CAHIERS DE TOPOLOGIE ET GEOMETRIE DIFFERENTIELLE CATEGORIQUES Vol. L-1 (2009)

ON BOUNDEDNESS AND SMALL-ORTHOGONALITY CLASSES

Dedicated to Jiří Adámek on the occasion of his sixtieth birthday

by Lurdes SOUSA

Abstract

Une caractérisation des catégories localement bornées et un critère pour identifier les sous-catégories α-orthogonales dans ces catégories (pour un cardinal régulier α) sont donnés.

1 Introduction

In [11], P. Gabriel and F. Ulmer proved that in locally presentable categories the orthogonal subcategory $\mathcal{N}^\perp$ is reflective for any set $\mathcal{N}$ of morphisms. The key point of the proof is the fact that for any object of the base category there is some infinite regular cardinal α such that the object is α-small, where α-smallness means α-presentability. In [10] and [15], P. Freyd and M. Kelly gave a generalization of this property for a wider range of categories, using a different concept of smallness for objects: boundedness. They showed that in a locally bounded category (as defined in [14] and [17]) the subcategory of all objects orthogonal to a set of morphisms is reflective. (In fact they went further: they proved that $\mathcal{N}^\perp$ is reflective for every class $\mathcal{N}$ which is the union of a set of morphisms with a class of epimorphisms.)

In a cocomplete category $\mathcal{A}$ an object A is said to be α-bounded if the hom-functor $\mathcal{A}(A,-)$ preserves α-directed unions. A locally bounded category (see [14]) is a complete and cocomplete category $\mathcal{A}$ with a proper factorization system $(\mathcal{E},\mathcal{M})$ and an $\mathcal{E}$-generator $\mathcal{G}$ such that (i) $\mathcal{A}$ has $\mathcal{E}$-cointersections and (ii) there is a regular cardinal α such that each object of $\mathcal{G}$ is α-bounded. We call these categories *locally α-bounded* when they are $\mathcal{E}$-cowellpowered and α is a regular cardinal which fits the condition (ii). Locally presentable categories and epi-reflective subcategories of the category of topological spaces are examples of locally bounded categories. We

Financial support by the Centre for Mathematics of the University of Coimbra and by the School of Technology of Viseu is acknowledged.

show that a cocomplete and cowellpowered category is locally bounded precisely when there is a regular cardinal α and a set $\mathcal{H}$ of α-bounded objects such that any object A of $\mathcal{A}$ is an α-directed union of objects of $\mathcal{H}$. This characterization will be useful in the study of small-orthogonality classes, that is, subcategories of the form $\mathcal{N}^\perp$ for $\mathcal{N}$ a set of morphisms.

In [13] the α-orthogonality classes of a locally α-presentable category were proved to be exactly the subcategories closed under limits and α-directed colimits, for all uncountable regular cardinals α. (Recall that, following [4], an α-orthogonality class is a subcategory of the form $\mathcal{N}^\perp$ for some set $\mathcal{N}$ of morphisms whose domains and codomains are α-presentable.) This characterization does not work for $\alpha = \aleph_0$, as was shown in [20] and [12]. A description of the $\aleph_0$-orthogonality classes in locally finitely presentable categories in terms of closure properties was given in [5]: they are the subcategories $\mathcal{A}$ closed under products, directed colimits and $\mathcal{A}$-pure subobjects. In the context of locally bounded categories we shall adopt the terminology *α-orthogonality class* as expected: the meaning is as in [4], just replacing "presentable" by "bounded". The aim of this paper is to characterize the reflective subcategories of locally bounded categories which are small-orthogonality classes. In cowellpowered locally bounded categories a subcategory is a small-orthogonality class iff it is an α-orthogonality class for some α. We are going to restrict ourselves to reflective subcategories whose reflector preserves $\mathcal{M}$-monomorphisms. For example, reflective subcategories of $\mathbf{Top}$ whose closure under subspaces is the category $\mathbf{Top}_0$ of T_0 spaces have an $\mathcal{M}$-preserving reflector, for $\mathcal{M} = \{\text{embeddings}\}$. Also the reflector from the category $\mathbf{Norm}$ of normed spaces and linear contractions into its subcategory $\mathbf{Ban}$ of Banach spaces preserves embeddings. In [18] Ringel studied the properties of $\mathcal{M}$-preserving reflectors for $\mathcal{M}$ the class of monomorphisms. We show that, in locally α-bounded categories, a reflective subcategory with an $\mathcal{M}$-preserving reflector is an α-orthogonality class iff it is closed under α-directed unions and α-$\mathcal{B}$-neat subobjects. (The notion of α-$\mathcal{B}$-neat morphism is parallel to the one of α-$\mathcal{B}$-pure morphism, used in [5]: If $\mathcal{B}$ is a subcategory of $\mathcal{A}$, a morphism $f : A \to B$ of $\mathcal{A}$ is said to be *α-$\mathcal{B}$-neat* provided that, if we have morphisms e, u and v such that $f \cdot u = v \cdot e$ and e is a $\mathcal{B}$-epimorphism, then there exists a morphism u' such that $u' \cdot e = u$.) For instance, the category $\mathbf{Top}_0$ is an $\aleph_0$-orthogonality class of $\mathbf{Top}$, but the category $\mathbf{Sob}$ of sober spaces is not an $\aleph_0$-orthogonality class of $\mathbf{Top}_0$. The category $\mathbf{Ban}$ is an $\aleph_1$-orthogonality class of $\mathbf{Norm}$.

