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ON REGULAR AND HOMOLOGICAL CLOSURE OPERATORS

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Dedicated to Francis Borceux on the occasion of his sixtieth birthday

Résumé

En ayant remarqué que la propriété d'hérédité faible des opérateurs réguliers de fermeture dans Top et des opérateurs de fermeture homologiques dans les catégories homologiques permet d'identifier les théories de torsion, nous étudions ces opérateurs de fermeture en parallèle, en montrant que les opérateurs réguliers de fermeture jouent en topologie le même rôle que les opérateurs de fermeture homologiques jouent en algèbre.

Abstract

Observing that weak heredity of regular closure operators in Top and of homological closure operators in homological categories identifies torsion theories, we study these closure operators in parallel, showing that regular closure operators play the same role in topology as homological closure operators do algebraically.

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Introduction

Homological categories were introduced by Borceux and Bourn [2], and have since then been studied by several authors, as the right non-abelian setting to study homology. As shown by Bourn and Gran [6], these categories provide also a suitable setting to study torsion theories. In [6] the authors introduce torsion theories in homological categories and show that they are identifiable by weak heredity of their homological closure operators. This result resembles the characterization of disconnectednesses of topological spaces via weak heredity of their regular closure operators, and encompasses the characterization of torsion-free subcategories of abelian categories via weak heredity of their regular closure operators obtained in [7] (see also [12]). Having as starting point this common property, we establish parallel properties of regular and homological closure operators, in topological spaces and in homological categories, respectively. Since in abelian categories regular closure operators are exactly the homological ones, this study raises the question of finding in which cases these closure operators coincide in homological categories. We show that it is necessary that they are induced by a subcategory of abelian objects. Moreover, in semi-abelian categories regular and homological closures coincide exactly when they are induced by a regular-epireflective subcategory of abelian objects.

In Section 1 we describe briefly disconnectednesses of topological spaces and torsion theories in homological categories. In Section 2 we introduce regular and homological closure operators, showing that the latter ones can be described as maximal closure operators. In Section 3 we establish parallel results for regular and homological closures, based on the results obtained in [6]. In Theorem 3.1.4 we show the validity of the corresponding topological version of the characterization of hereditary torsion theories via hereditary homological closure operators. Next we investigate openness and closedness of regular epimorphisms, with respect to the regular closure, showing that these properties are unlikely topological; see Propositions 3.2.2 and 3.3.2. Finally, in Corollary 3.4.2, we characterise the regular-epireflective subcategories of semi-abelian categories for which the regular and the homological closures coincide, generalising the result obtained in [13] for abelian categories.

1 (Dis)connectednesses and Torsion Theories

1.1 (Dis)connectednesses in Topology

Given a subcategory **A** of the category **Top** of topological spaces and continuous maps, we define the full subcategories

$$l\mathbf{A} := \{X \in \mathbf{Top} \mid \text{ if } f : X \to A \text{ and } A \in \mathbf{A}, \text{ then } f \text{ is constant}\},$$

$$r\mathbf{A} := \{X \in \mathbf{Top} \mid \text{ if } f : A \to X \text{ and } A \in \mathbf{A}, \text{ then } f \text{ is constant}\}.$$

A subcategory of the form $l\mathbf{A}$ for some **A** is said to be a *connectedness*, while a subcategory of the form $r\mathbf{A}$ is said to be a *disconnectedness*. Connectednesses and disconnectednesses of **Top** were thoroughly studied by Arhangel'skiĭ and Wiegandt in [1]. We list here some properties of these subcategories we will need throughout.

1.1.1 Proposition

(1) *Every disconnectedness is a regular-epireflective subcategory of* **Top**.

(2) **Top**, *the subcategory of T_0-spaces* $\mathbf{Top}_0$, *the subcategory of T_1-spaces* $\mathbf{Top}_1$ *and the subcategory* **Sgl** *consisting of the empty and the singleton spaces are disconnectednesses.*

(3) *Let* **A** *be a disconnectedness. If* **A** *is different from* **Top** *and from* $\mathbf{Top}_0$, *then* $A \subseteq \mathbf{Top}_1$. *If* **A** *is different from* **Sgl**, *then* **A** *contains the subcategory* **TDisc** *of totally disconnected spaces.*

(4) **Sgl**, *the subcategory* **Ind** *of indiscrete spaces, and* **Top** *are connectednesses. These are the only connectednesses closed under subspaces.*

