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SATURATION FOR CLASSES OF MORPHISMS

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Résumé. Une classe Σ extérieurement saturée de morphismes d'une catégorie $\mathcal{C}$ est une classe de morphismes qui sont inversés par un foncteur $F : \mathcal{C} \to \mathcal{D}$. Par ailleurs, Σ est intérieurement saturée si elle coincide avec son double orthogonal au sens de Freyd-Kelly. Dans cette courte note nous prouvons qu'une classe $\Sigma \subset Mor\,\mathcal{C}$ est intérieurement saturée si et seulement si elle est extérieurement saturée et admet un calcul de fractions á gauche.

Abstract. An externally saturated class Σ of morphisms in a category $\mathcal{C}$ is the class of morphisms that are inverted by some functor $F : \mathcal{C} \to \mathcal{D}$. On the other hand, Σ is internally saturated if it coincides with its double orthogonal in the sense of Freyd-Kelly. In this short note we prove that $\Sigma \subset Mor\,\mathcal{C}$ is an internally saturated class if and only if it is externally saturated and admits a calculus of left fractions.

Keywords. orthogonality, saturation, calculus of fractions, shape, shape equivalences..
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1. Introduction

Given a class Σ of morphisms in a category $\mathcal{C}$, there are currently at hand wo different notions of saturation for it. Σ is called externally saturated if it s saturated in the sense of category of fractions, e.g. as defined in the book by Gabriel and Zisman [6]. On the other hand Σ is called internally saturated f it coincides with its double orthogonal in the sense of Freyd and Kelly [5]. The internal-external terminology is due, as far as we know, to Casacuberta and Frei [1]. In that paper it was shown that every internal saturated class is also externally saturated, using a suitable shape functor. There the main result asserts that if Σ is the class of morphisms inverted by some functor $F : \mathcal{C} \to \mathcal{D}$ having a right adjoint, then Σ is is both internally and externally saturated. This is the case when F is part of a monad. In a previous paper the second author pointed out that every internally saturated class has also a calculus of left fractions, here we prove that the converse holds true provided that the category $\mathcal{C}$ has finite colimits and a terminal object, that is, internally saturated classes are the same as externally saturated classes admitting a calculus of left fractions. It turns out that internally saturated classes of morphisms are exactly what was needed in [4] in the context of the Adams completion with respect to a homology theory.

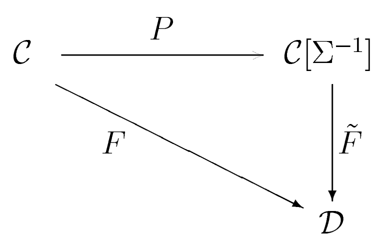
2. Preliminaries

Let $\mathcal{C}$ be be a fixed category, that we assume to be finitely cocomplete and endowed with a terminal object $\mathtt{T}$, throughout the paper. Moreover, given a class Σ morphisms of $\mathcal{C}$, let us denote by $\mathcal{C}[\Sigma^{-1}]$ the category of fractions of $\mathcal{C}$ with respect to Σ and let $P : \mathcal{C} \to \mathcal{C}[\Sigma^{-1}]$ be the canonical functor [6]. Let us recall that the pair $(\mathcal{C}[\Sigma^{-1}], P)$ is uniquely determined by the following properties:

(a) $P(s)$ is an isomorphism, for all $s \in \Sigma$,

(b) if $F : \mathcal{C} \to \mathcal{D}$ is a functor such that $F(s)$ is an isomorphism, for all $s \in \Sigma$, then there is a unique functor $\tilde{F} : \mathcal{C}[\Sigma^{-1}] \to \mathcal{D}$ such that $\tilde{F} \circ P = F$.

Diagrammatically



The *external saturation* of Σ is the class $\overline{\Sigma}$ of all morphisms in $\mathcal{C}$ that are taken to isomorphisms by P. Σ is *externally saturated* when $\Sigma = \overline{\Sigma}$.

It is easy to realize that Σ is externally saturated iff there is some functor $F : \mathcal{C} \to \mathcal{D}$ such that

$$\Sigma = \mathcal{S}(F) = \{s \in Mor\,\mathcal{C} \mid F(s) \text{ is an isomorphism}\},$$

see, for instance [4], Prop.1.1.

Proposition 2.1. *An externally saturated class Σ has the following properties :*

(a) it contains all isomorphisms of $\mathcal{C}$,

(b) it has the "two out of three property", that is, given morphisms $u : A \to B$ and $v : B \to C$, if two of $u, v, v \circ u$ are in Σ, then the third is also in Σ,

(c) given morphisms $u : A \to B, v : B \to C$ and $t : C \to D$, if $v \circ u, t \circ v \in \Sigma$, then $v \in \Sigma$.