2 Properties of locally bounded categories

Let $\mathcal{A}$ be a category with a proper factorization system $(\mathcal{E}, \mathcal{M})$ (where proper means that $\mathcal{E}$ and $\mathcal{M}$ consist of epimorphisms and monomorphisms, respectively). Recall that $\mathcal{E}$ and $\mathcal{M}$ determine each other: $\mathcal{E} = \mathcal{M}^{\uparrow}$ and $\mathcal{M} = \mathcal{E}^{\downarrow}$ ([10]).

A set $\mathcal{G}$ is said to be an *$\mathcal{E}$-generator* of $\mathcal{A}$ if for each object A there is some subset $\{G_i,\ i \in I\}$ of $\mathcal{G}$ and an $\mathcal{E}$-morphism $e : \amalg_{i\in I} G_i \to A$. (A detailed study of $\mathcal{E}$-generators is made in, e.g., [6] and [7].)

Let $m_i : A_i \to A,\ i \in I$, be a diagram in $\mathcal{A}$ with all $m_i \in \mathcal{M}$. The *$\mathcal{M}$-union* (or just *union*) of $(m_i)_{i\in I}$ is the supremum of $(m_i)_{i\in I}$, up to isomorphism, in the class of all $\mathcal{M}$-subobjects of A. It coincides with the $\mathcal{M}$-part $m : B \to A$ of the $(\mathcal{E}, \mathcal{M})$-factorization of the canonical morphism $\amalg_{i\in I} A_i \to A$. We shall often write $\cup_{i\in I} m_i = m$ or $\cup_{i\in I} A_i = B$ for short.

Let α be an infinite regular cardinal. An object A is said to be *α-bounded* if the hom-functor $\mathcal{A}(A, -)$ preserves α-directed unions.

2.1. Definition (1) ([14], [17]) A category $\mathcal{A}$ is said to be *locally bounded* if it is cocomplete, has a proper factorization system $(\mathcal{E}, \mathcal{M})$, and there is an infinite regular cardinal α such that:

(i) $\mathcal{A}$ has $\mathcal{E}$-cointersections;

(ii) $\mathcal{A}$ has an $\mathcal{E}$-generator all of whose objects are α-bounded.

(2) By a *locally α-bounded category with respect to $\mathcal{M}$* we shall mean a category under the conditions of (1), for a given α, which moreover is $\mathcal{E}$-cowellpowered. The reference to $\mathcal{M}$ will often be omitted.

2.2. Remark Every locally bounded category is complete. In [14] and [17], the authors include completeness in the definition of locally bounded category. However the completeness comes for free, since any $\mathcal{E}$-cocomplete category with an $\mathcal{E}$-generator is complete. This follows from the fact that any such category is total (see [7]), that is, the Yoneda embedding $\mathcal{A} \hookrightarrow [\mathcal{A}^{\mathrm{op}}, \mathbf{Set}]$ has a left adjoint ([16]); and any total category is complete and $\mathcal{M}$-complete (see [7] and [8]).

2.3. Examples (1) Every locally presentable category is locally bounded with respect to monomorphisms, and also with respect to strong monomorphisms (see [10] and [2]).

(2) The category **Top** of topological spaces is locally $\aleph_0$-bounded with respect to strong monomorphisms (= embeddings). And every epi-reflective subcategory

of **Top** is locally $\aleph_0$-bounded with respect to embeddings. More generally, any $\mathcal{E}$-reflective subcategory $\mathcal{B}$ of a locally α-bounded category with respect to $\mathcal{M}$ is also locally α-bounded with respect to $\mathcal{M} \cap Mor(\mathcal{B})$ ([10], [2]).

(3) Any topological category over **Set** (see [3]) is locally $\aleph_0$-bounded with respect to strong monomorphisms.

(4) The category **Ban** of Banach spaces and linear contractions is locally $\aleph_1$-bounded ([14], [17]).

2.4. Remark The following properties are easily verified:

(i) In a locally bounded category, for every object A there is an infinite regular cardinal α such that A is α-bounded ([10], 3.1.2).

(ii) In a cocomplete category if β and γ are regular cardinals such that $\beta \leq \gamma$, then every β-bounded object is also γ-bounded; consequently, the fulfillment of 2.1 for $\alpha = \beta$ ensures that it also holds for $\alpha = \gamma$.

