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# THE ROLE OF SYMMETRIES IN CUBICAL SETS AND CUBICAL CATEGORIES
## (On weak cubical categories, I)

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**Abstract.** Symmetric weak cubical categories were introduced in [G3, G5], as a basis for the study of cubical cospans and higher cobordism. Such cubical structures are equipped with an action of the symmetric groups, which simplifies the coherence conditions. We give now a deeper study of the role of symmetries. While *ordinary* cubical sets have a tensor product which is non symmetric and biclosed, the *symmetric* ones have a *symmetric* monoidal closed structure (and *one* internal hom). Similar facts hold for cubical categories and the symmetric ones, and should play a relevant role in the sequel, the study of cubical limits and adjunctions. Weak double categories are a cubical truncation of the present structures.

**Résumé.** Les catégories cubiques symétriques faibles ont été introduites en [G3, G5], pour l'étude des cospans cubiques et du cobordism de dimension supérieure. Ces structures sont équipées d'une action des groupes symétriques qui simplifie les conditions de cohérence. On donne ici une étude plus approfondie du rôle des symétries. Les ensembles cubiques *ordinaires* ont un produit tensoriel qui est non symétrique et fermé, à gauche et à droite; mais les ensembles cubiques *symétriques* ont une structure monoïdale fermée *symétrique* (et *un seul* hom interne). Des faits semblables se vérifient pour les catégories cubiques ordinaires et symétriques; ils devraient jouer un rôle important dans la suite, c'est-à-dire l'étude des limites et des adjonctions cubiques. Les catégories doubles faibles sont un tronquement cubique des structures considérées ici.

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## Introduction

A *weak cubical category* [G3-G5] has a cubical structure, with faces and degeneracies; moreover, there are weak compositions in countably many directions, which we call *cubical* (or *geometric*), and one strict composition, in the *transversal* (or *structural*) direction.

As a basic example, one can think of the weak cubical category $\omega\mathbb{C}\mathrm{osp}(\mathbf{X})$ of cubical cospans in a category with pushouts $\mathbf{X}$. An n-dimensional *object* is a functor $x\colon \boldsymbol{\wedge}^n \to \mathbf{X}$, where $\boldsymbol{\wedge}$ is the 'formal cospan' category

$$
(1) \qquad -1 \to 0 \leftarrow 1 \qquad \boldsymbol{\wedge} \qquad\qquad
\begin{array}{ccccc}
(-1,-1) & \longrightarrow & (0,-1) & \longleftarrow & (1,-1) \\
\downarrow & & \downarrow & & \downarrow \\
(-1,0) & \longrightarrow & (0,0) & \longleftarrow & (1,0) \\
\uparrow & & \uparrow & & \uparrow \\
(-1,1) & \longrightarrow & (0,1) & \longleftarrow & (1,1)
\end{array} \qquad \boldsymbol{\wedge}^2.
$$

An n-dimensional *transversal map*, or *structural map*, is a natural transformation $f\colon x \to y\colon \boldsymbol{\wedge}^n \to \mathbf{X}$ of such functors. Their composition is also called *transversal*, or *structural*.

The ordinary categories $\mathbb{C}\mathrm{osp}_n(\mathbf{X}) = \mathbf{Cat}(\boldsymbol{\wedge}^n, \mathbf{X})$ form a cubical object in $\mathbf{Cat}$, with obvious faces and degeneracies. Moreover, n-dimensional objects (and maps) have *cubical*, or *geometric*, composition laws $x +_i y$ in each direction $i = 1,..., n$, which are constructed with pushouts; these compositions are consistent with faces and degeneracies, *but only behave well up to suitable transversal maps,* which yield invertible comparisons for their associativity, unitarity and interchange.

Actually, as already stressed in [G3], $\omega\mathbb{C}\mathrm{osp}(\mathbf{X})$ is a *symmetric* weak cubical category, when equipped with the obvious action of the symmetric group $S_n$ on $\mathbf{Cat}(\boldsymbol{\wedge}^n, \mathbf{X})$; namely, the action of permuting the factors of $\boldsymbol{\wedge}^n$, i.e. the directions of n-cubical cospans in $\mathbf{X}$. These symmetries allow one to only consider the faces, degeneracies and cubical compositions in a single direction (see 2.2), *which greatly simplifies the coherence conditions*. Notice also that cubical 1-truncation, keeping *one* weak direction *and* the strict transversal one, yields the weak *double* category $\mathbb{C}\mathrm{osp}(\mathbf{X})$ of ordinary cospans and their transversal maps, studied in [GP1]; here, symmetries 'disappear', since the groups $S_0$ and $S_1$ are trivial.

We begin here a study of the theory of symmetric weak cubical categories, or weak sc-categories, with the aim of extending the study of weak double categories developed in [GP1-GP4].

The present paper analyses the role of symmetries, beyond the simplification of coherence properties. Symmetric cubical sets - where the n-dimensional symmetric group $S_n$ operates on the n-dimensional component, form a *symmetric monoidal* *closed* category s**Cub**, while the ordinary ones have a *non-symmetric* monoidal biclosed structure. Even more relevant to the sequel should be the (related) fact that ordinary cubical sets have a *left* and a *right* path functor (1.6), whose liftings to symmetric cubical sets are isomorphic, yielding - essentially - *one* path functor (2.3); the latter will be crucial in defining and studying cubical limits.

In the first two sections, after reviewing the (*non* symmetric) monoidal biclosed structure of ordinary cubical sets [BH2], we study symmetric cubical sets and their path functor. Then, Section 3 recalls the definition of a cubical category and of a symmetric one, from [G3], and introduces the path functors of these two structures (3.6). Again, in the symmetric case there is *one* path functor, which produces *one* internal hom (3.7(c)). The definition of the weak case, introduced in [G3], is only sketched here (in 3.5) as it would be too long to completely rewrite it.

The next two sections are devoted to examples: after recalling the weak sc-categories $\omega\mathbb{S}\mathrm{p}(\mathbf{X})$ and $\omega\mathbb{C}\mathrm{osp}(\mathbf{X})$ of cubical (co)spans, also introduced in [G3], we construct the strict sc-category of cubical relations $\omega\mathbb{R}\mathrm{el}$ (4.2, 4.3). Then, we reconsider the passage from $\omega\mathbb{S}\mathrm{p}(\mathbf{Set})$ to $\omega\mathbb{R}\mathrm{el}$, abstracting the notion of a quotient of weak sc-categories modulo transversal maps *of reduction*, a kind of rewriting procedure with normal forms (Section 5). The dual procedure allows us to construct a strict sc-category $\omega\mathbb{C}\mathrm{at}$ of (small) categories and cubical profunctors (5.7), from a quotient of the weak sc-category $\omega\mathbb{C}\mathrm{osp}(\mathbf{Emb})$ of cospans of full embeddings of categories.

We end in Section 6 by defining the *symmetric tensor product* of symmetric cubical sets (and categories); after the internal hom of Section 3, this completes the symmetric monoidal closed structure; it also yields a cylinder functor, by tensoring with an obvious 'standard interval'. However, the computation of the tensor product (and of the cylinder) is complicated and - likely - not really needed.

References to the rich literature on higher categories can be found in two recent books, by T. Leinster [Le] and E. Cheng - A. Lauda [CL]; but this literature is mostly developed in the globular geometry, rather than the cubical one. Strict cubical categories with connections (and without transversal maps) have been studied in [ABS], and proved to be equivalent to (globular) ω-categories.

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Size aspects (for categories of cubical categories, for instance) can be easily settled working with a suitable hierarchy of universes.

## 1. Cubical sets and internal homs

While reviewing the (*non* symmetric) monoidal biclosed structure of the category **Cub** of ordinary cubical sets [BH2], we want to stress the role of the 'transposer' S, which reverses the order of faces; in particular, we are interested in the *external symmetry* $s: S(X \otimes Y) \to (SY) \otimes (SX)$, which somewhat *surrogates* here the 'internal' symmetry of a symmetric tensor product. The binary index $\alpha$ takes values 0, 1, written $-, +$ in superscripts.

**1.1. Cubical sets.** A cubical set $X = ((X_n), (\partial_i^\alpha), (e_i))$, in the usual sense [K1, K2, BH1, BH2], has *faces* $(\partial_i^\alpha)$ and *degeneracies* $(e_i)$

(1) $\partial_i^\alpha: X_n \rightleftarrows X_{n-1} :e_i$ $\qquad (i = 1,..., n;\ \alpha = \pm)$,

satisfying the cubical relations :

(2) $\partial_i^\alpha.\partial_j^\beta = \partial_j^\beta.\partial_{i+1}^\alpha \quad (j \leq i), \qquad e_j.e_i = e_{i+1}.e_j \quad (j \leq i),$

$\partial_i^\alpha.e_j = e_j.\partial_{i-1}^\alpha \quad (j < i), \qquad$ *or* $\ \mathrm{id} \quad (j = i), \qquad$ *or* $\ e_{j-1}.\partial_i^\alpha \quad (j > i).$

Elements of $X_n$ are called n-*cubes*; *vertices* and *edges* for $n = 0$ or 1, respectively. Every n-cube $x \in X_n$ has $2^n$ vertices: $\partial_1^\alpha \partial_2^\beta \partial_3^\gamma(x)$ for $n = 3$ and $\alpha, \beta, \gamma = \pm$.

A *morphism* $f = (f_n): X \to Y$ is a sequence of mappings $f_n: X_n \to Y_n$ which commute with faces and degeneracies.

Small cubical sets and their morphisms form a category **Cub**, which has all limits and colimits and is cartesian closed. In fact, it is the presheaf category of functors $X: \mathbb{I}^{op} \to$ **Set**, where $\mathbb{I}$ is the subcategory of **Set** consisting of the *elementary cubes* $2^n = \{0, 1\}^n$, together with the maps $\{0, 1\}^m \to \{0, 1\}^n$ which delete some coordinates and insert some 0's and 1's, without modifying the order of the remaining coordinates [GM].

The terminal object $\top$ is freely generated by one vertex $*$ and will also be written $\{*\}$; but notice that *each* of its components is a singleton. The initial object is empty, i.e., all its components are; the other cubical sets have a non-empty component in each degree.

The category **Cub** has two (covariant) involutive endofunctors, which we shall call *reversor* and *transposer*

(3) $R$: **Cub** $\to$ **Cub**, $\qquad RX = X^{op} = ((X_n), (\partial_i^{-\alpha}), (e_i))$ $\qquad$ (*reversor*),

(4) $S$: **Cub** $\to$ **Cub**, $\qquad SX = ((X_n), (\partial_{n+1-i}^\alpha), (e_{n+1-i}))$ $\qquad$ (*transposer*).

The first reverses the 1-dimensional direction; the second can be thought to reverse 'the 2-dimensional one', in a sense which will appear below (see 1.6.5). If $x \in X_n$, the corresponding element in $(RX)_n = X_n$ will often be written as $x^{op}$, so that $\partial_i^-(x^{op}) = (\partial_i^+ x)^{op}$.

A cubical set X is said to be *reversive* if $RX \cong X$ and *permutative* if $SX \cong X$.

**1.2. Tensor product.** The category **Cub** has a monoidal structure [K1, BH2]

(1) $(X\otimes Y)_n = (\Sigma_{p+q=n} X_p{\times}Y_q)/{\sim_n}$,

where $\sim_n$ is the equivalence relation generated by identifying $(e_{r+1}x, y)$ with $(x, e_1 y)$, for all $(x, y) \in X_p \times Y_q$ (where $p+q = n-1$).

Writing $x\otimes y$ the equivalence class of $(x, y)$, faces and degeneracies are defined as follows, when x is of degree p and y of degree q

(2) $\partial_i^\alpha(x\otimes y) = \begin{cases} (\partial_i^\alpha x)\otimes y, & \text{if } 1 \leq i \leq p, \\ x\otimes(\partial_{i-p}^\alpha y), & \text{if } p < i \leq n, \end{cases}$

(3) $e_i(x\otimes y) = \begin{cases} (e_i x)\otimes y, & \text{if } 1 \leq i \leq p+1, \\ x\otimes(e_{i-p} y), & \text{if } p+1 \leq i \leq n+1. \end{cases}$

Note that $e_{p+1}(x\otimes y) = (e_{p+1}x)\otimes y = x\otimes(e_1 y)$ is well defined precisely because of the previous equivalence relation.

The identity of the tensor product is the terminal object $\top = \{*\}$ (1.1), obviously reversive and permutative.

**1.3. The external symmetry.** *The tensor product is not symmetric*, but is related to reversor and transposer (1.1.3, 1.1.4) as follows

(1) $R(X\otimes Y) = RX\otimes RY$,

(2) $s(X, Y)\colon S(X\otimes Y) \cong (SY)\otimes(SX), \qquad x\otimes y \mapsto y\otimes x$.

(R is a strict isomorphism of the monoidal structure, while the pair (S, s) is an anti-isomorphism.) Therefore, reversive objects are stable under tensor product while permutative objects are stable under tensor powers: if $SX \cong X$, then $S(X^{\otimes n}) \cong (SX)^{\otimes n} \cong X^{\otimes n}$.

Notice that the symmetry $s\colon A{\times}B \to B{\times}A$ of the cartesian product of sets (i.e., 0-truncated cubical sets) becomes, here, the *external symmetry* (2), which is an isomorphism of functors **Cub** × **Cub** → **Cub**. Its inverse is $S(s(SY, SX))$.

**1.4. The standard interval.** The (*elementary*) *directed interval*, or *standard interval*, $\uparrow\mathbf{i} = \mathbf{2}$ is the cubical set freely generated by a 1-cube, u

(1) $\quad 0 \xrightarrow{u} 1 \qquad\qquad\qquad \partial_1^-(u) = 0, \quad \partial_1^+(u) = 1.$

This cubical set is reversive and permutative. It is the representable presheaf $y(2) = \mathbb{I}(-, 2): \mathbb{I}^{op} \to \mathbf{Set}$.

The *directed* n-*cube* is its n-th tensor power $\uparrow\mathbf{i}^{\otimes n} = \uparrow\mathbf{i}\otimes ... \otimes\uparrow\mathbf{i}$ (for $n \geq 0$), freely generated by its n-cube $u^{\otimes n}$, still reversive and permutative. It is the representable presheaf $y(2^n) = \mathbb{I}(-, 2^n): \mathbb{I}^{op} \to \mathbf{Set}$. The *directed square* $\uparrow\mathbf{i}^2 = \uparrow\mathbf{i}\otimes\uparrow\mathbf{i}$ can be represented as follows, showing the generator $u\otimes u$ and its faces

$$(2) \qquad \begin{array}{ccc} 00 & \xrightarrow{u\otimes 0} & 10 \\ {\scriptstyle 0\otimes u}\downarrow & u\otimes u & \downarrow{\scriptstyle 1\otimes u} \\ 01 & \xrightarrow[u\otimes 1]{} & 11 \end{array} \qquad\qquad \begin{array}{ll} \bullet \longrightarrow & {\scriptstyle 1} \\ \downarrow {\scriptstyle 2} & \end{array}$$

(The face $\partial_1^-(u\otimes u) = 0\otimes u$ is *orthogonal* to direction 1, a criterion which works in every dimension.) By the Yoneda Lemma, $\mathbf{Cub}(\uparrow\mathbf{i}^{\otimes n}, X) = X_n$.