Proof. (a) and (b) are obvious. For (c) see [9], Proposition 19.3.3 (a). □

The *orthogonal* of a class $\Sigma \subset Mor\,\mathcal{C}$ is the class of objects in $\mathcal{C}$ defined as follows

$$\Sigma^{\perp} = \{P \in Ob\,\mathcal{C} \mid \mathcal{C}(s, P) \text{ is bijective for all } s \in \Sigma\}$$

where $s : X \to Y$ and $\mathcal{C}(s, P) : \mathcal{C}(Y, P) \to \mathcal{C}(X, P)$ is defined by composition with s.
Symmetrically, the orthogonal of a class $\mathcal{K} \subset Ob\,\mathcal{C}$ is given by

$$\mathcal{K}^{\top} = \{s \in Mor\,\mathcal{C} \mid \mathcal{C}(s, P) \text{ is bijective for all } P \in \mathcal{K}\}.$$

Notice that, in general, the following relations hold:

1. $\Sigma \subseteq \Sigma^{\perp\top}$,
2. $\mathcal{K} \subseteq \mathcal{K}^{\top\perp}$.

Σ (resp. $\mathcal{K}$) will be called *internally saturated* whenever $\Sigma = \Sigma^{\perp\top}$ (resp. $\mathcal{K} = \mathcal{K}^{\top\perp}$) [1, 2].
By a standard abuse of notation we often denote by $\mathcal{K}$ the class of objects and the full subcategory of $\mathcal{C}$ with these objects.
If $\mathcal{K}$ is a subcategory of $\mathcal{C}$, the shape category [7] of the pair $(\mathcal{C}, \mathcal{K})$ is the category $SH(\mathcal{C}, \mathcal{K})$ having the same objects as $\mathcal{C}$ and morphisms given by

$$SH(\mathcal{C}, \mathcal{K})(X, Y) = Nat(\mathcal{C}(Y, E(-)), \mathcal{C}(X, E(-))),$$

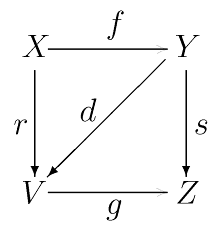
where $E : \mathcal{K} \to \mathcal{C}$ is the inclusion functor. There is a *shape functor*

$$Sh : \mathcal{C} \to SH(\mathcal{C}, \mathcal{K})$$

which takes objects fixed and sends a morphism $f : X \to Y$ to the natural transformation $\mathcal{C}(f, E(-)) : \mathcal{C}(Y, E(-)) \to \mathcal{C}(X, E(-))$ defined by composition with f. f is called a *shape equivalence* for the pair $(\mathcal{C}, \mathcal{K})$ whenever $Sh(f) = \mathcal{C}(f, E(-))$ is a natural isomorphism. It follows at once that the class of shape equivalences for $(\mathcal{C}, \mathcal{K})$ coincides with the orthogonal class $\mathcal{K}^{\top}$. On the other hand, it is clear that, given an internally saturated class Σ of morphisms of $\mathcal{C}$, then $\Sigma = \mathcal{S}(Sh)$ is the class of shape equivalences for the pair $(\mathcal{C}, \Sigma^{\perp})$ [1, 10]. Such arguments show that

Proposition 2.2. *[1] Every internally saturated class $\Sigma \subset Mor\,\mathcal{C}$ is also externally saturated.*

Given morphisms $f : X \to Y$ and $g : V \to Z$ in $\mathcal{C}$, we write $f \uparrow g$ to mean that every commutative diagram



has a unique diagonal $d : Y \to V$ such that $g \circ d = s$ and $d \circ f = r$.
For $\Sigma \subset Mor\,\mathcal{C}$ one defines

$$\Sigma^{\uparrow} = \{f \in Mor\,\mathcal{C} \mid f \uparrow g, \text{ for all } g \in \Sigma\}$$

Let $\mathcal{K} \subset \mathcal{C}$ and denote by $\mathcal{K}_{\mathtt{T}}$ the class of all morphisms $K \longrightarrow \mathtt{T}$, for $K \in \mathcal{K}$. Then, it is clear that

Proposition 2.3. $\mathcal{K}^{\perp} = \mathcal{K}_{\mathtt{T}}^{\uparrow}$.

It follows that, for any subcategory $\mathcal{K} \subset \mathcal{C}$, the class $\mathcal{K}^{\perp}$ is the left part of a *prefactorization system* on $\mathcal{C}$ and consequently it has a number of properties, among which we record the fact that $\mathcal{K}^{\perp}$ is closed under pushouts [5], Propositions 2.1.1 and 2.1.3.

3. Calculus of Fractions

Let us recall [9, 6] that a class of morphisms $\Sigma \subset Mor\,\mathcal{C}$ is said to admit *a calculus of left fractions* when it has the following properties

(a) contains all identities,

(b) is closed under composition,

(c) every span

$$X \xleftarrow{\ s\ } X' \xrightarrow{\ f\ } Y$$

with $s \in \Sigma$, can be inserted in a commutative diagram

$$\begin{array}{ccc} X' & \xrightarrow{\ f\ } & Y \\ {\scriptstyle s}\downarrow & & \downarrow{\scriptstyle s'} \\ X & \xrightarrow[\ f'\ ]{} & Y' \end{array}$$

where $s' \in \Sigma$,

(d) if $f \circ s = g \circ s,\ s \in \Sigma$, then there exists a $t \in \Sigma$ such that $t \circ f = t \circ g$.