2.5. Lemma *In a cocomplete category with a proper factorization system $(\mathcal{E}, \mathcal{M})$ any $\mathcal{E}$-quotient of an α-bounded object is α-bounded.*

Proof Let B be α-bounded, let $e : B \to E$ belong to $\mathcal{E}$ and let

$$C_i \xrightarrow{\quad n_i \quad} C \quad (i \in I)$$

be an α-directed $\mathcal{M}$-union, that is, $1_C = \cup_{i \in I} n_i$. Given $f : E \to C$, there are some i and some morphism $f' : B \to C_i$ such that $f \cdot e = n_i \cdot f'$. Then, since $n_i \in \mathcal{M}$ and $e \in \mathcal{M}^{\uparrow}$, there exists $f'' : E \to C_i$ such that $f = n_i \cdot f''$. □

2.6. Remark The property stated in Lemma 2.5 is in contrast to the case of α-presentability: a quotient of an α-presentable object is not necessarily α-presentable (see Remark 1.3 of [4]).

2.7. Lemma *In a cocomplete category with a proper $(\mathcal{E}, \mathcal{M})$ factorization system:*

(i) any α-small colimit of α-bounded objects is α-bounded;

(ii) any α-small union of α-bounded objects is α-bounded.

Proof (i) We are going to prove the statement for the particular case of coproducts. Then the result follows for colimits taking into account Lemma 2.5 and the fact that $\mathcal{M} \subseteq Mono$ implies that $RegEpi \subseteq \mathcal{E}$.

Let A_k $(k \in K)$ be an α-small set of α-bounded objects. Let $c_i : C_i \to C$ $(i \in I)$ be an α-directed union, and consider a morphism $d : \amalg_{k \in K} A_k \to C$. Since every A_k is α-bounded, there are morphisms $f_k : A_k \to C_{i_k}$ such that $d \cdot \nu_k = c_{i_k} \cdot f_k$ for all k (where ν_k are the injections of the coproduct). Since K is α-small and

I is α-directed, there is some $i \in I$ such that $i_k \leq i$, $k \in K$. Then, putting $g_k = (\ A_k \xrightarrow{f_k} C_{i_k} \longrightarrow C_i\ )$, we obtain $c_i \cdot g_k = d \cdot \nu_k$. Let $h : \amalg A_k \to C_i$ be the morphism determined by the morphisms g_k and the universality of the coproduct. Then we have $d = c_i \cdot h$.

(ii) Let $m_k : A_k \to A$ $(k \in K)$ be a union (not necessarily α-directed) with K α-small and all A_k α-bounded. Let $c_i : C_i \to C$ $(i \in I)$ be an α-directed union, and consider a morphism $f : A \to C$. Since $1_A = \cup_{k \in K} m_k$, the induced canonical morphism $e : \amalg A_k \to A$ belongs to $\mathcal{E}$. Put

$$d = f \cdot e$$

and let i and $h : \amalg A_k \to C_i$ be obtained as in (i). Then, we have the following commutative diagram:

$$\begin{array}{ccc} \amalg A_k & \xrightarrow{e} & A = \cup A_k \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle f} \\ C_i & \xrightarrow{c_i} & C \end{array}$$

By the diagonal fill-in property, there exists a morphism $t : A \to C_i$ such that $c_i \cdot t = f$. $\square$

2.8. Theorem *Let $\mathcal{A}$ be a cocomplete and $\mathcal{E}$-cowellpowered category with a proper factorization system $(\mathcal{E}, \mathcal{M})$. The following conditions are equivalent:*

(i) $\mathcal{A}$ is locally α-bounded with respect to $\mathcal{M}$.

(ii) There is a set $\mathcal{H}$ of α-bounded objects such that any object of $\mathcal{A}$ is an α-directed $\mathcal{M}$-union of objects of $\mathcal{H}$.

Proof (ii) $\Rightarrow$ (i): It is clear that if $\mathcal{H}$ is a set as in (ii), then it is an $\mathcal{E}$-generator of $\mathcal{A}$. In fact, given $A \in \mathcal{A}$, let $H_i \xrightarrow{m_i} A$ $(i \in I)$ be an α-directed $\mathcal{M}$-union, with all H_i in $\mathcal{H}$. This means exactly that the induced canonical morphism $\amalg H_i \to A$ belongs to $\mathcal{E}$.

(i) $\Rightarrow$ (ii): Let $\mathcal{G}$ be an $\mathcal{E}$-generator of $\mathcal{A}$ with all objects α-bounded. The class of objects

$$\mathcal{H} = \{\ \mathcal{E}\text{-quotients of } \alpha\text{-small coproducts of objects of } \mathcal{G}\}$$

is essentially small, because $\mathcal{G}$ is small and $\mathcal{A}$ is $\mathcal{E}$-cowellpowered. Moreover, from 2.5 and 2.7, the objects of $\mathcal{H}$ are α-bounded. We show that $\mathcal{H}$ fulfils (ii).

