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THE UNIVERSAL OPERAD ACTING ON LOOP SPACES, AND GENERALISATIONS

Eugenia CHENG and Todd TRIMBLE

Résumé. Dans cet article, nous étudions l'opérade globulaire utilisée par Batanin pour définir l'ω-groupoïde fondamental d'un espace. Nous identifions une propriété universelle de cette opérade et nous construisons un cadre catégoriel général des opérades universelles agissant sur une structure donnée. Un exemple motivant est l'opérade universelle agissant sur les espaces de lacets. D'autres exemples comprennent des versions n-dimensionelles de l'ω-groupoïde fondamental—la version à homotopie près, la version enrichie dans les espaces topologiques, ou tout simplement la version tronquée. Identifier la propriété universelle de l'opérade de Batanin nous aide à trouver d'autres opérades convenables à reconnaître les ω-groupoïdes fondamentaux. Nous espérons que ces opérades non-universelles et plus petites nous permettent de démontrer que les ω-groupoïdes définis par les opérades globulaires modélisent les types d'homotopie.
Abstract. In this paper we analyse the globular operad used by Batanin to define the fundamental ω-groupoid of a space. We identify a universal property of this globular operad and give a general categorical framework for universal operads acting on structures. A motivating example is the universal operad acting on loop spaces. Other examples include n-dimensional versions of the fundamental ω-groupoid—up-to-homotopy, enriched in spaces, or simply truncated. Identifying the universal property of Batanin's operad helps us to find other suitable operads for recognising fundamental ω-groupoids. The hope is that these smaller, non-universal operads will enable a proof that

globular operadic ω-groupoids model homotopy types.
Keywords. Operad, globular operad, fundamental ω-groupoid, loop spaces, enriched categories, homotopy types.
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Contents

Introduction

One of the earliest motivations for studying higher-dimensional algebra was Grothendieck's suggestion of modelling homotopy types by "ω-groupoids" [8]. There have been many different approaches to this, as there are many different approaches to defining ω-groupoids. One approach is first to define ω-categories and then to pick out the ω-groupoids among them as those in which every element is "weakly invertible". This is in contrast to the "direct" approach in which non-invertible elements are never considered, for example with the notion of Kan complexes. This and other simplicial approaches (for example [17, 16, 15]) are "non-algebraic" in that they do not *specify* all the operations of the algebraic structure. Some other ways of thinking about this are:

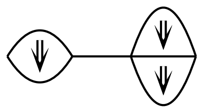
- they demand that composites *exist* rather than specifying them, thus
- they cannot be defined as algebras for a monad or models for an algebraic theory, and
- they can be thought of as nerves of algebraic structures, rather than algebraic structures themselves.

A different family of approaches seeks to specify the operations of an ω-groupoid explicitly, using operads. Operads achieved great success in the study of iterated loop spaces, as a tool for parametrising multiplication of loops [13]. This multiplication is associative and unital only up to homotopy; a similar phenomenon occurs for ω-groupoids but is generalised to all types of composition at every dimension.

One reason that operads are so efficacious for the study of loop spaces is that we can pick combinatorially convenient operads for different situations; the theory tells us how the resulting structures on loop spaces are equivalent. The operads that are useful in practice (for example the little disks operad) often do not have good universal properties, as the universal ones are much too large for practical use.

By contrast, Category Theory tends to seek objects with nice universal properties. One of the aims of this paper is to show that the operads used by Trimble [18] and Batanin [1] in their definitions of n-category have a nice universal property. We take the view that the main purpose of identifying this universal property is to help us find smaller, non-universal operads for practical use. One such "practical use" is the modelling of homotopy types.

There are (at least) two ways of using operads in higher-dimensional algebra. Trimble proceeds inductively, using a classical operad at each dimension. Batanin on the other hand parametrises all dimensions at once, using a more general form of operad called "globular operad", in which the arities of operations are no longer just natural numbers but "globular pasting diagrams" such as



Note that although Trimble's definition, being inductive, can *a priori* only achieve finite n dimensions, a coinductive argument may be used to provide the ω-dimensional version [7].

In both cases, general ω-categories are defined, with the ω-groupoids being identified among them afterwards. This is analogous to the fact that while certain operads may be used to recognise loop spaces, it is not the case that all the algebras for such operads are loop spaces—only those among them that are group-like.

Batanin's definition, or at least, the variant we shall use, says *an ω-category is a globular set equipped with the structure of an algebra for any contractible globular operad*. Then an ω-groupoid is an ω-category in which every cell is weakly invertible (we will recall the precise definitions in Section 4.1). Thus to give the fundamental ω-groupoid of a space X we must

1. identify its underlying globular set UX,
2. find a contractible globular operad that acts on UX, and
3. show that every cell is weakly invertible.

Step (1) is straightforward—the n-cells of UX are found essentially by mapping the topological n-ball B^n into X (with a little care over sources and targets).

Batanin achieves (2) by identifying a particular globular operad K that acts naturally on the underlying globular set of any space. Note that "acting naturally" here means two things—the action is canonical, but also, more technically, the action is natural in X.

Essentially, given an n-pasting diagram α, the operations of K of arity α are the boundary-preserving maps from B^n to the geometric realisation of α, where "boundary-preserving" must be carefully interpreted to take into account all dimensions of boundary. The following facts are then immediately true.

1. For any space X, UX is a K-algebra.

2. Any K-algebra is an ω-category but not necessarily an ω-groupoid.
3. There might be ω-categories that are not K-algebras.

Batanin further shows that for any space X, UX is a K-algebra in which every cell is weakly invertible (in a sense to be made precise). That is, it is an ω-groupoid.

Crucially, there are other globular operads that act naturally on all globular sets UX (that is, naturally in X), and we will prove the following result as an instance of the main theorem.

Theorem 1. *A natural action of a globular operad P on underlying globular sets UX is precisely given by a map $P \longrightarrow K$ of globular operads.*

This result exhibits the action of K as *universal* (in fact terminal) among such operad actions. Another way of saying this is: "Any such natural action factors uniquely through the canonical action of K."

Note that U extends to a functor $\mathsf{Top} \longrightarrow \mathsf{GSet}$ (where we write GSet for the category of globular sets and their morphisms), and the naturality of the actions in question means that in effect we should think of our operads as acting on the *functor* U. In fact we prove the universality result in general for suitable functors $U \colon \mathcal{S} \longrightarrow \mathcal{G}$. Other examples are as follows; here n-GSet denotes the category of n-dimensional globular sets.

1. Loop spaces, using the functor

$$\begin{array}{rcccc} \Omega & : & \mathsf{Top}_* & \longrightarrow & \mathsf{Top} \\ & & X & \longmapsto & \mathsf{Top}_*(S^1, X) \end{array}$$

2. Fundamental n-groupoids, given by the functor

$$\Pi_n : \mathsf{Top} \longrightarrow n\text{-}\mathsf{GSet}$$

which agrees with U at all dimensions less than n, and takes homotopy classes at dimension n (see Section 4.3 for a precise definition).

3. The "incoherent" version of Π_n, a functor

$$U_n : \mathsf{Top} \longrightarrow n\text{-}\mathsf{GSet}$$

which simply truncates UX to n dimensions.

4. The "path space" version of Π_n, a functor
$$\mathcal{P}_n : \mathsf{Top} \longrightarrow \mathsf{Top}\text{-}n\text{-}\mathsf{Gph}.$$
Here Top-n-Gph denotes the category of "n-graphs enriched in Top"; we will treat these as n-graphs in Top where for all $k < n$ the space of k-cells is set-like. Then $\mathcal{P}_n$ agrees with U (and Π_n at all dimensions less than n); for $k = n$ we have a *space* of k-cells of arity α, given by the *space* of boundary-preserving maps from B^k to the geometric realisation of α, where at lower dimensions we took the *set* of such maps.

In each case we have a cartesian monad T on $\mathcal{G}$ giving us a pertinent notion of T-operad, and the main theorem gives us a universal T-operad E_U acting on the functor in question. In (1), T is the free topological monoid monad (giving rise to classical operads in Top); for (2) and (3) we use the free strict n-category monad, and for (4) we use the free topologically-enriched n-category monad.

Note that symmetric operads do not fit into this framework as the free commutative monoid monad is not cartesian; it is however *weakly* cartesian, so symmetric operads do fit into the more general framework of Weber [19]. A direct examination of symmetric operads acting on loop spaces yields a universal symmetric operad analogous to the non-symmetric one, suggesting that the main theorem could be extended to Weber's weakly cartesian framework. However this is beyond the scope of this work.

Once we have identified this simple universal property, we have an obvious method of finding smaller non-universal operads for the given purpose. That is, having constructed the universal operad E_U we just have to look for any T-operad P equipped with an operad map $P \longrightarrow E_U$.

In the case of globular operads we can make use of the work of [4] in which we prove that every Trimble n-category is a Batanin n-category. Part of the proof produces a functor
$$\left\{\begin{array}{l}\text{Classical operads}\\ \text{acting on path spaces}\end{array}\right\} \longrightarrow \left\{\begin{array}{l}\text{Globular operads}\\ \text{acting on } \omega\text{-path spaces}\end{array}\right\}$$
Even the universal operad on the left yields a non-universal operad on the right, and applying the functor to non-universal operads on the left gives further non-universal examples on the right. We will discuss this in Section 4.4.

The structure of this paper is as follows. In Section 1 we prove the main universality theorem in sufficient generality to cover the several key examples to which the remainder of the paper is devoted. In Section 2 we prove some technical results needed to address sizes issues when constructing internal homs; we are mostly working in categories of functors between large categories, the construction of internal homs takes some care. In Section 3 we discuss operads acting on loop spaces. Among other things this serves to emphasise the unwieldy nature of the universal operad in question, and the importance of finding non-universal ones for calculations, as is done in the theory of loop spaces. In Section 4 we discuss the motivating example, globular operads for defining the fundamental ω-groupoids of spaces, together with the various n-dimensional versions described above. We end with a brief discussion of future work.

Note for experts

Experts who wish to read the paper quickly might wish to proceed directly as follows.

1. The definition of E_U, the universal operad acting on a functor U is given in Definition 1.17.
2. The main theorem, giving the universal property of E_U, is Theorem 1.21.
3. The technical theorem addressing size issues is Theorem 2.4.
4. The loop space example is given in Theorem 3.3.
5. The fundamental ω-groupoid example is given in Theorem 4.8.

Terminology and notation

1. Top will denote a category of topological spaces that is complete, cocomplete and cartesian closed, for example the category of compactly generated weakly Hausdorff spaces.
2. $\mathbb{N}$ will denote the natural numbers including 0.
3. By "classical operad" we will always mean non-symmetric operad.

4. We will use the equivalent categories n-GSet and n-Gph more-or-less interchangeably, although technically the former is defined as a presheaf category and the latter by iterated enrichment.

5. We will write T-operads as their underlying collection $(P \longrightarrow T1)$ or as their associated monad P. In other work we refer to these as "$(\mathcal{E}, T)$-operads".

6. We will write n-Pd for the set of n-dimensional globular pasting diagrams (that is, the n-cells of the free strict ω-category on the terminal globular set), and Pd for the set of all globular pasting diagrams.

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1. The main theorem

In this section we will prove the main theorem, exhibiting a universal T-operad acting on a given functor U. There is a slight subtlety involved to ensure that our framework can support all the examples we have in mind, as listed in the introduction. We will need to fix some suitable categories and functors:

- a category $\mathcal{S}$ with initial object; in our examples this will be Top or Top_*;
- $\mathcal{G}$ a cartesian category, for example GSet, n-GSet or Top;
- a functor $U\colon \mathcal{S} \longrightarrow \mathcal{G}$, for example the loop space functor or ω-path space functor;
- a cartesian monad T on $\mathcal{G}$, typically the free ω-category monad or n-dimensional version.

Finally we require that in the slice category $[\mathcal{S}, \mathcal{G}]/\Delta_{T1}$ a particular internal hom defining the endomorphism operad on U exists, where Δ_{T1} denotes the constant functor at $T1$. Some of the results are stated by Leinster for $\mathcal{G}$ a presheaf category, or slightly weaker, for $[\mathcal{S}, \mathcal{G}]$ locally cartesian closed. (Recall that a category $\mathcal{C}$ is called *cartesian* if it has all pullbacks, and *locally cartesian closed* if for all $X \in \mathcal{C}$ the slice category $\mathcal{C}/X$ is cartesian closed.) In fact, either of these requirements is excessive for us, both in the sense of being, abstractly, not necessary and in the sense of excluding the examples we have in mind. We do not need every slice to be cartesian closed, only the one above; in fact we don't even need this slice to be cartesian closed as we are only interested in one particular internal hom, the one defining the endomorphism operad on U. In our examples some care is needed about internal homs because of size issues; this technical issue is resolved in Section 2, and requires the following further conditions:

- The monad $T\colon \mathcal{G} \longrightarrow \mathcal{G}$ is not only cartesian but familially representable (as happens when T is polynomial).
- The functor $U\colon \mathcal{S} \longrightarrow \mathcal{G}$ is a right adjoint from a cocomplete category $\mathcal{S}$ to a presheaf category $\mathcal{G}$, i.e., to a free cocompletion of a small category.

When appropriate, these assumptions are adjusted to fit within an enriched category context; in fact our examples will only be enriched in Set or Top, which helps the technical details go through without many changes.

Our range of examples is summed up in Table 1; for the full definitions see Sections 3, 4.2 and 4.3 for loop spaces, ω-path spaces, and finite-dimensional cases respectively.

1.1 Background

We begin with some background theory which is found in [12], but is simplified here by the fact that we are only considering operads, whereas Leinster provides the more general theory for multicategories.

The content of the following definitions is that, given a cartesian monad S on a cartesian category $\mathcal{E}$, there is a monoidal structure on the category $\mathcal{E}/S1$ of "S-collections", and an S-operad is a monoid in this monoidal category. Moreover, under, suitable conditions $\mathcal{E}$ is enriched in $\mathcal{E}/S1$ and tensored over it, enabling us to define endomorphism operads and use them to express algebra actions.

Table 1: Examples

	U	$\mathcal{S}$	$\mathcal{G}$	T
loop spaces	Ω	Top_*	Top	free Top-monoid
ω-path spaces	U	Top	GSet	free ω-category
n-truncated path spaces	U_n	Top	n-Gph	free n-category
n-homotopy path spaces	Π_n	Top	n-Gph	free n-category
n-topological path spaces	$\mathcal{P}_n$	Top	Top-n-Gph	free Top-n-category

In all that follows, $\mathcal{E}$ is a cartesian category and S is a cartesian monad on it, that is, the functor part preserves pullbacks and the naturality squares for the unit and multiplication are pullbacks. In our examples $\mathcal{E}$ will either be $\mathcal{G}$ or $[\mathcal{S}, \mathcal{G}]$, and S will be the monad T on $\mathcal{G}$ or the induced monad T_* on $[\mathcal{S}, \mathcal{G}]$ respectively.

Definition 1.1. The category of *S-collections* is the slice category $\mathcal{E}/S1$. There is a monoidal structure on $\mathcal{E}/S1$ given as follows. $\begin{array}{c} A \\ \big\downarrow{\scriptstyle p} \\ S1 \end{array} \otimes \begin{array}{c} B \\ \big\downarrow{\scriptstyle q} \\ S1 \end{array}$ is the left-hand edge of the diagram:

$$\begin{tikzcd}
 & & & \cdot \arrow[dl] \arrow[dr] \arrow[dd, phantom, very near start, "\lrcorner" rotate=-45] & \\
 & & SA \arrow[dl, "Sp"'] \arrow[dr, "S!"] & & B \arrow[dl, "Sq"] \\
 & S^2 1 \arrow[dl, "\mu_1"'] & & S1 & \\
S1 & & & &
\end{tikzcd}$$

Note that we will sometimes write a collection $(P \longrightarrow S1)$ simply as P to simplify the notation. We will sometimes write the tensor product as $(A \otimes B \longrightarrow S1)$ if there is no danger of ambiguity.

Definition 1.2. An *S-operad* is a monoid in the monoidal category $\mathcal{E}/S1$. A morphism of S-operads is a monoid map. S-operads and their morphisms form a category S-Opd.

The following hom and tensor structures are given by the special case $E = 1$ of [12, Prop. 6.4.1].

Definition 1.3. Let S be a cartesian monad on a cartesian category $\mathcal{E}$, where $\mathcal{E}/S1$ is cartesian closed.

