CAHIERS DE TOPOLOGIE ET GEOMETRIE DIFFERENTIELLE CATEGORIQUES
Vol. LI-4 (2010)

ON CONNECTIVITY SPACES

by Stéphane DUGOWSON

Résumé. Cet article présente les bases d'une théorie des espaces connectifs. Il étudie notamment l'engendrement des structures, l'existence des (co)limites dans les catégories concernées, le produit tensoriel et la structure de catégorie monoïdale fermée associée. On y définit une notion d'homotopie ainsi que le *smash product* des espaces connectifs intègres pointés et la structure de catégorie monoïdale fermée associée. On étudie ensuite les espaces connectifs finis et l'on introduit un nouvel invariant numérique pour les entrelacs : l'ordre connectif. On présente enfin le théorème peu connu de Brunn-Debrunner-Kanenobu, qui affirme que tout espace connectif fini intègre peut être représenté par un entrelacs.

Abstract. This paper presents some basic facts about connectivity spaces. In particular, it explains how to generate connectivity structures, the existence of limits and colimits in the main categories of connectivity spaces, the closed monoidal category structure given by the tensor product of integral connectivity spaces; it defines homotopy for connectivity spaces and mentions briefly some related difficulties; it defines the smash product of pointed integral connectivity spaces and shows that this operation results in a closed monoidal category with such spaces as objects. Then, it studies finite connectivity spaces, associating a directed acyclic graph with each such space and then defining a new numerical invariant for links: the connectivity order. Finally, it mentions the not very well-known Brunn-Debrunner-Kanenobu theorem which asserts that every finite integral connectivity space can be represented by a link.

Keywords: Connectivity. Closed Monoidal Categories. Links. Borromean. Brunnian.

Mathematics Subject Classification 2000: 54A05, 54B30, 57M25.

Connectivity spaces are topological objects which have not yet received much attention. This paper presents results we have recently obtained relating to them. In the first section we recall their definition. The second section explains how to generate a connectivity structure from a given family of subsets to be regarded as connected. The third section is about categorical constructions in the main categories of connectivity spaces, by seeing them as particular cases of "categories with lattices of structures". The fourth section studies the closed monoidal category structure given by the tensor product of integral connectivity spaces. The fifth section defines homotopy for connectivity spaces and briefly mentions some difficulties related to this notion. The sixth section is devoted to pointed integral connectivity spaces and to the smash product of such spaces. In the last section we study finite connectivity spaces, associating a directed acyclic graph with each such space, and then defining a new numerical invariant for links: the connectivity index. Finally, we discuss the not very well-known Brunn-Debrunner-Kanenobu theorem, which asserts that every finite integral connectivity space can be represented by a link in the space $\mathbf{R}^3$ (or in $\mathbf{S}^3$).

Notations

If X is a set, the set of subsets of X is denoted by $\mathcal{P}(X)$ or $\mathcal{P}_X$, and the set $\mathcal{P}(\mathcal{P}_X)$ by $\mathcal{Q}_X$. For any $\mathcal{A} \in \mathcal{Q}_X$, $\mathcal{A}^\bullet$ denotes the set $\{A \in \mathcal{A}, \mathrm{card}(A) \geq 2\}$. If $\sim$ is an equivalence relation on X, the equivalence class of $x \in X$ is denoted by $\tilde{x}$. If Y is a subset of X, $\sim_Y$ denotes the equivalence relation defined on X by $a \sim_Y b$ if and only if $a = b$ or $(a, b) \in Y^2$, and X/Y denotes the quotient $X/\sim_Y$.

1 Definitions, Examples

Let us recall the definition of connectivity spaces and connectivity morphisms [2, 7].