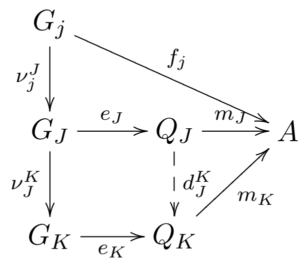
Let $A \in \mathcal{A}$, and let

$$\{f_i : G_i \to A,\, i \in I\} = \bigcup_{G \in \mathcal{G}} \mathcal{A}(G, A)\,.$$

Let

$$\mathcal{J} = \{J \subseteq I \,:\, J \text{ is } \alpha\text{-small}\}.$$

Consider the following commutative diagram



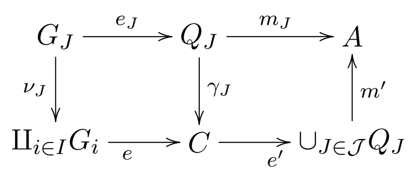
where:

- $G_J = \amalg_{j \in J} G_j$ and the morphisms ν_j^J are the corresponding injections;
- for each $J \subseteq K$, $\nu_J^K : G_J \to G_K$ is the obvious canonical morphism;
- $f_J : G_J \to A$ is the morphism determined by f_j, $j \in J$;
- $m_J \cdot e_J$ is the $(\mathcal{E}, \mathcal{M})$ factorization of $f_J : G_J \to A$;
- for each $J \subseteq K$, $d_J^K : Q_J \to Q_K$ is the morphism given by the diagonal fill-in property applied to the equality $(m_K \cdot e_K) \cdot \nu_J^K = m_J \cdot e_J$.

For $\mathcal{J}$ equipped with the inclusion order, both the diagrams

$$\left(\nu_J^K : G_J \to G_K\right)_{J \subseteq K,\, J,K \in \mathcal{J}} \qquad \text{and} \qquad \left(d_J^K : Q_J \to Q_K\right)_{J \subseteq K,\, J,K \in \mathcal{J}}$$

are α-directed. Moreover the colimit of the former one is $\amalg_{i \in I} G_i$. Let $\gamma_J : Q_J \to C = \operatorname{Colim} Q_J$ be the colimit cocone of the latter one. Then there is a morphism $e : \amalg_{i \in I} G_i \to C$ making the left-hand square of the following diagram commutative.



The morphism e belongs to $\mathcal{E}$, since all e_J do. Let $m' \cdot e'$ be the $(\mathcal{E}, \mathcal{M})$ factorization of the canonical morphism from C to A determined by the morphisms m_J. By hypothesis, $m' \cdot (e' \cdot e) : \amalg_{i \in I} G_i \to A$ belongs to $\mathcal{E}$ (because $\mathcal{G}$ is an $\mathcal{E}$-generator). Consequently, m' lies in $\mathcal{E}$, and, since it also belongs to $\mathcal{M}$, is an isomorphism, that is, A is an union of the $\mathcal{M}$-subobjects

$$m_J : Q_J \to A,\ J \in \mathcal{J}. \qquad \square$$

2.9. Corollary *A locally bounded category is $\mathcal{E}$-cowellpowered iff for every regular infinite cardinal β the class of all β-bounded objects is essentially small.*

Proof Let $\mathcal{A}$ be locally α-bounded. Without loss of generality we assume that $\beta \geq \alpha$. Then $\mathcal{A}$ is also locally β-bounded and has a set $\mathcal{H}$ of β-bounded objects such that any object of $\mathcal{A}$ is an $\mathcal{M}$-union of objects of $\mathcal{H}$. Given a β-bounded object A let $m_i : H_i \to A \ \ (i \in I)$ be that existing union. The β-boundedness of A implies the equality $m_i \cdot t = 1_A$ for some $t : A \to H_i$. But then $A \simeq H_i$.

Conversely, let $\mathcal{A}$ be a category fulfilling the conditions of 2.1(1), and such that for every regular infinite cardinal β the class of all β-bounded objects is essentially small. Given an object X of $\mathcal{A}$, there is some regular infinite cardinal β such that X is β-bounded (see 2.4(i)). Consequently, by 2.5, the class of $\mathcal{E}$-quotients of X has a representative set. $\square$

3 Small-orthogonality classes

In this section we study the following problem: When is a reflective subcategory[1] $\mathcal{B}$ of a locally bounded category $\mathcal{A}$ a *small-orthogonality class*, i.e., a category of the form $\mathcal{N}^{\perp}$, for $\mathcal{N}$ a set of morphisms? In this study we restrict ourselves to the particular case of the reflector $R : \mathcal{A} \to \mathcal{B}$ preserving $\mathcal{M}$-monomorphisms. More precisely, we characterize those reflective subcategories of a locally α-bounded category with an $\mathcal{M}$-preserving reflector which are of the form $\mathcal{N}^{\perp}$ with all morphisms of $\mathcal{N}$ having α-bounded domains and codomains.

In the case of locally presentable categories the subcategories of the form $\mathcal{N}^{\perp}$ for $\mathcal{N}$ a set of morphisms with α-presentable domains and codomains were characterized in [13] and [5] (see Introduction).

Throughout this section by an *α-orthogonality class* of a locally bounded category we shall mean a subcategory of the form $\mathcal{N}^{\perp}$ for some set $\mathcal{N}$ whose all morphisms have α-bounded domains and codomains. We borrow this terminology from [4] using boundedness instead of presentability.

3.1. Remark Recall that, for a subcategory $\mathcal{B}$ of $\mathcal{A}$, a morphism $g : C \to D$ of $\mathcal{A}$ is said to be a *$\mathcal{B}$-epimorphism* if for any pair of morphisms $a, \, b : D \to B$ with $B \in \mathcal{B}$, the equality $a \cdot g = b \cdot g$ implies $a = b$.