1.2 Torsion theories in homological categories

A pointed category **C** is *homological* if it is

(1) *(Barr-)regular*, that is if it is finitely complete and (regular epimorphisms, monomorphisms) is a pullback-stable factorization system in **C**, and

(2) *protomodular*, that is given a commutative diagram

$$\begin{array}{ccccc} A & \longrightarrow & B & \longrightarrow & C \\ \downarrow & 1 & \vdots & 2 & \downarrow \\ D & \longrightarrow & E & \longrightarrow & F \end{array}$$

where the dotted vertical arrow is a regular epimorphism, if $\boxed{1}$ and the whole rectangle are pullbacks, then $\boxed{2}$ is a pullback as well.

C is said to be *semi-abelian* if it is pointed, exact and protomodular. That is, in addition to (1) and (2) the pointed category **C** also satisfies

(3) every equivalence relation is effective, i.e. a kernel pair relation.

A *torsion theory* in a homological category is a pair $(\mathbf{T}, \mathbf{F})$ of full and replete subcategories of **C** such that:

1. If $T \in \mathbf{T}$ and $F \in \mathbf{F}$, then the only morphism $T \to F$ is the zero morphism.
2. For each $X \in \mathbf{C}$ there is a short exact sequence

$$0 \longrightarrow T \longrightarrow X \xrightarrow{\rho_X} F \longrightarrow 0$$

with $T \in \mathbf{T}$ and $F \in \mathbf{F}$.

If $(\mathbf{T}, \mathbf{F})$ is a torsion theory, the subcategory **T** is called the *torsion subcategory*, and **F** is called the *torsion-free subcategory*. Every torsion-free subcategory **F** is regular-epireflective, with the **F**-reflection of X given by ρ_X as above.

Torsion theories in homological categories were introduced by Bourn and Gran in [6], encompassing the properties of Dickson's torsion theories in abelian categories [10].

The notion of *abelian object* has been studied in non-abelian settings (see [2]). In homological categories they can be defined as those objects which have an internal abelian group structure. As shown by Bourn in [5]:

1.2.1 Proposition

(1) *In a homological category, the following conditions are equivalent for an object X:*

(i) *X has an internal abelian group structure;*

(ii) *the diagonal $\delta_X : X \to X \times X$ is an equaliser.*

(2) *In a semi-abelian category, the following conditions are equivalent for an object X:*

(i) *X has an internal abelian group structure;*

(ii) *the diagonal $\delta_X : X \to X \times X$ is a kernel.*

2 Regular and homological closure operators

2.1 Closure operators

Throughout $\mathbf{C}$ is a finitely complete category with cokernel pairs and $\mathcal{M}$ is a pullback-stable class of monomorphisms of $\mathbf{C}$. This means that $\mathbf{C}$ *has inverse* $\mathcal{M}$*-images*, that is for each morphism $f : X \to Y$ there is a change-of-base functor

$$f^{-1}(\ ) : \mathcal{M}/Y \to \mathcal{M}/X$$

where $\mathcal{M}/X$ is the (preordered) category of $\mathcal{M}$*-subobjects of* X, that is of morphisms in $\mathcal{M}$ with codomain X. When, for each morphism $f : X \to Y$, the functor $f^{-1}(\ ) : \mathcal{M}/Y \to \mathcal{M}/X$ has a left adjoint $f(\ ) : \mathcal{M}/X \to \mathcal{M}/Y$, we say that $\mathbf{C}$ *has direct* $\mathcal{M}$*-images*.

A *closure operator* c on $\mathbf{C}$ with respect to $\mathcal{M}$ assigns to each $m : M \to X$ in $\mathcal{M}$ a morphism $c_X(m) : c_X(M) \to X$ in $\mathcal{M}$ such that, for every object X,

(C1) c_X is *extensive*: $m \leq c_X(m)$ for every $m : M \to X$ in $\mathcal{M}$;

(C2) c_X is *monotone*: $m \leq m' \;\Rightarrow\; c_X(m) \leq c_X(m')$, for every $m : M \to X$, $m' : M' \to X$ in $\mathcal{M}$;

(C3) morphisms are *c-continuous*: $c_X(f^{-1}(n)) \leq f^{-1}(c_Y(n))$ for every morphism $f : X \to Y$ and every $n : N \to Y$ in $\mathcal{M}$.

