopen upset U of X_M.*

We next provide a dual characterization of the Frobenius condition

$$\ell(a \wedge h(b)) = \ell(a) \wedge b$$

for each $a \in M$ and $b \in L$.

Lemma 2.7. *Let $h\colon L \to M$ be a frame homomorphism with Priestley dual $f\colon X_M \to X_L$. The following are equivalent:*

(1) *h has a left adjoint ℓ and $\ell(a \wedge h(b)) = \ell(a) \wedge b$ for all $a \in M$ and $b \in L$.*

(2) *$\uparrow f[U]$ is clopen and $\uparrow(f[U] \cap V) = \uparrow f[U] \cap V$ for all $U \in \mathsf{ClopUp}(X_M)$ and $V \in \mathsf{ClopUp}(X_L)$.*

Proof. By Lemma 2.5, h has a left adjoint ℓ iff $\uparrow f[U]$ is clopen for each $U \in \mathsf{ClopUp}(X_M)$. It is left to show that $\ell(a \wedge h(b)) = \ell(a) \wedge b$ for each $a \in M$ and $b \in L$ iff $\uparrow(f[U] \cap V) = \uparrow f[U] \cap V$ for each $U \in \mathsf{ClopUp}(X_M)$ and $V \in \mathsf{ClopUp}(X_L)$. Letting $U = \phi(a)$ and $V = \phi(b)$, since $\uparrow f[U] = \phi\ell(a)$ by (d), we have

$$\phi(\ell(a) \wedge b) = \phi\ell(a) \cap \phi(b) = \uparrow f[U] \cap V.$$

On the other hand, since $f[U \cap f^{-1}(V)] = f[U] \cap V$, by (a) we have

$$\begin{aligned} \phi\ell(a \wedge h(b)) &= \uparrow f[\phi(a \wedge h(b))] = \uparrow f[\phi(a) \cap \phi h(b)] \\ &= \uparrow f[\phi(a) \cap f^{-1}\phi(b)] = \uparrow f[U \cap f^{-1}(V)] \\ &= \uparrow(f[U] \cap V). \end{aligned}$$

Thus,

$$\begin{aligned} \ell(a \wedge h(b)) = \ell(a) \wedge b &\iff \phi\ell(a \wedge h(b)) = \phi(\ell(a) \wedge b) \\ &\iff \uparrow(f[U] \cap V) = \uparrow f[U] \cap V. \end{aligned}$$

□

We thus have translated the three conditions of Theorem 2.1 into the dual conditions in the language of Priestley spaces. We next prove that the translated conditions are equivalent.

Theorem 2.8. *Let $f\colon X \to Y$ be a Priestley morphism between L-spaces. The following are equivalent*:

(1) *If $U \in \mathsf{ClopUp}(X)$, then $f[U] \in \mathsf{ClopUp}(Y)$.*

(2) *f is a p-morphism and $\uparrow f[U]$ is clopen for all $U \in \mathsf{ClopUp}(X)$.*

(3) *$\uparrow f[U]$ is clopen and $\uparrow(f[U] \cap V) = \uparrow f[U] \cap V$ for all $U \in \mathsf{ClopUp}(X)$ and $V \in \mathsf{ClopUp}(Y)$.*

Proof. (1)⇒(2). Let $U \in \mathsf{ClopUp}(X)$. By (1), $f[U]$ is an upset of Y, so $\uparrow f[U] = f[U]$. Therefore, $\uparrow f[U]$ is clopen in Y by (1). It is left to prove that f is a p-morphism. For this it suffices to show that $f(\uparrow x)$ is an upset for each $x \in X$ (see, e.g, [6, Prop 1.4.12]). By Lemma 2.2(2),

$$\uparrow x = \bigcap\{U \in \mathsf{ClopUp}(X) : x \in U\},$$

so by Lemma 2.2(4),

$$\begin{aligned} f[\uparrow x] &= f\left[\bigcap\{U \in \mathsf{ClopUp}(X) : x \in U\}\right] \\ &= \bigcap\{f[U] : x \in U \in \mathsf{ClopUp}(X)\}. \end{aligned}$$

Thus, $f[\uparrow x]$ is an upset by (1).

(2)⇒(3). It is sufficient to show that $\uparrow(f[U] \cap V) = \uparrow f[U] \cap V$ for each $U \in \mathsf{ClopUp}(X)$ and $V \in \mathsf{ClopUp}(Y)$. But since f is a p-morphism, $\uparrow f[U] = f[U]$, so $\uparrow f[U] \cap V = f[U] \cap V = \uparrow(f[U] \cap V)$ because $f[U] \cap V$ is an upset.