**1.5. Left and right cylinder functors.** Let us start from the standard interval $\uparrow\mathbf{i}$, and work with the monoidal structure recalled above, with unit $\{*\}$ and reversor R. Recall that u denotes the 1-dimensional generator of $\uparrow\mathbf{i}$, and $u^{op}$ is the corresponding edge of $\uparrow\mathbf{i}^{op}$ (1.1).

The cubical set $\uparrow\mathbf{i}$ has a structure consisting of two *faces* $(\partial^\alpha)$, a *degeneracy* (e) and a reflection or external reversion (r):

(1) $\quad \partial^\alpha: \{*\} \to \uparrow\mathbf{i}, \qquad \partial^\alpha(0) = \alpha \qquad (\alpha = 0, 1),$

$\quad e: \uparrow\mathbf{i} \to \{*\}, \qquad e(t) = *, \quad e(u) = e_1(*),$

$\quad r: \uparrow\mathbf{i} \to \uparrow\mathbf{i}^{op}, \qquad r(0) = 1^{op}, \quad r(1) = 0^{op}, \quad r(u) = u^{op}.$

Since the tensor product is not symmetric, the elementary directed interval yields a *left (elementary) cylinder* $\uparrow\mathbf{i}\otimes X$ and a *right cylinder* $X\otimes\uparrow\mathbf{i}$. These functors are not isomorphic, but each of them determines the other, using the transposer S (1.1) and the property $S(\uparrow\mathbf{i}) = \uparrow\mathbf{i}$

(2) $\quad I: \mathbf{Cub} \to \mathbf{Cub}, \qquad IX = \uparrow\mathbf{i}\otimes X,$

$\quad SIS: \mathbf{Cub} \to \mathbf{Cub}, \qquad SIS(X) = S(\uparrow\mathbf{i} \otimes SX) = X \otimes \uparrow\mathbf{i}.$

The last equality is actually the canonical isomorphism $s(\uparrow\mathbf{i}, SX)$ (1.3.2); but we will realise SIS as described above.

The left cylinder, $IX = \uparrow\mathbf{i}\otimes X$, inherits from the structure of $\uparrow\mathbf{i}$ (1.5.1) two faces, a degeneracy and a reflection, as follows

$$\begin{aligned}
(3)\quad & \partial^\alpha = \partial^\alpha\otimes X\colon X \to IX, && \partial^\alpha(x) = \alpha\otimes x && (\alpha = 0, 1),\\
& e = e\otimes X\colon IX \to X, && e(u\otimes x) = e_1(*)\otimes x = *\otimes e_1(x) = e_1(x), &&\\
& r = r\otimes RX\colon IRX \to RIX, && && \\
& r(\alpha\otimes x^{op}) = ((1-\alpha)\otimes x)^{op}, && r(u\otimes x^{op}) = (u\otimes x)^{op} && (\alpha = 0, 1).
\end{aligned}$$

**1.6. Left and right path functors.** The category **Cub** has a *left path* functor P, right adjoint to the left cylinder functor $IX = \uparrow\mathbf{i}\otimes X$.

The functor P shifts down all components discarding the faces and degeneracies of index 1; the latter are then used to build three natural transformations, the *faces* and *degeneracy* of P

$$\begin{aligned}
(1)\quad & P\colon \mathbf{Cub} \to \mathbf{Cub}, && PY = ((Y_{n+1}), (\partial^\alpha_{i+1}), (e_{i+1})),\\
& \partial^\alpha = \partial^\alpha_1\colon PY \to Y, && e = e_1\colon Y \to PY.
\end{aligned}$$

The transposer S (1.1.4) produces the *right path* functor SPS, right adjoint to the right cylinder $SIS(X) = X\otimes\uparrow\mathbf{i}$. Explicitly, SPS shifts down all components and discards the faces and degeneracies of highest index (used again to build the corresponding three natural transformations)

$$\begin{aligned}
(2)\quad & SPS\colon \mathbf{Cub} \to \mathbf{Cub}, && SPS(Y) = ((Y_{n+1}), (\partial^\alpha_i), (e_i)),\\
& \partial^\alpha\colon SPS(Y) \to Y, && \partial^\alpha = (\partial^\alpha_{n+1}\colon Y_{n+1} \to Y_n)_{n\geq 0},\\
& e\colon Y \to SPS(Y), && e = (e_{n+1}\colon Y_n \to Y_{n+1})_{n\geq 0}.
\end{aligned}$$

An (elementary or immediate) *left homotopy* $f\colon f^- \to_L f^+\colon X \to Y$ is defined as a map $f\colon X \to PY$ with $\partial^\alpha f = f^\alpha$. This leads immediately to a simple expression of f as a family of mappings

$$\begin{aligned}
(3)\quad & f_n\colon X_n \to Y_{n+1}, && \partial^\alpha_{i+1} f_n = f_{n-1}\,\partial^\alpha_i, \quad e_{i+1} f_{n-1} = f_n\, e_i,\\
& && \partial^\alpha_1 f_n = f^\alpha \qquad (\alpha = \pm;\ i = 1,..., n).
\end{aligned}$$

Similarly, an (elementary) *right homotopy* $f\colon f^- \to_R f^+\colon X \to Y$ is a map $f\colon X \to SPS(Y)$ with faces $\partial^\alpha f = f^\alpha$. This amounts to a family $(f_n)$ such that

$$\begin{aligned}
(4)\quad & f_n\colon X_n \to Y_{n+1}, && \partial^\alpha_i f_n = f_{n-1}\,\partial^\alpha_i, \quad e_i f_{n-1} = f_n\, e_i,\\
& && \partial^\alpha_{n+1} f_n = f^\alpha \qquad (\alpha = \pm;\ i = 1,..., n).
\end{aligned}$$

The transposer can be viewed as an isomorphism $S: \mathbf{Cub}_L \to \mathbf{Cub}_R$ between the left and the right structure. One can define an *external transposition* s (replacing, from a formal point of view, the transposition $s: P^2 \to P^2$ of topological spaces, which permutes the two variables); it is actually an identity $PP' = P'P$

(5) $s: PSPS \to SPSP, \qquad s_n = idY_{n+2},$

since both functors shift down all components of two degrees, discarding the faces and degeneracies of lowest and highest index.

**1.7. Internal homs.** The category **Cub** has left and right internal homs [BH2].

The *right* internal hom CUB(A, Y) can be built with the *left* cocylinder functor P and its natural transformations (which give a cubical object $P^{\bullet}Y$ in **Cub**)

(1) $- \otimes A \dashv CUB(A, -), \qquad CUB_n(A, Y) = \mathbf{Cub}(A, P^nY).$

The natural bijection

(2) $\varphi(X, Y): \mathbf{Cub}(X \otimes A, Y) \to \mathbf{Cub}(X, CUB(A, Y)),$

is constructed as follows, on an arbitrary morphism $f = (f_n): X \otimes A \to Y$. Its n-component $f_n$ decomposes into a family of mappings

(3) $f_{pq}: X_p \times A_q \to Y_{p+q},$

consistent with the equivalence relations $\sim_n$ (1.2.1). By the exponential law in **Set**, these amount to mappings $g_{pq}: X_p \to \mathbf{Set}(A_q, Y_{p+q})$. At fixed p, we get a mapping

(4) $g_p = (g_{pq}): X_p \to \mathbf{Cub}(A, P^pY) = CUB_p(A, Y) \subset \Pi_q \mathbf{Set}(A_q, Y_{p+q}),$

whose family forms a morphism of cubical sets $g = (g_p): X \to CUB(A, Y)$.

**1.8. Higher path functors.** We have seen that the two path functors $P, P': \mathbf{Cub} \to \mathbf{Cub}$ commute (1.6.5). Therefore, every composition of n occurrences of them can be written as

(1) $P^n_i = P^{n-i}.P'^i = P^{n-i}.SP^iS: \mathbf{Cub} \to \mathbf{Cub} \qquad (i = 0,..., n).$

$P^n_i(X)$ has p-component $X_{p+n}$; its faces and degeneracies $X_{p+n} \rightleftarrows X_{p+n-1}$ are part of those of X, corresponding to the directions n–i+1,..., n–i+p (renumbered as 1,..., p). In particular, $P^1_0 = P$ and $P^1_1 = P'$.

There are generalised faces linking higher path functors

(2) $P^i.\partial^\alpha.P^j.SP^kS: P^{i+j+1}.SP^kS \to P^{i+j}.SP^kS,$

$P^i.SP^j\partial^\alpha P^kS: P^i.SP^{j+k+1}S \to P^{i+j}.SP^kS,$

and similar generalised degeneracies.

## 2. Symmetric cubical sets and their closed structure

We consider now *symmetric* cubical sets, equipped with transpositions. In the singular cubical set of a topological space, this amounts to transposing variables.

Lifting the previous left or right path functors (1.6) to the symmetric case, we get isomorphic functors, and essentially *one* path functor. The latter produces *one* internal hom, and a *symmetric monoidal closed structure*. The real points of interest are the path functor and the internal hom, which in the next section will allow us to define the cubical transformations of sc-functors.

On the other hand, the *symmetric tensor product* (of symmetric cubical sets) and the corresponding cylinder functor are complicated and - perhaps - not really needed; they will be sketched in Section 6.

**2.1. Symmetric cubical sets.** As in [G3], a *symmetric cubical* set, or *sc-set*, is a cubical set which is further equipped with mappings, called *transpositions*

(1) $s_i: X_n \to X_n \qquad (i = 1,..., n-1;\ n \geq 2).$

These have to satisfy the Moore relations

(2) $s_i.s_i = 1, \qquad s_i.s_j.s_i = s_j.s_i.s_j \quad (i = j-1), \qquad s_i.s_j = s_j.s_i \quad (i < j-1),$

and the following equations of coherence with faces and degeneracies:

| | | $j < i$ | $j = i$ | $j = i+1$ | $j > i+1$ |
|---|---|---|---|---|---|
| (3) | $\partial^\alpha_j.s_i =$ | $s_{i-1}.\partial^\alpha_j$ | $\partial^\alpha_{i+1}$ | $\partial^\alpha_i$ | $s_i.\partial^\alpha_j$, |
| | $s_i.e_j =$ | $e_j.s_{i-1}$ | $e_{i+1}$ | $e_i$ | $e_j.s_i$. |

Assigning the mappings (1) under conditions (2) amounts to letting the symmetric group $S_n$ operate on $X_n$. Indeed, it is well known that $S_n$ is generated, under the Moore relations, by the 'ordinary' transpositions $s_1,..., s_{n-1}$, where $s_i$, acting on the set $\{1,..., n\}$, exchanges $i$ with $i+1$ (see Coxeter-Moser [CM], 6.2; or Johnson [Jo], Section 5, Thm. 3).

A *morphism* $f = (f_n): X \to Y$ is a sequence of mappings $f_n: X_n \to Y_n$ which commute with faces, degeneracies and transpositions. The category s**Cub**, of small sc-sets and their morphisms, is again a category of presheaves $X: \mathbb{I}_s{}^{op} \to$ **Set**, for the *symmetric cubical site* $\mathbb{I}_s$. The latter can be defined as the subcategory of **Set** consisting of the elementary cubes $2^n = \{0, 1\}^n$ together with the maps $2^m \to 2^n$ which delete some coordinates, permute the remaining ones and insert some 0's and 1's. It is a subcategory of the *extended cubical site* $\mathbb{K}$ of [GM], which also contains the 'connections' (higher degeneracies).

The reversor and transposer of **Cub** (1.1.3, 1.1.4) have obvious liftings

(4) R: s**Cub** $\to$ s**Cub**, $RX = X^{op} = ((X_n), (\partial_i^{-\alpha}), (e_i), (s_i))$ (*reversor*),

(5) S: s**Cub** $\to$ s**Cub**, $SX = ((X_n), (\partial^\alpha_{n+1-i}), (e_{n+1-i}), (s_{n+1-i}))$ (*transposer*).

But here the transpositions make S *isomorphic to the identity functor*, and not essential (as we will see in 2.6).

**2.2. Reduced presentations of symmetric cubical sets.** In a symmetric cubical set, the presence of transpositions makes all faces and degeneracies determined by the 1-directed ones, $\partial_1^-$, $\partial_1^+$ and $e_1$. In fact, from $\partial^\alpha_{i+1} = \partial^\alpha_i.s_i$ and $e_{i+1} = s_i.e_i$, we deduce that:

(1) $\partial^\alpha_i = \partial^\alpha_1.\mathbf{s}'_i$, $e_i = \mathbf{s}_i.e_1$ $(i = 2,..., n;\ \alpha = \pm)$,

where we are using the inverse 'permutations' $\mathbf{s}_i$ and $\mathbf{s}'_i$

(2) $\mathbf{s}_i = s_{i-1}. ... .s_1$, $\mathbf{s}'_i = s_1. ... .s_{i-1}$.

This leads to a more economical presentation of our structure, as proved in [G5]. Namely, an sc-set can be equivalently defined as a system

(3) $X = ((X_n), \partial_1^-, \partial_1^+, e_1, (s_i))$,

under the Moore relations for transpositions (2.1.2) and the axioms:

(4) $\partial^\alpha_1.\partial^\beta_1 = \partial^\beta_1.\partial^\alpha_1.s_1$, $e_1.e_1 = s_1.e_1.e_1$, $\partial^\alpha_1.e_1 = id$,
$s_i.\partial^\alpha_1 = \partial^\alpha_1.s_{i+1}$, $e_1.s_i = s_{i+1}.e_1$.

In other words, X can be presented as a system $((X_n), \partial_1^-, \partial_1^+, e_1)$ where each $X_n$ is an $S_n$-set (equipped with an action of the symmetric group $S_n$) and the axioms (4) are satisfied.

**2.3. The symmetric path functor.** We define now the *path functor* P of sc-sets, by lifting the left path functor of ordinary cubical sets (1.6): P (acting on an sc-set in the complete presentation, 2.1) shifts down all components, discarding the faces, degeneracy and transpositions *of index* 1

(1) P: s**Cub** $\to$ s**Cub**,

$PX = ((X_{n+1}), (\partial^{\alpha}_{i+1}), (e_{i+1}), (s_{i+1})), \qquad (Pf)_n = f_{n+1}.$

Again, the discarded faces and degeneracy are used to build three natural transformations, the faces and degeneracy of $P$ (while the discarded $s_1$ will give the transposition of $P^2$, cf. 2.4)

(2) $\partial^{\alpha}: PX \to X, \qquad \partial^{\alpha} = (\partial^{\alpha}_1: X_{n+1} \to X_n)_{n\geq 0},$

$e: X \to PX, \qquad e = (e_1: X_n \to X_{n+1})_{n\geq 0}.$

Also here the transposer $S$ (2.1.5) yields the *right path* functor $P' = SPS$, which shifts down all components discarding the mappings *of highest index*. But $S \cong \mathrm{id}$ and $SPS \cong P$ (as we will see in 2.6), so that *one* path functor is sufficient.