When $\mathbf{C}$ has direct $\mathcal{M}$-images, condition (C3) can be equivalently expressed by

(C3′) $f(c_X(m)) \leq c_Y(f(m))$, for every $\mathcal{M}$-subobject m of X.

Extensivity of c says that every $m : M \to X \in \mathcal{M}$ factors as

$$\begin{array}{ccc} M & \xrightarrow{\quad m \quad} & X \\ & {\scriptstyle j_m} \searrow \quad \nearrow {\scriptstyle c_X(m)} & \\ & c_X(M) & \end{array}$$

The morphism $m : M \to X$ is *c-closed* if $c_X(m) \cong m$, and *c-dense* if $c_X(m) \cong 1_X$.

A closure operator c is said to be

- *idempotent* if $c_X(m)$ is c-closed for every $m : M \to X \in \mathcal{M}$;
- *weakly hereditary* if j_m is c-dense for every $m \in \mathcal{M}$;
- *hereditary* if, for $m : M \to X$, $l : X \to Y$ and $l \cdot m$ in $\mathcal{M}$,

$$c_X(m) \cong l^{-1}(c_Y(l \cdot m)).$$

It is immediate that hereditary closure operators are in particular weakly hereditary.

Closure operators with respect to $\mathcal{M}$ can be preordered by

$$c \leq d \;:\Leftrightarrow\; \forall m : M \to X \in \mathcal{M} \; c_X(m) \leq d_X(m).$$

2.2 Regular versus homological closure operators

For any such class $\mathcal{M}$ of monomorphisms containing the regular monomorphisms, every reflective subcategory $\mathbf{A}$ of $\mathbf{C}$ induces a *regular closure operator* $\mathrm{reg}^{\mathbf{A}}$ on $\mathbf{C}$ with respect to $\mathcal{M}$, assigning to each $m : M \to X$ in $\mathcal{M}$ the equaliser of the following diagram

$$X \underset{v}{\overset{u}{\rightrightarrows}} Y \xrightarrow{\rho_Y} RY \ ,$$

where (u, v) is the cokernel pair of m and ρ_Y is the **A**-reflection of Y; that is,

$$\mathrm{reg}^{\mathbf{A}}_X(m) = \mathrm{eq}(\rho_Y \cdot u, \rho_Y \cdot v).$$

Regular closure operators are idempotent but not weakly hereditary in general.

When the category **C** is pointed, replacing equalisers by kernels in the construction above gives rise to another interesting closure operator. Let $\mathcal{M}$ be a pullback-stable class of monomorphisms containing the kernels, and let **A** be a reflective subcategory of **C**. The *homological closure operator* $\mathrm{h}^{\mathbf{A}}$ *induced by* **A** in $\mathcal{M}$ assigns to each $m : M \to X$ the kernel of the following composition of morphisms

$$X \xrightarrow{\pi_M} Y \xrightarrow{\rho_Y} RY \ ,$$

where π_M is the cokernel of m and ρ_Y is the **A**-reflection of Y; that is,

$$\mathrm{h}^{\mathbf{A}}_X(m) = \ker(\rho_Y \cdot \pi_M).$$

Homological closure operators are idempotent but not weakly hereditary in general.

If **C** has direct $\mathcal{M}$-images, then $\mathrm{reg}^{\mathbf{A}}$ is completely determined by its restriction to **A**, via the formula

$$\mathrm{reg}^{\mathbf{A}}_X(m) \cong \rho_X^{-1}(\mathrm{reg}^{\mathbf{A}}_{RX}(\rho_X(m))), \tag{$\star$}$$

for any $m : M \to X$ in $\mathcal{M}$, with $\rho_X : X \to RX$ the **A**-reflection of X.