(3)⇒(1). It suffices to show that $f[U]$ is an upset. If not, then there exist $x \in U$ and $y \in Y$ with $f(x) \le y$ but $y \notin f[U]$. This yields $y \notin \downarrow(\downarrow y \cap f[U])$, so there is a clopen upset V of Y such that $y \in V$ and $V \cap \downarrow y \cap f[U] = \varnothing$ (see Lemma 2.2(2)). Therefore, $y \notin \uparrow(f[U] \cap V)$ but $y \in \uparrow f[U] \cap V$, a contradiction to (3). Thus, $f[U]$ is an upset. □

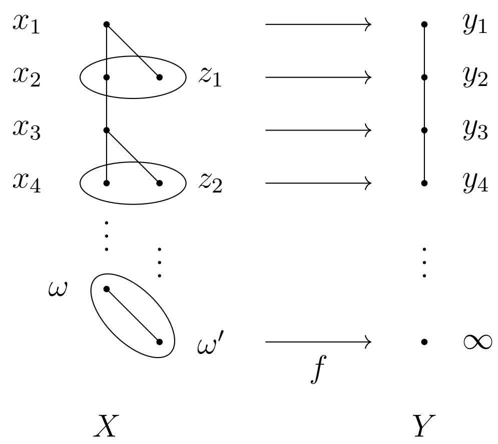
By Lemmas 2.4, 2.6 and 2.7, the three conditions of Theorem 2.8 are equivalent to the corresponding three conditions of Theorem 2.1. Hence, the Joyal-Tierney Theorem is a consequence of Theorem 2.8. We conclude this section with the following observation.

Remark 2.9. Condition (1) of Theorem 2.8 is equivalent to:

(1′) If U is an open upset of X, then $f[U]$ is an open upset of Y.

Clearly (1′) implies (1) since if U is clopen, then $f[U]$ is closed, hence a clopen upset of Y by (1′). Conversely, if U is an open upset, then $U = \bigcup\{V \in \mathsf{ClopUp}(X) : V \subseteq U\}$ by Lemma 2.2(2). Therefore, $f[U] = \bigcup\{f[V] : V \in \mathsf{ClopUp}(X),\ V \subseteq U\}$ is a union of clopen upsets of Y by (1). Thus, $f[U]$ is an open upset of Y. Consequently, (1) is equivalent to f being an open map with respect to the open upset topologies.

On the other hand, this does not imply that f is an open map with respect to the Stone topologies. To see this, we use the space defined in [1, p. 32]. Let X be the 2-point compactification of the discrete space $\{x_n, z_n : n \geq 1\}$ with ω the limit point of $\{x_n : n \geq 1\}$ and ω' the limit point of $\{z_n : n \geq 1\}$. Let Y be the 1-point compactification of the discrete space $\{y_n : n \geq 1\}$. We order X and Y and define the map $f\colon X \to Y$ as shown in the diagram below.



It is straightforward to see that X and Y are L-spaces and f is an L-morphism such that $f[U]$ is a clopen upset of Y for each clopen upset U of X. However, f is not an open map since $U := \{z_n : n \geq 1\} \cup \{\omega'\}$ is an open subset of X whose image $\{y_{2n} : n \geq 1\} \cup \{\infty\}$ is not an open subset of Y.

3. The subfit case

As was shown in [8, Prop. V.1.8], if in the Joyal-Tierney Theorem we assume that L is subfit, then the localic map $r\colon M \to L$ is open iff its left adjoint $h\colon L \to M$ is a complete lattice homomorphism (so h being a Heyting homomorphism becomes redundant). We will give an alternative proof of this result in the language of Priestley spaces.

We recall that a frame L is *subfit* if for all $a, b \in L$ we have

$$a \not\leq b \Longrightarrow (\exists c \in L)(a \vee c = 1 \text{ and } b \vee c \neq 1).$$

We next give a dual characterization of when L is subfit. As usual, for a poset X we write $\min X$ for the set of minimal points of X.

Lemma 3.1. *Let L be a frame and X_L its Priestley space. Then L is subfit iff $\min X_L$ is dense in X_L.*

Proof. First suppose that $\min X_L$ is dense in X_L. To see that L is subfit, let $a, b \in L$ with $a \not\leq b$. Then $\phi(a) \not\subseteq \phi(b)$, so $\phi(a) \setminus \phi(b)$ is a nonempty clopen subset of X. Therefore, there is $x \in (\phi(a) \setminus \phi(b)) \cap \min X_L$. Let $U = X_L \setminus \{x\}$. Then U is an open upset of X_L. Since $\phi(a) \cup U = X_L$ and U is a union of clopen upsets (see Lemma 2.2(2)), compactness of X_L implies that there is a clopen upset $U' \subseteq U$ with $\phi(a) \cup U' = X_L$. Because $U' = \phi(c)$ for some $c \in L$, we have $a \vee c = 1$. On the other hand, since $x \notin \phi(b) \cup U' = \phi(b \vee c)$, it follows that $b \vee c \neq 1$. Thus, L is subfit.