Let $\mathcal{A} = \mathbf{Top}$. If $\mathcal{B} = \mathbf{Haus}$ the $\mathcal{B}$-epimorphisms are just the dense morphisms of $\mathbf{Top}$. If $\mathcal{B} = \mathbf{Top}_0$ the $\mathcal{B}$-epimorphisms are the b-dense morphisms, i.e., the continuous maps $f : X \to Y$ such that $\overline{\{y\}} \cap H \cap f(X) \neq \emptyset$ for each $y \in Y$ and

[1] Throughout this paper all subcategories are assumed to be full and isomorphism-closed.

each open set H of Y containing y. More generally, if $\mathcal{A}$ has equalizers and a proper factorization system $(\mathcal{E}, \mathcal{M})$, then for any subcategory $\mathcal{B}$ of $\mathcal{A}$ the $\mathcal{B}$-epimorphisms are the morphisms which are dense with respect to the regular closure operator induced in $\mathcal{A}$ by $\mathcal{B}$ ([9]).

If $\mathcal{B}$ is reflective in $\mathcal{A}$ it is easy to see that the $\mathcal{B}$-epimorphisms are just those morphisms of $\mathcal{A}$ whose image by the reflector is an epimorphism in $\mathcal{B}$.

3.2. Definition Let $\mathcal{A}$ be a locally bounded category and let $\mathcal{B}$ be a subcategory of $\mathcal{A}$. A morphism $f : A \to B$ of $\mathcal{A}$ is said to be *α-$\mathcal{B}$-neat* provided that in each commutative diagram

$$\begin{array}{ccc} C & \xrightarrow{g} & D \\ {\scriptstyle u}\downarrow & & \downarrow{\scriptstyle v} \\ A & \xrightarrow{f} & B \end{array}$$

with C and D α-bounded and g a $\mathcal{B}$-epimorphism, u factorizes through g, i.e., $u = u' \cdot g$ for some u'.

3.3. Remark The following properties are easily established (compare with the properties of $\mathcal{B}$-pure morphisms in [5]):

(i) The composition of α-$\mathcal{B}$-neat morphisms is an α-$\mathcal{B}$-neat morphism.

(ii) If $f \cdot g$ is α-$\mathcal{B}$-neat than g is α-$\mathcal{B}$-neat.

(iii) Every γ-$\mathcal{B}$-neat morphism is α-$\mathcal{B}$-neat for $\gamma \geq \alpha$.

(iv) All α-$\mathcal{B}$-neat morphisms are monomorphisms; and every equalizer is an α-$\mathcal{B}$-neat morphism.

(v) If $\mathcal{B}$ is cogenerating in $\mathcal{A}$, then

$$StrongMono(\mathcal{A}) \subseteq \{\alpha\text{-}\mathcal{B}\text{-neat morphisms}\}.$$

The last statement follows from the fact that, in this case, every $\mathcal{B}$-epimorphism is an epimorphism in $\mathcal{A}$.

3.4. Proposition *Let $\mathcal{A}$ be a locally α-bounded category with respect to $\mathcal{M}$. Then any α-orthogonality class of $\mathcal{A}$ is a reflective subcategory of $\mathcal{A}$ which is*

(i) closed under α-directed $\mathcal{M}$-unions;

(ii) locally α-bounded with respect to $\mathcal{M}' = \mathcal{M} \cap Mor(\mathcal{B})$;

(iii) closed under α-$\mathcal{B}$-neat subobjects.

Proof Let $\mathcal{B} = \mathcal{N}^{\perp}$ for $\mathcal{N}$ a set of morphisms in $\mathcal{A}$ with α-bounded domains and codomains. From [10], we know that $\mathcal{B}$ is reflective and has an $(\mathcal{E}', \mathcal{M}')$ proper factorization system, with $\mathcal{E}' = (\mathcal{M}')^{\uparrow}$. Moreover, cowellpoweredness of $\mathcal{A}$ with respect to $\mathcal{E}$ implies $\mathcal{E}'$-cowellpoweredness of $\mathcal{B}$. Let $R : \mathcal{A} \to \mathcal{B}$ be the reflector.

(i) Let

$$b_i : B_i \to Z \qquad (i \in I)$$

be an α-directed $\mathcal{M}$-union in $\mathcal{A}$ with all $B_i \in \mathcal{B}$. We want to show that $Z \in \mathcal{B} = \mathcal{N}^{\perp}$. Let $h : X \to Y$ be a morphism of $\mathcal{N}$ and let $f : X \to Z$. Since X is α-bounded there is some i and some $f' : X \to B_i$ such that $b_i \cdot f' = f$. The morphism f' factorizes through h, because $B_i \in \mathcal{B}$, and, hence, so does the morphism f. To show the uniqueness of the last factorization, let $y, y' : Y \to Z$ be such that $y \cdot h = y' \cdot h$. Since Y is α-bounded, we can find $k \in I$ and $t, t' : Y \to B_k$ such that $y = b_k \cdot t$ and $y' = b_k \cdot t'$. Now the equality $b_k \cdot t \cdot h = b_k \cdot t' \cdot h$, the orthogonality of B_k to h and the fact that $b_k \in \mathcal{M}$ imply that $t = t'$, thus $y = y'$.