1. Given $\begin{array}{c} P \\ \downarrow \\ S1 \end{array} \in \mathcal{E}/S1$ and $A \in \mathcal{E}$ we define $\begin{array}{c} P \\ \downarrow \\ S1 \end{array} \odot A \in \mathcal{E}$ as the vertex of the following pullback:
$$\begin{tikzcd} & \cdot \arrow[dl] \arrow[dr] \arrow[dd, phantom, "\lrcorner" rotate=-45, very near start] & \\ P \arrow[dr] & & SA \arrow[dl, "S!"] \\ & S1 & \end{tikzcd}$$
This assignation on objects extends to a functor $\mathcal{E}/T1 \times \mathcal{E} \longrightarrow \mathcal{E}$.

2. Given $A, B \in \mathcal{E}$ we define
$$\mathsf{Hom}(A,B) = \left[\begin{array}{c} SA \\ \downarrow{\scriptstyle S!} \\ S1 \end{array} , \begin{array}{c} S1 \times B \\ \downarrow{\scriptstyle \pi_1} \\ S1 \end{array} \right] \in \mathcal{E}/S1$$
where the square brackets denote the exponential in $\mathcal{E}/S1$ where it exists, and π_1 denotes projection onto the first component. This assignation on objects extends to a functor $\mathcal{E}^{\mathrm{op}} \times \mathcal{E} \longrightarrow \mathcal{E}/T1$.

Proposition 1.4. *(Leinster) There is an isomorphism*
$$\mathcal{E}\left(\begin{array}{c} P \\ \downarrow \\ S1 \end{array} \odot A\,,\ B \right) \ \cong\ \mathcal{E}/S1 \left(\begin{array}{c} P \\ \downarrow \\ S1 \end{array},\ \mathsf{Hom}(A,B) \right)$$
natural in $\begin{array}{c} P \\ \downarrow \\ S1 \end{array}$, *$A$ and B.*

Remark 1.5. Leinster demands that $\mathcal{E}$ be locally cartesian closed but we see that this is excessive for our construction—it does not hold in our examples, but we only need one particular hom in one particular slice to exist.

Leinster uses this result to define the notion of *endomorphism S-operad*; again he does this for S-multicategories, but we only need the operad case.

Proposition 1.6. [12, Proposition 6.4.2] *Let S be a cartesian monad on a cartesian category $\mathcal{E}$. Then given any object $A \in \mathcal{E}$, if the S-collection*

$$\mathsf{End}\,(A) = \mathsf{Hom}\,(A, A)$$

exists then it naturally has the structure of an S-operad.

Definition 1.7. We call End (A) the *endomorphism operad* of A.

Definition 1.8. An *algebra* for an S-operad P is given by an object $A \in \mathcal{E}$ together with a map

$$\begin{array}{c} P \\ \downarrow \\ S1 \end{array} \odot A \longrightarrow A$$

compatible with the operad structure of P; equivalently it is an algebra for the associated monad.

We can equivalently express this using the endomorphism operad.

Proposition 1.9 (Leinster)**.** *let P be an S-operad. If the endomorphism operad on an object $A \in \mathcal{E}$ exists then a P-algebra structure on A is an operad map*

$$\begin{array}{c} P \\ \downarrow \\ S1 \end{array} \longrightarrow \mathsf{End}\,(A).$$

This duality will play a key part in our proof of the main theorem.

1.2 Operad actions

We now seek to abstract the notion of an operad acting on loop spaces, path spaces, or ω-path spaces.

Definition 1.10. Let T be a cartesian monad on a cartesian category $\mathcal{G}$, let P be a T-operad, and let $U \colon \mathcal{S} \longrightarrow \mathcal{G}$ be a functor. An *action of P on U-objects* is given by, for all $X \in \mathcal{S}$ a morphism

$$\alpha_X \colon \begin{array}{c} P \\ \downarrow \\ T1 \end{array} \odot\ UX \longrightarrow UX$$

in $\mathcal{G}$ such that

1. operad compatibility: for all $X \in \mathcal{S}$, α_X exhibits UX as a P-algebra, and
2. naturality: the α_X are the components of a natural transformation

$$\begin{array}{c} P \\ \downarrow \\ T1 \end{array} \odot U(-) \longrightarrow U(-).$$

The aim of this work is to show that under the right hypotheses there is a universal T-operad E_U with such an action, characterised by the following universal property: an action of a T-operad P on U-objects is uniquely and completely determined by a T-operad map $P \longrightarrow E_U$.

The operad E_U will be defined as $\mathsf{ev}_\emptyset(\mathsf{End}\,(U))$ and the universality result holds whenever this definition makes sense. The next few results will build up towards making sense of this formula. Here $\emptyset$ is the initial object of $\mathcal{S}$ if it has one. Now we put $\mathcal{E} = [\mathcal{S}, \mathcal{G}]$ which is cartesian if $\mathcal{G}$ is cartesian, with pullbacks computed pointwise. We use the following monad on $\mathcal{E}$, induced by composition with T.

Lemma 1.11. *Given a cartesian monad T on a cartesian category $\mathcal{G}$ we have a cartesian monad $T_* = T \circ -$ on $[\mathcal{S}, \mathcal{G}]$. Explicitly, given $A\colon \mathcal{S} \longrightarrow \mathcal{G}$ we have*

$$\begin{array}{rccc} T_*A : & \mathcal{S} & \longrightarrow & \mathcal{G} \\ & X & \longmapsto & (TA)X \end{array}$$

Proof. The multiplication and unit for T_* are constructed from those of T and the naturality squares are pullbacks since pullbacks in the functor category are computed pointwise; the pointwise squares are all naturality squares for the multplication and unit of T hence are themselves pullbacks.

Preservation of pullbacks also comes from the fact that pullbacks in the functor category are computed pointwise; the pointwise squares we need to check are all pullbacks in $\mathcal{G}$ with T applied, so are pullbacks since T preserves pullbacks.

$\square$

In our proof of the main theorem we are going to move back and forth between T-operads and T_*-operads using "constant" and "evaluation" functors. Before we define these, it is useful to make a few observations about the structure of the slice category $[\mathcal{S}, \mathcal{G}]/T_*1$.

Remarks 1.12.

1. Given $\begin{array}{c} A \\ \downarrow{\scriptstyle p} \\ T_*1 \end{array}, \begin{array}{c} B \\ \downarrow{\scriptstyle q} \\ T_*1 \end{array} \in [\mathcal{S}, \mathcal{G}]/T_*1$ their tensor product is given by a certain pullback in $[\mathcal{S}, \mathcal{G}]$. This is computed pointwise, and its component at $X \in \mathcal{S}$ is the left-hand edge of

$$\begin{tikzcd}
 & & & \cdot \arrow[dl] \arrow[dr] & \\
 & & TAX \arrow[dl, "Tp_X"'] \arrow[dr, "T!"] & & BX \arrow[dl, "q_X"] \\
 & T^2 1 \arrow[dl, "\mu_1"'] & & T1 & \\
 T1 & & & &
\end{tikzcd}$$

which we see is the collection $\begin{array}{c} AX \\ \downarrow{\scriptstyle p_X} \\ T1 \end{array} \otimes \begin{array}{c} BX \\ \downarrow{\scriptstyle q_X} \\ T1 \end{array}$.

2. The tensor product $\begin{array}{c} A \\ \downarrow{\scriptstyle p} \\ T_*1 \end{array} \odot B$ is also given by a pullback; this time the component at $X \in \mathcal{S}$ is the vertex of the pullback

$$\begin{tikzcd}
 & \cdot \arrow[dl] \arrow[dr] & \\
 AX \arrow[dr, "p_X"'] & & TBX \arrow[dl, "T!"] \\
 & T1 &
\end{tikzcd}$$

which we see is the tensor product $\begin{array}{c} AX \\ \downarrow{\scriptstyle p_X} \\ T1 \end{array} \odot BX$.

We now define the "constant" and "evaluation" functors.

Lemma 1.13. *Let T be a cartesian monad on a cartesian category $\mathcal{G}$. We have a "constant" functor acting as follows:*

$$\begin{array}{ccc} T\text{-}\mathsf{Opd} & \longrightarrow & T_*\text{-}\mathsf{Opd} \\ P & & \Delta_P \\ \downarrow & \longmapsto & \downarrow \\ T1 & & T_*1 \end{array}$$

where $\Delta_P : \mathcal{S} \longrightarrow \mathcal{G}$ is the constant functor evaluating everywhere at P.

Proof. Given a T-operad P we also write its associated monad (on $\mathcal{G}$) as P. Suppose the operad P is given by the cartesian natural transformation

$$\alpha : P \Longrightarrow T.$$

Then we also have a cartesian natural transformation

$$\alpha_* : P_* \Longrightarrow T_*$$

i.e. a T_*-operad. The components of α_* are given by, for each $A \in [\mathcal{S}, \mathcal{G}]$ the natural transformation

$$\alpha A : PA \Longrightarrow TA.$$

It is easy to check that the naturality squares are pullbacks by examining them pointwise. □

Remark 1.14. It is useful to note that the operad P_* has underlying T_*-collection $(P_*1 \longrightarrow T_*1)$. Here 1 is the terminal object in $[\mathcal{S}, \mathcal{G}]$, so it is the constant functor that sends every object to 1 (the terminal object in $\mathcal{G}$) and every morphism to the identity. Thus, evaluated at X the above collection is just $(P1 \longrightarrow T1)$, that is, the underlying collection of the operad P.

Thus the operad P_* can be thought of as Δ_P, the "constant operad" in $[\mathcal{S}, \mathcal{G}]$ that evaluates everywhere as P. We will sometimes write it in this way, and will often think of it in this way. Similarly it is useful to note that the functor T_*1 is the constant functor Δ_{T1}.

The following corollary is barely more than a matter of notation, but is useful for the proof of the main theorem.

Corollary 1.15. *A natural transformation*

$$\begin{array}{c} P \\ \downarrow \\ T1 \end{array} \odot U(-) \longrightarrow U(-)$$

is precisely a natural transformation

$$\begin{array}{c} P_* \\ \downarrow \\ T_*1 \end{array} \odot U \longrightarrow U.$$

We now define the "evaluation" functor; this is a straightforward generalisation of the evaluation functor used by Leinster [12, Section 9.2].

Definition 1.16. For any $X \in \mathcal{S}$ we have a functor

$$\mathsf{ev}_X : T_*\text{-}\mathsf{Opd} \longrightarrow T\text{-}\mathsf{Opd}$$

which we define in steps as follows.

1. We have a functor
$$\begin{array}{rccc} \mathsf{ev}_X : & [\mathcal{S}, \mathcal{G}] & \longrightarrow & \mathcal{G} \\ & F & \longmapsto & FX \end{array}$$

2. We know $\mathsf{ev}_X(T_*1) = T1$ for all X, so the above functor ev_X extends to a functor
$$\begin{array}{rccc} \mathsf{ev}_X : & [\mathcal{S}, \mathcal{G}]/T_*1 & \longrightarrow & \mathcal{G}/T1 \\ & \begin{array}{c} A \\ \downarrow \\ T_*1 \end{array} & \longmapsto & \begin{array}{c} AX \\ \downarrow \\ T1 \end{array} \end{array}$$

3. It is straightforward to check that this functor is monoidal (this is essentially the content of the first of Remarks 1.12) hence it maps operads to operads. That is, we get a functor
$$\mathsf{ev}_X : T_*\text{-}\mathsf{Opd} \longrightarrow T\text{-}\mathsf{Opd}.$$

We now define our putative universal operad, which is again a generalisation of the one used by Leinster in [12, Section 9.2]; however note that it will take a considerable amount of technical work to show that it exists in the cases of interest.

Definition 1.17. Let T be a cartesian monad on a cartesian category $\mathcal{G}$, $\mathcal{S}$ a category with an initial object $\emptyset$, and U a functor $\mathcal{S} \longrightarrow \mathcal{G}$. If the internal hom

$$\mathsf{Hom}\,(U, U) = \begin{bmatrix} T_*U & & T_*1 \times U \\ \downarrow & , & \downarrow \\ T_*1 & & T_*1 \end{bmatrix}$$

exists in $[\mathcal{S}, \mathcal{G}]/T_*1$ then this gives the endomorphism T_*-operad $\mathsf{End}\,(U)$. We define the *universal T-operad acting on U-objects* to be the T-operad

$$E_U = \mathsf{ev}_\emptyset(\mathsf{End}\,(U)).$$

Note that evaluating the collection $\mathsf{Hom}\,(U, U)$ at $\emptyset$ gives us the underlying collection of E_U, and the operad structure is inherited.

The internal hom certainly exists if $[\mathcal{S}, \mathcal{G}]$ is locally cartesian closed, or, more specifically, if the slice $[\mathcal{S}, \mathcal{G}]/T_*1$ is cartesian closed. However, in some of our key examples this demand is too stringent, mostly for size reasons; we will address this in Section 2.

The rest of the section will be devoted to identifying the universal property of this operad; the universal property is not studied by Leinster. The next three lemmas show how we can use the initial object of $\mathcal{S}$ to simplify all our calculations. For the rest of this section, T is a cartesian monad on a cartesian category $\mathcal{G}$, P is a T-operad, and $\mathcal{S}$ is a category with an initial object $\emptyset$.

Lemma 1.18. *Let $\mathcal{S}$ have an initial object $\emptyset$ and let $\Delta_V : \mathcal{S} \longrightarrow \mathcal{G}$ denote the constant functor evaluating at $V \in \mathcal{G}$. Then a natural transformation*

$$\alpha \colon \Delta_V \Longrightarrow F$$

is completely determined by its component at $\emptyset$, which has the form

$$\alpha_\emptyset : V \longrightarrow F\emptyset.$$

Proof. Simple diagram chase: the component α_X is determined by the naturality square

$$\begin{tikzcd} V \arrow[r, "\alpha_\emptyset"] \arrow[d, equal] & F\emptyset \arrow[d, "F!"] \\ V \arrow[r, "\alpha_X"'] & FX. \end{tikzcd}$$

□

Lemma 1.19. *With notation as above, a map of T_*-collections*

$$\begin{tikzcd} \Delta_V \arrow[rr, "\alpha"] \arrow[dr] & & F \arrow[dl] \\ & T_*1 & \end{tikzcd}$$

is completely determined by a map of T-collections

$$\begin{tikzcd} V \arrow[rr, "\alpha_\emptyset"] \arrow[dr] & & F\emptyset \arrow[dl] \\ & T1. & \end{tikzcd}$$

Proof. *A priori* a map α of T_*-collections as shown is a natural transformation $\alpha\colon \Delta_V \Longrightarrow F$ making the triangle commute. By Lemma 1.18 the natural transformation α is completely determined by its component at $\emptyset$; it remains to check that the commutativity of the triangle at $\emptyset$ ensures the commutativity of every triangle

$$\begin{tikzcd} V \arrow[rr, "\alpha_X"] \arrow[dr] & & FX \arrow[dl] \\ & T1 & \end{tikzcd}$$

which is accomplished by a simple diagram chase. □

Lemma 1.20. *Let E be a T_*-operad. A map*

$$\begin{tikzcd} P_* \arrow[rr, "\beta"] \arrow[dr] & & E \arrow[dl] \\ & T_*1 & \end{tikzcd}$$

is a map of T_*-operads if and only if applying $\mathsf{ev}_\emptyset$ gives a map of T-operads

$$\begin{tikzcd} P \arrow[rr, "\beta_\emptyset"] \arrow[dr] & & \mathsf{ev}_\emptyset(E) \arrow[dl] \\ & T1. & \end{tikzcd}$$

Proof. We have to check that β respects the monoid structure if $\beta_\emptyset$ does. For multiplication, we check that the following diagram commutes.

$$\begin{tikzcd}
\begin{pmatrix} P_* & & P_* \\ \downarrow & \otimes & \downarrow \\ T_*1 & & T_*1 \end{pmatrix} \arrow[r, "\beta \otimes \beta"] \arrow[d, "\mu"'] & \begin{pmatrix} E & & E \\ \downarrow & \otimes & \downarrow \\ T_*1 & & T_*1 \end{pmatrix} \arrow[d, "\mu"] \\
\begin{pmatrix} P_* \\ \downarrow \\ T_*1 \end{pmatrix} \arrow[r, "\beta"'] & \begin{pmatrix} E \\ \downarrow \\ T_*1 \end{pmatrix}
\end{tikzcd}$$

In principle we have to check that this commutes at every $X \in \mathcal{S}$, which amounts to checking that the following diagram commutes (where by a slight abuse of notation we only write the "variable" part of each collection):

$$\begin{tikzcd}
P \otimes P \arrow[r, "\beta_X \otimes \beta_X"] \arrow[d, "\mu^P"'] & EX \otimes EX \arrow[d, "\mu^E_X"] \\
P \arrow[r, "\beta_X"'] & EX.
\end{tikzcd}$$

However, we see that commutativity of this diagram at any X follows from commutativity at $\emptyset$ using the following diagram:

$$\begin{tikzcd}
P \otimes P \arrow[rr, "\beta_X \otimes \beta_X"] \arrow[d, "\mu^P"'] \arrow[ddr, "\beta_\emptyset \otimes \beta_\emptyset"] & & EX \otimes EX \arrow[d, "\mu^E_X"] \\
P \arrow[rr, "\beta_X"] \arrow[ddr, "\beta_\emptyset"'] & & EX \\
& E\emptyset \otimes E\emptyset \arrow[uur, "E! \otimes E!"'] \arrow[d, "\mu^E_\emptyset"] & \\
& E\emptyset \arrow[uur, "E!"'] &
\end{tikzcd}$$

The top and bottom triangles are naturality "squares" for β (which are triangles as the source functor of β is constant) and the right hand square is a naturality square for μ. The diagrams for the unit work similarly. $\square$

1.3 The universal property

We are finally ready to prove the main theorem.