Conversely, suppose that $\min X_L$ is not dense in X_L. Then there is a nonempty clopen subset A of X_L such that $A \cap \min X_L = \varnothing$. We may assume that $A = U \setminus V$, where $U \not\subseteq V$ are clopen upsets of X_L (see Lemma 2.2(1)). From $A \cap \min X_L = \varnothing$ it follows that $U \cap \min X_L \subseteq V$. Let $a, b \in L$ be such that $U = \phi(a)$ and $V = \phi(b)$. Since $U \not\subseteq V$, we have $a \not\leq b$. Suppose $c \in L$ is such that $a \vee c = 1$. Let $W = \phi(c)$. Then $U \cup W = X_L$, so $\min X_L \subseteq U \cup W$. Because $U \cap \min X_L \subseteq V$, this yields $\min X_L \subseteq V \cup W$, which forces $V \cup W = X_L$ because $\uparrow \min X_L = X_L$ (see, e.g., [6, Thm. 3.2.1]). Thus, $b \vee c = 1$, and hence L is not subfit. □

Lemma 3.2. *Let $f\colon X \to Y$ be a Priestley morphism between L-spaces. If $\min Y$ is dense in Y and $\uparrow f[U]$ is clopen for each $U \in \mathsf{ClopUp}(X)$, then f is a p-morphism.*

Proof. It is sufficient to show that Condition (1) of Theorem 2.8 holds, which amounts to showing that $f[U]$ is an upset for each $U \in \mathsf{ClopUp}(X)$. If not, then $\uparrow f[U] \setminus f[U] \neq \varnothing$ for some $U \in \mathsf{ClopUp}(X)$. Let $V = \uparrow f[U] \setminus f[U]$. Since $\uparrow f[U]$ is open and $f[U]$ is closed, V is a nonempty open subset of Y. Thus, $V \cap \min Y \neq \varnothing$ because $\min Y$ is dense in Y. On the other hand,

$$V \cap \min Y \subseteq \uparrow f[U] \cap \min Y = f[U] \cap \min Y.$$

This is a contradiction since $V \cap f[U] = \varnothing$. Consequently, $f[U]$ is an upset. □

As an immediate consequence of Lemma 3.2, we obtain:

Theorem 3.3. *Let $f\colon X \to Y$ be a Priestley morphism between L-spaces. If $\min Y$ is dense in Y, then Condition (2) in Theorem 2.8 is equivalent to*

$(2')$ $\uparrow f[U]$ *is clopen for each* $U \in \mathsf{ClopUp}(X)$.

Theorems 2.8 and 3.3 together with Lemmas 2.4 and 2.5 yield the following version of the Joyal-Tierney Theorem:

Corollary 3.4. [8, Prop. V.1.8] *Let $r\colon M \to L$ be a localic map with left adjoint h. If L is subfit, then r is open iff h is a complete lattice homomorphism.*

References

[1] G. Bezhanishvili, *Varieties of monadic Heyting algebras. II. Duality theory*, Studia Logica **62** (1999), no. 1, 21–48.

[2] G. Bezhanishvili and N. Bezhanishvili, *Profinite Heyting algebras*, Order **25** (2008), no. 3, 211–227.

[3] G. Bezhanishvili, N. Bezhanishvili, D. Gabelaia, and A. Kurz, *Bitopological duality for distributive lattices and Heyting algebras*, Math. Structures Comput. Sci. **20** (2010), no. 3, 359–393.

[4] B. A. Davey and H. A. Priestley, *Introduction to lattices and order*, second ed., Cambridge University Press, New York, 2002.

[5] L. L. Esakia, *Topological Kripke models*, Soviet Math. Dokl. **15** (1974), 147–151.

[6] L. L. Esakia, *Heyting algebras. Duality theory*, Translated from the Russian by A. Evseev. Edited by G. Bezhanishvili and W. Holliday. Trends in Logic, vol. 50, Springer, 2019.

[7] J. Picado and A. Pultr, *Locales treated mostly in a covariant way*, Textos de Matemática. Série B, vol. 41, Universidade de Coimbra, Departamento de Matemática, Coimbra, 2008.

[8] J. Picado and A. Pultr, *Frames and locales: Topology without points*, Frontiers in Mathematics, Birkhäuser/Springer Basel AG, Basel, 2012.

[9] H. A. Priestley, *Representation of distributive lattices by means of ordered Stone spaces*, Bull. London Math. Soc. **2** (1970), 186–190.

[10] H. A. Priestley, *Ordered topological spaces and the representation of distributive lattices*, Proc. London Math. Soc. (3) **24** (1972), 507–530.

[11] H. A. Priestley, *Ordered sets and duality for distributive lattices*, Orders: description and roles (L'Arbresle, 1982), North-Holland Math. Stud., vol. 99, North-Holland, Amsterdam, 1984, pp. 39–60.

[12] A. Pultr and J. Sichler, *Frames in Priestley's duality*, Cahiers Topologie Géom. Différentielle Catég. **29** (1988), no. 3, 193–202.

G. Bezhanishvili
Department of Mathematical Sciences
New Mexico State University
Las Cruces NM 88011 USA
guram@nmsu.edu

L. Carai
Department of Philosophy
University of Barcelona
Carrer de Montalegre 6
08001 Barcelona, Spain
luca.carai.uni@gmail.com

P. J. Morandi
Department of Mathematical Sciences
New Mexico State University
Las Cruces NM 88011 USA
pmorandi@nmsu.edu