There is an alternative way of replacing equalisers by kernels in the definition of regular closure operator. Indeed, $\mathrm{reg}^{\mathbf{A}}$ is the maximal closure such that every equaliser in **A** is closed. In particular:

2.2.1 Lemma *If* **A** *is a reflective subcategory of* **Top** *and* X *is an object of* **A**, *then:*

(1) *the diagonal* $\delta_X : X \to X \times X$ *is* $\mathrm{reg}^{\mathbf{A}}$*-closed;*

(2) *For every* $x \in X$, *the inclusion* $\{x\} \to X$ *is* $\mathrm{reg}^{\mathbf{A}}$*-closed.*

In a pointed finitely-complete category **C**, given a pullback-stable class of monomorphisms $\mathcal{M}$ containing the zero-subobjects and a reflective subcategory **A**, one calls *maximal closure operator induced by* **A**, the maximal closure operator $\max^{\mathbf{A}}$ with $0_A : 0 \to A$ closed, for every $A \in \mathbf{A}$ (or, equivalently, with kernels of **A**-morphisms closed). It is easily verified that:

2.2.2 Proposition *If* **A** *is a reflective subcategory of a pointed and finitely-complete category* **C** *with cokernels, then* $\mathrm{h}^{\mathbf{A}} = \max^{\mathbf{A}}$.

While regular closure operators were introduced by Salbany [15] more than 30 years ago, and widely studied since then, homological closure operators were introduced more recently by Bourn and Gran [6] in the context of homological categories.

For comprehensive accounts on closure operators and homological categories we refer the reader to [12] and [2, 14] respectively.

3 How close are regular and homological closure operators

3.1 (Weak) heredity

The study of weak heredity of regular closure operators presented in [7] encompasses the following topological and algebraic results.

3.1.1 Theorem

(1) *For a regular-epireflective subcategory* **A** *of* **Top**, *the following assertions are equivalent:*

   (i) $\mathrm{reg}^{\mathbf{A}}$ *is weakly hereditary;*

   (ii) **A** *is a disconnectedness.*

(2) *For a (regular-)epireflective subcategory* **A** *of an abelian category* **C**, *the following conditions are equivalent:*

   (i) $\mathrm{reg}^{\mathbf{A}}$ *is weakly hereditary;*

   (ii) **A** *is a torsion-free subcategory.*

Disconnectedness in topological spaces and torsion-free subcategories in abelian categories are particular cases of right-constant subcategories (see [9] for details), hence the two theorems above are instances of a more general result. Moreover, as shown in [13], if **C** is an abelian category, then the regular closure operator induced by an epireflective subcategory **A** coincides with the maximal closure operator induced by **A**. This shows, moreover, that Theorem 3.1.1.2 is a particular case of the following result, due to Bourn and Gran [6].

3.1.2 Theorem *For a regular-epireflective subcategory* **A** *of a homological category* **C***, the following conditions are equivalent:*

(i) $\max^{\mathbf{A}}$ *is weakly hereditary;*

(ii) **A** *is a torsion-free subcategory.*

In [6] Bourn and Gran show also that heredity of $\max^{\mathbf{A}}$ identifies hereditary torsion theories, that is those torsion theories with hereditary torsion part.

3.1.3 Theorem *For a regular-epireflective subcategory* **A** *of a homological category* **C***, the following conditions are equivalent:*

(i) $\max^{\mathbf{A}}$ *is hereditary;*

(ii) **A** *is a hereditary torsion-free subcategory.*

As for weak heredity there is a corresponding result in topology.

3.1.4 Theorem *For a regular-epireflective subcategory* **A** *of* **Top***, the following conditions are equivalent:*

(i) $\mathrm{reg}^{\mathbf{A}}$ *is hereditary;*

(ii) **A** *is an hereditary disconnectedness (that is, its connectedness counterpart* $\mathrm{l}(\mathbf{A})$ *is hereditary);*

(iii) **A** *is either* **Top** *or the category* $\mathbf{Top}_0$ *of* T_0*-spaces or the category* **Sgl** *consisting of singletons and the empty set.*

Proof. First we remark that (ii)⇔(iii) follows from Proposition 1.1.1(4).

(iii)⇔(i): If $\mathbf{A} = \mathbf{Top}$, then $\mathrm{reg}^{\mathbf{A}}$ is the discrete closure, which is trivially hereditary. If $\mathbf{A} = \mathbf{Top}_0$, then $\mathrm{reg}^{\mathbf{A}}$ is the b-closure, with, for $A \subseteq X$,

$$\mathrm{b}_X(A) = \{x \in X \,|\, \text{for every neighbourhood } U \text{ of } x, \overline{\{x\}} \cap U \cap A \neq \emptyset\},$$

which is known to be hereditary (see for instance [12]). If $\mathbf{A} = \mathbf{Sgl}$, then $\mathrm{reg}^{\mathbf{A}}$ is the indiscrete closure, that is