Theorem 1.21 (Main theorem). *Let T be a cartesian monad on a cartesian category $\mathcal{G}$, $\mathcal{S}$ a category with an initial object $\emptyset$, and U a functor $\mathcal{S} \longrightarrow \mathcal{G}$. Suppose further that* $\mathsf{End}\,(U)$ *exists, so we can define*

$$E_U = \mathsf{ev}_\emptyset(\mathsf{End}\,(U)).$$

Let P be a T-operad. Then an action of P on U-objects is precisely a map of T-operads $P \longrightarrow E_U$.

Proof. We write $\mathcal{E} = [\mathcal{S}, \mathcal{G}]$. An action of P on U-objects is by definition a natural transformation

$$\alpha : \begin{array}{c} P_* \\ \downarrow \\ T_*1 \end{array} \odot U \longrightarrow U$$

such that for all X, the component

$$\alpha_X : \begin{array}{c} P \\ \downarrow \\ T1 \end{array} \odot UX \longrightarrow UX$$

exhibits UX as a P-algebra.

By the tensor structure (Proposition 1.4), specifying such an α amounts to specifying a morphism

$$\overline{\alpha} : \begin{array}{c} P_* \\ \downarrow \\ T_*1 \end{array} \longrightarrow \mathsf{Hom}\,(U, U)$$

in $\mathcal{E}/T_*1$ or, writing it out more fully,

$$\begin{tikzcd} P_* \arrow[rr, "\bar{\alpha}"] \arrow[dr] & & \mathsf{Hom}\,(U,U) \arrow[dl] \\ & T_*1. & \end{tikzcd}$$

Now $P_* = \Delta_P$ is a constant functor, so by Lemma 1.19 the natural transformation $\overline{\alpha}$ is completely determined by the component at the initial object $\emptyset$, that is, a map of T-collections

$$\begin{tikzcd} P \arrow[r, "\bar{\alpha}_\emptyset"] \arrow[dr] & \mathsf{ev}_\emptyset(\mathsf{Hom}\,(U,U)) \arrow[d] \\ & T1 \end{tikzcd}$$

thus our natural transformation α is completely determined by a map of underlying T-collections

$$\overline{\alpha}_\emptyset : \begin{array}{c} P \\ \downarrow \\ T1 \end{array} \longrightarrow \mathsf{ev}_\emptyset(\mathsf{End}\,(U)) = E_U.$$

It remains to show that this is a map of operads if and only if for all $X \in \mathcal{S}$, α_X exhibits UX as a P-algebra. We proceed in steps, by proving that the following are equivalent.

1. For all $X \in \mathcal{S}$, $\alpha_X : \begin{array}{c} P \\ \downarrow \\ T1 \end{array} \odot UX \longrightarrow UX$ exhibits UX as a P-algebra.

2. $\alpha : \begin{array}{c} P_* \\ \downarrow \\ T_*1 \end{array} \odot U \longrightarrow U$ exhibits U as a P_*-algebra.

3. $\overline{\alpha} : \begin{array}{c} P_* \\ \downarrow \\ T_*1 \end{array} \longrightarrow \mathsf{End}\,(U)$ is an operad map.

4. $\overline{\alpha}_\emptyset : \begin{array}{c} P \\ \downarrow \\ T1 \end{array} \longrightarrow \mathsf{ev}_\emptyset(\mathsf{End}\,(U)) = E_U$ is an operad map.

- 1 $\iff$ 2 is Corollary 1.15.
- 2 $\iff$ 3 is Proposition 1.9.
- 3 $\iff$ 4 is Lemma 1.20.

□

Remark 1.22. In this proof it is tempting to use the fact that (1) is equivalent to the assertion that for all $X \in \mathcal{S}$,

$$\overline{\alpha_X} : \begin{array}{c} P \\ \downarrow \\ T1 \end{array} \longrightarrow \mathsf{End}\,(UX)$$

is an operad map; this is true but not helpful at this point, as

$$\mathsf{ev}_X(\mathsf{End}\,(U)) \neq \mathsf{End}\,(UX).$$

1.4 Topological version

The main theorem also holds in a topologically enriched version which we will use for several of our examples. The theorem is essentially the same, provided that the categories and functors involved are interpreted in an enriched sense.

As usual Top will denote a category of topological spaces that is complete, cocomplete and cartesian closed, for example the category of compactly generated weakly Hausdorff spaces. We will be considering $\mathcal{V}$-enriched categories where $\mathcal{V} = \mathsf{Top}$.

The starting point is that we now want the categories $\mathcal{S}$ and $\mathcal{G}$ to be $\mathcal{V}$-categories rather than just plain categories. In our examples, $\mathcal{S}$ will be Top or Top_* and $\mathcal{G}$ will be Top or $[\mathbb{G}^{\text{op}}, \mathsf{Top}]$. We also want the monad T on $\mathcal{G}$ to be enriched: as a functor, T is $\mathcal{V}$-enriched, and the multiplication and unit structures on T should be $\mathcal{V}$-natural transformations. And likewise, the requirement of familial representability of T now means that the functor T is to be an enriched coproduct of (enriched) representable functors.

In the case $\mathcal{V} = \mathsf{Top}$, such adjustments tend to be mild. For example, because the forgetful functor $\mathsf{hom}(1, -) \colon \mathsf{Top} \longrightarrow \mathsf{Set}$ is faithful, $\mathcal{V}$-natural transformations are, in this case, the same as ordinary natural transformations. And in this case, enrichment of functors is a property-like structure: an ordinary functor between the underlying categories of Top-categories, either is or isn't Top-enriched. That is, for a functor $F \colon \mathcal{C} \longrightarrow \mathcal{D}$ the functions $\mathcal{C}(X, Y) \longrightarrow \mathcal{D}(FX, FY)$ giving the action on morphisms either are or are not continuous, and that is the criterion for F to be enriched.

There is an *a priori* distinction between enriched (co)limits and ordinary (co)limits, but under mild assumptions on $\mathcal{S}$ and $\mathcal{G}$ (e.g., they are $\mathcal{V}$-tensored and cotensored; see below), the distinction is one we can ignore entirely. To set the context, we introduce some notation:

- If $\mathcal{C}$ is a $\mathcal{V}$-enriched category, write $\mathcal{C}_0$ for the underlying (plain) category of $\mathcal{C}$, obtained by applying $\mathcal{V}(1, -) : \mathcal{V} \longrightarrow \mathsf{Set}$ to the homs of $\mathcal{C}$ as objects in $\mathcal{V}$, with 1 the monoidal unit of $\mathcal{V}$.

- If D is an ordinary category, we write $\overline{D}$ for the free $\mathcal{V}$-enriched category generated by D. The objects of $\overline{D}$ are those of D, and the enriched homs of D as objects in $\mathcal{V}$ are defined by the formula

  $$D(d, d') = \coprod_{f \colon d \longrightarrow d'} 1.$$

  In the case $\mathcal{V} = \mathsf{Top}$, it simply means we interpret the hom-sets of D as discrete topological spaces, and eventually by abuse of notation we will write the resulting $\mathcal{V}$-category just as D.

- If $\mathcal{A}$ and $\mathcal{B}$ are $\mathcal{V}$-categories, we write $[\mathcal{A}, \mathcal{B}]$ for the (plain) category of $\mathcal{V}$-functors and $\mathcal{V}$-transformations. We write $\mathsf{Cat}(D, E)$ for the category of functors and transformations between plain categories D, E. The freeness property of $\overline{D}$ then asserts

  $$[\overline{D}, \mathcal{C}] \cong \mathsf{Cat}(D, \mathcal{C}_0)$$

  for any $\mathcal{V}$-category $\mathcal{C}$.

- If in addition $\mathbb{A}$ is small, there is a canonical way of endowing $[\mathbb{A}, \mathcal{B}]$ with enriched structure, and we write $\mathcal{B}^{\mathbb{A}}$ for this $\mathcal{V}$-category.

Under this notation, for any diagram category D we intend to take (co)limits over, and for any enriched category $\mathcal{C}$, we have a $\mathcal{V}$-enriched $\mathcal{V}$-functor category

$$[\overline{D}, \mathcal{C}]$$

and we have a $\mathcal{V}$-functor $\overline{D} \longrightarrow \overline{1}$, which induces a " diagonal" $\mathcal{V}$-functor

$$\Delta_D = \left(\mathcal{C} \cong [\overline{1}, \mathcal{C}] \longrightarrow [\overline{D}, \mathcal{C}] \right).$$

If Δ_D has an enriched right adjoint $\mathcal{V}\text{-}\lim_D$ (a right adjoint in the 2-category $\mathcal{V}$-Cat), then we say that $\mathcal{C}$ has enriched D-limits. Or, if Δ_D has an enriched left adjoint, then we say that $\mathcal{C}$ has enriched D-colimits.

Enriched D-limits in $\mathcal{C}$, assuming they exist, *are* ordinary D-limits when viewed in $\mathcal{C}_0$. That is to say, if there is an enriched right adjoint

$$\mathcal{V}\text{-}\lim_D \colon [\overline{D}, \mathcal{C}] \longrightarrow \mathcal{C}$$

then applying the forgetful 2-functor $\mathcal{V}$-$\mathsf{Cat} \longrightarrow \mathsf{Cat}$ (which preserves adjunctions as all 2-functors do) we get an ordinary right adjoint in Cat,

$$[\overline{D}, \mathcal{C}]_0 \longrightarrow C_0 \quad \cong \quad \mathsf{Cat}(D, \mathcal{C}_0) \longrightarrow C_0$$

that is right adjoint to an ordinary diagonal functor.

The distinction between enriched D-limits in $\mathcal{C}$ and ordinary D-limits in $\mathcal{C}_0$ is that the former might not exist even if the latter do. The distinction disappears if we know in advance that $\mathcal{C}$ is $\mathcal{V}$-complete, and indeed in our examples $\mathcal{S}$ and $\mathcal{G}$ will be $\mathcal{V}$-complete and $\mathcal{V}$-cocomplete, for straightforward reasons.

Alternatively, if $\mathcal{C}$ is $\mathcal{V}$-tensored, then $\mathcal{C}$ will have enriched D-limits if it has ordinary D-limits [10, p.50]. And if $\mathcal{C}$ is $\mathcal{V}$-cotensored, then $\mathcal{C}$ will have enriched D-colimits if it has ordinary D-colimits. Again, existence of tensors and cotensors is straightforwardly observed in our examples of $\mathcal{S}$ and $\mathcal{G}$.

Under some such niceness assumptions on $\mathcal{S}$ and $\mathcal{G}$ that allow us to forget about distinguishing between enriched and ordinary (co)limits, we may say:

1. Pullbacks in $\mathcal{G}$ are simply pullbacks in $\mathcal{G}_0$.

2. Pullbacks in $[\mathcal{S}, \mathcal{G}]$ are computed pointwise, as in $\mathsf{Cat}(\mathcal{S}_0, \mathcal{G}_0)$.

3. We can consider T as a cartesian monad on $\mathcal{G}_0$ and consider a T-operad P; then the underlying cartesian functor $P\colon \mathcal{G}_0 \longrightarrow \mathcal{G}_0$ turns out to be a $\mathcal{V}$-functor $\mathcal{G} \longrightarrow \mathcal{G}$. This follows from the above, as the action of P can be defined entirely from $P1$ using pullbacks.

This last point means that in effect we do not have to change our definition of T-operads and their maps, but that their underlying functors will all turn out to be $\mathcal{V}$-enriched "for free". This in turn means that the evaluation functor

$$\mathsf{ev}_X \colon T_*\text{-}\mathsf{Opd} \longrightarrow T\text{-}\mathsf{Opd}$$

still makes sense even though at the level of functors the evaluation functor

$$\mathsf{ev}_X \colon [\mathcal{S}, \mathcal{G}] \longrightarrow \mathcal{G}_0$$

necessarily now has only the underlying category $\mathcal{G}_0$ as its codomain.

We can now re-state the main theorem as follows.

Theorem 1.23 (Main theorem, topological version). *Let $\mathcal{S}$ and $\mathcal{G}$ be* Top*-categories, where $\mathcal{S}$ has an initial object $\emptyset$. Let T be a monad on $\mathcal{G}$ that is cartesian in the* Top*-enriched sense, and let U be a* Top*-functor $\mathcal{S} \longrightarrow \mathcal{G}$. Suppose further that* $\mathsf{End}\,(U)$ *exists, so we can define*

$$E_U = \mathsf{ev}_\emptyset(\mathsf{End}\,(U)).$$

Let P be a T-operad. Then an action of P on U-objects is precisely a map of T-operads $P \longrightarrow E_U$.

Our main theorem gives a universal operad acting on U-objects whenever $\mathsf{End}\,(U)$ exists; in the next section we will show that it does exist in the cases we're interested in. This existence typically involves some fairly technical considerations.

2. Internal hom constructions

In this section we will address some size issues that arise when we construct the endomorphism operad $\mathsf{End}\,(U)$ in practice. These issues arise on account of us seeking an internal hom in the category $[\mathcal{S}, \mathcal{G}]/T_*1$ where the category $\mathcal{S}$ is not small. However, in our examples we are helped by some specific properties of the categories and functors in question, and we will now provide the technical results that make this work.

It may be worth elucidating this issue in the case of one of our motivating examples, $\mathcal{G} = [\mathbb{G}^{\text{op}}, \mathsf{Set}]$. In that case we could invoke equivalences

$$\big[\mathcal{S},\ [\mathbb{G}^{\text{op}}, \mathsf{Set}]\big]/T_*1 \ \simeq\ [\mathcal{S} \times \mathbb{G}^{\text{op}},\ \mathsf{Set}]/T_*1 \ \simeq\ \big[(\mathcal{S} \times \mathbb{G}^{\text{op}})/T_*1,\ \mathsf{Set}\big]$$

and then if $\mathcal{S}$ were small we could use the usual construction of an exponential in a presheaf category $[\mathbb{C}^{\mathrm{op}}, \mathsf{Set}]$:

$$(Y^X)(c) = [\mathbb{C}^{\mathrm{op}}, \mathsf{Set}]\big(\mathbb{C}(-, c) \times X\,,\, Y\big).$$

However, this functor evaluates each object c at a collection of natural transformations, and when the category we're taking presheaves over isn't small, this collection is not guaranteed to be a set. That is, if $\mathcal{S}$ is not small, this construction is not guaranted to produce an object in the required functor category.

In this section we are going to show that this does work in particular circumstances which cover all the examples we have in mind. The main technical result we will use is that we can define internal homs between right adjoints in an enriched functor category of the form

$$\big[\mathcal{S}, \mathcal{V}^{\mathbb{C}^{\mathrm{op}}}\big]$$

where $\mathbb{C}$ is small. We then "translate" from our slice category into one of this form via an equivalence

$$\big[\mathcal{S}\,,\, \mathcal{V}^{\mathbb{B}^{\mathrm{op}}}\big]/T_*1 \;\simeq\; \big[\mathcal{S}\,,\, \mathcal{V}^{(\mathbb{B}/T1)^{\mathrm{op}}}\big]$$

and the fact that the functors whose internal hom we are now taking are right adjoints will follow from U being a right adjoint and T being familially representable.

All of the material in this section is developed in the generality of enriched category theory, relying heavily on [10]. Throughout this section $\mathcal{V}$ will be a locally small, complete, cocomplete, cartesian closed category, or a "cartesian cosmos". In fact for our examples we only use the cases Set and Top, a convenient category of small topological spaces.