Definition 1 (Connectivity spaces)**.** *A* connectivity space *is a pair* $(X, \mathcal{K})$ *where* X *is a set and* $\mathcal{K}$ *is a set of subsets of* X *such that* $\emptyset \in \mathcal{K}$ *and*

$$\forall \mathcal{I} \in \mathcal{P}(\mathcal{K}), \bigcap_{K \in \mathcal{I}} K \neq \emptyset \Longrightarrow \bigcup_{K \in \mathcal{I}} K \in \mathcal{K}.$$

The set X *is called the* carrier *of the space* $(X, \mathcal{K})$*, the set* $\mathcal{K}$ *is its* connectivity structure*. The elements of* $\mathcal{K}$ *are called the* connected subsets *of the space. The morphisms between two connectivity spaces are the functions which transform connected subsets into connected subsets. They are called the* connectivity morphisms*, or the* connecting maps[1]*. A connectivity space is called* integral *if every singleton subset is connected. The connected subsets with cardinality greater than one will be called the non-trivial connected subsets. A connectivity space is called* finite *if its carrier is a finite set.*

If X is a connectivity space, $|X|$ will denote its carrier, and $\kappa(X)$ its connectivity structure, so $X = (|X|, \kappa(X))$.

Remark 1. Instead of supposing that the empty set is always a member of connectivity structures, we could suppose without any substantial change that it is never such a member. But it seems preferable to choose one or the other of those two assumptions, to avoid "doubling" the involved categories.

Remark 2. Each point of an integral connectivity space belongs to a maximal connected subset. Those subsets are the connected components of the space; they constitute a partition of it.

In [2], Börger denotes *Zus* the category of integral connectivity spaces, because of the German word *Zusammenhangsräume*. We propose here to use rather **Cnc** to denote the category of connectivity spaces, **Cnct** to denote the category of integral connectivity spaces and **fCnct** to denote the category of finite integral connectivity spaces.

[1] Though *non-disconnecting maps* would be more accurate.

Example 1. Let $U_T : \mathbf{Top} \to \mathbf{Cnct}$ be the functor whose value is defined on each topological space (X, τ) as the connectivity space $(X, \mathcal{K})$ with $\mathcal{K}$ the set of connected subsets (in ordinary topological sense) of (X, τ). Then U_T is not full and not surjective (up to isomorphism) on objects ; it is faithful but is neither strictly injective nor injective up to isomorphism on objects : for example, if $X = \{a, b\}$, $\tau_1 = \{\emptyset, \{a\}, X\}$ and $\tau_2 = \{\emptyset, X\}$, then (X, τ_1) and (X, τ_2) are not isomorphic but $U_T(X, \tau_1) = U_T(X, \tau_2)$.

Example 2. Let **Grf** be the topological construct[2] whose objects are the simple undirected graphs and whose morphisms are the functions which send edges to edges or singletons. More precisely, such a graph can be defined as a pair $(X, \mathcal{G})$ with $\mathcal{G} \in \mathcal{Q}_X$ such that

$$\{A \in \mathcal{P}_X, \mathrm{card}A = 1\} \subseteq \mathcal{G} \subseteq \{A \in \mathcal{P}_X, \mathrm{card}A = 2\},$$

and morphisms $f : (X, \mathcal{G}) \to (Y, \mathcal{H})$ are functions $f : X \to Y$ such that $\forall A \in \mathcal{G}, f(A) \in \mathcal{H}$. A subset K of such a graph $(X, \mathcal{G})$ is said to be connected if for every pair (x, x') of elements of K, there exists a finite path $x = x_0, x_1, \cdots, x_n = x'$ such that each x_i is in K and each $\{x_i, x_{i+1}\}$ is in $\mathcal{G}$. The forgetful functor $U_G : \mathbf{Grf} \to \mathbf{Cnct}$, whose value is defined for each simple undirected graph $(X, \mathcal{G})$ as $(X, \mathcal{K})$ with $\mathcal{K}$ the set of connected subsets of X, is a full embedding.

Example 3. With each tame link[3] L in $\mathbf{R}^3$ or $\mathbf{S}^3$, we associate an integral connectivity space S_L taking the components of the link L as points of S_L, the connected subsets of it being defined by the nonsplittable sublinks of L. The connectivity structure $\kappa(S_L)$ will be called the *splittability structure* of L.

---

[2] Following [1], §5.1, p. 61, a category of structured sets and structure preserving functions between them is called a *construct*. More precisely, a construct is a concrete category over the category **Set** of sets, that is a pair $(\mathbf{A}, U)$ where $\mathbf{A}$ is a category and $U : \mathbf{A} \to \mathbf{Set}$ is a faithful functor (forgetful functor). A *topological* construct is then a construct $(\mathbf{A}, U)$ such that the functor U is topological, *i.e.* such that every U-structured source $(f_i : E \to UA_i)_I$ has a unique U-initial lift $(\bar{f}_i : A \to A_i)_I$(see [1], 10.57, p. 182 and §21.1, p. 359, and *infra*, the section 3.1 of the present article).