**2.4. The transposition of the path functor.** The 'second order' path functor is computed as:

(1) $P^2: s\mathbf{Cub} \to s\mathbf{Cub},$

$P^2X = ((X_{n+2}), (\partial^{\alpha}_{i+2}), (e_{i+2}), (s_{i+2})), \qquad (P^2f)_n = f_{n+2}.$

It has two pairs of faces $P\partial^{\alpha}$, $\partial^{\alpha}P: P^2 \to P$ and two degeneracies $Pe, eP: P \to P^2$ $(i = 1, 2)$

(2) $P(\partial^{\alpha}X) = (\partial^{\alpha}_1: X_{n+2} \to X_{n+1})_{n\geq 0}, \qquad \partial^{\alpha}(PX) = (\partial^{\alpha}_2: X_{n+2} \to X_{n+1})_{n\geq 0},$

$P(eX) = (e_1: X_{n+1} \to X_{n+2})_{n\geq 0}, \qquad e(PX) = (e_2: X_{n+1} \to X_{n+2})_{n\geq 0}.$

Because of these formulas, it would be appropriate to label the faces $P\partial^{\alpha}$ as $\partial^{\alpha}_1$, and the faces $\partial^{\alpha}P$ as $\partial^{\alpha}_2$. Similarly for degeneracies. (But such labels are not really necessary here and will be avoided. In various papers on homotopy theory, we have made the opposite choice, guided by the path functor of topological spaces.)

The important fact is that we have a *transposition* for the path functor $P$

(3) $s: P^2 \to P^2, \qquad x \mapsto s_1.x.$

First, $s: P^2X \to P^2X$ is indeed a morphism of sc-sets, as it follows immediately from the symmetric cubical relations (2.1)

(4)
$$\begin{array}{ccc} X_{n+2} & \xrightarrow{s_1} & X_{n+2} \\ {\scriptstyle \partial^{\alpha}_{i+2}}\downarrow\uparrow{\scriptstyle e_{i+2}} & & {\scriptstyle \partial^{\alpha}_{i+2}}\downarrow\uparrow{\scriptstyle e_{i+2}} \\ X_n & \xrightarrow[s_1]{} & X_n \end{array} \qquad \begin{array}{ccc} X_{n+2} & \xrightarrow{s_1} & X_{n+2} \\ {\scriptstyle s_{i+2}}\downarrow & & \downarrow{\scriptstyle s_{i+2}} \\ X_{n+2} & \xrightarrow[s_1]{} & X_{n+2} \end{array}$$

Second, always because of the symmetric cubical relations (more precisely, of some of them, not applied above), the involution s interchanges the 1-directed faces (or degeneracies) of $P^2$ with the 2-directed ones, independently of how we choose to label them

(5) $\partial^\alpha P.s \; = \; P\partial^\alpha, \qquad\qquad s.eP \; = \; Pe.$

**2.5. Internal homs.** We define the internal-hom functor as

(1) $\text{sCUB}\colon \; \text{s}\mathbf{Cub}^{\text{op}} \times \text{s}\mathbf{Cub} \to \text{s}\mathbf{Cub}, \qquad \text{sCUB}_n(A, Y) \; = \; \text{s}\mathbf{Cub}(A, P^nY).$

Notice that

(2) $\text{sCUB}(\{*\}, Y) \; = \; Y, \qquad\qquad \text{sCUB}(\uparrow\mathbf{i}, Y) \; = \; PY,$

where $\uparrow\mathbf{i} = y(2)$ is the (representable) symmetric cubical set freely generated by one 1-cube u. This is the same as the cubical set $\uparrow\mathbf{i}$ of Section 1, equipped with the unique symmetric structure which permutes its (degenerate) cubes of degree $\geq 2$.

**2.6. The right path functor.** This point is not technically needed for the sequel, but makes clear how the previous structure of ordinary cubical sets is simplified by the addition of symmetries.

(a) Firstly, *the transposer* S *is (here) isomorphic to the identity endofunctor*.

Consider the involutive permutation $\sigma_n \in S_n$

(1) $\sigma_n \; = \; (s_1 s_2 ... s_{n-1}) \; ... \; (s_1 s_2 s_3)(s_1 s_2) s_1,$

which, acting on $\{1,..., n\}$, reverses all indices. It is easy to verify that, for $i < n$

(2) $\partial^\alpha_n.\sigma_n \; = \; \sigma_{n-1}.\partial^\alpha_1, \qquad e_n.\sigma_{n-1} \; = \; \sigma_n.e_1, \qquad s_i.\sigma_n \; = \; \sigma_n.s_{n+1-i}.$

We have thus a natural isomorphism (according to the reduced presentation of symmetric cubical sets which was given in 2.2):

(3) $\sigma X\colon X \to SX, \qquad\qquad x \mapsto \sigma_n.x,$

$$\begin{array}{ccc} X_n & \xrightarrow{\sigma_n} & X_n \\ {\scriptstyle \partial^\alpha_1}\downarrow\uparrow{\scriptstyle e_1} & & {\scriptstyle \partial^\alpha_n}\downarrow\uparrow{\scriptstyle e_n} \\ X_{n-1} & \xrightarrow[\sigma_{n-1}]{} & X_{n-1} \end{array} \qquad\qquad \begin{array}{ccc} X_n & \xrightarrow{\sigma_n} & X_n \\ {\scriptstyle s_i}\downarrow & & \downarrow{\scriptstyle s_{n+1-i}} \\ X_n & \xrightarrow[\sigma_n]{} & X_n \end{array}$$

(b) Secondly, and as a consequence, *there is a natural isomorphism* $c\colon P \to SPS$ *between the left and the right path functors*, which is computed with the *main cyclic permutation* $\gamma_n = s_n. \dots .s_1 \in S_{n+1}$

(4) $\gamma = \sigma PS.P\sigma\colon\ P \to SPS, \qquad \gamma_n\colon X_{n+1} \to X_{n+1},$

$x \mapsto (\sigma PSX)_n.(P(\sigma X))_n(x) = \sigma_n.\sigma_{n+1}.x = s_n. \dots s_1.x = \gamma_n.x \qquad (x \in X_{n+1}).$

Its inverse is computed with the inverse cyclic permutation $\gamma'_n = s_1. \dots .s_n \in S_{n+1}$

(5) $\gamma'\colon SPS \to P, \qquad \gamma'_n\colon X_{n+1} \to X_{n+1},\quad x \mapsto \gamma'_n.x.$

(c) Finally, combining these isomorphisms with the external transposition of 1.6.5:

(6) $s\colon PSPS \to SPSP, \qquad s_n = idY_{n+2},$

we obtain again the transposition of the path functor $s\colon P^2 \to P^2$ (2.4.3)

(7) $s = \gamma'P.id.P\gamma\colon\ P^2 \to PSPS \to SPSP \to P^2,$

$x \mapsto (s_2 \dots s_{n+1})(s_{n+1} \dots s_1)x = s_1x.$

## 3. Cubical categories and the symmetric case

First, we recall the main definitions on cubical categories, introduced in [G3]; we refer to this paper for a complete formulation of the weak structures. Then we introduce the path functor of cubical categories and of symmetric cubical categories (strict or weak), which will be a crucial tool for further developments.

**3.1. Reduced cubical categories.** We begin by considering a cubical set equipped with compositions in all directions; these are assumed to be strictly categorical (i.e., strictly associative and unital) and to satisfy the interchange property.

(cub.1) A *reduced cubical category* $\mathbf{A}$ is, first of all, a cubical set (1.1):

(1) $\mathbf{A} = ((A_n), (\partial^\alpha_i), (e_i)).$

(cub.2) Moreover, for $1 \leq i \leq n$, the i-*concatenation* $x +_i y$ (or i-composition) of two n-cubes $x, y$ is defined when the latter are *i-consecutive*, i.e. $\partial^+_i(x) = \partial^-_i(y)$; the following 'geometrical' interactions with faces and degeneracies are required:

(2) $\partial^-_i(x +_i y) = \partial^-_i(x), \qquad \partial^+_i(x +_i y) = \partial^+_i(y),$

$$\partial^\alpha_j(x +_i y) = \begin{cases} \partial^\alpha_j(x) +_{i-1} \partial^\alpha_j(y), & \text{if } j < i, \\ \partial^\alpha_j(x) +_i \partial^\alpha_j(y), & \text{if } j > i, \end{cases}$$

$$(3) \qquad e_j(x +_i y) \;=\; \begin{cases} e_j(x) +_{i+1} e_j(y), & \text{if } j \le i \le n, \\ e_j(x) +_i e_j(y), & \text{if } i < j \le n+1. \end{cases}$$

(cub.3) For $1 \le i \le n$, we have a category $A^n_i = (A_{n-1}, A_n, \partial^-_i, \partial^+_i, e_i, +_i)$, where faces give domains and codomains, and degeneracy yields the identities. In other words, we have the following equations for i-consecutive n-cubes $x, y, z$:

$$(4) \quad (x +_i y) +_i z \;=\; x +_i (y +_i z), \qquad\qquad e_i\partial^-_i x +_i x \;=\; x \;=\; x +_i e_i\partial^+_i x.$$

(cub.4) For $1 \le i < j \le n$, and n-cubes $x, y, z, u$, we have

$$(5) \quad (x +_i y) +_j (z +_i u) \;=\; (x +_j z) +_i (y +_j u) \qquad\qquad (\textit{middle-four interchange}),$$

whenever these compositions make sense:

$$(6) \quad \partial^+_i(x) \;=\; \partial^-_i(y), \quad \partial^+_i(z) \;=\; \partial^-_i(u), \qquad\qquad \begin{array}{|c|c|} \hline x & y \\ \hline z & u \\ \hline \end{array} \qquad \begin{array}{l} \bullet \longrightarrow i \\ \downarrow j \end{array}$$

$$\partial^+_j(x) \;=\; \partial^-_j(z), \quad \partial^+_j(y) \;=\; \partial^-_j(u),$$

A *cubical functor* $F\colon \mathbf{A} \to \mathbf{B}$ between reduced cubical categories is a morphism of cubical sets which preserves all composition laws.

**3.2. Commutative cubes.** Let $\mathbf{X}$ be an ordinary category. As a simple example of the previous structure, we recall the construction of the reduced cubical category $\omega\mathrm{Cub}(\mathbf{X})$ of commutative cubical diagrams in $\mathbf{X}$.

An n-cube can be viewed as a functor $x\colon \mathbf{i}^n \to \mathbf{X}$, where $\mathbf{i} = \mathbf{2} = \{0 \to 1\}$ is the category corresponding to the ordinal two. This category has the basic structure of a *formal interval* (or reflexive cograph), with respect to the cartesian product in **Cat**: in other words, it comes equipped with two (obvious) faces $\partial^\alpha$, defined on the singleton category $\mathbf{1} = \{*\} = \mathbf{i}^0$ and a (uniquely determined) degeneracy $e$

$$(1) \qquad \{*\} \;\underset{e}{\overset{\partial^\alpha}{\rightleftarrows}}\; \mathbf{i} \qquad\qquad\qquad \partial^\alpha(*) \;=\; \alpha \qquad\qquad\qquad (\alpha = 0, 1).$$

These maps (trivially) satisfy the equations $e\partial^\alpha = \mathrm{id}$. A 1-cube $x\colon \mathbf{i} \to \mathbf{X}$ amounts to an arrow $x\colon x_0 \to x_1$ and has faces $\partial^\alpha(x) = x.\partial^\alpha = x_\alpha$, while the degeneracy, or identity, of an object $x$ is $e(x) = x.e\colon \mathbf{i} \to \mathbf{X}$.

Then, as usual in abstract homotopy theory based on a formal interval (with respect to the cartesian product), the functors

(2) $(-)^n_i \;=\; \mathbf{i}^{i-1} \times - \times \mathbf{i}^{n-i}\colon\; \mathbf{Cat} \to \mathbf{Cat} \qquad\qquad (1 \leq i \leq n),$

produce the higher faces and degeneracies of the interval

(3) $\partial^\alpha_i\colon \mathbf{i}^{n-1} \to \mathbf{i}^n, \qquad \partial^\alpha_i(t_1,..., t_{n-1}) \;=\; (t_1,..., \alpha,..., t_{n-i}),$

$\quad\;\; e_i\colon \mathbf{i}^n \to \mathbf{i}^{n-1}, \qquad e_i(t_1,..., t_n) \;=\; (t_1,..., \hat{t}_i,..., t_n) \qquad\qquad (t_j = 0, 1).$

(Note that these functors between order-categories are determined by their action on objects. The dimension $n$ is generally omitted.)

By a contravariant action, we get the faces and degeneracies of the cubical set $\omega\mathrm{Cub}(\mathbf{X})$, denoted by the same symbols

(4) $\partial^\alpha_i(x) \;=\; x.\partial^\alpha_i, \qquad\qquad e_i(x) \;=\; x.e_i \qquad\qquad (i = 1,..., n;\;\; \alpha = \pm).$

Concatenation of 1-cubes is the ordinary composition in $\mathbf{X}$. But it will be useful to give a formal construction, based on the *concatenation pushout* $\mathbf{i}_2 = \mathbf{3}$ (in $\mathbf{Cat}$), equipped with a *concatenation map* $c$

(5)
$$\begin{array}{ccc} \{*\} & \xrightarrow{\;\partial^+\;} & \mathbf{i} \\ {\scriptstyle \partial^-}\downarrow & & \downarrow{\scriptstyle c^-} \\ \mathbf{i} & \xrightarrow[\;c^+\;]{} & \mathbf{i}_2 \end{array} \qquad\qquad \begin{array}{l} \mathbf{i}_2 \;=\; \mathbf{3} \;=\; \{0 \to 1 \to 2\}, \\ c\colon \mathbf{i} \to \mathbf{i}_2, \\ c(0) \;=\; 0, \;\; c(1) \;=\; 2. \end{array}$$

And indeed, given two consecutive 1-cubes $x, y\colon \mathbf{i} \to \mathbf{X}$, their (ordinary) composite $z = yx$ can be expressed with the functor $[x, y]\colon \mathbf{i}_2 \to \mathbf{X}$ determined by the pushout, and the map $c$

(6) $z \;=\; [x, y].c\colon \mathbf{i} \to \mathbf{i}_2 \to \mathbf{X} \qquad\qquad (\partial^+_1 x = \partial^-_1 y).$

Now, acting on the concatenation pushout and the concatenation map $c$, the functors $(-)^n_i$ produce the *n-dimensional i-concatenation pushout* $\mathbf{i}^{ni}_2$ and the *n-dimensional i-concatenation map* $c_i\colon \mathbf{i}^n \to \mathbf{i}^{ni}_2$

(7)
$$\begin{array}{ccc} \mathbf{i}^{n-1} & \xrightarrow{\;\partial^+_i\;} & \mathbf{i}^n \\ {\scriptstyle \partial^-_i}\downarrow & & \downarrow{\scriptstyle c^-_i} \\ \mathbf{i}^n & \xrightarrow[\;c^+_i\;]{} & \mathbf{i}^{ni}_2 \end{array} \qquad\qquad \begin{array}{l} \mathbf{i}^{ni}_2 \;=\; \mathbf{i}^{i-1} \times \mathbf{i}_2 \times \mathbf{i}^{n-i}, \\ \\ c_i \;=\; \mathbf{i}^{i-1} \times c \times \mathbf{i}^{n-i}\colon\; \mathbf{i}^n \to \mathbf{i}^{ni}_2. \end{array}$$

Given two *i-consecutive* n-cubes $x, y: \mathbf{i}^n \to \mathbf{X}$ (with $\partial_i^+ x = \partial_i^- y$), their *i-concatenation* $z = x +_i y$ is computed using the functor $[x, y]: \mathbf{i}_2^{ni} \to \mathbf{X}$ determined by the pushout in $\mathbf{X}$

(8) $z = [x, y].c_i: \mathbf{i}^n \to \mathbf{i}_2^{ni} \to \mathbf{X}$.