2.1 Preliminaries on enriched category theory

First we fix our terminology and notation for the enriched setting. Let $\mathbb{C}$ be a small $\mathcal{V}$-category. We write $\mathcal{V}^{\mathbb{C}^{\mathrm{op}}}$ for the $\mathcal{V}$-category of $\mathcal{V}$-presheaves on $\mathbb{C}$, with hom-objects given by the usual end formula:

$$\mathcal{V}^{\mathbb{C}^{\mathrm{op}}}(F, G) = \int_{c:\mathbb{C}} Gx^{Fc}.$$

Note that we use the notation $c : \mathbb{C}$ rather than the more common $c \in C$ in all our (co)end formulae. The exponential here denotes the internal hom in the cartesian closed category V; we might also write internal hom in the form $\mathcal{V}(Fc, Gc)$ when the exponential notation becomes arduous, as there is no ambiguity.

We may refer to the hom-object $\mathcal{V}^{\mathbb{C}^{\mathrm{op}}}(F, G)$ as the enriched hom, to distinguish it from the internal hom: $\mathcal{V}^{\mathbb{C}^{\mathrm{op}}}$ is cartesian closed, with internal hom given by the usual formula

$$\begin{aligned} G^F(c) &= \mathcal{V}^{\mathbb{C}^{\mathrm{op}}}\big(\mathbb{C}(-, c) \times F,\ G\big) \\ &= \int_{d:\mathbb{C}} \mathcal{V}\big(\mathbb{C}(d, c) \times Fd,\ Gd\big) \end{aligned}$$

Now, ultimately we are interested in internal homs in a (plain) category $[\mathcal{S}, \mathcal{V}^{\mathbb{C}^{\mathrm{op}}}]$ of $\mathcal{V}$-functors and $\mathcal{V}$-transformations. Given $\mathcal{V}$-functors $F, G \colon \mathcal{S} \longrightarrow \mathcal{V}^{\mathbb{C}^{\mathrm{op}}}$, we write their internal hom in $[\mathcal{S}, \mathcal{V}^{\mathbb{C}^{\mathrm{op}}}]$ as $[F, G]$. The following thought experiment may elucidate the situation. We could use the equivalence

$$[\mathcal{S}, \mathcal{V}^{\mathbb{C}^{\mathrm{op}}}] \simeq [\mathcal{S} \times \mathbb{C}, \mathcal{V}]$$

and attempt to define the internal hom $[F, G] : \mathcal{S} \times \mathbb{C} \longrightarrow \mathcal{V}$ by the end formula:

$$[F, G](s, c) = \int_{t:\mathcal{S}, d:\mathbb{C}} \mathcal{V}\big(\mathcal{S}(s, t) \times \mathbb{C}(d, c) \times F(t, d),\ G(t, d)\big) \tag{2.1}$$

In general this end might not exist as $\mathcal{S}$ is not small, but when it does exist it will be an internal hom as we will carefully verify later in the course of proving Theorem 2.4. This main technical result is to show that this end does exist under some mild conditions on the categories, when F and G are right adjoints. We proceed in two steps.

1. Give circumstances in which the enriched hom between right adjoints in $[\mathcal{S}, \mathcal{V}^{\mathbb{C}^{\mathrm{op}}}]$ exists.
2. Show that if F and G are right adjoints in $[\mathcal{S}, \mathcal{V}^{\mathbb{C}^{\mathrm{op}}}]$ then the above end is an instance of an enriched hom between (some other) right adjoints, and so the above end exists and gives the internal hom $[F, G]$.

We will then apply this result to the examples we're interested in.

2.2 Internal homs between right adjoints

Our main strategy is to express the end (2.1) as an enriched hom between right adjoints. The following lemma ensures that such an enriched hom will exist. In all that follows when we speak of a right adjoint $\mathcal{S} \longrightarrow \mathcal{V}^{\mathbb{C}^{\mathrm{op}}}$ we mean a right adjoint in the 2-category of $\mathcal{V}$-categories, $\mathcal{V}$-functors, and $\mathcal{V}$-transformations.

Lemma 2.1. *Suppose $\mathbb{C}$ is small, $\mathcal{S}$ is $\mathcal{V}$-cocomplete, and the functors*

$$\mathcal{S} \underset{G'}{\overset{G}{\rightrightarrows}} \mathcal{V}^{\mathbb{C}^{\mathrm{op}}}$$

have respective left adjoints F, F'. Write $\bar{F} = Fy_C$, the restriction of F along the Yoneda embedding $y_C \colon \mathbb{C} \longrightarrow \mathcal{V}^{\mathbb{C}^{\mathrm{op}}}$ and similarly $\bar{F}' = F'y_C$. Then the enriched hom $[\mathcal{S}, \mathcal{V}^{\mathbb{C}^{\mathrm{op}}}](G, G')$, as an end $\int_{x:\mathcal{S}} \mathcal{V}^{\mathbb{C}^{\mathrm{op}}}(Gx, G'x)$, exists and is given by

$$[\mathcal{S}, \mathcal{V}^{\mathbb{C}^{\mathrm{op}}}](G, G') \cong \mathcal{S}^{\mathbb{C}}(\bar{F}', \bar{F}).$$

Proof. By [10, Theorem 4.51], G and G' are given by

$$Gx \cong \mathcal{S}(\bar{F}-, x)$$

$$G'x \cong \mathcal{S}(\bar{F}'-, x)$$

We have

$$\begin{aligned}
\mathcal{S}^{\mathbb{C}}(\bar{F}', \bar{F}) &\cong \int_{c:\mathbb{C}} \mathcal{S}(\bar{F}'c, \bar{F}c) && \text{as } \mathbb{C} \text{ is small} \\
&\cong \int_{c:\mathbb{C}} \int_{x:\mathcal{S}} \mathcal{S}(\bar{F}'c, x)^{\mathcal{S}(\bar{F}c, x)} && \text{by enriched Yoneda, [10, 2.31]} \\
&\cong \int_{x:\mathcal{S}} \int_{c:\mathbb{C}} \mathcal{S}(\bar{F}'c, x)^{\mathcal{S}(\bar{F}c, x)} && \text{by the Fubini theorem, [10, 2.8]} \\
&\cong \int_{x:\mathcal{S}} V^{\mathbb{C}^{\mathrm{op}}}\big(\mathcal{S}(\bar{F}-, x), \mathcal{S}(\bar{F}'-, x)\big)
\end{aligned}$$

as required. $\square$

We now aim to define an internal hom $[F, G]\colon \mathcal{S} \longrightarrow \mathcal{V}^{\mathbb{C}^{\mathrm{op}}}$ between right adjoints. It is essentially given by the same formula (2.1) as would be expected if $\mathcal{S}$ were small, except we express it in such a way that the enriched hom we need to invoke is between right adjoints so that Lemma 2.1 ensures it exists. This may give the following constructions an air of over-complication, but the aim is to express something familiar as a composite of right adjoints, which takes a little manœuvring. In the following constructions we use the same hypotheses as in Lemma 2.1.

The following will be the first right adjoint in our enriched hom, derived from F. Given an object $s \in \mathcal{S}$ we write $\mathcal{S}(s, -) \cdot F$ for the following composite; here δ denotes the diagonal, Δ produces the constant functor, and Π denotes the functor taking products.

$$\begin{array}{ccccccccc} \mathcal{S} \xrightarrow{\delta} \mathcal{S}\times\mathcal{S} & \xrightarrow{\mathcal{S}(s,-)\times F} & \mathcal{V}\times\mathcal{V}^{\mathbb{C}^{\mathrm{op}}} & \xrightarrow{\Delta\times 1} & \mathcal{V}^{\mathbb{C}^{\mathrm{op}}}\times\mathcal{V}^{\mathbb{C}^{\mathrm{op}}} & \xrightarrow{\Pi} & \mathcal{V}^{\mathbb{C}^{\mathrm{op}}} \\ x \longmapsto (x, x) & \longmapsto & \big(\mathcal{S}(s, x),\ F(x)\big) & \longmapsto & \big(\Delta_{\mathcal{S}(s,x)},\ F(x)\big) & \longmapsto & \mathcal{S}(s, x)\times F(x) \end{array}$$

Here $\mathcal{S}(s, x) \times F(x)$ denotes the functor

$$\begin{array}{ccc} C^{\mathrm{op}} & \longrightarrow & V \\ c & \longmapsto & \mathcal{S}(s, x) \times F(x)(c) \end{array}$$

Proposition 2.2. *If F is a right adjoint in $\mathcal{V}$-Cat then so is the above composite.*

Proof. Let I denote the unit $\mathcal{V}$-category. There is an evident $\mathcal{V}$-category $I + I$, and there is a unique $\mathcal{V}$-functor $! : I + I \longrightarrow I$. The diagonal functor $\delta\colon \mathcal{S} \longrightarrow \mathcal{S} \times \mathcal{S}$ may be identified with the functor given by pre-composition with !, which we write as $\mathcal{S}^!$; this has an enriched left adjoint (which, at the underlying category level, is just the coproduct):

$$\begin{tikzcd} \mathcal{S}^I \arrow[r, shift left, "\mathcal{S}^!"] \arrow[r, phantom, "\top"] & \mathcal{S}^{I+I} \arrow[l, shift left, "+ = \mathsf{Lan}_!"] \end{tikzcd}$$

Similarly, we have a unique $\mathcal{V}$-functor $!_{\mathbb{C}}\colon \mathbb{C}^{\mathrm{op}} \longrightarrow I$, and Δ above may be identified with $\mathcal{V}^{!_{\mathbb{C}}}$. This too has an enriched left adjoint:

$$\begin{tikzcd} \mathcal{V}^I \arrow[r, shift left, "\mathcal{V}^{!_{\mathbb{C}}}"] \arrow[r, phantom, "\top"] & \mathcal{V}^{\mathbb{C}^{\mathrm{op}}} \arrow[l, shift left, "\mathsf{colim}_{\mathbb{C}} = \mathsf{Lan}_{!_{\mathbb{C}}}"] \end{tikzcd}$$

The second map has left adjoint $(- \otimes s) \times F'$ where $(- \otimes s) : V \longrightarrow S$ is tensoring with an object s (which is left adjoint to the representable $S(s, -) : S \longrightarrow V$), and F' is the left adjoint of F.

Finally, the enriched left adjoint of Π is the diagonal:

$$\begin{tikzcd} \mathcal{V}^{\mathbb{C}^{\mathrm{op}}} \times \mathcal{V}^{\mathbb{C}^{\mathrm{op}}} \arrow[r, shift left, "\Pi"] \arrow[r, phantom, "\top"] & \mathcal{V}^{\mathbb{C}^{\mathrm{op}}} \arrow[l, shift left, "\delta"] \end{tikzcd}$$

□

The following will be the second right adjoint in our enriched hom, derived from G. For every object $c \in \mathbb{C}$ we write $G^{\mathbb{C}(-,c)}$ for the following composite:

$$\begin{array}{ccccc} \mathcal{S} & \xrightarrow{G} & \mathcal{V}^{\mathbb{C}^{\mathrm{op}}} & \xrightarrow{(-)^{\mathbb{C}(-,c)}} & \mathcal{V}^{\mathbb{C}^{\mathrm{op}}} \\ x & \longmapsto & Gx & \longmapsto & Gx^{\mathbb{C}(-,c)} \end{array}$$

Here $Gx^{\mathbb{C}(-,c)}$ denotes the exponential (in the cartesian closed category $\mathcal{V}^{\mathbb{C}^{\mathrm{op}}}$) of the functors $G(x)$ and the representable $\mathbb{C}(-, c)$.

Proposition 2.3. *If G is a right adjoint in $\mathcal{V}$-Cat then so is the above composite.*

Proof. Recall that $\mathcal{V}^{\mathbb{C}^{\mathrm{op}}}$ is cartesian closed in the enriched sense, so that we have a $\mathcal{V}$-natural isomorphism

$$\mathcal{V}^{\mathbb{C}^{\mathrm{op}}}(X \times Y\,,\ Z) \;\cong\; \mathcal{V}^{\mathbb{C}^{\mathrm{op}}}(X\,,\ Z^Y)$$

between enriched functors valued in $\mathcal{V}$. In particular, the enriched functor $(-)^Y$ is an enriched right adjoint; we are using the case $Y = \mathbb{C}(-, c)$. □

We are now ready to prove the main technical theorem we need, constructing internal homs between right adjoints via the above composites.

Theorem 2.4. *Let $\mathcal{V}$ be a locally small, complete, cocomplete cartesian closed category, let $\mathcal{S}$ be a $\mathcal{V}$-cocomplete $\mathcal{V}$-category, and let $\mathbb{C}$ be a small $\mathcal{V}$-category. Let $F, G \colon \mathcal{S} \longrightarrow \mathcal{V}^{\mathbb{C}^{\mathrm{op}}}$ be two right adjoints. Then, in the category of $\mathcal{V}$-functors $[\mathcal{S}, \mathcal{V}^{\mathbb{C}^{\mathrm{op}}}]$ there is an internal hom $[F, G]$ constructed according to the formula*

$$[F, G](s)(c) \;=\; \big[\mathcal{S}, \mathcal{V}^{\mathbb{C}^{\mathrm{op}}}\big]\big(\mathcal{S}(s, -) \cdot F\ ,\ G^{\mathbb{C}(-,c)}\big).$$

Proof. First note that this enriched hom may be legitimately formed as an object of $\mathcal{V}$ by Lemma 2.1, since $\mathcal{S}(s,-)\cdot F$ and $G^{\mathbb{C}(-,c)}$ are right adjoints. Thus,

$$\begin{aligned}
[F,G](s)(c) &= \int_{t:\mathcal{S}} \mathcal{V}^{\mathbb{C}^{\mathrm{op}}}\big(\mathcal{S}(s,t)\times F(t)\,,\ G(t)^{\mathbb{C}(-,c)}\big)\\
&\cong \int_{t:\mathcal{S}} \mathcal{V}^{\mathbb{C}^{\mathrm{op}}}\big(\mathcal{S}(s,t)\times F(t)\times \mathbb{C}(-,c)\,,\ \ G(t)\big)\\
&\cong \int_{t:\mathcal{S}} \ \int_{d:\mathbb{C}} \mathcal{V}\big(\mathcal{S}(s,t)\times F(t)(d)\times \mathbb{C}(d,c)\,,\ \ G(t)(d)\big)
\end{aligned}$$

This is the formula expected as in (2.1).

For completeness we now show that $[F,G]$ satisfies the requisite universal property. Thus, suppose we are given a $\mathcal{V}$-functor $X\colon\ \mathcal{S}\longrightarrow\mathcal{V}^{\mathbb{C}^{\mathrm{op}}}$. We establish a natural bijection between the family of maps of the form $\psi\colon\ X\longrightarrow[F,G]$ and those of the form $\phi : X\times F\longrightarrow G$.

Let us equivalently regard F,G,X as V-functors $\mathcal{S}\times\mathbb{C}^{\mathrm{op}}\longrightarrow\mathcal{V}$, to avail ourselves of more pleasant notation such as $X(s,c)$. Using the definitions of end and coend, and instances of $\times$–hom adjunctions, we have a natural bijection between natural transformations $\psi\colon\ X\longrightarrow[F,G]$ and the following extranatural families of maps:

$$\psi(s,c) : X(s,c) \longrightarrow [F,G](s,c) = \int_{t:\mathcal{S}} \int_{d:\mathbb{C}} G(t,d)^{\mathcal{S}(s,t)\times F(t,d)\times \mathbb{C}(d,c)} \quad (1)$$

$$X(s,c) \longrightarrow G(t,d)^{\mathcal{S}(s,t)\times F(t,d)\times \mathbb{C}(d,c)} \quad (2)$$

$$\mathcal{S}(s,t) \times X(s,c) \times \mathbb{C}(d,c) \longrightarrow G(t,d)^{F(t,d)} \quad (3)$$

$$\int^{(s,c)\,:\,\mathcal{S}\times\mathbb{C}^{\text{op}}} \mathcal{S}(s,t) \times X(s,c) \times \mathbb{C}(d,c) \longrightarrow G(t,d)^{F(t,d)} \quad (4)$$

$$X(t,d) \longrightarrow G(t,d)^{F(t,d)} \quad (5)$$

$$X(t,d) \times F(t,d) \longrightarrow G(t,d) \quad (6)$$

$$(X \times F)(t,d) \longrightarrow G(t,d) \quad (7)$$

$$\phi : X \times F \longrightarrow G \quad (8)$$

where line (2) is achieved by definition of extranaturality and coends, line (3) via a $\times$–hom adjunction, (4) by definition of extranaturality and ends, (5) by Yoneda, (6) via a $\times$–hom adjunction, and (7) and (8) by definition. □

Remark 2.5. Note that in practice we will express the formula for the internal hom in the usual format in a functor category, once we know that the end in question exists, that is:

$$\begin{aligned}[F,G](s,c) &= \int_{t:\mathcal{S}} \int_{d:\mathbb{C}} G(t,d)^{\mathcal{S}(s,t)\times F(t,d)\times \mathbb{C}(d,c)} \\ &= [\mathcal{S}\times\mathbb{C}^{\text{op}}, \mathcal{V}](H^{(s,c)} \times F,\ G)\end{aligned}$$

where $H^{(s,c)}$ denotes the appropriate representable $\mathcal{S}(s,-) \times \mathbb{C}(-,c)$.