[3] A link is called *tame* if it is not *wild*, that is if it is (ambient) isotopic to a polygonal link (or to a smooth link, see [4]).

Example 4. The simplest integral connectivity space which is neither in $U_T(\mathbf{Top})$ nor in $U_G(\mathbf{Grf})$ is the *Borromean* space $\mathbf{B}_3$, defined by $|\mathbf{B}_3| = 3 = \{0,1,2\}$ and $\kappa(\mathbf{B}_3) = \mathcal{B}_3$ such that $\mathcal{B}_3^\bullet = \{|\mathbf{B}_3|\}$. More generally, for each integer $n \in \mathbf{N}$, the n-points *Brunnian* space $\mathbf{B}_n$ is the integral connectivity space defined by $|\mathbf{B}_n| = n$ and $\kappa(\mathbf{B}_n) = \mathcal{B}_n$ such that $\mathcal{B}_n^\bullet = \{|\mathbf{B}_n|\}$. The names *Borromean* and *Brunnian* are justified by the fact that the corresponding spaces are the ones associated with the links with the same names.

Example 5. More generally, for each set X and each cardinal ν, there is a unique integral connectivity space whose non-trivial connected subsets are those with cardinal greater than ν.

Example 6. Let p be an integer. The *hyperbrunnian space* $\mathbf{HB}_p$ is the integral connectivity space such that $|\mathbf{HB}_p| = \{0,1,\cdots,p-1\}^{\mathbf{N}}$ and with non-trivial connected subsets all the $K \subseteq |\mathbf{HB}_p|$ for which there exist $k \in \mathbf{N}$ and $a \in |\mathbf{HB}_p|$ such that K be of the form

$$K = \{x \in |\mathbf{HB}_p|, \forall n < k, x_n = a_n\}.$$

The space $\mathbf{HB}_3$ will be called the *hyperborromean* space. For each $k \in \mathbf{N}$, the function $\phi_k : \mathbf{HB}_p \to \mathbf{B}_p$ defined by $f(x) = x_k$ is a connectivity morphism. If $p \geq 2$, the function $f : \mathbf{HB}_p \to \mathbf{I}$ defined by

$$f(x) = \sum_{n=0}^{n=\infty} \frac{x_n}{p^{n+1}}$$

is a surjective connectivity morphism onto $\mathbf{I} = [0,1]$, the connectivity space associated with the usual topological interval $[0,1]$.

Example 7. More generally, if X is a set and $(T,\leq)$ is a totally ordered set, we define the integral connectivity space $\mathbf{B}_T(X)$ by $|\mathbf{B}_T(X)| = X^T$ and $\kappa(\mathbf{B}_T(X))^\bullet = \{K_{f,t}, (f,t) \in X^T \times T\}$ where $K_{f,t} = \{g \in X^T, \forall s \in T, s < t \Rightarrow g(s) = f(s)\}$. Then $\mathbf{B}_p = \mathbf{B}_{\{*\}}(p)$, and $\mathbf{HB}_p = \mathbf{B}_{\mathbf{N}}(p)$. If $\mathrm{card}(X) \geq 2$, then $\mathbf{B}_T(X)$ is a connected space iff T has a least element.

Example 8. Let $(X,\leq)$ be a totally ordered set. The set of all intervals (of any form) of X constitutes an integral connectivity structure on X, called the *order connectivity structure*. In particular, ordinal numbers define connectivity spaces, called the *ordinal connectivity spaces*.

2 Generating Connectivity Structures

2.1 The Theorem of Generation

Proposition 1. *Let X be a set, and Cnc_X (resp. $Cnct_X$) the set of connectivity structures on X (resp. the set of integral connectivity structures on X). For the order defined by*

$$\mathcal{X}_1 \leq \mathcal{X}_2 \Leftrightarrow \mathcal{X}_1 \subseteq \mathcal{X}_2,$$

$(Cnc_X, \leq)$ and $(Cnct_X, \leq)$ are complete lattices.