$$\mathrm{reg}_X^{\mathbf{Sgl}}(A) = X \text{ for every } \emptyset \neq A \subseteq X \text{ and } \mathrm{reg}_X^{\mathbf{Sgl}}(\emptyset) = \emptyset,$$

which is hereditary. Conversely, assume that $\mathbf{A}$ is none of these three subcategories. By Proposition 1.1.1(3), $\mathbf{TDisc} \subseteq \mathbf{A} \subseteq \mathbf{Top}_1$. Consider the Sierpinski space $S = \{0,1\}$, with $\{0\}$ the only non-trivial open subset, and its product $S \times S$. The two-point discrete space $D = \{(0,1),(1,0)\}$ is a subspace of $S \times S$; the $\mathbf{A}$-reflection of $S \times S$ is a singleton, while $D \in \mathbf{A}$. Hence $\mathrm{reg}_D^{\mathbf{A}}(0,1) = (0,1)$ while $\mathrm{reg}_{S\times S}^{\mathbf{A}}(0,1) = S \times S$, and therefore $\mathrm{reg}^{\mathbf{A}}$ is not hereditary.

□

3.2 Openness of regular epimorphisms

Another interesting feature of homological closure operators pointed out by Bourn and Gran [6] is to make regular epimorphisms open. Recall that, given a closure operator c, a morphism $f : X \to Y$ is *c-open* if, for every $n : N \to Y \in \mathcal{M}$,

$$c_X(f^{-1}(n)) \cong f^{-1}(c_Y(n));$$

that is, the inequality in the c-continuity condition (C3) becomes an isomorphism. It was shown in [8] that:

3.2.1 Proposition *For an idempotent closure operator c in a homological category $\mathbf{C}$ the following conditions are equivalent:*

(i) $c = \max^{\mathbf{A}}$ *for some regular-epireflective subcategory* $\mathbf{A}$*;*

(ii) *regular epimorphisms in* $\mathbf{C}$ *are c-open.*

It is easy to check that in general this is not a common property of regular closure operators in **Top**.

3.2.2 Proposition *For a closure operator c in* **Top** *the following conditions are equivalent:*

(i) *c is regular, and every regular epimorphism is c-open;*

(ii) *c is either the discrete or the indiscrete closure operator.*

Proof. (ii)⇒(i) is trivial.

(i)⇒(ii): Let c be a regular closure operator induced by a regular-epireflective subcategory $\mathbf{A}$ different from $\mathbf{Top}$. Then either $\mathbf{A} = \mathbf{Top}_0$ or $\mathbf{A} \subseteq \mathbf{Top_1}$. If $\mathbf{A} = \mathbf{Top_0}$, then $\mathrm{reg}^{\mathbf{A}}$ is the b-closure, which does not satisfy (i): take $X = \{0,1,2,3\} \to Y = \{0,1,2\}$ with $f(i) = i$ if $i \le 2$ and $f(3) = 2$, where the only non-trivial open subset of X is $\{1,2\}$, hence the quotient topology is indiscrete; then $f^{-1}(\mathrm{b}(0)) = X$ and $\mathrm{b}(f^{-1}(0)) = \{0,3\}$. If $A \subseteq \mathbf{Top_1}$, then $\mathrm{reg}^{\mathbf{A}}$ is indiscrete in the Sierpinski space. Hence, for every closed, non-open, subset C of a space Z, since $\chi_C : Z \to S$ is a quotient map, hence $\mathrm{reg}^{\mathbf{A}}$-open, one has $\mathrm{reg}^{\mathbf{A}}_Z(C) = \chi_C^{-1}(\mathrm{reg}^{\mathbf{A}}_S(1)) = Z$. Therefore, if Z is T_1 and non-discrete, it has a non-open point z, and so $\mathrm{reg}^{\mathbf{A}}_Z(z) = Z$, which implies that $Z \notin \mathbf{A}$. This means then that $\mathbf{A}$ has only discrete spaces, and then $\mathbf{A} \subset \mathbf{TDisc}$, which implies $\mathbf{A} = \mathbf{Sgl}$ by Proposition 1.1.1. □

3.3 Closedness of regular epimorphisms

Closed morphisms with respect to a closure operator are defined analogously to open morphisms, replacing inverse images by direct images. When $\mathbf{C}$ has direct $\mathcal{M}$-images, a morphism $f : X \to Y$ is said to be *c-closed* if, for every $m \in \mathcal{M}/X$,

$$f(c_X(m)) \cong c_Y(f(m)).$$

(As said before, the inequality $f(c_X(m)) \le c_Y(f(m))$ is equivalent to c-continuity of f.)