2.3 Application to the endomorphism operad

We now show how to apply Theorem 2.4 to show that $\mathsf{End}\,U$ may be formed in our cases of interest. Recall that in our examples:

- $\mathcal{V}$ is Set or Top.

- $\mathcal{S}$ is Top or Top_*.
- $\mathbb{B}$ is a small (plain) category which we may then regard as a Top-category in which all hom objects are discrete spaces. In our examples $\mathbb{B}$ is the globular category $\mathbb{G}$ or the finite version $\mathbb{G}_n$ (or indeed the terminal category $dir0o$.
- T is a familially representable $\mathcal{V}$-monad on $\mathcal{V}^{\mathbb{B}^{\mathrm{op}}}$ (so in particular is cartesian); we will come back to this definition shortly.
- T_* is the induced cartesian monad

$$\left[\mathcal{S},\ \mathcal{V}^{\mathbb{B}^{\mathrm{op}}}\right] \xrightarrow{T\circ -} \left[\mathcal{S},\ \mathcal{V}^{\mathbb{B}^{\mathrm{op}}}\right]$$

- U is a $\mathcal{V}$-functor $\mathcal{S} \longrightarrow \mathcal{V}^{\mathbb{B}^{\mathrm{op}}}$; all our examples of U are constructed via a functor $\mathbb{B} \xrightarrow{D} \mathcal{S}$ with

$$U(X) = \mathcal{S}(D-, X) \in \mathcal{V}^{\mathbb{B}^{\mathrm{op}}}$$

thus U is a right adjoint; its left adjoint is the left Kan extension of D along the Yoneda embedding

$$\begin{tikzcd} \mathbb{B} \arrow[r, "y_{\mathbb{B}}"] \arrow[d, "D"'] & \mathcal{V}^{\mathbb{B}^{\mathrm{op}}} \arrow[dl, shift right, ""] \arrow[dl, phantom, "\perp"] \\ \mathcal{S} \arrow[ur, shift right, "U"'] & \end{tikzcd}$$

We seek to construct the endomorphism operad $\mathsf{End}\,U$ as the following internal hom in the (plain) slice category $[\mathcal{S}, \mathcal{V}^{\mathbb{B}^{\mathrm{op}}}]/T_*1$:

$$\mathsf{Hom}\,(U, U) = \begin{bmatrix} T_*U & & T_*1 \times U \\ \downarrow & , & \downarrow \\ T_*1 & & T_*1 \end{bmatrix}$$

We are going to use Theorem 2.4. Our first step here is to re-express the slice category as an equivalent category of the form $[\mathcal{S}, \mathcal{V}^{\mathbb{C}^{\mathrm{op}}}]$, and our next step will be to show that under that equivalence, the objects whose internal hom we're taking become right adjoints.

The first step is straightforward for $\mathcal{V} = \mathsf{Set}$ so we cover that case first; it requires a little more effort for $\mathcal{V} = \mathsf{Top}$.

Lemma 2.6. *Let $\mathcal{S}$ be locally small and cocomplete, and $\mathbb{B}$ small, and T a monad on $[\mathbb{B}^{\text{op}}, \mathsf{Set}]$. There are equivalences of categories:*

$$\Big[\mathcal{S},\ [\mathbb{B}^{\text{op}}, \mathsf{Set}]\Big]/T_*1 \ \simeq\ \Big[\mathcal{S},\ [\mathbb{B}^{\text{op}}, \mathsf{Set}]/T1\Big] \ \simeq\ \Big[\mathcal{S},\ \big[(\mathbb{B}/T1)^{\text{op}}, \mathsf{Set}\big]\Big]$$

where $\mathbb{B}/T1$ denotes the category of elements.

Proof. First note that the functor T_*1 is the composite

$$\begin{array}{ccccc} \mathcal{S} & \xrightarrow{1} & [\mathbb{B}^{\text{op}}, \mathsf{Set}] & \xrightarrow{T} & [\mathbb{B}^{\text{op}}, \mathsf{Set}] \\ c & \longmapsto & 1 & \longmapsto & T1 \end{array}$$

so it is the constant functor Δ_{T1}. Thus an object $(F \xrightarrow{\alpha} T_*1)$ in the first category amounts to a cocone $(F\_ \longrightarrow T1)$. This gives the first equivalence. The second follows from the fact that slices of presheaf toposes are equivalent to presheaf toposes as follows:

$$[\mathbb{B}^{\text{op}},\ \mathsf{Set}]/T1 \ \simeq\ \big[(\mathbb{B}/T1)^{\text{op}},\ \mathsf{Set}\big]$$

□

For the case $\mathcal{V} = \mathsf{Top}$ we deal with the two equivalences separately; the first follows easily, with the only subtlety being that $\mathcal{V}^{\mathbb{B}^{\text{op}}}/T1$ is now an enriched slice category. However as we are only considering enrichment in Top this amounts to the same as the ordinary slice but with a topology on the homs, and that topology is inherited. We will express this lemma in simpler terms to emphasise the fact that nothing very special is going on, but what we have in mind here is $\mathcal{G} = \mathcal{V}^{\mathbb{B}^{\text{op}}}$ and $X = T1$.

Lemma 2.7. *Let $\mathcal{V} = \mathsf{Top}$, let $\mathcal{S}$ and $\mathcal{G}$ be $\mathcal{V}$-categories. Consider $X \in \mathcal{G}$ and write $\Delta_X : \mathcal{S} \longrightarrow \mathcal{G}$ for the constant functor. Then there is an equivalence of categories*

$$\Big[\mathcal{S},\ \mathcal{G}\Big]/\Delta_X \ \simeq\ \Big[\mathcal{S},\ \mathcal{G}/X\Big]$$

Proof. As in the previous proof, an object $(F \xrightarrow{\alpha} \Delta_X)$ in the first category amounts to a cocone $(F\_ \longrightarrow X)$, that is, an object in the second category. The only extra subtlety here is that the enriched structure of $\mathcal{G}/X$ is inherited from $\mathcal{G}$, so F being a $\mathcal{V}$-functor on the left ensures that the stated corrspondence does produce a $\mathcal{V}$-functor on the right. □

We now deal with the second equivalence. In what follows we will sometimes realise plain categories as Top-categories in which all the hom-spaces are discrete; by abuse of notation we will not change the notation for this.

Lemma 2.8. *Let $\mathcal{V} = \mathsf{Top}$. Let $\mathbb{A}$ be a small (plain) category. Let $F\colon \mathbb{A} \longrightarrow \mathsf{Set}$ be any functor and $i : \mathsf{Set} \longrightarrow \mathsf{Top}$ be the functor each set to the discrete space on it. Then there is a $\mathcal{V}$-equivalence of $\mathcal{V}$-categories:*

$$\mathcal{V}^{\mathbb{A}}/iF \simeq \mathcal{V}^{\mathbb{A}/F}$$

where $\mathbb{A}/F$ denotes the category of elements of F.

Proof. We borrow the standard proof that

$$[\mathbb{A}, \mathsf{Set}]/F \simeq [\mathbb{A}/F, \mathsf{Set}]$$

—we just have to check continuity in a few key places. We know that we have a functor

$$[\mathbb{A}, \mathsf{Set}]/F \xrightarrow{\alpha} [\mathbb{A}/F, \mathsf{Set}]$$

that is full, faithful and essentially surjective on objects. Recall that $\mathbb{A}/F$ is the category of elements of F given as follows.

- Objects are pairs $(a \in \mathbb{A}, x \in Fa)$.
- A morphism $(a, x) \longrightarrow (a', x')$ is a morphism $f\colon a \longrightarrow a' \in \mathbb{A}$ such that $Ff(x) = x'$.

First we recall the action of α. Given an element $(S, \theta) = \begin{array}{c} S \\ \downarrow{\scriptstyle\theta} \\ F \end{array} \in [\mathbb{A}, \mathsf{Set}]/F$

with components $\begin{array}{c} Sa \\ \downarrow{\scriptstyle\theta_a} \\ Fa \end{array}$, we will write $\alpha(S, \theta) = \bar{S} \in [\mathbb{A}/F, \mathsf{Set}]$, and its action is as follows:

- On objects: $(a, x) \in \mathbb{A}/F$ is sent to the set ${\theta_a}^{-1}(x) \subseteq Sa$.

- On morphisms: the morphism
$$(a, x) \xrightarrow{f} (a', x')$$
is sent to
$$\theta_a^{-1}(x) \longrightarrow \theta_{a'}^{-1}(x')$$
the restriction of Sf to the fibre $\theta_a^{-1}(x)$; this works because of naturality of θ.

The functor α is full, faithful and essentially surjective; it has (pseudo)-inverse β given as follows. Given $R \in [\mathbb{A}/F, \mathsf{Set}]$, the element

$$\beta(R) = \begin{array}{c} \hat{R} \\ \downarrow{\scriptstyle \theta^R} \\ F \end{array} \in [\mathbb{A}, \mathsf{Set}]/F$$

is given by
$$\hat{R}a = \coprod_{x \in Fa} R(a, x). \tag{2.2}$$
The map ${\theta^R}_a : R(a, x) \longrightarrow Fa$ sends everything to $x \in Fa$. The rest of the data is induced by the universal property of the coproduct (2.2).

In order to modify this proof for the topological case, we need to check that

1. If S is a Top-functor $\mathbb{A} \longrightarrow \mathsf{Top}$ then $\bar{S}$ becomes a Top-functor $\mathbb{A}/F \longrightarrow \mathsf{Top}$,
2. If R is a Top-functor $\mathbb{A}/F \longrightarrow \mathsf{Top}$ $\hat{R}$ becomes a Top-functor $\mathbb{A} \longrightarrow \mathsf{Top}$,
3. the components of θ^R are continuous, and
4. α and β are themselves Top-functors.

These all follow by making the preimage, coproduct and restriction maps in Top instead of in Set. $\Box$

Corollary 2.9. *Under the usual hypotheses we have the following equivalences of categories:*

$$\big[\mathcal{S}, \mathcal{V}^{\mathbb{B}^{\mathrm{op}}}\big]/T_*1 \;\simeq\; \big[\mathcal{S}, \mathcal{V}^{\mathbb{B}^{\mathrm{op}}}/T1\big] \;\simeq\; \big[\mathcal{S}, \mathcal{V}^{(\mathbb{B}/T1)^{\mathrm{op}}}\big]$$

Our next task is to take the objects whose internal hom we want to calculate in the slice category $\left[\mathcal{S}, \mathcal{V}^{\mathbb{B}^{op}}\right]/T_*1$, and "translate" them into the functor category $\left[\mathcal{S}, \mathcal{V}^{(\mathbb{B}/T1)^{op}}\right]$ and show that they are right adjoints so that we may take their internal hom.

We first briefly recall some helpful results and definitions; we will not state these in very great generality.

Lemma 2.10. *Let $\mathcal{V}$ be* Set *or* Top *and keep the usual hypotheses. Consider the canonical morphism $T1 \xrightarrow{!} 1$ in $\mathcal{V}^{\mathbb{B}^{op}}$. Then there is a $\mathcal{V}$-adjunction which we will write as*

$$\begin{tikzcd} \mathcal{V}^{\mathbb{B}^{op}}/T1 \arrow[r, shift left, "\Sigma_{T1}"] \arrow[r, phantom, "\perp"] & \mathcal{V}^{\mathbb{B}^{op}}/1 \arrow[l, shift left, "(T1)^*"] \end{tikzcd} \simeq \mathcal{V}^{\mathbb{B}^{op}}$$

where $(T1)^*$ is given by pullback along $!$ (so in this case, effectively it is just a product) and Σ_{T1} is $! \circ -$ (sometimes called the dependent sum).

Lemma 2.11. *The $\mathcal{V}$-functor $T : \mathcal{V}^{\mathbb{B}^{op}} \longrightarrow \mathcal{V}^{\mathbb{B}^{op}}$ can be canonically factorised as:*

$$\begin{array}{ccccccccc} \mathcal{V}^{\mathbb{B}^{op}} & \simeq & \mathcal{V}^{\mathbb{B}^{op}}/1 & \xrightarrow{\hat{T}} & \mathcal{V}^{\mathbb{B}^{op}}/T1 & \xrightarrow{\Sigma_{T1}} & \mathcal{V}^{\mathbb{B}^{op}}/1 & \simeq & \mathcal{V}^{\mathbb{B}^{op}} \\ X & \longmapsto & \begin{array}{c} X \\ \downarrow ! \\ 1 \end{array} & \longmapsto & \begin{array}{c} TX \\ \downarrow T! \\ T1 \end{array} & \longmapsto & \begin{array}{c} TX \\ \downarrow ! \\ 1 \end{array} & \longmapsto & TX \end{array}$$

Definition 2.12. The functor T is called a *parametric right adjoint* (p.r.a.) if $\hat{T}$ is a right adjoint. A monad is called *parametric right adjoint* if its functor part is p.r.a. and its unit and multiplication are cartesian. Any familially representable monad is p.r.a.

We are finally ready to tackle the internal hom in question.

Theorem 2.13. *Under the usual hypotheses, including that T is a parametric right adjoint and that U is a right adjoint, the following internal hom in $[\mathcal{S}, \mathcal{V}^{\mathbb{B}^{op}}]/T_*1$ exists:*

$$\left[\begin{array}{c} T_*U \\ \downarrow T_*1 \\ T_*1 \end{array} , \begin{array}{c} T_*1 \times U \\ \downarrow p \\ T_*1 \end{array} \right]$$

where p denotes projection onto the first component.

Proof. We "translate" each of these objects from the slice category above into the functor category

$$\left[\mathcal{S},\ \mathcal{V}^{(\mathbb{B}/T1)^{\mathrm{op}}}\right]$$

as in Lemma 2.7 and then express them as right adjoints. We then apply Theorem 2.4 with $\mathbb{C} = \mathbb{B}/T1$.

First note that according to the first equivalence of Corollary 2.9, the object $(T_*U \xrightarrow{T_*!} T_*1)$ becomes the cocone $(TU\_ \xrightarrow{T!} T1)$. So we take $\hat{T} \circ U$, the following composite of right adjoints, giving the cocone required:

$$\begin{array}{ccccccc}
\mathcal{S} & \xrightarrow{U} & \mathcal{V}^{\mathbb{B}^{\mathrm{op}}} & \xrightarrow{\hat{T}} & \mathcal{V}^{\mathbb{B}^{\mathrm{op}}}/T1 & \simeq & \mathcal{V}^{(\mathbb{B}/T1)^{\mathrm{op}}} \\
x & \longmapsto & Ux & \longmapsto & \begin{array}{c} TUx \\ \downarrow{\scriptstyle T!} \\ T1 \end{array} & &
\end{array}$$

For the second object of our internal hom, note that according to the first equivalence of Corollary 2.9 the object $(T_*1 \times U \xrightarrow{p} T_*1)$ becomes the cocone $(T1 \times U\_ \xrightarrow{p} T1)$. We consider $(T1)^* \circ U$, the following composite of right adjoints, giving the cocone required:

$$\begin{array}{ccccccc}
\mathcal{S} & \xrightarrow{U} & \mathcal{V}^{\mathbb{B}^{\mathrm{op}}} & \xrightarrow{(T1)^*} & \mathcal{V}^{\mathbb{B}^{\mathrm{op}}}/T1 & \simeq & \mathcal{V}^{(\mathbb{B}/T1)^{\mathrm{op}}} \\
x & \longmapsto & Ux & \longmapsto & \begin{array}{c} T1 \times Ux \\ \downarrow{\scriptstyle p} \\ T1 \end{array} & &
\end{array}$$

Here $(T1)^*$ is a right adjoint with left adjoint given by Σ_{T1}. Theorem 2.4 now applies, and we can compute the internal hom of right adjoints:

$$\left[\hat{T} \circ U, (T1)^* \circ U.\right]$$

□

3. Loop spaces

In this section we discuss the first of our motivating examples—operads acting on loop spaces. Most of the work here is just in unravelling the definitions to show that the universal operad acting on loop spaces is the one we are expecting from [14].