A functor $F: \mathbf{X} \to \mathbf{Y}$ can obviously be extended to a cubical functor $F_*$, which coincides with $F$ in degree 0 (identifying $\mathbf{X}$ with $\mathrm{Cub}_0(\mathbf{X})$)

(9) $F_*: \omega\mathrm{Cub}(\mathbf{X}) \to \omega\mathrm{Cub}(\mathbf{Y}), \qquad F_*(x: \mathbf{i}^n \to \mathbf{X}) = F \circ x: \mathbf{i}^n \to \mathbf{Y}$.

Our formal interval $\mathbf{i}$, in $\mathbf{Cat}$, has no reversion (a peculiar fact of *directed* algebraic topology, see [G1, G2] and references therein). But it has transpositions

(10) $s: \mathbf{i}^2 \to \mathbf{i}^2, \qquad s(t_1, t_2) = (t_2, t_1),$

$s_i = \mathbf{i}^{i-1} \times s \times \mathbf{i}^{n-1-i}: \mathbf{i}^n \to \mathbf{i}^n \qquad (i = 1,..., n-1).$

They operate, contravariantly, on every category $\mathrm{Cub}_n(\mathbf{X}) = \mathbf{Cat}(\mathbf{i}^n, \mathbf{X})$

(11) $s_i(x) = x.s_i: \mathbf{i}^n \to \mathbf{X}$,

generating an action of the whole symmetric group $S_n$.

**3.3. Cubical categories.** The reduced cubical category $\omega\mathrm{Cub}(\mathbf{X})$ has a natural extension $\omega\mathbb{C}\mathrm{ub}(\mathbf{X})$ (notice the different notation), where we introduce *transversal maps* $f: x \to x'$ of n-cubes (also called n-*maps*, or *(n+1)-cells*, or *structural maps*) as natural transformations $f: x \to x': \mathbf{i}^n \to \mathbf{X}$, so that the n-th component $\mathbb{C}\mathrm{ub}_n(\mathbf{X}) = \mathbf{Cat}(\mathbf{i}^n, \mathbf{X})$ is now a category. The new faces, degeneracy and composition are written

(1) $\partial_0^- f = x, \quad \partial_0^+ f = x', \qquad e_0 x = \mathrm{id}(x), \qquad c_0(f, g) = gf: x \to x''$,

where $gf$ is the ordinary (vertical) composition of natural transformations.

The new structure we are interested in, a *cubical category* $\mathbb{A}$ [G3], is a category object within reduced cubical categories (and their cubical functors)

$$(2) \qquad \mathbb{A}^{(0)} \underset{e_0}{\overset{\partial_0^\alpha}{\leftleftarrows\!\!\rightarrow}} \mathbb{A}^{(1)} \overset{c_0}{\longleftarrow} \mathbb{A}^{(2)} \qquad (\alpha = \pm),$$

or, equivalently, a reduced cubical category within categories

(3) $\mathbb{A} = ((\mathrm{tv}_n\mathbb{A}), (\partial_i^\alpha), (e_i), (+_i)), \qquad \mathrm{tv}_n\mathbb{A} = (A_n, M_n, \partial_0^\alpha, e_0, c_0).$

Explicitly, this statement means that $\mathbb{A}$ is a reduced cubical category where each component $tv_n\mathbb{A}$ is a category (namely, the category of n-cubes of $\mathbb{A}$ and their transversal maps, called the *transverse category of* $\mathbb{A}$ *of degree* $n$), while the cubical faces, degeneracies and concatenations are functors

(4) $\partial_i^\alpha: tv_n\mathbb{A} \rightleftarrows tv_{n-1}\mathbb{A} : e_i, \qquad +_i: tv_n\mathbb{A} \times_i tv_n\mathbb{A} \to tv_n\mathbb{A}.$

(The pullback $tv_n\mathbb{A} \times_i tv_n\mathbb{A}$ is the category of pairs of i-consecutive n-cubes.)

A *cubical functor* $F: \mathbb{A} \to \mathbb{B}$ between cubical categories strictly preserves the whole structure. A reduced cubical category amounts to a cubical category all of whose n-maps are identities.

A *transversal* (or *structural*) *transformation* $h: F \to G: \mathbf{A} \to \mathbf{B}$ between cubical functors assigns, to every n-cube $x$ of $\mathbf{A}$, a transversal map in $\mathbb{B}$

(5) $h(x): F(x) \to G(x),$

consistently with faces, degeneracies, concatenations, and satisfying the naturality condition

(nat) $hy \circ Ff = Gf \circ hx, \qquad$ (for every n-map $f: x \to y$ in $\mathbb{A}$).

In a cubical category, as well as in all the weaker cases considered below, a transversal n-map $f: x \to x'$ is said to be *special* if its $2^n$ vertices are identities

(6) $\partial^{\boldsymbol{\alpha}}f: \partial^{\boldsymbol{\alpha}}x \to \partial^{\boldsymbol{\alpha}}x' \qquad \partial^{\boldsymbol{\alpha}} = \partial_1^{\alpha_1}\partial_2^{\alpha_2} ... \partial_n^{\alpha_n} \qquad (\alpha_i = \pm).$

In degree 0, this just means an identity.

Recalling that a k-map between k-cubes is viewed as a *(k+1)-dimensional cell*, an n-truncated cubical category is called an *(n+1)-cubical category*. For instance $2\mathbb{C}ub_*(\mathbf{X}) = tr_2\mathbb{C}ub_*(\mathbf{X})$ is a *3-cubical* categories; and, indeed, its 2-maps are commutative 3-dimensional cubes.

Thus, a 1-cubical category is a category, a 2-cubical category amounts to a (strict) double category, and a 3-cubical category amounts to a (strict) triple category of a particular kind, with:

- objects (of one type);
- arrows in directions 0, 1 and 2, *where the last two types coincide*;
- 2-dimensional cells in directions 01, 02, 12, *where the first two types coincide*;
- and 3-dimensional cells (of one type).

**3.4. Symmetric cubical categories.** A *symmetric cubical category*, or *sc-category*

(1) $\mathbb{A} = ((tv_n\mathbb{A}), (\partial_i^\alpha), (e_i), (+_i), (s_i)),$

is a cubical category (3.3) equipped with cubical functors $s_i\colon tv_n\mathbb{A} \to tv_n\mathbb{A}$ (1 = 1,..., n–1) called *transpositions*, which make it a symmetric cubical set. Furthermore, concatenations and transpositions must be consistent, in the following sense

(2) $s_{i-1}(x +_i y) = s_{i-1}(x) +_{i-1} s_{i-1}(y), \qquad s_i(x +_i y) = s_i(x) +_{i+1} s_i(y),$
$s_j(x +_i y) = s_j(x) +_i s_j(y) \qquad (j \neq i-1, i).$

As for symmetric cubical sets (see 2.2), all faces, degeneracies and concatenations are now determined by the 1-directed ones $(\partial_1^\alpha, e_1, +_1)$, together with transpositions.

$\omega\mathbb{C}ub(\mathbf{X})$ is a symmetric cubical category, with transpositions defined as above (3.2.11). The *involutive* case, further equipped with reversions under axioms which can be easily deduced from [GM], is also of interest, e.g. for higher relations and higher spans or cospans; however, we will not go here into such details.

A *symmetric cubical functor*, or *sc-functor*, is a cubical functor which also preserves transpositions. A *symmetric transversal* (or *structural*) *transformation* $h\colon F \to G\colon \mathbf{A} \to \mathbf{B}$ between sc-functors is defined as above (3.3), by further requiring that the transversal maps $h(x)\colon F(x) \to G(x)$ commute with all transpositions.

**3.5. Symmetric weak cubical categories.** (a) First, a *reduced symmetric pre-cubical category*

(1) $\mathbf{A} = ((A_n), (\partial_i^\alpha), (e_i), (s_i), (+_i)),$

is a symmetric cubical set with compositions, satisfying the consistency axioms (cub.1-2) of 3.1, where transpositions and compositions agree (in the sense of 3.4.2). We are not (yet) assuming that i-compositions behave in a categorical way or satisfy interchange, in any sense, even weak; and there are no transversal maps.

(This notion has been introduced in [G3], 3.4, under the name of 'symmetric pre-cubical category'; but here this term will be used for the stronger notion below, which was also introduced in [G3], 4.1, without a specific name.)

(b) Next, a *symmetric pre-cubical category* will be a *category object* $\mathbb{A}$ within the category of reduced symmetric pre-cubical categories and their (structure-preserving) morphisms

(2) $$\mathbb{A}^{(0)} \underset{e_0}{\overset{\partial_0^\alpha}{\leftrightarrows}} \mathbb{A}^{(1)} \overset{c_0}{\longleftarrow} \mathbb{A}^{(2)} \qquad\qquad (\alpha = \pm).$$

We have thus:

(wcub.1) A reduced symmetric pre-cubical category $\mathbb{A}^{(0)} = ((A_n), (\partial_i^\alpha), (e_i), (s_i), (+_i))$, whose entries are called *n-cubes*, or *n-dimensional objects* of $\mathbb{A}$.

(wcub.2) A reduced symmetric pre-cubical category $\mathbb{A}^{(1)} = ((M_n), (\partial_i^\alpha), (e_i), (s_i), (+_i))$, whose entries are called *n-maps*, or *(n+1)-cells*, of $\mathbb{A}$.

(wcub.3) Morphisms $\partial_0^\alpha$ (0-*faces*) and $e_0$ (0-*degeneracy*), with $\partial_0^\alpha.e_0 = \mathrm{id}$.

(wcub.4) A composition law $c_0$ which assigns to two 0-consecutive n-maps $f\colon x \to x'$, $h\colon x' \to x''$ (of the same dimension), an n-map $hf\colon x \to x''$ (also written $h.f$). This composition law is (strictly) categorical, and forms a category $\mathrm{tv}_n\mathbb{A} = (A_n, M_n, \partial_0^\alpha, e_0, c_0)$, for every $n \geq 0$. It is also consistent with the symmetric pre-cubical structure, in the following sense

(3) $$\partial_i^\alpha(hf) = (\partial_i^\alpha h).(\partial_i^\alpha f), \qquad e_i(hf) = (e_ih)(e_if), \qquad s_i(hf) = (s_ih)(s_if),$$

$$(h +_i k).(f +_i g) = hf +_i kg, \qquad \begin{array}{ccccc} \bullet & \xrightarrow{\partial_i^- f} & \bullet & \xrightarrow{\partial_i^- h} & \bullet \\ x\downarrow & -f\to & \downarrow & -h\to & \downarrow x'' \\ \bullet & \longrightarrow & \bullet & \longrightarrow & \bullet \\ y\downarrow & -g\to & \downarrow & -k\to & \downarrow y'' \\ \bullet & \xrightarrow[\partial_i^+ g]{} & \bullet & \xrightarrow[\partial_i^+ k]{} & \bullet \end{array} \qquad \begin{array}{cc} \bullet \longrightarrow & 0 \\ \downarrow & i \end{array}$$

(c) Finally, a *symmetric weak cubical category* $\mathbb{A}$ [G3, 4.2] is a symmetric pre-cubical category, as defined above in (b), which is further equipped with assigned invertible *special* transversal maps (see 3.3.6). The latter play the role of comparisons for units, associativity (in direction 1) and cubical interchange (in direction 1, 2), the other comparisons being generated by transpositions.

Essentially, we have the following additional structure and conditions:

(wcub.5) For every n-cube $x$ $(n > 0)$, we have an invertible special n-map $\lambda_1 x$, natural on n-maps

(4) $$\lambda_1 x\colon (e_1\partial_1^- x) +_1 x \to x \qquad\qquad (\textit{left-unit 1-comparison}).$$

(wcub.6) For every n-cube $x$ $(n > 0)$, we have an invertible special n-map $\rho_1 x$, natural on n-maps

(5) $\rho_1 x\colon x \to x +_1 (e_1 \partial_1^+ x)$, (*right-unit 1-comparison*).

(wcub.7) For every three n-cubes $x, y, z$, which are consecutive in direction 1, we have an invertible special n-map $\kappa_1(x, y, z)$, natural on n-maps

(6) $\kappa_1(x, y, z)\colon x +_1 (y +_1 z) \to (x +_1 y) +_1 z$ (*associativity 1-comparison*).

(wcub.8) Given four n-cubes $x, y, z, u$ making the following concatenations legitimate, we have an invertible n-map $\chi_1(x, y, z, u)$, which is natural on n-maps

(7) $\chi_1(x, y, z, u)\colon (x +_1 y) +_2 (z +_1 u) \to (x +_2 z) +_1 (y +_2 u)$

(*interchange 1-comparison*).

(wcub.9) Finally, these comparisons are coherent (*coherence axiom*).

The complete axioms (wcub.5-9), written in [G3], 4.2-4.3, give conditions on the cubical faces of these comparisons and an explicit list of coherence conditions.

Truncation works as described at the end of 3.3. Since the symmetric groups $S_0$ and $S_1$ are trivial, *a 1-truncated symmetric weak cubical category has no transpositions and is the same as a weak double category*.

**3.6. Path functors of cubical categories.** We will write cb**Cat** the 2-category of (small) cubical categories, cubical functors and their transversal transformations; we will write sc**Cat** the symmetric analogue.

Cubical categories have a left and a right path 2-functor, which are obvious liftings of the ones of cubical sets

(1) $P\colon \mathrm{cb}\mathbf{Cat} \to \mathrm{cb}\mathbf{Cat}$, $\quad P' = SPS\colon \mathrm{cb}\mathbf{Cat} \to \mathrm{cb}\mathbf{Cat}$.

In every degree, $P$ discards faces, degeneracies and concatenations in direction 1 while $P'$ discards the ones in the last direction. Again, $P$ and $P'$ are linked by the transposer $S\colon \mathrm{cb}\mathbf{Cat} \to \mathrm{cb}\mathbf{Cat}$, which in every degree reverses the order of faces, degeneracies and concatenations.