Proof. These ordered sets have $\mathcal{P}_X$ as a maximal element, and for each nonempty family $(\mathcal{X}_i)_{i\in I}$ of (integral) connectivity structures on X, $\bigcap_i \mathcal{X}_i$ is again an (integral) connectivity structure on X.

□

If $\mathcal{X}_1 \leq \mathcal{X}_2$, we say that $\mathcal{X}_1$ is *finer* than $\mathcal{X}_2$, or that $\mathcal{X}_2$ is *coarser* than $\mathcal{X}_1$. $\mathcal{P}_X$, the coarsest structure on X, is called the *indiscrete* structure on X. The finest connectivity structure contains only the empty set; it is called the *discrete* connectivity structure. The finest integral connectivity structure contains only the empty set and the singletons; it is called the *discrete* integral connectivity structure, or simply the discrete structure.

Remark 3. The lattices Cnc_X and $Cnct_X$ are not distributive, unless X has no more than two points. For example, if $X = \{1,2,3\}$ and, for each $i \in X$, $\mathcal{X}_i$ is the integral connectivity structure on X with $(X \setminus \{i\})$ as the only non trivial connected set, then $\bigvee_i(\mathcal{X}_i) = \mathcal{P}_X$, so $\mathcal{B}_3 \wedge (\bigvee_i(\mathcal{X}_i)) = \mathcal{B}_3$, while $\bigvee_i(\mathcal{B}_3 \wedge \mathcal{X}_i)$ is the discrete integral connectivity structure on X.

Definition 2. *Let X be a set, and $\mathcal{A} \in \mathcal{Q}_X$ a set of subsets of X. The finest connectivity structure (resp. integral connectivity structure) on X which contains $\mathcal{A}$ is called the* connectivity structure *(resp.* integral connectivity structure*)* generated *by $\mathcal{A}$ and is denoted by $[\mathcal{A}]_0$ (resp. $[\mathcal{A}]$).*

Thus, $[\mathcal{A}]_0 = \bigwedge\{\mathcal{X} \in Cnc_X, \mathcal{A} \subseteq \mathcal{X}\}$ and $[\mathcal{A}] = \bigwedge\{\mathcal{X} \in Cnct_X, \mathcal{A} \subseteq \mathcal{X}\}$.

Proposition 2. *Let X be a set, $\mathcal{A}$ a set of subsets of X, $(Y, \mathcal{Y})$ a connectivity space (resp. integral connectivity space) and $f : X \to Y$ a function. Then f is a connectivity morphism from $(X, [\mathcal{A}]_0)$ (resp. $(X, [\mathcal{A}])$) to $(Y, \mathcal{Y})$ if and only if $f(A) \in \mathcal{Y}$ for all $A \in \mathcal{A}$.*

Proof. $\{A \in \mathcal{P}_X, f(A) \in \mathcal{Y}\}$ is a connectivity structure on X containing $\mathcal{A}$ and then containing $[\mathcal{A}]_0$ (resp. $[\mathcal{A}]$).

□

The expression "*generated structure*" is justified by the next theorem, in which ω_0 denotes the smallest infinite ordinal.

Theorem 3 (Generation of connectivity structures)**.** *Let X be a set and $\mathcal{A} \in \mathcal{Q}_X$ a set of subsets of X. Then there exists an ordinal $\alpha_0 \leq \omega_0 + 1$ such that*

$$[\mathcal{A}]_0 = \Phi^\alpha(\mathcal{A}) \text{ for all } \alpha \geq \alpha_0,$$

where the Φ^α are the operators $\mathcal{Q}_X \to \mathcal{Q}_X$ defined by induction for every ordinal α by

- $\Phi^0 = id_{\mathcal{Q}_X}$,
- *if there is an ordinal β such that $\alpha = \beta + 1$, then $\Phi^\alpha = \Phi \circ \Phi^\beta$*
- *otherwise, for all $\mathcal{U} \in \mathcal{Q}_X$, $\Phi^\alpha(\mathcal{U}) = \bigcup_{\beta<\alpha} \Phi^\beta(\mathcal{U})$,*

and with Φ the operator defined for all $\mathcal{U} \in \mathcal{Q}_X$ by

$$\Phi(\mathcal{U}) = \{\emptyset\} \cup \{\bigcup_{A \in \mathcal{E}} A, \mathcal{E} \in \mathcal{L}_\mathcal{U}\},$$

where $\mathcal{L}_\mathcal{U} = \{\mathcal{E} \in \mathcal{P}(\mathcal{U}), \bigcap_{A \in \mathcal{E}} A \neq \emptyset\}$.