(ii) Of course $\mathcal{B}$ is cocomplete. Moreover:

(a) If X is an α-bounded object of $\mathcal{A}$, then RX is an α-bounded object of $\mathcal{B}$. This is clear since, from (i), every α-directed $\mathcal{M}'$-union in $\mathcal{B}$ is an α-directed $\mathcal{M}$-union in $\mathcal{A}$.

(b) If $\mathcal{G}$ is an $\mathcal{E}$-generator of $\mathcal{A}$ then it is well known that $R(\mathcal{G})$ is an $\mathcal{E}'$-generator of $\mathcal{B}$ ([10]). In fact, let $A \in \mathcal{B}$, and let $e : \amalg_{i\in I} G_i \to A$ be a morphism of $\mathcal{E}$ with all G_i in $\mathcal{G}$. Then the morphism $Re : \amalg_{i\in I} RG_i \to A$ belongs to $\mathcal{E}'$ since, as it is easily seen, $R(\mathcal{E}) \subseteq (\mathcal{M}')^{\uparrow}$.

(iii) Let $m : Z \to B$ be an α-$\mathcal{B}$-neat morphism with $B \in \mathcal{B}$. We want to show that $Z \in \mathcal{B}$. Let $h : X \to Y$ lay in $\mathcal{N}$. Given a morphism $f : X \to Z$, since $B \in \mathcal{N}^{\perp}$, we get f' such that $f' \cdot h = m \cdot f$. Because m is α-$\mathcal{B}$-neat, there is f'' such that $f'' \cdot h = f$. The uniqueness of f'' follows from the fact that $m \cdot f$ factors uniquely through h and m is a monomorphism. □

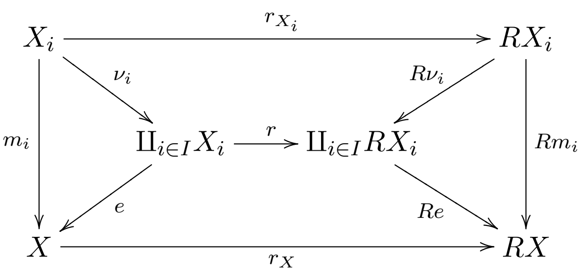
3.5. Remark Let $\mathcal{A}$ be a locally α-bounded category with respect to $\mathcal{M}$. Let $\mathcal{B}$ be a subcategory of $\mathcal{A}$ which is locally α-bounded with respect to $\mathcal{M} \cap Mor(\mathcal{B})$ and closed under limits and under α-directed $\mathcal{M}$-unions. Then $\mathcal{B}$ is reflective. In fact, the inclusion functor $\mathcal{B} \hookrightarrow \mathcal{A}$ fulfils the solution set condition: Given $A \in \mathcal{A}$, there is some regular cardinal $\lambda \geq \alpha$ such that A is λ-bounded in $\mathcal{A}$ and $\mathcal{B}$ is a locally λ-bounded category. Consequently, there is a set $\{B_i,\, i \in I\}$ of λ-bounded objects of $\mathcal{B}$ such that every object of $\mathcal{B}$ is a λ-directed $\mathcal{M} \cap Mor(\mathcal{B})$-union of B_i's. But, being closed in $\mathcal{A}$ under α-directed unions, $\mathcal{B}$ is also closed under λ-directed unions. Then, any morphism $g : A \to B$ with codomain in $\mathcal{B}$ factorizes through some of the objects B_i.

Next we want to characterize the reflective subcategories of a locally bounded category which are small-orthogonality classes. We restrict ourselves to reflective subcategories whose reflector preserves $\mathcal{M}$-monomorphisms. This kind of reflectors were studied by Ringel in [18], for $\mathcal{M} = \{\text{monomorphisms}\}$. $\mathbf{Top}_0$ and $\mathbf{Sob}$ are examples of subcategories of $\mathbf{Top}$ whose reflector preserves embeddings. Let $\mathbf{Sob}_\alpha$ denote the limit-closure in $\mathbf{Top}$ of the ordinal α regarded as a topological space with the Alexandrov topology. Both $\mathbf{Top}$ and $\mathbf{Top}_0$ have an $\{$embeddings$\}$-preserving reflector into $\mathbf{Sob}_\alpha$ (see [19]). Also the inclusion functor of the category $\mathbf{Ban}$ of Banach spaces into the category $\mathbf{Norm}$ of normed spaces and linear contractions has a reflector which preserves embeddings.

3.6. Theorem *Let $\mathcal{A}$ be a locally α-bounded category with respect to $\mathcal{M}$. Let $\mathcal{B}$ be a reflective subcategory of $\mathcal{A}$ whose reflector preserves morphisms of $\mathcal{M}$. Then $\mathcal{B}$ is an α-orthogonality class in $\mathcal{A}$ iff it is closed under α-directed $\mathcal{M}$-unions and α-$\mathcal{B}$-neat subobjects.*

Proof The necessity was proved in 3.4.