Also here, $P$ and $P'$ have isomorphic liftings to the symmetric case (where $S \cong \mathrm{id}$), and we will only use the path 2-functor which discards direction 1, written

(2) $P\colon \mathrm{sc}\mathbf{Cat} \to \mathrm{sc}\mathbf{Cat}$.

The symmetric weak case is similar, and has a path 2-functor

(3) $P\colon \mathrm{wsc}\mathbf{Cat} \to \mathrm{wsc}\mathbf{Cat}$,

for the 2-category of weak sc-categories, their (strict) sc-functors and their transversal transformations.

For cb**Cat**, sc**Cat** and wsc**Cat**, we always have the relation

(4) $\mathrm{tv}_n \circ P = \mathrm{tv}_{n+1}$.

**3.7. Exponentials.** (a) First, for a small *ordinary* category **X** and a symmetric pre-cubical category $\mathbb{A}$, we have the *symmetric pre-cubical category of level functors and their natural transformations*

(1) $\mathbb{A}^{\mathbf{X}} = \mathbb{L}\mathrm{v}(\mathbf{X}, \mathbb{A})$, $\quad\quad \mathrm{tv}_n(\mathbb{L}\mathrm{v}(\mathbf{X}, \mathbb{A}) = \mathbf{Cat}(\mathbf{X}, \mathrm{tv}_n\mathbb{A})$.

An n-cube is an ordinary functor $F: \mathbf{X} \to \mathrm{tv}_n\mathbb{A}$, and will also be called an *n-level functor* with values in $\mathbb{A}$; an n-map is a natural transformation $f: F \to G: \mathbf{X} \to \mathrm{tv}_n\mathbb{A}$. Their faces, degeneracies, transpositions and concatenations are obtained by post-composition with the structural functors of $\mathbb{A}$ (3.3, 3.4)

(2) $\partial^{\alpha}_{i}: \mathrm{tv}_n\mathbb{A} \rightleftarrows \mathrm{tv}_{n-1}\mathbb{A} :e_i$,

$\quad s_i: \mathrm{tv}_n\mathbb{A} \rightleftarrows \mathrm{tv}_n\mathbb{A}$, $\quad\quad +_i: \mathrm{tv}_n\mathbb{A} \times_i \mathrm{tv}_n\mathbb{A} \to \mathrm{tv}_n\mathbb{A}$.

(b) If $\mathbb{A}$ is a weak sc-category, also $\mathbb{A}^{\mathbf{X}}$ is, with comparisons obtained from the ones of $\mathbb{A}$.

(c) Now, let $\mathbb{X}$ be a small weak sc-category and $\mathbb{A}$ a weak sc-category. We define the weak sc-category of *higher sc-functors from* $\mathbb{X}$ to $\mathbb{A}$ *and their transversal* (or *structural*) *transformations*

(3) $\mathbb{A}^{\mathbb{X}} = \mathbb{W}\mathrm{sc}(\mathbb{X}, \mathbb{A})$.

An n-cube is an sc-functor $F: \mathbb{X} \to P^n\mathbb{A}$, an n-map is a transversal transformation of such functors. Faces, degeneracies, transpositions are obtained by post-composition with the structure of the path functor $P$ of weak sc-categories (3.6.3)

(4) $\partial^{\alpha}_{i}: P^n\mathbb{A} \rightleftarrows P^{n-1}\mathbb{A} :e_i$, $\quad\quad s_i: P^n\mathbb{A} \to P^n\mathbb{A}$.

Similarly, one obtains concatenations and the comparisons of the weak structure of $\mathbb{A}^{\mathbb{X}}$.

If $\mathbb{X}$ is the free weak sc-category on a small category **X**, we obtain the 'same' structure $\mathbb{A}^{\mathbf{X}}$ considered above, in (b).

(d) *Cubical transformations*. The cubical geometry allows us to view these n-cubes as *higher homotopies*. Thus, a 1-cube $F\colon \mathbb{X} \to P\mathbb{A}$ can be viewed as a *cubical* (or *geometric*) transformation of sc-functors

(5) $F\colon F^- \to F^+\colon \mathbb{X} \to \mathbb{A}, \qquad F^\alpha = \partial^\alpha F,$

and more generally an n-cube $F\colon \mathbb{X} \to P^n\mathbb{A}$ is an *n-dimensional cubical transformation*, with $2n$ faces $\partial_i^\alpha F$.

Notice that in the *non*-symmetric case, the presence of two non-isomorphic path functors makes things much more complicated. We would have *left* and *right* cubical transformations

(6) $\mathbb{X} \to P\mathbb{A}, \qquad \mathbb{X} \to P'\mathbb{A},$

and, in higher degree, we should replace $P^n$ with n-ary compositions $P_i^n = P^{n-i}.P'^i$ (1.8.1).

**3.8. Transversal invariance.** Extending a property of double categories (introduced in [GP1], 2.4, under the name of *horizontal invariance*, and characterised in [GP2], 1.5, we say that the symmetric pre-cubical category $\mathbb{A}$ is *transversally invariant* if, for every n-cube $x$ and every pair of transversal (n–1)-isomorphisms $f^\alpha\colon \partial_1^\alpha x \to y^\alpha$ $(\alpha = \pm)$, there exist some transversal n-isomorphism $f\colon x \to y$ with $\partial_1^\alpha f = f^\alpha$ (and therefore $\partial_1^\alpha y = y^\alpha$)

(1)
$$\begin{array}{ccc} \bullet & \xrightarrow{\;f^-\;} & y^- \\ {\scriptstyle x}\downarrow & f & \downarrow{\scriptstyle y} \\ \bullet & \xrightarrow[\;f^+\;]{} & y^+ \end{array} \qquad\qquad \begin{array}{cc} \bullet & \longrightarrow \; 0 \\ \downarrow{\scriptstyle 1} & \end{array}$$

Of course, because of symmetries, the same property holds for every pair of faces $\partial_i^\alpha$, with $i = 1,..., n$.

**3.9. Weak double categories and coskeleton.** A weak double category will generally be viewed in wsc**Cat** *via the coskeleton functor*, right adjoint to cubical 1-truncation wsc**Cat** $\to$ w**Dbl**

(1) $\mathrm{cosk}_1\colon \mathrm{w}\mathbf{Dbl} \to \mathrm{wsc}\mathbf{Cat}, \qquad \mathrm{tr}_1 \dashv \mathrm{cosk}_1.$

Concretely, if $\mathbb{A}$ is a weak double category, the weak sc-category $\mathbb{B} = \mathrm{cosk}_1(\mathbb{A})$ coincides with $\mathbb{A}$ in cubical degree 0 and 1. Then, in the component $\mathrm{tv}_2\mathbb{B}$, a 2-cube is a 'shell' of 1-cubes of $\mathbb{A}$

$$\text{(2)} \qquad \begin{array}{ccc} A & \xrightarrow{\;v\;} & B \\ {\scriptstyle u}\downarrow & \# & \downarrow{\scriptstyle u'} \\ C & \xrightarrow[\;v'\;]{} & D \end{array} \qquad \begin{array}{l} \bullet \longrightarrow 1 \\ \downarrow 2 \end{array} \qquad \begin{array}{ll} \partial_1^-u = \partial_1^-v, & \partial_1^+u' = \partial_1^+v', \\ & \\ \partial_1^+u = \partial_1^-v', & \partial_1^+v = \partial_1^-u', \end{array}$$

under no further condition. A transversal 2-map is a similar 'shell' of 1-maps of $\mathbb{A}$. Notice that the *#-marked* square (2) is *not* assumed to commute under concatenation of 1-cubes, in any sense (strict, weak or lax). Similarly, one defines all the higher components, by n-dimensional shells of 1-cubes and 1-maps of $\mathbb{A}$.

Faces and degeneracies are obvious: for instance, for the 2-cube $U$ represented above, $\partial_1^-U = u$ and $\partial_2^-U = v$. Concatenations are also obvious, and computed with the concatenation of 1-cubes (or 1-maps) in $\mathbb{A}$; thus, in dimension 2 and direction 1, we get

$$\text{(3)} \qquad \begin{array}{ccccc} A & \xrightarrow{\;v\;} & B & \xrightarrow{\;w\;} & B' \\ {\scriptstyle u}\downarrow & \# & \downarrow{\scriptstyle u'} & \# & \downarrow{\scriptstyle u''} \\ C & \xrightarrow[\;v'\;]{} & D & \xrightarrow[\;w'\;]{} & D' \end{array} \qquad \mapsto \qquad \begin{array}{ccc} A & \xrightarrow{\;v+w\;} & B' \\ {\scriptstyle u}\downarrow & \# & \downarrow{\scriptstyle u''} \\ C & \xrightarrow[\;v'+w'\;]{} & D' \end{array}$$

Finally, the comparisons for associativity and units are families of comparisons of $\mathbb{A}$, while interchange is necessarily strict.

Viewing weak double categories in this way leads us to define a *cubical* (or *geometric*) *transformation of double functors* (between weak double categories) $F\colon F^- \to F^+\colon \mathbb{X} \to \mathbb{A}$ as a cubical transformation of the corresponding 1-coskeletons

(4) $F\colon \mathrm{cosk}_1F^- \to \mathrm{cosk}_1F^+\colon \mathrm{cosk}_1\mathbb{X} \to \mathrm{cosk}_1\mathbb{A}$.

Explicitly, this means to assign:

(a) to every object (0-cube) $X$ of $\mathbb{X}$ a 1-cube $FX\colon F^-X \to\!\!\!\!\!\bullet\;\; F^+X$ of $\mathbb{A}$,

(b) to every 0-map $f\colon X \to Y$ of $\mathbb{X}$, a 1-map $Ff\colon F^-f \to\!\!\!\!\!\bullet\;\; F^+f$ of **A**,

consistently with the transversal structure (faces, degeneracies and composition):

(5) $F(\partial_0^\alpha f) = \partial_0^\alpha(Ff), \qquad F(e_0X) = e_0(FX), \qquad F(gf) = Fg.Ff.$

Notice that there is *no* 'naturality' condition based on a 1-cube $u: X \to X'$ of $\mathbf{X}$: the latter is simply sent to a 2-dimensional *shell*, with 1-directed faces $F^{\alpha}(u)$ and 2-directed faces $FX$, $FX'$

$$(6) \qquad \begin{array}{ccc} F^{-}X & \xrightarrow{FX} & F^{+}X' \\ {\scriptstyle F^{-}u}\downarrow & \# & \downarrow{\scriptstyle F^{+}u} \\ F^{-}X' & \xrightarrow[FX']{} & F^{+}X' \end{array} \qquad\qquad \begin{array}{l} \bullet \longrightarrow 1 \\ \downarrow 2 \end{array}$$

Moreover, the consistency with concatenation of 1-cubes is simply 'managed' by the cubical functors $F^{-}, F^{+}$.

More generally, we define in the same way a *cubical transformation* $F^{-} \to F^{+}: \mathbb{X} \to \mathbb{A}$ *of weak* (or *lax*, or *colax*) *cubical functors* $F^{\alpha}$: the only comparisons which we need are those of the latter.

This notion is certainly simpler than a 'strong vertical transformation of lax double functors', as defined in [GP1], 7.4 - where a weak naturality condition was assumed. Further study will show whether the present notion does work better.

## 4. Examples of symmetric weak or strict cubical categories

After the strict sc-category $\omega\mathbb{C}\mathrm{ub}(\mathbf{X})$, described in 3.2, we describe here the weak sc-categories $\omega\mathbb{C}\mathrm{osp}(\mathbf{X})$ and $\omega\mathbb{S}\mathrm{p}(\mathbf{X})$ of cubical (co)spans. Then we construct the strict sc-category $\omega\mathbb{R}\mathrm{el}$ of cubical relations of sets, and we end with a sketch of a strict sc-category $\omega\mathbb{C}\mathrm{at}$ of cubical profunctors. These two constructions will be made precise in the next section.

All these examples are transversally invariant (3.8).

**4.1. Cubical cospans.** We begin our examples by reviewing the symmetric weak cubical category $\omega\mathbb{C}\mathrm{osp}(\mathbf{X})$ of higher cubical cospans, introduced in [G3] to study cubical cospans in Algebraic Topology and higher (cubical) cobordism. Its cubical 1-truncation (3.3, 3.5) is the weak double category $\mathbb{C}\mathrm{osp}(\mathbf{X})$ studied in [GP1].

Let $\mathbf{X}$ be a category *with a full choice of distinguished pushouts*: in other words, to every span $(f, g)$ we assign *one distinguished* pushout $(f', g')$, under the *unitarity constraint* for which the distinguished pushout of the span $(f, 1)$ is $(1, f)$ (and symmetrically)

$$
(1)\qquad
\begin{array}{ccc}
\bullet & \xrightarrow{f} & \bullet \\
{\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f'} \\
\bullet & \xrightarrow[g']{} & \bullet
\end{array}
\qquad\qquad
\begin{array}{ccc}
x & \xrightarrow{f} & x' \\
{\scriptstyle 1}\downarrow & & \downarrow{\scriptstyle 1} \\
x & \xrightarrow[f]{} & x'
\end{array}
$$

The 'geometric model' of the construction of our cubical structure is the category $\mathbf{\wedge}$, called the *formal cospan*, and its cartesian powers

$$
(2)\qquad -1 \longrightarrow 0 \longleftarrow 1 \qquad \mathbf{\wedge},
$$

$$
\begin{array}{ccccc}
(-1,-1) & \longrightarrow & (0,-1) & \longleftarrow & (1,-1) \\
\downarrow & & \downarrow & & \downarrow \\
(-1,0) & \longrightarrow & (0,0) & \longleftarrow & (1,0) \\
\uparrow & & \uparrow & & \uparrow \\
(-1,1) & \longrightarrow & (0,1) & \longleftarrow & (1,1)
\end{array}
\qquad \mathbf{\wedge}^2.
\qquad\qquad
\begin{array}{cc}
\bullet & \longrightarrow \; 1 \\
\downarrow{\scriptstyle 2} &
\end{array}
$$

An n-cube of $\omega\mathbb{C}\mathrm{osp}(\mathbf{X})$ is a functor $x: \mathbf{\wedge}^n \to \mathbf{X}$, and an n-map is a natural transformation $f: x \to y: \mathbf{\wedge}^n \to \mathbf{X}$; these objects and maps form the category

(3) $\mathrm{Cosp}_n(\mathbf{X}) = \mathbf{Cat}(\mathbf{\wedge}^n, \mathbf{X})$.