We recall that an epireflective subcategory is said to be *Birkhoff* if it is closed under regular epimorphisms.

Next we analyse the topological counterpart of the following result.

3.3.1 Proposition [6] *If* $\mathbf{A}$ *is a regular-epireflective subcategory of a semi-abelian category* $\mathbf{C}$*, the following assertions are equivalent:*

(i) *regular epimorphisms are* $\max^{\mathbf{A}}$*-closed;*

(ii) $\mathbf{A}$ *is a Birkhoff subcategory.*

3.3.2 Proposition *For a regular-epireflective subcategory* $\mathbf{A}$ *of* $\mathbf{Top}$ *the following conditions are equivalent:*

(i) *regular epimorphisms are* $\mathrm{reg}^{\mathbf{A}}$*-closed;*

(ii) $\mathbf{A}$ *is a Birkhoff subcategory;*

(iii) $\mathbf{A} = \mathbf{Top}$ *or* $\mathbf{A} = \mathbf{Sgl}$.

Proof. Trivially (iii)⇒(ii). To show that (ii)⇒(iii), first note that $\mathbf{Top}_0$ is not closed under quotients, hence it is not a Birkhoff subcategory. Now, if $\mathbf{A} \subseteq \mathbf{Top}_1$ and $\mathbf{A}$ contains a non-discrete space Z, hence with a closed non-open subset C, then $\chi_C : Z \to S$ is a quotient map although the Sierpinski space S does not belong to $\mathbf{A}$. Hence every object of $\mathbf{A}$ is discrete, which implies that $\mathbf{A} = \mathbf{Sgl}$.

(iii)⇒(i) is clear, since $\mathrm{reg}^{\mathbf{Top}}$ is the discrete closure and $\mathrm{reg}^{\mathbf{Sgl}}$ is the indiscrete closure, both making regular epimorphisms c-closed.

(i)⇒(iii): If $\mathbf{A} = \mathbf{Top}_0$, $\mathrm{reg}^{\mathbf{A}}$ is the b-closure. The quotient map $X \to Y$ used in the proof of Proposition 3.2.2 is not b-closed since

$$f(\mathrm{b}(0)) = f(\{0,3\}) = \{0,2\} \text{ and } \mathrm{b}(f(0)) = \mathrm{b}(0) = \{0,1,2\}.$$

If $\mathbf{A} \subseteq \mathbf{Top}_1$ and C is a closed, non-open, subset of $Z \in \mathbf{A}$, then $\chi_C : Z \to S$ is a quotient map. Moreover, $\mathrm{reg}^{\mathbf{A}}$ is indiscrete in S, since the $\mathbf{A}$-reflection of S is a singleton, and every point in Z is $\mathrm{reg}^{\mathbf{A}}$-closed, because $Z \in \mathbf{A}$ (see Lemma 2.2.1). For any $z \in C$ one has

$$\chi_C(\mathrm{reg}^{\mathbf{A}}_Z(z)) = \chi_C(z) = 1 \neq \mathrm{reg}^{\mathbf{A}}_S(\chi_C(z)) = \mathrm{reg}^{\mathbf{A}}_S(1) = S.$$

Therefore every object of $\mathbf{A}$ is discrete, and so $\mathbf{A} = \mathbf{Sgl}$. □

3.4 When regular and homological closures coincide

Finally, it is natural to ask in which pointed regular categories regular and maximal closure operators coincide. Until the end of this section, we will assume that these closure operators are defined in the class of monomorphisms of $\mathbf{C}$.

3.4.1 Theorem *Let* $\mathbf{A}$ *be a regular-epireflective subcategory of a pointed regular category with cokernels. The following assertions are equivalent:*

(i) *when restricted to* $\mathbf{A}$, $\mathrm{reg}^{\mathbf{A}}$ *and* $\max^{\mathbf{A}}$ *coincide;*

(ii) $\mathrm{reg}^{\mathbf{A}} = \max^{\mathbf{A}}$*;*

(iii) *in* $\mathbf{A}$ *every equaliser is a kernel;*

(iv) *for every object* A *of* $\mathbf{A}$*, the diagonal* δ_A *is a kernel in* $\mathbf{A}$.