In this example we take

$$\begin{array}{rcl}
\mathcal{S} & = & \mathsf{Top}_* \\
\mathcal{G} & = & \mathsf{Top} \\
U & = & \Omega = \mathsf{Top}_*(S,-) \text{ where here } \mathsf{Top}_*(X,Y) \text{ denotes the unbased space of} \\
 & & \text{based maps } X \longrightarrow Y\text{, and } S \text{ is the unit circle} \\
T & = & \text{free topological monoid monad on } \mathsf{Top}\text{, thus } TX = \coprod_{k\in\mathbb{N}} X^k \text{ (see [9])}
\end{array}$$

Note that the initial object in Top_* is the one-point space; we will still write it as $\emptyset$ although it is not empty.

Thus $T1$ is the space $\mathbb{N}$ with the discrete topology, a T-operad is just a classical (non-Σ) operad, and operads acting on U-objects are just operads acting on loop spaces. We seek to understand the universal operad

$$E_\Omega = \mathsf{ev}_\emptyset(\mathsf{End}\,(U)).$$

We will show that

$$E_\Omega(k) = \mathsf{Top}_*(S, S^{\vee k}),$$

the operad which has been called the "universal operad acting on loop spaces" by Salvatore and others [14, 12, 2].

First we use the results of Section 2 to show that $\mathsf{End}\,(\Omega)$ exists.

Proposition 3.1. *With the definitions as above, we can form* $\mathsf{End}\,(\Omega)$ *as an internal hom in the slice category* $[\mathcal{S}, \mathcal{G}]/T_*1$.

Proof. The monad T is familially representable and U is a right adjoint: we identify Top with $\mathsf{Top}^{dir0o^{\mathrm{op}}}$, and note that U can be regarded as being constructed via the functor $dir0o \xrightarrow{S} \mathsf{Top}_*$ picking out the circle S. Then

$$U = \mathsf{Top}_*(S,-) \in \mathsf{Top}$$

is a right adjoint; its left adjoint is the left Kan extension of S along the Yoneda embedding

$$\begin{tikzcd}
dir0o \arrow[r, "y_{dir0o}"] \arrow[d, "S"'] & \mathsf{Top}^{dir0o^{\mathrm{op}}} \simeq \mathsf{Top} \arrow[dl, shift right] \arrow[dl, phantom, "\perp"] \\
\mathsf{Top}_* \arrow[ur, shift right, "U"'] &
\end{tikzcd}$$

We can thus apply Theorem 2.4 to form the required internal hom as

$$[\hat{T} \circ U, (T1)^* \circ U].$$

□

We now set about unravelling what $E_\Omega = \mathsf{ev}_\emptyset(\mathsf{End}\,(U))$ is. The first step is to understand the slice category in question, re-expressed as a functor category.

Now note that $T1$ in this case is $\mathbb{N}$, the discrete category made into a Top-category, and $\mathsf{Top}/\mathbb{N} \equiv \mathsf{Top}^{\mathbb{N}}$. So the first equivalence of Corollary 2.9 in this case becomes:

$$\begin{aligned} [\mathsf{Top}_*, \mathsf{Top}]/T_*1 &\simeq [\mathsf{Top}_*, \mathsf{Top}/T1] \\ &\simeq [\mathsf{Top}_*, \mathsf{Top}^{\mathbb{N}}] \\ &\simeq [\mathsf{Top}_* \times \mathbb{N}, \mathsf{Top}] \end{aligned}$$

Example 3.2. An element $\begin{array}{c} S \\ \downarrow \\ T_*1 \end{array}$ in $[\mathsf{Top}_*, \mathsf{Top}]/T_*1$ consists of, for all $X \in \mathsf{Top}_*$ a continuous map $\begin{array}{c} SX \\ \downarrow \\ \mathbb{N} \end{array}$ such that for all $f : X \longrightarrow X'$ the following diagram commutes

$$\begin{tikzcd} SX \arrow[rr, "Sf"] \arrow[dr] & & SX' \arrow[dl] \\ & \mathbb{N}. & \end{tikzcd}$$

Since $\mathbb{N}$ is discrete, we know $SX = \coprod_n S_n X$, say, where each S_n is a functor $\mathsf{Top}_* \longrightarrow \mathsf{Top}$. Thus we have a functor

$$\begin{array}{ccc} \mathsf{Top}_* \times \mathbb{N} & \longrightarrow & \mathsf{Top} \\ (X, n) & \longmapsto & S_n(X). \end{array}$$

Conversely, given a functor $S : \mathsf{Top}_* \times \mathbb{N} \longrightarrow \mathsf{Top}$ we have for all n a functor

$$S_n = S(n, -) : \mathsf{Top}_* \longrightarrow \mathsf{Top}.$$

This corresponds to $\begin{array}{c}\bar{S}\\ \downarrow\\ \mathbb{N}\end{array}$ by $\bar{S}X = \coprod_n S_n(X)$.

Theorem 3.3. *For all $k \geq 0$ we have $E_\Omega(k) = \mathsf{Top}_*(S, S^{\vee k})$.*

Proof. We write $\mathcal{S} = \mathsf{Top}_*$. We must calculate $\mathsf{ev}_\emptyset(\mathsf{Hom}\,(\Omega,\Omega))$. Recall that $\mathsf{Hom}\,(\Omega,\Omega) \in [\mathcal{S},\mathsf{Top}]/T_*1$ is given by

$$\left[\begin{array}{cc} T_*\Omega & T_*1\times\Omega \\ \big\downarrow{\scriptstyle T_*!}\ , & \big\downarrow{\scriptstyle \pi_1} \\ T_*1 & T_*1 \end{array}\right] = \left[\begin{array}{cc} A & B \\ \downarrow\ , & \downarrow \\ \Delta_{\mathbb{N}} & \Delta_{\mathbb{N}} \end{array}\right],$$

say, where the square brackets denote the exponential in the slice category $[\mathcal{S},\mathsf{Top}]/T_*1$. To calculate this hom we express it in the equivalent category $[\mathcal{S}\times\mathbb{N},\mathsf{Top}]$; then to evaluate it at $\emptyset$ we evaluate it at $(\emptyset,k)$ for each $k\in\mathbb{N}$.

Now

$$\begin{array}{rclclcl} AX &=& T\Omega X &=& \coprod_k (\Omega X)^k &=& \coprod_k \mathcal{S}(S^{\vee k}, X) \\ BX &=& \mathbb{N}\times\Omega X &=& \coprod_k \Omega X &=& \coprod_k \mathcal{S}(S, X) \end{array}$$

so

$$\begin{array}{rcl} \bar{A}(X,k) &=& \mathcal{S}(S^{\vee k}, X) \\ \bar{B}(X,k) &=& \mathcal{S}(S, X). \end{array}$$

We now use the formula for the internal hom in $[\mathcal{S}\times\mathbb{N},\mathsf{Top}]$ as in Remark 2.5:

$$\mathsf{Hom}\,(\bar{A},\bar{B})(\emptyset,k) \quad = \quad \big[\mathcal{S}\times\mathbb{N},\mathsf{Top}\big]\big(H^{(\emptyset,k)}\times\bar{A},\bar{B}\big) \tag{3.1}$$

where

$$H^{(\emptyset,k)}\big(Y,m\big) = \big(\mathcal{S}\times\mathbb{N}\big)\big((\emptyset,k),(Y,m)\big) = \begin{cases} 1 & k=m \\ \emptyset & \text{otherwise} \end{cases}$$

where here 1 and $\emptyset$ are terminal and initial respectively in Set.

Now $\mathcal{S}\times\mathbb{N}$ is a coproduct $\coprod_{m\in\mathbb{N}}\mathcal{S}$, so in general

$$[\mathcal{S}\times\mathbb{N},\mathsf{Top}](F,G) \cong \prod_{m\in\mathbb{N}} \big[\mathcal{S},\mathsf{Top}\big]\big(F(-,m),G(-,m)\big).$$

So, using (3.1) above, we have:

$$\begin{aligned}\mathsf{Hom}\,(\bar{A},\bar{B})(\emptyset,k) &= \prod_{m\in\mathbb{N}}\big[\mathcal{S},\mathsf{Top}\big]\Big(H^{(\emptyset,k)}(-,m)\times\mathcal{S}(S^{\vee m},-),\mathcal{S}(S,-)\Big)\\ &= \big[\mathcal{S},\mathsf{Top}\big]\big(\mathcal{S}(S^{\vee k},-),\mathcal{S}(S,-)\big)\\ &= \mathcal{S}(S,S^{\vee k}) \quad \text{by the enriched Yoneda Lemma.}\end{aligned}$$

□

Remark 3.4. This operad is often thought of as the "coendomorphism operad" on S in Top_*; we now see that it is derived from the *endomorphism* operad in the functor category $[\mathsf{Top}_*,\mathsf{Top}]$, on the representable functor at S.

Example 3.5. Let D be the non-Σ version of the little intervals operad, so $D(k)$ is the space of configurations of k disjoint intervals inside the unit interval. It is well-known that D acts naturally on loop spaces; in fact the action is explicitly defined via the action of the universal operad E. Given an element of $D(k)$ we derive an element of $E(k)$, that is a based continuous map $S \longrightarrow S^{\vee k}$, as follows. We identify the endpoints of the (big) unit interval to make the circle S of the domain; we then map any point outside the little intervals to the basepoint of $S^{\vee k}$, and map the ith little interval to the ith circle in the wedge $S^{\vee k}$. The element of $D(k)$ is then considered to act on loop spaces via the action of this derived element of $E(k)$.

Note that there are many operations of E that do not arise in this way. Broadly these fall into three types:

- maps $S \longrightarrow S^{\vee k}$ that are not surjective, so "omit" some loops,
- maps that involve going "backwards" around a loop, or
- maps that involve going more than once around loops.

In this sense the universal operad is "too big", and the operads that have proved efficacious in loop space theory are much smaller.

Examples 3.6. Let $D(k)$ be the space of continuous, endpoint-preserving maps $[0,1] \longrightarrow [0,k]$. These act on loop spaces naturally as they act on path spaces naturally. Other examples arise as suboperads of this one, for example by using only increasing maps, or piecewise linear increasing maps.

One might wish to restrict further to smooth maps (for example in order to reparametrise cobordisms). This is more complicated as one would have to ensure that composites remained smooth; this is related to some issues tackled using collars in [3].

4. Fundamental ω-groupoids

In this section we discuss our motivating example, the functor

$$U : \mathsf{Top} \longrightarrow \mathsf{GSet}$$

giving the "fundamental globular set" or "ω-path space" of a space. In a sense there is no further calculation to be done as Leinster has already worked out what the globular operad $\mathsf{ev}_\emptyset(\mathsf{End}\,(U))$ is; see [12, Example 9.2.7], in which Leinster writes the operad in question as $P' = (\mathsf{ev}_\emptyset)_*(\mathsf{End}\,(\Pi_\omega))$. Thus, the history of this operad may be summarised as follows.

1. Batanin defines the operad directly [1].
2. Leinster expresses the operad as $\mathsf{ev}_\emptyset(\mathsf{End}\,(U))$ [12].
3. The present work establishes that $\mathsf{End}\,(U)$ does exist, so that this expression for P' makes sense, and exhibits the universal property of P'.

In fact as usual the story is slightly more complicated as we use the "non-algebraic Leinster variant" of contractible globular operads (as used by Cisinski [5]). In this section we will give some of the details of Leinster's calculation and then show how to modify the proof to achieve the n-dimensional versions in Section 4.3.

4.1 Globular theory

We first recall some theory from [1]; for an alternative treatment see [12] or [6].

Definition 4.1. Let T be the free strict ω-category monad on $\mathsf{GSet} = [\mathbb{G}^{\mathrm{op}}, \mathsf{Set}]$; T is familially representable and in particular cartesian.

- A T-operad (in the sense of Definition 1.2) is called a *globular operad*.
- A globular operad is called *contractible* if its underlying T-collection is contractible.
- A T-collection $\begin{array}{c} A \\ \downarrow{\scriptstyle p} \\ T1 \end{array}$ is called *contractible* if

  1. given any 0-cells $a, b \in A$ and a 1-cell $y : pa \longrightarrow pb \in T1$, there exists a 1-cell $x : a \longrightarrow b \in A$ such that $px = y$, and
  2. for all $m \geq 1$, given any m-cells $a, b \in A$ that are "parallel" i.e. $sa = sb$ and $ta = tb$, and an $(m+1)$-cell $y : pa \longrightarrow pb \in T1$, there exists an $(m+1)$-cell $x : a \longrightarrow b \in A$ such that $px = y$.

  Note that for the finite n-dimensional version, we use the free strict n-category monad which we denote T_n, and need an extra condition at the nth dimension as follows: given any parallel n-cells $a, b \in A$ with $pa = pb \in T1$, we have $a = b$.
- A *weak ω-category* is any algebra for any contractible globular operad.
- A *weak ω-groupoid* is a weak ω-category in which every cell is weakly invertible. Batanin defines this via the n-coskeleta of the ω-category—the idea is that to be weakly invertible a k-cell in an ω-category should be weakly invertible in the n-coskeleton (the weak n-category formed by quotienting out by $(n+1)$-cells) for each $n \geq k$; weak invertibility in an n-category for finite n can be defined by induction. Since we will not need to use this definition we refer the reader to [1] for the full details.

The analogy with loop spaces should be clear; where for loop spaces we used operads with an operation of arity k for each $k \in \mathbb{N}$, we now have an operation of arity α for every pasting diagram α.

We wish to exhibit every globular set UX as an ω-groupoid, so first we must find a contractible globular operad that acts on each UX. Batanin proposes the following operad. Essentially the operations of arity $\alpha \in n\text{-}\mathsf{Pd}$ are the continuous, boundary-preserving maps from the topological n-ball to the geometric realisation of α. However we must be careful about exactly what boundary must be preserved. The idea is that the spaces in question should have globular "sources" and "targets" of each lower dimension, and these are the boundaries that should be preserved. This is expressed in Batanin's definition of "coglobular span in Top".