It is now easy to construct a symmetric cubical object in **Cat**, based on the structure of the category $\mathbf{\wedge}$ as a formal symmetric interval, with respect to the cartesian product (in **Cat**)

$$
(4)\qquad \partial^\alpha: \mathbf{1} \rightrightarrows \mathbf{\wedge}, \qquad e: \mathbf{\wedge} \to \mathbf{1}, \qquad s: \mathbf{\wedge}^2 \to \mathbf{\wedge}^2 \qquad (\alpha = \pm),
$$

$$
\partial^\alpha(*) = \alpha 1, \qquad\qquad s(t_1, t_2) = (t_2, t_1).
$$

Namely, faces, degeneracies and transpositions of n-cubes and n-maps are defined by pre-composition with the following maps between cartesian powers of $\mathbf{\wedge}$ (for $\alpha = \pm$ and $i = 1,..., n$)

$$
\begin{array}{lll}
(5) & \partial^\alpha_i: \mathbf{\wedge}^{n-1} \to \mathbf{\wedge}^n, & \partial^\alpha_i(t_1,..., t_{n-1}) = (t_1,..., \alpha 1,..., t_{n-1}), \\
 & e_i: \mathbf{\wedge}^n \to \mathbf{\wedge}^{n-1}, & e_i(t_1,..., t_n) = (t_1,..., \hat{t}_i,..., t_n), \\
 & s_i: \mathbf{\wedge}^{n+1} \to \mathbf{\wedge}^{n+1}, & s_i(t_1,..., t_{n+1}) = (t_1,..., t_{i+1}, t_i,..., t_n),
\end{array}
$$

so that the $2n$ faces of an n-cube $x: \mathbf{\wedge}^n \to \mathbf{X}$ are $\partial^\alpha_i(x) = x \circ \partial^\alpha_i: \mathbf{\wedge}^{n-1} \to \mathbf{X}$, and so on.

**4.2. The cubical category of relations.** We define now the strict sc-category of cubical relations of small sets $\omega\mathbb{R}\text{el} = \omega\mathbb{R}\text{el}(\mathbf{Set})$; of course, the same can be done in more general settings.

We will construct $\omega\mathbb{R}\text{el}$ as a quotient of $\omega\mathbb{S}\text{p}(\mathbf{Set})$, extending the construction of ordinary (binary) relations as equivalence classes of spans.

First, at the level of n-cubes, the equivalence relation is generated by pairs of cubical spans $x, x' \colon \mathbf{v}^n \to \mathbf{Set}$ which admit a special transversal map $p \colon x \to x'$ all whose components $p(\mathbf{t}) \colon x(\mathbf{t}) \to x'(\mathbf{t})$ are surjective mappings ($\mathbf{t} \in \{-1, 0, 1\}^n$).

Every equivalence class $[x]$ of n-cubes contains some representative which is a *jointly monic n-span*, in the sense that each pair of mappings with the same domain and having the same direction $i$ (for $i = 1,..., n$) is jointly monic. The existence of such representatives is trivial in degree 0, obvious and well known in degree 1. In degree 2, represented in the diagram below, we begin by choosing a jointly monic representative for each of the four faces $\partial_i^\alpha$ (the four spans at the boundary)

$$
\text{(1)} \qquad
\begin{array}{ccccc}
\bullet & \longleftarrow & \bullet & \longrightarrow & \bullet \\
\uparrow & & {\scriptstyle a'}\uparrow & & \uparrow \\
\bullet & \overset{a}{\longleftarrow} & \bullet & \overset{b}{\longrightarrow} & \bullet \\
\downarrow & & {\scriptstyle b'}\downarrow & & \downarrow \\
\bullet & \longleftarrow & \bullet & \longrightarrow & \bullet
\end{array}
\qquad\qquad
\begin{array}{cc}
\bullet & \longrightarrow \; 1 \\
\downarrow \; 2 &
\end{array}
$$

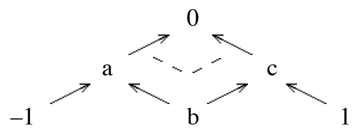
Then we choose a jointly monic span $(a, b)$ in direction 1 (say), which induces a consistent choice for the span $(a', b')$. The latter is also jointly monic, as it follows from the fact that the four composites from the centre to the vertices are jointly monic. In higher dimension one proceeds in the same way. A jointly monic representative of an equivalence class of n-cubes is determined up to an invertible special transversal map of cubical spans.

Second, at the level of n-maps, the equivalence relation is generated by pairs of n-maps of cubical spans $f \colon x \to y$, $f' \colon x' \to y'$ for which there exists a commutative diagram

$$
\text{(2)} \qquad
\begin{array}{ccc}
x & \overset{f}{\longrightarrow} & y \\
{\scriptstyle p}\downarrow & & \downarrow{\scriptstyle q} \\
x' & \underset{f'}{\longrightarrow} & y'
\end{array}
$$

The concatenation $x +_i y$ of two n-cubes which are i-consecutive (i.e., $\partial_i^+(x) = \partial_i^-(y)$) is computed in the obvious way, by $3^{n-1}$ distinguished pushouts whose 'vertices' are the ones of the common face. More precisely, concatenation can be given a formal definition (as in [G3], and along the same lines of 3.2.5), for which we only give here some hints. It is based on the *model of binary composition* (for ordinary cospans), the category $\wedge_2$ displayed below, with one non-trivial distinguished pushout

(6)

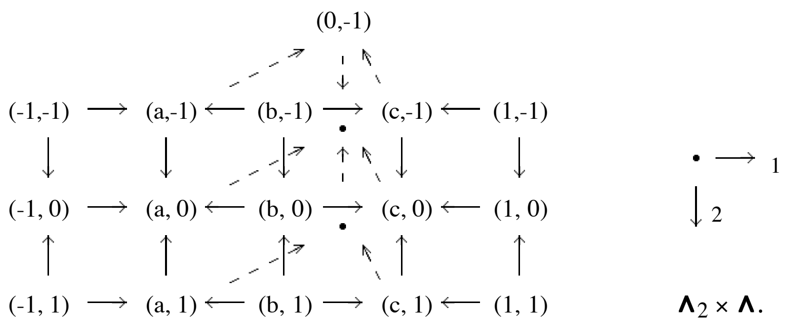


$\wedge_2$.

Indeed, given two consecutive cospans x, y in **X**, we get an obvious functor $[x, y]: \wedge_2 \to \mathbf{X}$, from which we deduce the concatenation $x +_1 y: \wedge \to \mathbf{X}$ by pre-composing [x, y] with the *concatenation map* $c: \wedge \to \wedge_2$, already displayed above by the labelling of objects in $\wedge_2$.

Then, 1-concatenation of n-cubes is based on the cartesian product $\wedge_2 \times \wedge^{n-1}$

(7)



$\wedge_2 \times \wedge$.

Comparisons for associativity and interchange can be defined taking advantage of this formal construction, see [G3], Section 3. On the other hand, degeneracies work as strict units, because of the unitarity constraint recalled above for the choice of pushouts.

Of course, cubical spans are obtained by the dual procedure, for a category **X** with assigned pullbacks:

(8) $\omega\mathbb{S}\mathrm{p}(\mathbf{X}) = \omega\mathbb{C}\mathrm{osp}(\mathbf{X}^{op})$, $\qquad \mathrm{Sp}_n(\mathbf{X}) = \mathbf{Cat}(\vee^n, \mathbf{X})$,

where the category $\vee$ is the *formal span*: $-1 \leftarrow 0 \rightarrow 1$.

where $p, q$ are special transversal maps whose components are surjective mappings.

Equivalently, since $p, q$ are the identity on each vertex, a transversal map $f\colon [x] \to [y]$ between n-cubical relations can be defined as a family of mappings between the vertices of $x$ and $y$

(3) $\quad f\mathbf{t}): x(\mathbf{t}) \to y(\mathbf{t}), \qquad\qquad \mathbf{t} \in \{-1, 1\}^n,$

which can be extended to a transversal map of cubical spans $x \to y$ between *jointly monic representatives*. This extension is unique, because of the cancellation property of such representatives.

Faces, degeneracies, transpositions and concatenations are induced on the quotient, by the ones of cubical spans. Since the comparisons of $\omega\mathbb{S}p(\mathbf{Set})$ are invertible transversal maps, all their components are surjective and the quotient we are considering is a *strict* symmetric cubical category.

The quotient procedure we have used will be abstracted in the next section, and its dual will be used to define cubical profunctors.

**4.3. Cubical relations as subsets of products.** One can give a more concrete description of $\omega\mathbb{R}el$, whose worse drawback is that the construction of degeneracies becomes cumbersome.

Items will be indexed on the three-element set $\{0, u, 1\}$ and its powers. A 1-cubical relation is an ordinary relation $a\colon a_0 \nrightarrow a_1$ of sets, viewed as a subset $a_u \subset a_0 \times a_1$, and will be written with a dot-marked arrow; their composition will be written in additive notation.

A 2-cubical relation $a$ consists of:

- four vertices $(a_{ij})\colon 2{\times}2 \to \mathbf{Set}$ (where $2{\times}2 = \{0, 1\}^2$ is a discrete category on four objects),
- four (binary) relations on the sides of a square, written $a_{uj}$ and $a_{iu}$ (see the diagram below, no condition of commutativity is assumed)
- and one *quaternary relation* $a_{uu} \subset \prod a_{ij}$ whose projection on each side is contained in the corresponding binary relation

$$(1)\qquad \begin{array}{ccc} a_{00} & \xrightarrow{\;a_{u0}\;} & a_{10} \\ {\scriptstyle a_{0u}}\downarrow & a_{uu} & \downarrow{\scriptstyle a_{1u}} \\ a_{01} & \xrightarrow[\;a_{u1}\;]{} & a_{11} \end{array} \qquad \begin{array}{cc} \bullet \longrightarrow & 1 \\ \downarrow 2 & \end{array} \qquad \begin{array}{l} a_{uu} \subset a_{00} \times a_{01} \times a_{10} \times a_{11}, \\ (p_{0j}, p_{1j})(a_{uu}) \subset a_{uj}, \\ (p_{i0}, p_{i1})(a_{uu}) \subset a_{iu}. \end{array}$$

(We write $p_{ij}$ the four cartesian projections of $\Pi\, a_{ij}$.) The 1-concatenation $c = a +_1 b$ is defined when the 2-cubes $a, b$ are consecutive in direction 1, i.e. $a_{1u} = b_{0u}$, and is computed below, at the right

$$(2) \qquad \begin{array}{ccccccccccc} & a_{u0} & & & & b_{u0} & & & & a_{u0} + b_{u0} & \\ a_{00} & \longrightarrow & a_{10} & = & b_{00} & \longrightarrow & b_{10} & \quad & a_{00} & \longrightarrow & b_{10} \\ a_{0u} \downarrow & a_{uu} & \downarrow a_{1u} & = & b_{0u} \downarrow & b_{uu} & \downarrow b_{1u} & & a_{0u} \downarrow & c_{uu} & \downarrow b_{1u} \\ a_{01} & \longrightarrow & a_{11} & = & b_{01} & \longrightarrow & b_{11} & & a_{01} & \longrightarrow & b_{11} \\ & a_{u1} & & & & b_{u1} & & & & a_{u1} + b_{u1} & \end{array}$$

Obviously, the subset

(3) $\quad c_{uu} \;=\; a_{uu} +_1 b_{uu} \;\subset\; a_{00} \times a_{01} \times b_{10} \times b_{11},$

contains those 4-tuples $(x_{00}, x_{01}, z_{10}, z_{11})$ for which there is some pair $(y, y') \in a_{10} \times a_{11} = b_{00} \times b_{01}$ such that $(x_{00}, x_{01}, y, y') \in a_{uu}$ and $(y, y', z_{10}, z_{11}) \in b_{uu}$. In other words, $a_{uu} +_1 b_{uu}$ *is an ordinary composition of relations*, provided we view $a_{uu}$ and $b_{uu}$ as binary relations, as follows:

(4) $\quad a_{uu}\colon a_{00} \times a_{01} \rightarrow a_{10} \times a_{11}, \qquad\qquad b_{uu}\colon b_{00} \times b_{01} \rightarrow b_{10} \times b_{11}.$

This proves that 1-concatenation is strictly associative, with strict units provided by the following degeneracies $e_1(a)$ of ordinary relations

$$(5) \qquad \begin{array}{ccc} & \mathrm{id} & \\ a_0 & \longrightarrow & a_0 \\ a_u \downarrow & (e_1a)_{uu} & \downarrow a_u \\ a_1 & \longrightarrow & a_1 \\ & \mathrm{id} & \end{array}$$

$$(e_1a)_{uu} \;=\; \{(x_0, x_1, x_0, x_1) \in a_0 \times a_1 \times a_0 \times a_1 \mid (x_0, x_1) \in a_u\}.$$

The same holds for 2-concatenation, which can be defined in the symmetric way, or by transposition of the previous operation:

(6) $\quad a \;+_2 a' \;=\; s_1(s_1a \;+_1 s_1a').$

We proceed analogously in higher dimension. The definition of degeneracies, extending (5), looks unnatural.

**4.4. Cospans of embeddings of categories.** Cubical profunctors can be constructed by a quotient procedure, whose formal aspects are transversally dual to the procedure sketched above (4.2): we will start from cubical *co*spans of full embeddings of categories, and identify them when they have the same reduced form, in a suitable sense. This will be done at the end of the next section, after formalising this kind of quotients.

The crucial point is the fact that an ordinary profunctor $x: x_{-1} \nrightarrow x_1$ has a *collage* $x_0$, which consists of the sum of the categories $x_{-1}$ and $x_1$, supplemented with new homs $x_0(a, b) = x(a, b)$, for $a$ in $x_{-1}$ and $b$ in $x_1$. (Formally, the collage of a profunctor is a double colimit, the *cotabulator*, in the weak double category of categories, functors and profunctors, see [GP1].)

Thus, the profunctor $x$ can be described as a cospan

(1) $$x_{-1} \xrightarrow{x^-} x_0 \xleftarrow{x^+} x_1$$

characterised by the following conditions (which imply that $x^-, x^+$ have disjoint images)

(i) $x^-, x^+$ are full embeddings,

(ii) there are no arrows in $x_0$ going from an object of $x_1$ to an object of $x_{-1}$,

(iii) the embeddings $x^-, x^+$ cover (together) all the objects of $x_0$.

We have already seen how a profunctor yields such a cospan. Conversely, given the cospan (1), the profunctor is reconstructed as:

(2) $x: (x_{-1})^{op} \times x_1 \to \mathbf{Set}, \qquad (a, b) \mapsto x_0(x^-(a), x^+(b)).$

Now, the first two conditions above are closed under concatenation, but the third, a sort of *jointly-epi* condition, is not (in the same way as *jointly monic* spans are not closed under concatenation). Which is why we will obtain (cubical) profunctors as equivalence classes of (cubical) cospans, with reduced representatives satisfying the (cubical extension of the) third condition (5.7).

## 5. Weak cubical categories, cubical rewriting and quotients

We formalise the procedure which has been used above to pass from the weak sc-category of cubical spans to the strict sc-category of cubical relations (4.2), as a 'quotient' which forces certain transversal maps to become identities. Then, the dual procedure is used to construct a strict sc-category of cubical profunctors. These techniques are a sort of *term rewriting*, based on the existence of 'normal forms', determined up to transversal isomorphism.

**5.1. Rewriting and normal forms.** Let us first recall a classical case of term rewriting, the description of the free group on a set X.