The integral connectivity structure $[\mathcal{A}]$ generated by $\mathcal{A}$ is obtained in the same way, adding the singletons of X at any stage of the process.

Proof. We only have to prove the part of the theorem concerning the generation of connectivity structures, the last claim about *integral* connectivity structure being then obvious.

For every $\mathcal{U}$ and $\mathcal{V}$ in $\mathcal{Q}_X$, the following three properties are easy to check:

- $\mathcal{U} \subseteq \Phi(\mathcal{U})$,
- $\mathcal{U} \subseteq \mathcal{V} \Rightarrow \Phi(\mathcal{U}) \subseteq \Phi(\mathcal{V})$,
- $\mathcal{U} \in Cnc_X \Leftrightarrow \Phi(\mathcal{U}) = \mathcal{U}$.

The first two properties then imply by induction that for all ordinal numbers α and β with $\alpha \leq \beta$, one has $\Phi^\alpha(\mathcal{U}) \subseteq \Phi^\beta(\mathcal{U})$, and the last two properties imply $\Phi(\mathcal{U}) \subseteq [\mathcal{U}]_0$ and, by induction, $\Phi^\alpha(\mathcal{U}) \subseteq [\mathcal{U}]_0$ for all ordinal numbers α. Then, if for an ordinal number α_0 the set $\Phi^{\alpha_0}(\mathcal{A})$ is a connectivity structure on X, it coincides with $[\mathcal{A}]_0$. So, to complete the proof, it suffices to verify that the set $\mathcal{C} = \Phi^{\omega_0+1}(\mathcal{A})$ is such a structure, *i.e.* $\Phi(\mathcal{C}) = \mathcal{C}$. For this, let $\mathcal{W}$ be the set $\Phi^{\omega_0}(\mathcal{A})$, so that $\mathcal{C} = \Phi(\mathcal{W})$. Then $\mathcal{W}$ is stable by union of *finite* families with nonempty intersections since $\Phi^{\omega_0}(\mathcal{A}) = \bigcup_{n\in\mathbf{N}} \Phi^n(\mathcal{A})$ so every such family is included in $\Phi^n(\mathcal{A})$ for some integer n, and its union is again in $\mathcal{W}$. Now, let $(S_u)_{u\in U}$ be any family of subsets of X belonging to $\mathcal{C}$ and such that $\bigcap_{u\in U} S_u \neq \emptyset$. We want to verify that $\bigcup_{u\in U} S_u \in \mathcal{C}$. For each $u \in U$, $S_u \in \mathcal{C}$ implies that there exists a family $(S_{u,i})_{i\in I_u}$ of subsets of X belonging to $\mathcal{W}$ such that $\bigcap_{i\in I_u} S_{u,i} \neq \emptyset$ and $\bigcup_{i\in I_u} S_{u,i} = S_u$. Let x be an element of $\bigcap_{u\in U} S_u$. For each $u \in U$, there exists an index $i_u \in I_u$ such that $x \in S_{u,i_u}$. For all $u \in U$ and $i \in I_u$, let $T_{u,i}$ be the set $S_{u,i} \cup S_{u,i_u}$. We have $S_{u,i} \in \mathcal{W}$, $S_{u,i_u} \in \mathcal{W}$ and $S_{u,i} \cap S_{u,i_u} \neq \emptyset$ (since $\bigcap_{i\in I_u} S_{u,i} \neq \emptyset$) so $T_{u,i} \in \mathcal{W}$ by the property of $\mathcal{W}$ we emphasized. Then $\bigcap_{u\in U, i\in I_U} T_{u,i} \neq \emptyset$, so $\bigcup_{u\in U, i\in I_U} T_{u,i} \in \Phi(\mathcal{W})$, that is $\bigcup_{u\in U} S_u \in \mathcal{C}$.