Definition 4.2. A *coglobular n-span* in a category $\mathcal{C}$ is a commuting diagram of the following shape in $\mathcal{C}$.

$$\begin{tikzcd}
 & x & \\
a_{n-1} \arrow[ur] & & b_{n-1} \arrow[ul] \\
a_{n-2} \arrow[u] \arrow[urr] & & b_{n-2} \arrow[u] \arrow[ull] \\
a_{n-3} \arrow[u] \arrow[urr] & & b_{n-3} \arrow[u] \arrow[ull] \\
\vdots \arrow[u] \arrow[urr] & & \vdots \arrow[u] \arrow[ull] \\
a_1 & & b_1 \\
a_0 \arrow[u] \arrow[urr] & & b_0 \arrow[u] \arrow[ull]
\end{tikzcd}$$

Example 4.3. The topological n-ball B^n has maps

$$B^{n-1} \underset{t}{\overset{s}{\rightrightarrows}} B^n$$

given by the inclusions of the north and south hemispheres. This makes B^n into a coglobular n-span in Top as

$$\begin{tikzcd}
 & B^n & \\
B^{n-1} \arrow[ur, "s"] & & B^{n-1} \arrow[ul, "t"'] \\
B^{n-2} \arrow[u, "s"] \arrow[urr, "s"] & & B^{n-2} \arrow[u, "t"'] \arrow[ull, "t"'] \\
\vdots \arrow[u, "s"] \arrow[urr, "s"] & & \vdots \arrow[u, "t"'] \arrow[ull, "t"'] \\
B^1 & & B^1 \\
B^0 \arrow[u, "s"] \arrow[urr, "s"] & & B^0 \arrow[u, "t"'] \arrow[ull, "t"']
\end{tikzcd}$$

Example 4.4. Let α be an n-dimensional pasting diagram with source and target $\partial\alpha$. Then there are inclusions of the geometric realisations

$$|\partial\alpha| \underset{t}{\overset{s}{\rightrightarrows}} |\alpha|$$

into the "source" and "target". This makes $|\alpha|$ into a coglobular n-span in Top as

$$\begin{tikzcd}
 & |\alpha| & \\
|\partial\alpha| \arrow[ur, "s"] & & |\partial\alpha| \arrow[ul, "t"'] \\
|\partial^2\alpha| \arrow[u, "s"] \arrow[urr, "s"] & & |\partial^2\alpha| \arrow[u, "t"'] \arrow[ull, "t"'] \\
\vdots \arrow[u, "s"] \arrow[urr, "s"] & & \vdots \arrow[u, "t"'] \arrow[ull, "t"'] \\
|\partial^{n-1}\alpha| & & |\partial^{n-1}\alpha| \\
|\partial^{n}\alpha| \arrow[u, "s"] \arrow[urr, "s"] & & |\partial^{n}\alpha| \arrow[u, "t"'] \arrow[ull, "t"']
\end{tikzcd}$$

Definition 4.5. A *map of coglobular n-spans in $\mathcal{C}$*

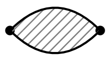
$$\begin{tikzcd}
 & x & \\
a_{n-1} \arrow[ur, "s"] & & b_{n-1} \arrow[ul, "t"'] \\
a_{n-2} \arrow[u, "s"] \arrow[urr, "s"] & & b_{n-2} \arrow[u, "t"'] \arrow[ull, "t"'] \\
\vdots \arrow[u, "s"] \arrow[urr, "s"] & & \vdots \arrow[u, "t"'] \arrow[ull, "t"'] \\
a_1 & & b_1 \\
a_0 \arrow[u, "s"] \arrow[urr, "s"] & & b_0 \arrow[u, "t"'] \arrow[ull, "t"']
\end{tikzcd}
\longrightarrow
\begin{tikzcd}
 & x' & \\
a'_{n-1} \arrow[ur, "s"] & & b'_{n-1} \arrow[ul, "t"'] \\
a'_{n-2} \arrow[u, "s"] \arrow[urr, "s"] & & b'_{n-2} \arrow[u, "t"'] \arrow[ull, "t"'] \\
\vdots \arrow[u, "s"] \arrow[urr, "s"] & & \vdots \arrow[u, "t"'] \arrow[ull, "t"'] \\
a'_1 & & b'_1 \\
a'_0 \arrow[u, "s"] \arrow[urr, "s"] & & b'_0 \arrow[u, "t"'] \arrow[ull, "t"']
\end{tikzcd}$$

is given by a map $v\colon x \longrightarrow x'$ and for all $0 \le i \le n-1$ maps

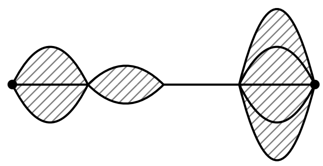
$$\begin{array}{rcl} f_i : a_i & \longrightarrow & a'_i \\ g_i : b_i & \longrightarrow & b'_i \end{array}$$

making everything commute.

Example 4.6. (Informal.) Consider the following coglobular 2-spans in Top: the ball B^2



and the geometric realisation $|\alpha|$



A map of coglobular spans $B^2 \longrightarrow |\alpha|$ is a map of the underlying spaces such that

- the top and bottom boundaries of B^2 are mapped to the top and bottom boundaries respectively of $|\alpha|$, and
- the endpoints of the top and bottom boundaries of B^2 are mapped to the endpoints of the top and bottom boundaries of $|\alpha|$.

4.2 The universal operad acting on ω-path spaces

We now invoke the results of Section 2 to show that $\mathsf{End}\,(U)$ in this case exists. As we are enriching in Set here we will revert to the more usual notation $[\mathbb{G}^{\mathrm{op}}, \mathsf{Set}]$ instead of $\mathsf{Set}^{\mathbb{G}^{\mathrm{op}}}$.

Theorem 4.7. *Let T be the free ω-category monad on $[\mathbb{G}^{\mathrm{op}}, \mathsf{Set}]$, and $U\colon \mathsf{Top} \longrightarrow [\mathbb{G}^{\mathrm{op}}, \mathsf{Set}]$ the ω-path space functor. Then the internal hom*

$$\mathsf{End}\,(U) = \begin{bmatrix} T_*U & & T_*1 \times U \\ \downarrow & , & \downarrow \\ T_*1 & & T_*1 \end{bmatrix}$$

exists in $\big[\mathsf{Top}, [\mathbb{G}^{\mathrm{op}}, \mathsf{Set}]\big]/T^*1$.

Proof. We know that T is familially representable and U is a right adjoint: U is constructed via a functor $\mathbb{G} \xrightarrow{D} \mathsf{Top}$ where $D(n) = B^n$ the topological n-ball. Then

$$U(X) = \mathcal{S}(D-, X) \in [\mathbb{G}^{\mathrm{op}}, \mathsf{Set}]$$

thus U is a right adjoint; its left adjoint is the left Kan extension of D along the Yoneda embedding

$$\begin{tikzcd} \mathbb{G} \arrow[r, "y_{\mathbb{G}}"] \arrow[d, "D"'] & {[\mathbb{G}^{\text{op}}, \mathsf{Set}]} \arrow[dl, shift right] \\ \mathsf{Top} \arrow[ur, shift right, "U"'] \arrow[ur, phantom, "\perp"] & \end{tikzcd}$$

We can thus apply Theorem 2.4 to form the required internal hom as

$$[\hat{T} \circ U, (T1)^* \circ U].$$

□

We now sketch the calculation of the universal globular operad acting on ω-path spaces. This can be found in [12, Example 9.2.7] but we give some of the details here as we will be modifying the calculation to give the finite-dimensional cases in the next section.

In order to calculate the operad in this case, we need to use the exponential in the slice category $\big[\mathsf{Top}, [\mathbb{G}^{\text{op}}, \mathsf{Set}]\big]/T_*1$, which we will do via Remark 2.5 and the equivalences of Lemma 2.6:

$$\begin{aligned} \Big[\mathsf{Top},\ [\mathbb{G}^{\text{op}}, \mathsf{Set}]\Big]/T_*1 &\simeq \Big[\mathsf{Top},\ \big[(\mathbb{G}/T1)^{\text{op}}, \mathsf{Set}\big]\Big] \\ &\simeq \Big[\mathsf{Top} \times (\mathbb{G}/T1)^{\text{op}},\ \mathsf{Set}\Big] \end{aligned}$$

The rest of the calculation is given by Leinster; we will sketch the main details here.

Theorem 4.8 (Leinster, [12, Example 9.2.7]). *Let $U\colon \mathsf{Top} \longrightarrow \mathsf{GSet}$ be the ω-path space functor. Then the operad $E_U = \mathsf{ev}_\emptyset(\mathsf{End}\,(U))$ has as operations of arity α the maps of coglobular spans $B^n \longrightarrow |\alpha|$. Here α is a pasting diagram of dimension n and $|\alpha|$ is its geometric realisation.*

Proof. (Sketch) Write $\mathcal{S} = \mathsf{Top}$. Now

$$\mathsf{End}\,(U) = \begin{bmatrix} T_*U & T_*1 \times U \\ \downarrow \ , & \downarrow \\ T_*1 & T_*1 \end{bmatrix} = \begin{bmatrix} A & B \\ \downarrow \ , & \downarrow \\ T_*1 & T_*1 \end{bmatrix},$$

say, where the square brackets denote the exponential in $\big[\mathcal{S}, [\mathbb{G}^{\mathrm{op}}, \mathsf{Set}]\big]/T_*1$, which we know to exist by Theorem 2.4. Here

$$T_*1(X) = T1 : \mathbb{G}^{\mathrm{op}} \longrightarrow \mathsf{Set}$$

for all $X \in \mathcal{S}$, and

$$\overline{T_*1}(X, n) = T1(n) \in \mathsf{Set}.$$

Note that $\mathbb{G}^{\mathrm{op}}/T1$ has as objects pairs $(n \in \mathbb{N}, \alpha \in T1(n) = n\text{-}\mathsf{Pd})$. Now given

$$\begin{array}{c} S \\ \big\downarrow{\scriptstyle p} \\ T_*1 \end{array} \in \big[\mathcal{S}, [\mathbb{G}^{\mathrm{op}}, \mathsf{Set}]\big]/T_*1,$$

with components

$$\begin{array}{c} SX \\ \big\downarrow{\scriptstyle p_X} \\ T1 \end{array},$$

we get

$$\bar{S} \in \big[\mathcal{S} \times (\mathbb{G}^{\mathrm{op}}/T1), \mathsf{Set}\big]$$

given by

$$\bar{S}(X, n, \alpha) = {p_X}^{-1}(\alpha) \subseteq SX(n).$$

Conversely given $S \in \big[\mathcal{S} \times (\mathbb{G}^{\mathrm{op}}/T1), \mathsf{Set}\big]$ we have $\begin{array}{c} \hat{S} \\ \big\downarrow{\scriptstyle p} \\ T_*1 \end{array} \in \big[\mathcal{S}, [\mathbb{G}^{\mathrm{op}}, \mathsf{Set}]\big]/T_*1$ given by

$$\hat{S}X(n) = \coprod_{\alpha \in T1(n)} S(X, n, \alpha)$$

thus the fibre of $\hat{S}X$ over α is $S(X, n, \alpha)$.

So we have

$$\begin{array}{rcl} \bar{A}(X, n, \alpha) & = & \{\, \text{pasting diagrams of shape } \alpha \text{ in } UX \,\} \\ & = & \mathcal{S}(|\alpha|, X) \end{array}$$

$$\begin{array}{rcl} \bar{B}(X, n, \alpha) & = & \{\, n\text{-cells in } UX \,\} \\ & = & \mathcal{S}(B^n, X) \end{array}$$

We can now use the usual internal hom formula in the functor category

$$\big[\mathcal{S} \times (\mathbb{G}^{\text{op}}/T1), \mathsf{Set}\big]$$

to get

$$\overline{\mathsf{End}\,(U)} = \big[\mathcal{S} \times (\mathbb{G}^{\text{op}}/T1), \mathsf{Set}\big]\big(H^{\bullet} \times \bar{A}, \bar{B}\big).$$

To find $\mathsf{ev}_{\emptyset}(\mathsf{End}\,(U))$ we can calculate fibre by fibre—the fibre over an n-pasting diagram α is

$$\overline{\mathsf{End}\,(U)}(\emptyset, n, \alpha) = \big[\mathcal{S} \times (\mathbb{G}^{\text{op}}/T1), \mathsf{Set}\big]\big(H^{(\emptyset,n,\alpha)} \times \bar{A}, \bar{B}\big)$$

as a set of natural transformations.

Note that

$$H^{(\emptyset,n,\alpha)}(X, m, \beta) = \begin{cases} 1 & \alpha = \beta \\ \{s,t\} & \beta = \partial\alpha \\ \emptyset & \text{otherwise} \end{cases}$$

Thus a natural transformation as above must have component at (X, m, β) of the form:

- if $\alpha = \beta$, $\mathcal{S}(|\alpha|, X) \longrightarrow \mathcal{S}(B^n, X)$
- if $\beta = \partial\alpha$
$$\{s,t\} \times \mathcal{S}(|\beta|, X) \longrightarrow \mathcal{S}(B^m, X)$$
hence a pair of maps $\mathcal{S}(|\beta|, X) \longrightarrow \mathcal{S}(B^m, X)$,
- otherwise: $\emptyset \longrightarrow \mathcal{S}(B^m, X)$ i.e. the trivial map.

We now examine naturality; as our domain is a product category we can examine naturality in X and (m, β) separately.

- Naturality in X tells us we must have a natural transformation
$$\mathcal{S}(|\alpha|, -) \longrightarrow \mathcal{S}(B^n, -)$$
and for each $0 \leq m < n$ two natural transformations
$$\mathcal{S}(|\partial^{n-m}\alpha|, -) \longrightarrow \mathcal{S}(B^m, -).$$
By Yoneda this is just an element of $\mathcal{S}(B^n, |\alpha|)$ and two elements of $\mathcal{S}(B^m, |\partial^{n-m}\alpha|)$ for each $0 \leq m < n$, that is, the underlying data for a morphism of coglobular spans.

- Naturality in (m, β) tells us that we have the necessary commuting conditions to be a morphism of coglobular spans as required.

$\square$

4.3 Finite-dimensional versions

There are three finite n-dimensional versions of this that follow immediately, one by taking truncations, one by taking homotopy classes at the top dimension, and one by taking path spaces at the top. The analogous result for the truncated version follows immediately, while the other versions follow with a little effort. First we recall the functors in question, described in the introduction. Note that they all agree on the first $(n-1)$ dimensions.

Definition 4.9. We will define the following functors for each $n \geq 0$.

- "n-truncation" $U_n : \mathsf{Top} \longrightarrow n\text{-}\mathsf{GSet} \simeq [\mathbb{G}_n^{\mathrm{op}}, \mathsf{Set}]$
- "fundamental n-groupoid" $\Pi_n : \mathsf{Top} \longrightarrow n\text{-}\mathsf{GSet}$
- "n-path space" $\mathcal{P}_n : \mathsf{Top} \longrightarrow \mathsf{Top}\text{-}n\text{-}\mathsf{Gph} \subset [\mathbb{G}_n^{\mathrm{op}}, \mathsf{Top}]$

$U_n X$ is the n-dimensional truncation of UX. $\Pi_n X$ agrees with UX for all dimensions up to $n-1$ but $(\Pi_n X)(n)$ is given by homotopy classes of n-cells in UX in the following sense: we identify any parallel n-cells $x, y \in UX(n)$ if there is an $(n+1)$-cell $f : x \longrightarrow y$ in $UX(n+1)$. That is, we apply the functor $q_n : \mathsf{GSet} \longrightarrow n\text{-}\mathsf{GSet}$ which is left adjoint to the functor

$$n\text{-}\mathsf{GSet} \xrightarrow{D_n} \mathsf{GSet}$$

that adds in putative identities at every dimension above n. (Note that in general the description of q_n would require us to generate an equivalence relation from the above relation; however in the case of globular sets of the form UX the above description suffices since we always have reverse and composite homotopies.)

For $\mathcal{P}_n$ we are thinking of a "Top-enriched n-graph" as an n-graph whose n-cells form a space but every lower dimension is just a set. However in order to apply Theorem 2.4 we are going to express these as n-globular spaces, that is, objects X of the enriched presheaf category $\mathsf{Top}^{\mathbb{G}_n^{\mathrm{op}}}$ such that for all

$k < n$, $X(k)$ is indiscrete. As with globular sets, given k-cells x, y we also write $X(x, y)$ for the subset (or subspace) of $X(k+1)$ of cells with domain x and codomain y.

Then $\mathcal{P}_0(X) = X$ and for $n > 0$ we have

- $\mathcal{P}_n(X)$ agrees with $\mathcal{P}_{n-1}(X)$ at all dimensions up to $n-2$.
- $\mathcal{P}_n(X)(n-1)$ is the set of points of the space $\mathcal{P}_{n-1}(X)(n-1)$ (more precisely, the indiscrete space on the underlying set of points).
- Given $x, y \in \mathcal{P}_n(X)(n-1)$, we have $\mathcal{P}_n(X)(x, y) = \mathcal{P}_{n-1}(X)(x, y)$ (the path space).

Remarks 4.10.

1. The functor Π_n will be used to find fundamental n-groupoids of spaces, while U_n is used in [7] when constructing ω-categories from "incoherent" n-categories. $\mathcal{P}_n$ can be thought of as an "(∞, n)" version, where algebraic information is extracted up to dimension n, with non-algebraic information remaining in higher dimensions.
2. We could give ω-dimensional versions of these functors, but in fact these would all be the same as U.

Corollary 4.11. *Let T_n be the free strict n-category monad on n-GSet. Then there is a universal n-globular operad (i.e. T_n-operad) acting on U_n given by the n-truncation of E_U.*

Proof. This is immediate, with proof as in the proof of Theorems 4.7 and 4.8. $\square$

Theorem 4.12. *There is a universal n-globular operad acting on Π_n given by $q_n E_U$.*

Proof. We prove this by adapting the proof of Theorem 4.8. For $m < n$ the mth dimension behaves exactly as for U; for the n-cells we must calculate $\Pi_n X$ and $T_n \Pi_n X$ so we must quotient $\mathcal{S}(B^n, X)$ and $\mathcal{S}(|\alpha|, X)$ by the equivalence relation demanded by our definition of Π_n.

It is useful to make this equivalence relation precise. $\Pi_n = q_n U$ so all parallel n-cells x, y of UX are to be identified if there is an $(n+1)$-cell $f\colon x \longrightarrow y$. In UX this means

$$B^n \overset{x}{\underset{y}{\rightrightarrows}} X$$

are identified if they can be expressed as composites

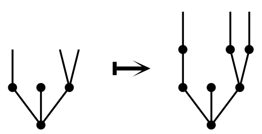
$$B^n \overset{s}{\underset{t}{\rightrightarrows}} B^{n+1} \xrightarrow{f} X \tag{4.1}$$

for some map f.