One begins with the disjoint union $Y = X \cup X^{-1}$ of the set X with a isomorphic copy, adding for each $x \in X$ an element $x^{-1}$ (which will become its inverse). Then, we let $Y^* = \Sigma\, Y^n$ be the free monoid on the set Y, consisting of words $w = y_1,..., y_n$ of elements of Y; they are multiplied by juxtaposition, and the unit is the empty word e.

We define an order relation $w' \prec w$, generated by the following (elementary) rewrite rule:

(1) the word w' can be obtained from w by omitting one occurrence of the sequence $x, x^{-1}$ or one occurrence of the sequence $x^{-1}, x$ (for some $x \in X$).

This order is obviously consistent with juxtaposition, and it spans an equivalence relation $w \sim w'$ which is a congruence of semigroups. It is easy to prove that the quotient semigroup $F = Y^*/\!\sim$ is a group, and actually the free group on the set X: the embedding $X \subset Y \subset Y^* \to Y^*/\!\sim$ satisfies the usual universal property.

Now, in order to better understand the construction, it is convenient to notice that an equivalence class of words in $Y^*$ is determined by a word in *normal form*, to which the rewrite rule (1) cannot be applied (i.e., a minimal element for the order relation).

Plainly, every equivalence class [w] contains some minimal word, which cannot be further reduced. But the crucial fact is that the ordered set [w] has a minimum, the *normal form* $\hat{w}$, which, therefore, does not depend on the reduction process which yields it. (One begins with proving that two distinct immediate predecessors of w always have a common immediate predecessor; since the number of immediate predecessors of a given word is finite, the existence of the minimum is an easy consequence.)

Notice that normal forms are not closed under multiplication: we only have that $(w.z)\hat{} = (\hat{w}.\hat{z})\hat{}$. Thus, the quotient $Y^*/\!\sim$ cannot be embedded as a subsemigroup of $Y^*$ (as soon as X is not empty).

Notice also that the fact of having a normal representative in every equivalence class is an effective way of describing the elements of the free group, but is not formally necessary for the construction of the latter. In other situations, as the ones considered below, it may happen that the existence of normal forms is crucial in order to prove that some operation passes to the quotient.

**5.2. Rewriting for cubical spans.** The passage from cubical spans to cubical relations (of sets) is based on a *preorder* $x' \prec x$ between cubical spans, given by the existence of a *reduction* $p: x \to x'$, or *domain-rewriting*: a special transversal map whose components are surjective mappings (so that we can think of $x'$ as a 'simpler form' of $x$). The following diagram shows the 1-dimensional case

$$\begin{array}{ccccc} x_{-1} & \xleftarrow{x^-} & x_0 & \xrightarrow{x^+} & x_1 \\ \| & & \downarrow p_0 & & \| \\ x_{-1} & \xleftarrow[x'^-]{} & x'_0 & \xrightarrow[x'^+]{} & x_1 \end{array} \qquad (1)$$

Now, in the equivalence relation between n-cubes $x \sim x'$ generated by this preorder:

(*) every equivalence class $[x]$ has a least representative $\hat{x}$, determined up to transversal isomorphism (and we choose one).

This representative will be called the *reduced form*, or *normal form*, of $x$. It is the essentially unique *jointly monic* n-span which belongs to $[x]$ (in the usual sense for $n = 1$, and in the sense of 4.2 for higher $n$).

As a second crucial fact,

(**) for every cubical span $x$, there is precisely one reduction $px: x \to \hat{x}$,

which will be called the *least reduction*, or *normal reduction*, of $x$. It is also characterised, directly (i.e. without using $\hat{x}$), as 'the' minimum in the set of reductions starting from $x$, with respect to the usual preorder of epimorphisms: $p' \prec p$ if $p'$ factors through $p$. (Actually, since $\hat{x}$ is jointly monic, there is a unique *special transversal map* $x \to \hat{x}$, and for every n-cube $y$ at most one special transversal map $y \to \hat{x}$, but these stronger facts will not be used.)

In the definition below, we are not assuming that each reduction be special, but this will always be the case in the present applications.

Notice also that, generally speaking, the transversal isomorphisms of a weak cubical category do *not* satisfy the uniqueness condition (**) and *cannot* be taken as reductions, by themselves. Thus, the quotient procedure which we are establishing cannot be used to 'strictify' an arbitrary weak cubical category. (In the same way as spans up to invertible cells do not form a 2-category.)

In the dual procedure, from cubical *co*spans to cubical *co*relations, the preorder relation $x' \prec x$ is given by the existence of a *coreduction*, or *codomain-rewriting* m: x' $\to$ x, i.e. a special transversal map whose components are injective mappings (so that, again, we can think of x' as a 'simpler form' of x, within cospans)

$$
(2) \qquad \begin{array}{ccccc}
x_{-1} & \xrightarrow{x^-} & x_0 & \xleftarrow{x^+} & x_1 \\
\| & & \uparrow m_0 & & \| \\
x_{-1} & \xrightarrow[x'^-]{} & x'_0 & \xleftarrow[x'^+]{} & x_1
\end{array}
$$

**5.3. Cubical reduction systems.** Let $\mathbb{A}$ be a symmetric *pre*-cubical category (3.5(b)), which is transversally invariant (3.8).

A (cubical) *reduction system* Rd$\mathbb{A}$ of $\mathbb{A}$ satisfies the following conditions:

(i) Rd$\mathbb{A}$ is a *wide substructure* of $\mathbb{A}$ (i.e., it is closed under faces, degeneracies, transpositions, concatenations and transversal composition, and contains all the cubes of $\mathbb{A}$); moreover, it contains all the invertible transversal maps of $\mathbb{A}$;

(ii) every transversal n-map p: x $\to$ x' which is a *reduction*, i.e. belongs to Rd$\mathbb{A}$, is an epimorphism (in the category $tv_n\mathbb{A}$);

(iii) for every n-cube x, the set of reductions p: x $\to$ x', preordered by the usual preorder relation of epimorphisms (p' $\prec$ p if p' factors through p) has a minimum px: x $\to$ $\hat{x}$ (and we choose one of them, determined up to transversal isomorphism); px will be called the *least reduction*, or *normal reduction*, of x, and $\hat{x}$ the *reduced form*, or *normal form*, of x;

(iv) for every n-cube x, px: x $\to$ $\hat{x}$ is the *unique* reduction from x to its reduced form;

(v) for every transversal n-map f: x $\to$ y of $\mathbb{A}$, there is a consistent map $\hat{f}$: $\hat{x}$ $\to$ $\hat{y}$ (in the sense that $\hat{f}$.px = py.f); since px is epi, $\hat{f}$ is uniquely determined (by the choice of px and py) and called the *reduced form* of f;

(vi) $\partial_i^\alpha\hat{x}$ is isomorphic to the reduced form of the face $\partial_i^\alpha x$.

**5.4. Lemma** (The associated congruence). In the situation described in 5.3, the following properties hold.

(a) For every n-cube x, if the reduced form $\hat{x}$ is transversally isomorphic to x', then there is a unique reduction x $\to$ x' and a unique reduction $\hat{x}$ $\to$ x'; the latter is invertible.

(b) The procedure f $\mapsto$ $\hat{f}$ is consistent with transversal composition

$$(1) \qquad \begin{array}{ccccc} x & \xrightarrow{f} & y & \xrightarrow{g} & z \\ {\scriptstyle px}\downarrow & & \downarrow{\scriptstyle py} & & \downarrow{\scriptstyle pz} \\ \hat{x} & \longrightarrow & \hat{y} & \longrightarrow & \hat{z} \end{array} \qquad\qquad (gf)\hat{} \ = \ \hat{g}.\hat{f}.$$

(c) The equivalence relation $x \sim x'$ generated by the existence of a reduction between n-cubes amounts to the existence of a transversal isomorphism $u: \hat{x} \to \hat{x}'$, which is uniquely determined (and a reduction)

$$(2) \qquad x \xrightarrow{px} \hat{x} \xrightarrow{u} \hat{x}' \xleftarrow{px'} x'$$

so that each equivalence class of cubes has precisely one reduced representative, up to isomorphism.

(d) The equivalence relation $f \sim f'$ between transversal n-maps generated by the existence of a commutative square $f'p = qf$, where $p, q$ are reductions, amounts to the fact that $\hat{f}$ and $\hat{f}'$ be transversally isomorphic, i.e. to the existence of a commutative square $\hat{f}'u = v\hat{f}$, where $u, v$ are transversal isomorphisms

$$(3) \qquad \begin{array}{ccccccc} x & \xrightarrow{px} & \hat{x} & \xrightarrow{u} & \hat{x}' & \xleftarrow{px'} & x' \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \hat{f}} & & \downarrow{\scriptstyle \hat{f}'} & & \downarrow{\scriptstyle f'} \\ y & \xrightarrow[py]{} & \hat{y} & \xrightarrow[v]{} & \hat{y}' & \xleftarrow[py'']{} & y' \end{array}$$

therefore, each equivalence class of transversal maps has precisely one reduced representative, up to transversal isomorphism;

(e) Reduced forms are consistent with faces, degeneracies, transpositions and composition of transversal maps.

(f) On the other hand, the reduced form of a concatenation is smaller than the concatenation of reduced forms: more precisely, for a concatenation $x +_1 y$, one can choose two least reductions $px: x \to \hat{x}$, $py: \ \to \hat{y}$ which are 1-consecutive, so that $px +_1 py: x +_1 y \to \hat{x} +_1 \hat{y}$ is a reduction (generally not the least one).

(g) The equivalence relations $x \sim x'$ and $f \sim f'$ are consistent with faces, degeneracies, transpositions, concatenations and composition of transversal maps.

**Proof**. (a) Let $u: \hat{x} \to x'$ be a transversal isomorphism and $v: \hat{x} \to x'$ any reduction; then $u^{-1}v.px: x \to \hat{x}$ is a reduction and must coincide with $px$, whence $u^{-1}v$

$= \mathrm{id}\hat{x}$ and $v = u$; moreover, a reduction $p: x \to x'$ necessarily factors through $px$, and coincides with $u.px$.

(b) Obvious, since reductions are epimorphisms.

(c) Relation (1) is obviously an equivalence relation. It implies $x \sim x'$, because every transversal isomorphism is a reduction. Conversely, if $p: x \to x'$ is a reduction, then $x, x'$ have the same reduced form, up to transversal isomorphism, hence they are in relation (1).

(d) Has a similar proof.

(e) For faces, this is assumed in point (vi) of the definition. For degeneracies, a reduction $p: x \to x'$ gives a reduction $e_ip: e_ix \to e_ix'$, and conversely a reduction $q: e_ix \to e_ix'$ gives $\partial_i^\alpha q: x \to x'$. For transpositions, the property follows from their being invertible. Composition of transversal maps has already been considered in (b).

(f) For a concatenation $x +_1 y$, with $\partial_1^+ x = \partial_1^- y$, the normal reductions $px: x \to \hat{x}$ and $py: y \to \hat{y}$ give, up to transversal isomorphism, the normal reductions $\partial_1^+ px$ and $\partial_1^- py$ of the common face $\partial_1^+ x = \partial_1^- y$; now, if $u: \partial_1^+ \hat{x} \to \partial_1^- \hat{y}$ is a transversal isomorphism, let $v: \hat{x} \to x'$ be a transversal isomorphism such that $\partial_1^+ v = u$ and $\partial_1^+ x' = \partial_1^- \hat{y}$ (it exists, by transversal invariance of $\mathbb{A}$ ). Then $v.px: x \to x'$ is a normal reduction of $x$ which can be concatenated with $py$, giving a reduction $v.px +_1 py: x +_1 y \to x' +_1 \hat{y}$. One can now rename $v.px: x \to x'$ as $px: x \to \hat{x}$.

The least reduction of $x +_1 y$ can be strictly smaller than the concatenated reduction. For instance, in the case of spans and relations (5.2), it is well known that the concatenation of two jointly monic spans need not be jointly monic.

(g) Follows from the last two points. The only non-obvious aspect being concatenation, let us suppose we have $x +_1 y$ and $x' +_1 y'$, with $x \sim x'$ and $y \sim y'$. By (f), we can choose the same reduced forms $\hat{x}$ and $\hat{y}$, in such a way that they are consecutive in direction 1, and we get two reductions with values in the same form

(4) $x +_1 y \ \to\ \hat{x} +_1 \hat{y} \ \leftarrow\ x' +_1 y'$.

Therefore $x +_1 y \sim x' +_1 y'$ (even if $\hat{x} +_1 \hat{y}$ need not be normal). □

**5.5. Theorem and Definition** (Quotients modulo cubical reductions). (a) Given a transversally invariant, symmetric pre-cubical category $\mathbb{A}$ (3.5(b)) and a reduction system $\mathrm{Rd}\mathbb{A}$ (5.3), one can form a symmetric pre-cubical category $\mathbb{B} = \mathbb{A}/\!\sim$, as a quotient modulo the equivalence relations of n-cubes and n-maps defined in 5.4. $\mathbb{B}$

will be written as $\mathbb{A}/\mathrm{Rd}\mathbb{A}$ and called the *quotient of* $\mathbb{A}$ *modulo reductions* (of $\mathrm{Rd}\mathbb{A}$).

The projection $Q: \mathbb{A} \to \mathbb{B}$ is a symmetric pre-cubical functor such that:

(i) all reductions of $\mathbb{A}$ are sent by $Q$ to transversal identities,

(ii) $Q$ is universal for this property.

If all the reductions of $\mathbb{A}$ are *special* transversal maps (3.3.6), one can identify $\mathrm{tv}_0\mathbb{A}$ and $\mathrm{tv}_0\mathbb{B}$.

(b) If **A** is a (transversally invariant) weak sc-category, then $\mathbb{B}$ is a strict sc-category.

**Proof.** (a) Form $\mathbb{B} = \mathbb{A}/\sim$ as specified above. We already know that the induced concatenation $[x] +_i [y] = [x +_i y]$ does not depend on the choice of a pair of *i-consecutive* representatives, but we must show that it is defined whenever the given classes $[x], [y]$ are i-consecutives. Indeed, if $\partial_i^+ x \sim \partial_i^- y$, then

(1) $\partial_i^+(\hat{x}) \sim (\partial_i^+(x))\hat{} \sim (\partial_i^- y)\hat{} \sim \partial_i^-(\hat{y})$,

and, up to modifying $x$ by a transversal isomorphism constructed with property (vii), we can assume that $\partial_i^+(\hat{x}) = \partial_i^-(\hat{y})$, so that

(2) $[x] +_i [y] = [\hat{x} +_i \hat{y}]$,

is defined.

Similarly, we already know from the previous lemma that the induced composition of transversal maps $[g].[f] = [gf]$ does not depend on the choice of a pair of *composable* representatives. But we must also show that, if the given classes $[f], [g]$ are transversally consecutive then they admit composable representatives. Indeed, if the transversal maps $f: x \to y$ and $g: y' \to z$ become composable in the quotient, i.e. $y' \sim y$, then $f$ is equivalent to $p_y.f: x \to \hat{y}$ and $g$ is equivalent to $\hat{g}.u: \hat{y} \to \hat{z}$, for the unique isomorphism $u: \hat{y} \to \hat{y}'$ (which is a reduction).