Proof. (i)⇒(ii): On one hand, since the maximal closure $\max^{\mathbf{A}}$ is the largest closure c with $0_A : 0 \to A$ c-closed for any $A \in \mathbf{A}$ and $\mathrm{reg}^{\mathbf{A}}$ and $\max^{\mathbf{A}}$ coincide in $\mathbf{A}$, $\mathrm{reg}^{\mathbf{A}} \leq \max^{\mathbf{A}}$.

On the other hand, denoting by ρ the $\mathbf{A}$-reflection, by ($\star$) of Section 2 we have that $\mathrm{reg}^{\mathbf{A}}_X(m) \cong \rho_X^{-1}(\mathrm{reg}^{\mathbf{A}}_{RX}(\rho_X(m)))$ is $\max^{\mathbf{A}}$-closed since, by (i), $\mathrm{reg}^{\mathbf{A}}_{RX}(\rho_X(m)) \cong \max^{\mathbf{A}}_{RX}(\rho_X(m))$, hence $\mathrm{reg}^{\mathbf{A}}_X \geq \max^{\mathbf{A}}_X$.

(ii)⇒(iii): Since every equaliser $m : M \to A$ in $\mathbf{A}$ is $\mathrm{reg}^{\mathbf{A}}$-closed, hence $\max^{\mathbf{A}}$-closed by (ii), and the $\max^{\mathbf{A}}$-closure of m in $\mathbf{A}$ is the kernel of

$$A \xrightarrow{\pi_M} Y \xrightarrow{\rho_Y} RY \ \in \mathbf{A},$$

$m \cong \max^{\mathbf{A}}_A(m) \cong \ker(\rho_Y \cdot \pi_M)$ is a kernel in $\mathbf{A}$ as claimed.

(iii)⇒(iv) is obvious, while (iv)⇒(iii) follows from the fact that the equaliser of $f, g : A \to B$ is the pullback of $\delta_B : B \to B \times B$ along $< f, g >: A \to B \times B$.

(iii)⇒(i): A monomorphism in $\mathbf{A}$ is $\mathrm{reg}^{\mathbf{A}}$-closed (resp. $\max^{\mathbf{A}}$-closed) if, and only if, it is an equaliser in $\mathbf{A}$ (resp. a kernel in $\mathbf{A}$). If equalisers are kernels, then, as idempotent closure operators, necessarily $\mathrm{reg}^{\mathbf{A}}$ and $\max^{\mathbf{A}}$ coincide in $\mathbf{A}$. □

If $\mathbf{A}$ is a regular-epireflective subcategory of a homological category, then $\mathbf{A}$ is homological as well (see [4]), and so in $\mathbf{A}$ every coequaliser is a cokernel. In the theorem above the dual property is required for **A** so that its homological and regular closure operators coincide. Indeed this condition leads us again to an abelian-like condition, as we show next.

3.4.2 Corollary

(1) *If* **A** *is a regular-epireflective subcategory of a homological category* **C** *with* $\mathrm{reg}^{\mathbf{A}} = \max^{\mathbf{A}}$*, then* **A** *consists of abelian objects.*

(2) *If* **A** *is a regular-epireflective subcategory of a semi-abelian category* **C***, then the following conditions are equivalent:*

 (i) $\mathrm{reg}^{\mathbf{A}} = \max^{\mathbf{A}}$*;*

 (ii) *every object in* **A** *is abelian.*

Proof. First we remark that both **C** and **A** are homological (semi-abelian resp.), and so the result follows from Proposition 1.2.1 since:

$$X \text{ is abelian } :\Leftrightarrow X \text{ has an internal abelian group structure}$$

If $\mathrm{reg}^{\mathbf{A}} = \max^{\mathbf{A}}$, then $\delta_A : A \to A \times A$ is a kernel, for every $A \in \mathbf{A}$. Hence, A is abelian. Conversely, if A is abelian then it has an internal abelian group structure in **C**, hence also in **A**, and so δ_A must be a kernel in **A** in case **A** is semi-abelian. □

We point out that there are non (semi-)abelian homological categories where every equaliser is a kernel. In fact such categories are necessarily additive but may fail to be exact. (We recall that an exact and additive category is abelian: see [14].) This is the case, for instance, for the category of topological abelian groups, which is regular and protomodular but not exact (see [3] for details.)

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