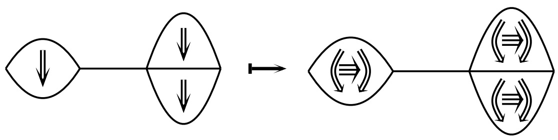
For $T_n q_n U$ we ask when we identify

$$|\alpha| \overset{x}{\underset{y}{\rightrightarrows}} X.$$

Write $\Sigma\alpha$ for the $(n+1)$-pasting diagram given by taking the tree for α and extending each leaf by one level. For example



or in pictures



Then we identify the n-cells x and y if they can be expressed as

$$|\alpha| \overset{|s|}{\underset{|t|}{\rightrightarrows}} |\Sigma\alpha| \xrightarrow{f} X \tag{4.2}$$

for some f. Note that the operation Σ is a form of suspension, and given a coglobular map

$$|\alpha| \longrightarrow |\beta|$$

there is a coglobular map

$$|\Sigma\alpha| \longrightarrow |\Sigma\beta|$$

making the following diagram commute

$$\begin{tikzcd} |\alpha| \arrow[r, shift left, "|s|"] \arrow[r, shift right, "|t|"'] \arrow[d, "p"'] & |\Sigma\alpha| \arrow[d, dashed] \\ |\beta| \arrow[r, shift left, "|s|"] \arrow[r, shift right, "|t|"'] & |\Sigma\beta| \end{tikzcd}$$

This is because the geometric realisation of an n-pasting diagram is homotopy equivalent to the n-ball, and the parallel pair of maps $|s|, |t|$ in each case gives, up to homotopy, the inclusion of the n-sphere (expressed as a pair of n-balls glued along their boundary) into the boundary of the $(n+1)$-ball.

Then we must ask the following questions.

1. Which natural transformations $\mathcal{S}(|\alpha|, -) \longrightarrow \mathcal{S}(B^n, -)$ respect this equivalence relation?

2. Which natural transformations $\mathcal{S}(|\alpha|, -) \longrightarrow \mathcal{S}(B^n, -)$ become the same on equivalence classes?

It is useful to note that the globular operad E_U is contractible (in the sense of 4.1). Now, we know that a natural transformation $\mathcal{S}(|\alpha|, -) \longrightarrow \mathcal{S}(B^n, -)$ is given by precomposition with a map $B^n \xrightarrow{p} |\alpha|$, and the naturality condition in $\mathbb{G}_n^{\mathrm{op}}/T_n 1$ ensures that this will have to be a map of coglobular spans as before. We must check when equivalent elements of $\mathcal{S}(|\alpha|, X)$ are mapped to equivalent elements of $\mathcal{S}(B^n, X)$. In fact this is the case for all $B^n \xrightarrow{p} |\alpha|$ as follows. Writing our equivalent elements of $\mathcal{S}(|\alpha|, X)$ as

$$\begin{tikzcd} |\alpha| \arrow[r, shift left, "|s|"] \arrow[r, shift right, "|t|"'] & |\Sigma\alpha| \arrow[r, "f"] & X \end{tikzcd}$$

we map them to $\mathcal{S}(B^n, X)$ by precomposition with p to give the two maps

$$\begin{tikzcd} B^n \arrow[r, "p"] & |\alpha| \arrow[r, shift left, "|s|"] \arrow[r, shift right, "|t|"'] & |\Sigma\alpha| \arrow[r, "f"] & X \end{tikzcd}$$

which are equivalent via

$$\begin{tikzcd} B^n = |b_n| \arrow[r, shift left, "|s|"] \arrow[r, shift right, "|t|"'] \arrow[d, "p"'] & |\Sigma b_n| = B^{n+1} \arrow[d, dashed] & \\ |\alpha| \arrow[r, shift left, "|s|"] \arrow[r, shift right, "|t|"'] & |\Sigma\alpha| \arrow[r, "f"] & X. \end{tikzcd}$$

Here we are writing b_n for the n-pasting diagram consisting of a single n-cell, thus $|b_n| = B^n$ and $|\Sigma b_n| = B^{n+1}$ so we have the maps

$$B^n \underset{t}{\overset{s}{\rightrightarrows}} B^{n+1} \xrightarrow{f} X$$

as in diagram (4.1) as required.

Next we show that all parallel maps $B^n \xrightarrow{p} |\alpha|$ induce the same map on equivalence classes. That is, given

$$B^n \underset{p'}{\overset{p}{\rightrightarrows}} |\alpha|$$

agreeing on all boundaries, we show that for any $|\alpha| \xrightarrow{f} X$ the induced maps

$$B^n \underset{p'}{\overset{p}{\rightrightarrows}} |\alpha| \xrightarrow{f} X$$

are equivalent elements of $\mathcal{S}(B^n, X)$ by expressing them as

$$|\alpha| \underset{|t|}{\overset{|s|}{\rightrightarrows}} |\Sigma\alpha| \xrightarrow{f} X$$

as in diagram (4.2).

In fact since $|\alpha|$ is contractible we have

$$\begin{tikzcd}
 & B^{n+1} \arrow[d, dashed, "f"] & \\
B^n \arrow[ur, shift left, "s"] \arrow[ur, shift right, "t"'] \arrow[r, shift left, "p"] \arrow[r, shift right, "p'"'] & {|\alpha|} \arrow[r, "f"'] & X
\end{tikzcd}$$

commuting serially giving an expression of the form of (4.2) as required. $\square$

We now turn our attention to the more topological case. We use the monad S_n for "free n-categories internal to Top"; this monad is constructed in the same way as the free strict n-category monad (for n-categories internal to Set), except that we take pullbacks in Top instead of in Set. It follows immediately that S_n is p.r.a. We will also call this monad the "free topological n-category monad".

Now note that we can construct $\mathcal{P}_n$ via the usual Kan extension construction as below: we start with a functor $\mathbb{G}_n \xrightarrow{D} \mathsf{Top}$ where $D(n) = B^n$

the topological n-ball, and form the usual induced functor which we will temporarily call V

$$V(X) = \mathcal{S}(D-, X) \in \mathsf{Top}^{\mathbb{G}_n^{\mathrm{op}}}$$

and we then post compose with a functor

$$\mathsf{Top}^{\mathbb{G}_n^{\mathrm{op}}} \longrightarrow \mathsf{Top}^{\mathbb{G}_n^{\mathrm{op}}}$$

which leaves the top dimension the same but at every lower dimension takes the indiscrete space on the underlying set of points. Note that this construction uses the functors O producing the underlying set of points and I producing the indiscrete space:

$$\begin{tikzcd} \mathsf{Top} \arrow[r, shift left, "O"] \arrow[r, phantom, "\perp"] & \mathsf{Set} \arrow[l, shift left, "I"] \end{tikzcd}$$

Given an n-globular space

$$X_n \overset{s}{\underset{t}{\rightrightarrows}} X_{n-1} \overset{s}{\underset{t}{\rightrightarrows}} \quad \dots \quad \overset{s}{\underset{t}{\rightrightarrows}} X_0$$

we produce the n-globular space

$$X_n \overset{s}{\underset{t}{\rightrightarrows}} OIX_{n-1} \overset{s}{\underset{t}{\rightrightarrows}} \quad \dots \quad \overset{s}{\underset{t}{\rightrightarrows}} OIX_0$$

and with the source and target maps on n-cells proceeding via the counit of the adjunction $O \dashv I$. We will call this functor $OI_{<n}$.

Thus we have the following situation giving the functor $\mathcal{P}_n$:

$$\begin{tikzcd} \mathbb{G}_n \arrow[r, "y_{\mathbb{G}}"] \arrow[d, "D"'] & \mathsf{Top}^{\mathbb{G}_n^{\mathrm{op}}} \arrow[r, "OI_{<n}"] & \mathsf{Top}^{\mathbb{G}_n^{\mathrm{op}}} \\ \mathsf{Top} \arrow[ur, "V"] \arrow[urr, dashed, "\mathcal{P}_n"'] & & \end{tikzcd}$$

Lemma 4.13. *The functor $\mathcal{P}_n$ is a right adjoint.*

Proof. As Top is complete, well-powered, and has a cogenerator, it suffices to check that $\mathcal{P}_n$ preserves limits. As limits in $\mathsf{Top}^{\mathbb{G}_n^{\mathrm{op}}}$ are computed pointwise it suffices to check that for each object $k \in \mathbb{G}_n^{\mathrm{op}}$ the composite

$$\mathsf{Top} \xrightarrow{\mathcal{P}_n} \mathsf{Top}^{\mathbb{G}_n^{\mathrm{op}}} \xrightarrow{\mathsf{eval}_k} \mathsf{Top}$$

preserves limits. When $k = n$ this is just V, which we know is a right adjoint so preserves limits. When $k < n$ this is OI, and this is a composite of right adjoints so preserves limits. $\square$

Theorem 4.14. *Write S_n for the free topological n-category monad on $\mathsf{Top}^{\mathbb{G}_n^{\mathrm{op}}}$. Then there is a universal S_n-operad acting on $\mathcal{P}_n$ whose m-cells are those of E_U for $m < n$, and whose space of n-cells of arity α is the* space *of coglobular maps $B^n \longrightarrow |\alpha|$.*

Note that $S_n 1$ is discrete at every dimension—it is in fact the same as $T_n 1$, just with each set of k-cells realised as a discrete space.

Proof. As S_n is p.r.a. and $\mathcal{P}_n$ is a right adjoint we may use Theorem 2.4 and compute the internal hom in the slice category

$$[\mathsf{Top}, \mathsf{Top}^{\mathbb{G}_n^{\mathrm{op}}}]/S_{n*}1.$$

Note that, under the equivalences of Corollary 2.9 and by Remark 2.5 we can use equivalences

$$\begin{aligned} [\mathsf{Top}, \mathsf{Top}^{\mathbb{G}_n^{\mathrm{op}}}]/S_{n*}1 &\simeq [\mathsf{Top}, \mathsf{Top}^{(\mathbb{G}_n^{\mathrm{op}}/S_n 1)}] \\ &\simeq [\mathsf{Top} \times \mathbb{G}_n^{\mathrm{op}}/S_n 1,\ \mathsf{Top}] \end{aligned}$$

Now $\mathcal{P}_n$ agrees with E_U everywhere except dimension n so this is the only dimension we need to consider here. In fact $\mathcal{P}_n$ agrees at dimension n as well if we simply "reinterpret" the notation

$$\mathcal{P}_n(X)(n) = \mathsf{Top}(B^n, X)$$

where this must now mean the *space* of maps $B^n \longrightarrow X$.

Now, following the proof of Theorem 4.8, to find the spaces of n-cells of arity α of $\mathsf{ev}_\emptyset(\mathsf{End}\,(U))$ we must calculate

$$\big[\mathsf{Top} \times (\mathbb{G}_n^{\mathrm{op}}/S_n 1), \mathsf{Top}\big]\big(H^{(\emptyset,n,\alpha)} \times \bar{A}, \bar{B}\big)$$

where

$$\begin{aligned} \bar{A}(X, m, \beta) &= \mathsf{Top}(|\beta|, X) \\ \bar{B}(X, m, \beta) &= \mathsf{Top}(B^m, X) \end{aligned}$$

interpreted as *spaces* of maps. We now have to calculate this as a space of enriched natural transformations using the end formula

$$\int_{(X,m,\beta)} \mathsf{Top}\Big(\ H^{(\emptyset,n,\alpha)}(X,m,\beta) \times \mathsf{Top}(|\beta|,X)\ ,\ \mathsf{Top}(B^m,X)\ \Big).$$

As before we have

$$H^{(\emptyset,n,\alpha)}(X,m,\beta) = \begin{cases} 1 & n=m, \alpha=\beta \\ \{s,t\} & n>m, \beta=\partial\alpha \\ \emptyset & \text{otherwise} \end{cases}$$

Fixing $\alpha = \beta$ we get

$$\int_{X\in\mathsf{Top}} \mathsf{Top}\ \big(\mathsf{Top}(|\alpha|,X)\ ,\ \mathsf{Top}(B^n,X)\ \big) = \mathsf{Top}(B^n,\alpha)$$

by enriched Yoneda. The rest of the end formula gives the same commuting conditions as before.

Note that evaluating the endomorphism operad $\mathsf{End}\,(\Pi_n)$ at (X,m,β) where $m<n$ gives the same answer as for $\mathsf{End}\,(U)$, expressed as a discrete space, so this internal hom is indeed in our full subcategory as required. □

Example 4.15. Operads acting on path spaces.

Note that the case $n = 1$ gives us operads acting on path spaces, but not in the most obvious sense as the operads in question will not be classical operads but S_1-operads.

The monad S_1 is the free topological category monad on what we might call $\mathsf{Gph(Top)}$, the category of graphs in spaces; $S_1 1$ has a single object, and its single hom-space is the discrete space $\mathbb{N}$. Thus an S_1-operad P has as its underlying data

- a set P_0 of objects, and
- for every pair a,b of objects and every arity $k \in \mathbb{N}$ a space of operations.

In particular any classical operad can be expressed as an S_1-operad with a single object; this is the "suspension" operation used in [4].

The functor $\mathcal{P}_1 : \mathsf{Top} \longrightarrow \mathsf{Top}^{\mathbb{G}_1^{\mathrm{op}}}$ takes a space X and produces the globular space with

- objects the indiscrete space on the points of X, and
- the hom-space is the space of paths in X, $\mathsf{Top}(I, X)$.

Note that this is not quite the same as a Top-enriched graph, which, essentially, would treat the hom-space as a disjoint union of individual hom-spaces $X(x, y)$.

The theorem then gives us the universal S_1-operad acting on this sense of path space, and examining the construction in that case shows that it is the suspension of the operad used by Trimble, which has $E(k) \in \mathsf{Top}$ is the space of continuous, endpoint-preserving maps $[0, 1] \longrightarrow [0, k]$. Thus we can say that Trimble's operad has the following universal property: its suspension is the universal S_1-operad acting on path spaces.

Note the notion of "operad acting on path spaces" can be defined directly without going via S_1-operads (see for example [11]), but making an abstract version of that approach is tricky. Trimble [18] uses the category Bip of bipointed spaces; however this is not a straightforward generalisation of the use of Top_* for loop spaces. For loop spaces we have

$$\begin{array}{rcl} \Omega X & = & \mathsf{Top}_*(S, X) \\ (\Omega X)^k & = & \mathsf{Top}_*(S^{\vee k}, X); \end{array}$$

for path spaces we can try to replace the circle S with the interval I regarded as a bipointed space via its endpoints, giving

$$X(x, y) = \mathsf{Bip}\big(\{I, 0, 1\}, \{X, x, y\}\big).$$

However raising this to the power of k does not give us a string of k composable paths as we require. Trimble instead expresses the action on path spaces using "operads in topological profunctors".

4.4 Non-universal examples

In this final section we discuss non-universal versions of the operads studied in the previous sections. One class of non-universal examples comes from applying the work of [4]. Recall that in this work we showed that every Trimble n-category is a Batanin n-category. One part of this takes a classical operad acting on path spaces and iteratively produces from it an n-globular operad acting on n-path spaces for any n.

Applying the construction to the universal operad E (regarded as a classical operad) gives a suboperad $\bar{E}$ of G. Thus, by the main theorem, $\bar{E}$ acts on n-path spaces. Given any non-universal example P we have a canonical morphism $P \longrightarrow E$ giving rise to a morphism $\bar{P} \longrightarrow \bar{E}$ and hence $\bar{P} \longrightarrow \bar{E} \longrightarrow G$. So $\bar{P}$ also acts on n-path spaces.

Thus for example we can apply this to the little intervals operad or other non-universal classical operads and get a smaller operad acting on n-path spaces. However this general method only allows us to control the operations at the lowest dimension. The 1-cells of $\bar{P}$ are formed from the points of $P(k)$, but the 2-cells also involve the paths of $P(k)$ and the 3-cells involve the homotopies between paths, and so on. This approach suffices for some purposes and in future work we will use it to show that doubly degenerate Trimble 3-categories, parametrised by the little intervals operad, give braided monoidal categories in a suitable sense.

However for more general results more control over the higher dimensions of the globular operad may be desirable. This cannot be done automatically using the machinery of [4], but the present work gives us a first step in the direction of being able to construct more tractable non-universal operads suitable for proving results about weak n-categories.

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Eugenia Cheng
School of the Art Institute of Chicago
112 S Michigan Avenue
Chicago
IL 60603, USA
info@eugeniacheng.com

Todd Trimble
Western Connecticut State University
181 White St
Danbury
CT 06810, USA
trimblet@wcsu.edu