Proposition 3.1. *[10] Every internally saturated class $\Sigma \subset Mor\,\mathcal{C}$ admits a calculus of left fractions.*

Proof. In view of Proposition 1.1, properties (a) and (b) for a calculus of left fractions are satisfied. Property (c) holds because of Proposition 1.3, since $\mathcal{C}$ has finite colimits and a terminal object. As for (d), consider a pair $f, g : Y \to Z$ and an $s : X \to Y, s \in \Sigma$ with $f \circ s = g \circ s$. One can form the pushout

$$\begin{array}{ccc} X & \xrightarrow{\;s\;} & Y \\ {\scriptstyle f \circ s = g \circ s}\downarrow & & \downarrow{\scriptstyle h} \\ Z & \xrightarrow[\;t\;]{} & W \end{array}$$

where $t \in \Sigma$, by Proposition 1.3. By the universal property of the pushout there are uniquely determined morphisms $u, v : W \to Z$ such that $u \circ t = 1_Z,\ u \circ h = g$ and $v \circ t = 1_Z,\ v \circ h = f$. Notice that, by Proposition 1.1 (b) it follows that $u, v \in \Sigma$, then in the pushout

$$\begin{array}{ccc} W & \xrightarrow{\;u\;} & Z \\ {\scriptstyle v}\downarrow & & \downarrow{\scriptstyle p} \\ Z & \xrightarrow[\;q\;]{} & Q \end{array}$$

also $p, q \in \Sigma$ and, moreover $p = q$, since $p = p \circ u \circ t = q \circ v \circ t = q$. Finally, $p \circ f = p \circ v \circ h = q \circ u \circ h = q \circ g$, which concludes the proof. We have taken the last argument from the proof of Theorem 1.3 of [4] for the sake of completeness. □

Corollary 3.2. *Every internally saturated class $\Sigma \subset Mor\,\mathcal{C}$ is externally saturated and admits a calculus of left fractions.*

Recall now the following result from [9], Theorem 19.3.1 (a)

Theorem 3.3. *Let $F : \mathcal{C} \to \mathcal{D}$ be a functor and let $\Sigma = \mathcal{S}(F)$. Let $\tilde{F} : \mathcal{C}[\Sigma^{-1}] \to \mathcal{D}$ be the unique functor such that $\tilde{F} \circ P = F$. If Σ admits a calculus of left fractions, then $\tilde{F}$ reflects isomorphisms.*

It allows us to prove our main result

Theorem 3.4. *Let $\Sigma \subset Mor\,\mathcal{C}$ be an externally saturated class admitting a calculus of left fractions. Then Σ is internally saturated.*

Proof. $\Sigma^{\perp\top}$ is the class of shape equivalences for the pair $(\mathcal{C}, \mathcal{K})$, where $\mathcal{K} = \Sigma^{\perp}$. Let $Sh : \mathcal{C} \to SH(\mathcal{C}, \mathcal{K})$ be the corresponding shape functor. Then there is a commutative diagram

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{\quad P \quad} & \mathcal{C}[\Sigma^{-1}] \\ & \overset{Sh}{\searrow} & \downarrow \tilde{Sh} \\ & & SH(\mathcal{C}, \mathcal{K}) \end{array}$$

where, by Theorem 2.3, the functor Sh reflects isomorphisms. Let s be a shape equivalence, then $Sh(s) = \tilde{Sh} \circ P(s)$ is iso in $SH(\mathcal{C}, \mathcal{K})$. It follows that $P(s)$ has to be an isomorphism in $\mathcal{C}[\Sigma^{-1}]$, hence $s \in \Sigma$, because Σ is externally saturated. This shows that $\Sigma^{\perp\top} \subseteq \Sigma$, hence the theorem. $\square$

From [4], Theorem 1.3 and Theorem 1.5 we obtain

Corollary 3.5. *An externally saturated class $\Sigma \subset Mor\,\mathcal{C}$ is internally saturated if and only if it satisfies the following "weak pushout property":*

every span

$$X \xleftarrow{\quad s \quad} X' \xrightarrow{\quad f \quad} Y$$

with $s \in \Sigma$, can be inserted in a weak pushout diagram

$$\begin{array}{ccc} X' & \xrightarrow{\quad f \quad} & Y \\ {\scriptstyle s}\downarrow & & \downarrow{\scriptstyle s'} \\ Y' & \xrightarrow[\quad f' \quad]{} & W \end{array}$$

where $s' \in \Sigma$.

Corollary 3.6. *Let $\Sigma \subset Mor\ \mathcal{C}$ be an internally saturated class of morphisms. The following are equivalent for an object $X \in \mathcal{C}$:*

(a) the functor $\mathcal{C}[\Sigma^{-1}](-,X) : \mathcal{C} \to Sets$ is represented by $Z \in \mathcal{C}$,

(b) $Z \in \Sigma^{\perp}$ and there is an $s : X \to Z,\ s \in \Sigma$,

(c) there is an $s : X \to Z,\ s \in \Sigma$, which is terminal in the comma category X/Σ.

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