Now, $\mathbb{B}$ is also a symmetric pre-cubical category and the universal property of the projection $Q$ is obvious. The last statement on special maps is obvious.

(b) In the new, stronger hypotheses on $\mathbb{A}$, comparisons and all the axioms pass to quotient. But all comparisons are invertible transversal maps, whence they are reductions and become identities in $\mathbb{B}$, which is therefore a strict sc-category. □

**5.6. From cubical spans to cubical relations.** We can now review 4.2 at the light of the previous points.

In the weak sc-category $\mathbb{S}p(\mathbf{Set})$, we say that a transversal map $f: x \to y: \mathsf{V}^n \to \mathbf{X}$ is a *reduction* if:

- all its components $f(\mathbf{t}): x(\mathbf{t}) \to y(\mathbf{t})$ $(\mathbf{t} \in \{-1, 0, 1\}^n)$ are surjective mappings,

- $f$ is special (i.e., all its vertices $f(\mathbf{t})$, for $\mathbf{t} \in \{-1, 1\}^n$ are identities).

The reduced form of an ordinary span is the associated jointly monic span; the same holds in higher dimension, according to the definition of a jointly monic cubical span given above (4.2). The conditions (i)-(vii) of 5.3 are satisfied.

The quotient of $\mathbb{S}p(\mathbf{Set})$ modulo these reductions is $\mathbb{R}el(\mathbf{Set})$, as a strict symmetric cubical category.

**5.7. Cubical profunctors.** Following the ideas of 4.4, we begin with considering the weak sc-category $\mathbb{E} = \omega\mathbb{C}osp(\mathbf{Emb})$ of cubical cospans of full embeddings of (small) categories.

This is legitimate, because the category **Emb**, of small categories and their full embeddings, has pushouts (which are also pushouts in **Cat**): given a span of such embeddings $A \leftarrow X \to B$, let us rename the items of $A$ and $B$ so that these functors are full *inclusions* and $X = A \cap B$. Then the pushout $W$ contains the obvious set-theoretical union $A \cup B$, supplemented with:

- new arrows $[\beta\alpha]: a \to x \to b$ (modulo the equivalence relation generated by identifying $\beta\alpha = \beta'\alpha'$ if there exists some $\xi: x \to x'$ in $X$ such that $\alpha' = \xi\alpha$ in $A$ and $\beta = \beta'\xi$ in $B$),

- and, symmetrically, new arrows $[\alpha\beta]: b \to x \to a$.

The composition in $W$ is easily defined, as in the following examples:

(1) $[\beta\alpha].\alpha' = [\beta(\alpha\alpha')]$, for $\alpha': a' \to a$, $\alpha: a \to x$, $\beta: x \to b$,

$[\alpha'.\beta'].[\beta\alpha] = \alpha'.(\beta'\beta).\alpha$, for $\alpha: a \to x$, $\beta: x \to b$, $\beta': b \to x'$, $\alpha': x' \to a'$,

where the last composition is in $A$ ($\beta'\beta: x \to x'$ belongs to the full subcategory $X = A \cap B$).

A *coreduction*, or rewriting of the codomain, will be any special transversal map of $\mathbb{E}$ (notice that all its components are full embeddings). This gives a wide substructure $\mathrm{Crd}\mathbb{E} \subset \mathbb{E}$, which satisfies the axioms transversally dual to those of 5.3.

The reduced form of an n-cospan $x: \mathsf{\Lambda}^n \to \mathbf{Emb}$ is obtained in a way similar to the construction for spans, in 4.2, actually simpler: the vertices of the cube (marked with bigger bullets, below) are unchanged, but we replace each other category $x(\mathbf{t})$

($\mathbf{t} \in \mathrm{Ob}\boldsymbol{\wedge}^n$) with the full subcategory determined by the objects which are reached by some object of a category occupying a vertex of the cube

$$(2) \qquad \begin{array}{ccccc} \bullet & \longrightarrow & \cdot & \longleftarrow & \bullet \\ \downarrow & & \downarrow & & \downarrow \\ \cdot & \longrightarrow & \cdot & \longleftarrow & \cdot \\ \uparrow & & \uparrow & & \uparrow \\ \bullet & \longrightarrow & \cdot & \longleftarrow & \bullet \end{array}$$

Now, the quotient $\mathbb{E}' = \mathbb{E}/\mathrm{Crd}\mathbb{E}$ is a symmetric cubical category, where each n-cube $[x]$ has precisely one reduced representative $\hat{x}$ (up to special transversal isomorphisms).

Coming back to the conditions (i)-(iii) of 4.4, we have taken (i) and (iii) into account, by restricting to cospans of full embeddings and passing to the quotient. We consider now condition (ii), by selecting some n-cubes of $\mathbb{E}'$, which are defined to be cubical profunctors, and taking all the transversal maps of $\mathbb{E}'$ between them.

We say that an n-cube $[x]$ of $\mathbb{E}'$ is an *n-profunctor* if it admits a representative $x$ such that, for each ordinary cospan which appears in $x$ in a given direction $i$

$$(3) \qquad x(\mathbf{t}') \to x(\mathbf{t}) \leftarrow x(\mathbf{t}'') \qquad (t'_i = -1,\ t_i = 0,\ t''_i = 1;\ \ t'_j = t_j = t''_j \ \text{ for } j \neq i\,),$$

condition (ii) of 4.4 is satisfied: there are no arrows in $x(\mathbf{t})$ going from an object of $x(\mathbf{t}'')$ to one of $x(\mathbf{t}')$. Equivalently, we can ask that this condition be satisfied by the normal form $\hat{x}$.

Degeneracies make some problems (as it is also the case within cospans in the domain of cobordism, see [G5]). Indeed, already in degree 1, the degenerate cospan of a (non-empty) category $e_1(x) = (x = x = x)$ is reduced and does not satisfy the previous condition. However (as in [G5]) we can replace degeneracies with *cylindrical degeneracies*: the 1-cube $E_1(x)$ on the category $x$ is the following cospan of *disjoint* embeddings (which is also reduced):

$$(4) \qquad x \xrightarrow{x^-} x_0 \xleftarrow{x^+} x$$

where the category $x_0 = x \times \mathbf{2}$ is the *collage* of the identity profunctor of $x$ (and $x^-, x^+$ are the obvious embeddings). It is easy to see that $[E_1(x)]$ is a strict identity for concatenation *with* 1-profunctors (but not with general 1-cubes of $\mathbb{E}'$).

Finally, we have obtained a strict sc-category $\omega\mathbb{C}\mathrm{at}$ of cubical profunctors, contained in $\mathbb{E}'$; the embedding preserves all the structure, except degeneracies, and is transversally full.

The fact that we have realised a *strict* cubical category should not surprise too much. The crucial point is the fact that a special transversal map $f: x \to x'$ between two profunctors is uniquely determined (while the same is not true of arbitrary cospans of categories). Another realisation of the bicategory of ordinary profunctors as a strict 2-category has been recalled in [GP1]: a profunctor $u: A \to\!\!\!\!\!\!+\; B$ can be defined as a colimit-preserving functor

(5) $\hat{u}: \mathbf{Set}^A \to \mathbf{Set}^B$, $\qquad \hat{u}(F)(b) = \int^a u(a, b) \times F(a).$

## 6. Complements on symmetric cubical sets

We prove now that the internal homs of symmetric cubical sets constructed in Section 2 come from a s*ymmetric monoidal closed structure*. The tensor product and the corresponding cylinder functor are complicated, which is why we preferred to work with the path functor.

**6.1. Remarks.** The *ordinary* tensor product $X{\otimes}Y$ of two symmetric cubical sets can not be directly equipped with transpositions. For instance, if $x, y$ are 2-cubes in $X$ and $Y$, we might define $s_1(x{\otimes}y) = (s_1x){\otimes}y$ and $s_3(x{\otimes}y) = x{\otimes}(s_1y)$, but $s_2(x{\otimes}y)$ cannot be obtained from the transposition $s_1$ of $X_2$ or $Y_2$, and has to be formally introduced.

Therefore, we will define a *symmetricy* tensor product $X \mathbin{\hat{\otimes}} Y$ with n-component the free $S_n$-set on $X{\otimes}Y$, modulo the identifications exemplified above.

We still write $2 = \{0, 1\}$, but the symmetric group $S_n$ will be viewed as the set of all bijections $u: \underline{n} \to \underline{n}$, where $\underline{n} = \{1,..., n\}$; in fact, this set of $n$ elements is here more convenient then the cardinal $n = \{0,..., n–1\}$.

**6.2. The action of permutations.** Recall, from 2.1, that $s\mathbf{Cub}$ is the category of functors $X: \mathbb{I}_s{}^{op} \to \mathbf{Set}$, where the *symmetric cubical site* $\mathbb{I}_s$ is (realised here as) the subcategory of $\mathbf{Set}$ consisting of the elementary cubes $2^{\underline{n}}$, together with the maps $2^{\underline{m}} \to 2^{\underline{n}}$ which delete some coordinates, permute the remaining ones and insert some 0's and 1's.

Let $X: \mathbb{I}_s{}^{op} \to \mathbf{Set}$ be a symmetric cubical set. It will be useful, for the sequel, to give an explicit description of the *left* action of the symmetric group $S_n$ on the component $X_n = X(2^{\underline{n}})$. In fact, the group $S_n$ (of permutations $u: \underline{n} \to \underline{n}$) acts contravariantly on the set $2^{\underline{n}} = \mathbf{Set}(\underline{n}, 2)$ and then covariantly on the set $X_n$

$$(1) \quad \hat{u}: 2^{\underline{n}} \to 2^{\underline{n}}, \qquad \hat{u}: t \mapsto t \circ u \qquad (t: \underline{n} \to 2),$$
$$u: X_n \to X_n, \qquad u.x = X(\hat{u})(x) \qquad (x \in X_n).$$

(For instance, let $X$ be the singular cubical set of a topological space $S$, with components $X_n = \mathbf{Top}([0,1]^{\underline{n}}, S)$; take $u, v \in S_n$, an n-cube $x: [0,1]^{\underline{n}} \to S$, and let $t: \underline{n} \to [0,1]$ denote the variable of $[0,1]^{\underline{n}}$. Then $(u.x)(t) = x(tu)$, where $(tu)_i = t_{u(i)}$. It is indeed a left action, since: $(v.(u.x))(t) = (u.x)(tv) = x(tvu) = (vu.x)(t)$.)

The permutation $u \in S_n$ acts as follows on the face $\partial_1^\alpha: X_n \to X_{n-1}$ and the degeneracy $e_1: X_{n-1} \to X_n$

$$(2) \quad \partial_1^\alpha \circ u = u' \circ \partial_i^\alpha, \qquad \text{where } u(i) = 1 \text{ and}$$
$$u'(j) = u(j) - 1 \text{ for } j < i, \; u'(j) = u(j+1) - 1 \text{ for } j \geq i,$$
$$e_1 \circ u = u'' \circ e_1, \qquad \text{where } u'' = id \times u: \underline{n} \to \underline{n}.$$

**6.3. Theorem and Definition.** The category $s\mathbf{Cub}$ of symmetric cubical sets has a symmetric monoidal closed structure, whose internal-hom is the functor sCUB defined in 2.5.

The n-th component of the *symmetric tensor product* $X \hat{\otimes} Y$ of symmetric cubical sets

$$(1) \quad (X \hat{\otimes} Y)_n = S_n((X \otimes Y)_n) / \approx_n,$$

is a quotient of the free $S_n$-set generated by the n-th component of the ordinary tensor product $X \otimes Y$, containing all the formal permutations $u.(x \otimes y)$ $(u \in S_n)$. The quotient is taken modulo the congruence $\approx_n$ of $S_n$-sets generated by the following 'identifications'

$$(2) \quad s_i(x \otimes y) = s_i(x) \otimes y \text{ for } i < p = \dim(x), \qquad s_i(x \otimes y) = x \otimes s_{i-p}(y) \text{ for } i > p.$$

Faces and degeneracies in direction 1 are defined as follows, letting $\partial_1^\alpha \circ u = u' \circ \partial_i^\alpha$ and $e_1 \circ u = u'' \circ e_1$ (with $u', u''$ as in 6.2.2)

$$(3) \quad \partial_1^\alpha[u.(x \otimes y)] = [u'.\partial_i^\alpha(x \otimes y)] = \begin{cases} [u'.((\partial_i^\alpha x) \otimes y))], & \text{if } i \leq p, \\ [u'.(x \otimes (\partial_{i-p}^\alpha y))], & \text{if } i > p, \end{cases}$$
$$e_1[u.(x \otimes y)] = [u''.e_1(x \otimes y)] = [u''.(e_1 x) \otimes y)].$$

This completes the definition of the symmetric cubical set $X \hat{\otimes} Y$, in the reduced form 2.2

(4) $X \hat{\otimes} Y \;=\; (((X \hat{\otimes} Y)_n), (\partial_1^\alpha), (e_1), (s_i)).$

**Proof.** The verification of the axioms 2.2.4 is left to the reader. To obtain the exponential law, as a natural bijection

(5) $s\mathbf{Cub}(X \hat{\otimes} A, Y) \to s\mathbf{Cub}(X, sCUB(A, Y)),$

let us take a morphism $f = (f_n): X \hat{\otimes} A \to Y$. Its n-component $f_n$ decomposes into a family of mappings

(6) $f_{pq}: S_n(X_p \times A_q) \to Y_n$ $\qquad (n = p + q),$

consistent with the action of $S_n$, the equivalence relation $\sim_n$ (1.2.1) and the new equivalence relation $\approx_n$. Their restrictions

(7) $f_{pq}: X_p \times A_q \to Y_n,$

amount to mappings $g_{pq}: X_p \to \mathbf{Set}(A_q, Y_{p+q})$. Keeping p fixed, we get mappings

(8) $g_p = (g_{pq}):\; X_p \to s\mathbf{Cub}(A, P^pY) \;=\; sCUB_p(A, Y) \subset \Pi_q \mathbf{Set}(A_q, Y_{p+q}),$

which form a morphism of symmetric cubical sets $g = (g_p): X \to sCUB(A, Y)$. □

**6.4. The cylinder functor.** We have already considered the representable symmetric cubical set $\uparrow\mathbf{i} = y(2)$ freely generated by one 1-cube u (2.5.2), and observed that $sCUB(\uparrow\mathbf{i}, Y) = PY$.

The cylinder functor

(1) $I: s\mathbf{Cub} \to s\mathbf{Cub}, \qquad I(X) \;=\; X \hat{\otimes} \uparrow\mathbf{i},$

is thus left adjoint to the path functor $P: s\mathbf{Cub} \to s\mathbf{Cub}$,

(2) $s\mathbf{Cub}(X \hat{\otimes} \uparrow\mathbf{i}, Y) \;=\; s\mathbf{Cub}(X, sCUB(\uparrow\mathbf{i}, Y)) \;=\; s\mathbf{Cub}(X, P(Y)).$

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