In order to prove the sufficiency, we first show that the reflector $R : \mathcal{A} \to \mathcal{B}$ preserves α-directed $\mathcal{M}$-unions. Given an α-directed $\mathcal{M}$-union $m_i : X_i \to X$ $(i \in I)$, we have commutative diagrams

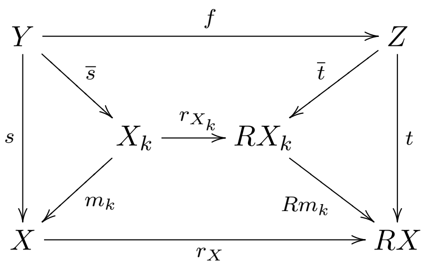


where $e \in \mathcal{E}$. But, as is easy to see, $R(\mathcal{E}) \subseteq \mathcal{E}' = (\mathcal{M}')^{\uparrow}$ for $\mathcal{M}' = \mathcal{M} \cap Mor(\mathcal{B})$. Then the morphisms $Rmi : RX_i \to RX$ form an $\mathcal{M}'$-union in $\mathcal{B}$.

To finish the proof, we show that, for

$$\mathcal{N} = \{h : X \to Y \text{ in } \mathcal{A},\ h \perp \mathcal{B},\ X, Y\ \alpha\text{-bounded}\},$$

$\mathcal{N}^{\perp} \subseteq \mathcal{B}$, and thus $\mathcal{B} = \mathcal{N}^{\perp}$. Let $X \in \mathcal{N}^{\perp}$. We show that the reflection $r_X : X \to RX$ of X in $\mathcal{B}$ is α-$\mathcal{B}$-neat; consequently, as $\mathcal{B}$ is closed under α-$\mathcal{B}$-subobjects, $X \in \mathcal{B}$. Let $f : Y \to Z$ be a $\mathcal{B}$-epimorphism with Y and Z α-bounded. Given morphisms $s : Y \to X$ and $t : Z \to RX$ such that $t \cdot f = r_X \cdot s$, let $m_i : X_i \to X$ be an α-directed $\mathcal{M}$-union in $\mathcal{A}$ with all X_i α-bounded. Then there is some $i \in I$ and $s' : Y \to X_i$ such that $m_i \cdot s' = s$. The closedness of $\mathcal{B}$ under α-directed $\mathcal{M}$-unions and the fact that Z is α-bounded implies the existence of some $j \in I$ and a morphism $t' : Z \to RX_j$ such that $Rm_j \cdot t' = t$. Since I is α-directed, we can then find $k \in I$ and morphisms $\overline{s}$ and $\overline{t}$ such that the following diagram is commutative (the commutativity of the upper quadrilateral is derived from the fact that Rm_k is monic):



Let $X_k \xrightarrow{f'} W \xleftarrow{s'} Z$ be the pushout of f along $\overline{s}$. Since $r_{X_k} \perp \mathcal{B}$, any morphism $g : X_k \to B$ with $B \in \mathcal{B}$ is factorizable through f'. Furthermore, as one easily sees, the pushout of a $\mathcal{B}$-epimorphism is also a $\mathcal{B}$-epimorphism. Hence $f' \perp \mathcal{B}$. The domain of f' is α-bounded, and from Lemma 2.7, also its codomain is α-bounded, then $f' \in \mathcal{N}$. Hence there is a morphism $n : W \to X$ such that $n \cdot f' = m_k$. Therefore, $n \cdot s'$ is the needed diagonal morphism, since $(n \cdot s') \cdot f = n \cdot f' \cdot \overline{s} = m_k \cdot \overline{s} = s$. □

3.7. Examples (1) The category $\mathbf{Top}_0$ is an $\aleph_0$-orthogonality class in $\mathbf{Top}$. In fact $\mathbf{Top}_0 = \{h\}^\perp$ where h is the map $h : \{0, 1\} \to \{0\}$, considering the two-elements set with the trivial topology.

(2) The category $\mathbf{Top}_1$ of T_1 topological spaces is an $\aleph_0$-orthogonality class of $\mathbf{Top}$. It is just the subcategory of all objects orthogonal to the quotient $S \hookrightarrow \{0\}$, where S is the Sierpiński space. In this case, the reflector does not preserve embeddings.

(3) $\mathbf{Sob}$ is not an $\aleph_0$-orthogonality class in $\mathbf{Top}_0$, and, consequently, it is not an $\aleph_0$-orthogonality class in $\mathbf{Top}$. This follows from the above theorem taking into account that $\mathbf{Sob}$ is not closed under $\aleph_0$-$\mathbf{Sob}$-neat subobjects in $\mathbf{Top}_0$.

For that, we show that every $\mathbf{Sob}$-epimorphism $e : X \to Y$ with X and Y finite is a surjection. (We recall that the $\mathbf{Sob}$-epimorphisms of $\mathbf{Top}_0$ are the b-dense morphisms, see 3.1.) Let $y \in Y$, let $\{H_i,\ i \in I\}$ be the set of all open neighbourhoods of y, and put $H = \bigcap_{i \in I} H_i$. Since I is finite, H is an open containing y, and, then, $H \cap e(X) \cap \overline{\{y\}} \neq \emptyset$. Let y' be an element of that intersection. Thus $\overline{\{y'\}} \subseteq \overline{\{y\}}$. But for all H_i we have $y' \in H_i$, hence $\overline{\{y\}} = \overline{\{y'\}}$. Since $Y \in \mathbf{Top}_0$, we conclude that $y = y'$, then $y \in e(X)$.

As a consequence we have that

$$\{embeddings\} \subseteq \{\ \aleph_0\text{-}\mathbf{Sob}\text{-neat morphisms}\}.$$

But then, if $\mathbf{Sob}$ were closed under $\aleph_0$-$\mathbf{Sob}$-neat subobjects, it would also be closed under embeddings, what is obviously false (since the reflections are embeddings).

(4) The category $\mathbf{Norm}$ of normed (real or complex) vectorial spaces and linear contractions is a locally $\aleph_0$-bounded category with respect to embeddings, and its $\aleph_0$-bounded objects are the spaces with finite dimension. Analogously, all spaces with countable dimension are $\aleph_1$-bounded. The subcategory $\mathbf{Ban}$ of all Banach spaces is an $\aleph_1$-orthogonality class of $\mathbf{Norm}$. In fact, it is easy to see that

$$\mathbf{Ban} = \mathcal{N}^{\perp}$$

where $\mathcal{N}$ is the class of all dense embeddings $X \hookrightarrow Y$ with X and Y with countable dimensions.

Acknowledgement I acknowledge the referee for the suggestion of the name α-$\mathcal{B}$-*neat*.

References

[1] J. Adámek, M. Hébert and L. Sousa, A Logic of Orthogonality, *Arch.Math.(Brno)* 42 (2006), 309-334.

[2] J. Adámek, M. Hébert and L. Sousa, The Orthogonal Subcategory Problem and the Small Object Argument, to appear in *Appl. Categorical Structures*.

[3] J. Adámek, H. Herrlich and G. E. Strecker, *Abstract and Concrete Categories*, John Wiley and Sons, New York 1990. Freely available at www.math.uni-bremen.de/∼dmb/acc.pdf

[4] J. Adámek and J. Rosický: *Locally presentable and accessible categories*, Cambridge University Press, 1994.

[5] J. Adámek and L. Sousa, On reflective subcategories of varieties, *J. Algebra* 276 (2004) 685-705.

[6] J. Adámek and W. Tholen, Total categories with generators, *J. Algebra* 133 (1990), 63-78.

[7] R. Börger, W. Tholen, Total categories and solid functors, *Canad. J. Math.* 42-1 (1990), 213-229.

[8] R. Börger, W. Tholen, M.B. Wischnewsky and H. Wolff, Compact and hypercomplete categories, *J. Pure Appl. Algebra* 21 (1981), 120-144.

[9] D. Dikranjan and W. Tholen, *Categorical Structure of Closure Operators*, Kluwer Academic Publishers (1995).

[10] P. J. Freyd and G.M. Kelly, Categories of continuous functors I, *J. Pure Appl. Algebra* 2 (1972), 169-191.

[11] P. Gabriel and F. Ulmer, *Local präsentierbare Kategorien*, Lect. Notes in Math. 221, Springer-Verlag, Berlin 1971.

[12] M. Hébert, J. Adámek, J. Rosický, More on orthogonality in locally prsentable categories, *Cahiers Topologie Géom. Différentielle Catég.* 42 (2001) 51-80.

[13] M. Hébert, J. Rosický, Uncountable orthogonality is a closure property, *Bull. London Math. Soc.* 33 (2001) 685-688.

[14] G. Janelidze and G. M. Kelly, The reflectiveness of covering morphisms in Algebra and Geometry, *Theory and Applications of Categories*, 3 (1997) 132-159.

[15] M. Kelly, A unified treatment of transfinite constructions for free algebras, free monoids, colimits, associated sheaves, and so on, *Bull. Austral. Math. Soc.* 22 (1980), 1-84.

[16] M. Kelly, A survey of totality for enriched and ordinary categories, *Cahiers Topologie Géom. Différentielle Catég.* 27 (1986).

[17] M. Kelly, *Basic concepts of enriched category theory*, Reprints in Theory and Applications of Categories, No. 10, 2005.

[18] C.M. Ringel, Monofunctors as reflectors, *Trans. Am. Math. Soc.* 161 (1971) 293-306.

[19] L. Sousa, α-sober spaces via the orthogonal closure operator, *Appl. Categ. Structures* 4 (1996) 87-95.

[20] H. Volger, Preservation theorems for limits of structures and global section of sheaves of structures, *Math. Z.* 166 (1979) 27-53.

Lurdes Sousa
School of Technology of Viseu, Campus Politecnico, 3504-510 Viseu, Portugal
CMUC, University of Coimbra, 3001-454 Coimbra, Portugal
sousa@estv.ipv.pt