□

Remark 4. In the above proof, the existence of the families $(I_u)_{u\in U}$, $((S_{u,i})_{i\in I_u})_{u\in U}$ and $(i_u)_{u\in U}$ depends on the axiom of choice.

Example 9. Let X be the connectivity space such that $|X| = \mathbf{R}^2 \simeq \mathbf{C}$ and $\kappa(X) = [\mathcal{D}]_0$, where $\mathcal{D}$ is the set of open disks of the Euclidean plane $\mathbf{R}^2$. For $k \in \{1, 2, 3\}$, let $r_k = (-\frac{1}{2} + \frac{\sqrt{3}}{2} i)^k$ be the cubic roots of unity. For each $(x_0, y_0) = z_0 \in \mathbf{C}$, let (z_n) be the sequence of complex numbers defined by the Newton's method for the equation $z^3 - 1 = 0$ and with first term z_0. If the sequence $(z_n)_{n\in\mathbf{N}}$ converges to r_k, we put $f(z_0) = k$, otherwise — in particular if the sequence (z_n) is defined only for a finite number of terms — we put $f(z_0) = 0$. Then the function $f : X \to \mathbf{B}_4$ defined in this way is a connectivity epimorphism. Indeed, the three basins of attraction $W_k = f^{-1}(k)$, $k \in \{1, 2, 3\}$, have the Wada property : their common boundary is the Julia set $W_0 = f^{-1}(0)$ (see [9]). If K is a nonempty element of the connectivity structure $[\mathcal{D}]_0$, it is open and connected for the usual topology of the plane and then either $K \subset W_k$ for a $k \in \{1, 2, 3\}$ and $f(K) = \{k\} \in \kappa(\mathbf{B}_4)$, or K intersects W_0 and then $f(K) = |\mathbf{B}_4|$ which is again in $\kappa(\mathbf{B}_4)$. Note that if we replace $\mathbf{B}_4$ by $(|\mathbf{B}_4|, \kappa(\mathbf{B}_4) \setminus \{\{0\}\})$, the function f is still a connectivity morphism. Moreover, it is easy to use this function f to define other surjective connectivity morphisms from the same connectivity plane X to the borromean space $\mathbf{B}_3$.

Example 10. There are several general ways to associate a connectivity space with each (partially) ordered set. We can for example define closed intervals of such a set exactly like in the totally ordered case, and then associate with each ordered set $(S, \leq)$ the connectivity space $(S, [\mathcal{J}])$ with $\mathcal{J}$ the set of closed intervals of S. In particular, for each topological construct and each set X, we obtain a connectivity space whose points are the structures on X.

2.2 Irreducibility

Definition 3. *Let X be a connectivity space. A connected subset K of $|X|$ is called* reducible *if it belongs to the connectivity structure generated by the others, that is*

$$K \in [\kappa(X) \setminus \{K\}]_0.$$

A nonempty connected subset of $|X|$ is said to be irreducible *if it is not reducible. The space X is said to be* irreducible *if $|X|$ is an irreducible connected subset of itself. It is said to be* distinguished *if each of its nonempty connected subsets is irreducible.*

Remark 5. With the notation of the theorem 3 we have either $\Phi(\kappa(X) \setminus \{K\}) = \kappa(X) \setminus \{K\}$, and then K is irreducible, or $\Phi(\kappa(X) \setminus \{K\}) = \kappa(X)$. In any case, $[\kappa(X) \setminus \{K\}]_0 = \Phi(\kappa(X) \setminus \{K\})$, and K is reducible iff there is a family $\mathcal{E}$ of proper connected subsets $A \subsetneq K$ such that $\bigcap_{A \in \mathcal{E}} A \neq \emptyset$ and $K = \bigcup_{A \in \mathcal{E}} A$.

Remark 6. A connected singleton is necessarily irreducible.

Example 11. If X is a finite connectivity space, a subset K of $|X|$ is reducible iff there are two connected subsets $A \subsetneqq \kappa(X)$ and $B \subsetneqq \kappa(X)$ such that

$$K = A \cup B \text{ and } A \cap B \neq \emptyset.$$

Example 12. The only irreducible connected subsets of $\mathbf{R}$ are the trivial ones.

Example 13. Brunnian spaces and hyperbrunnian spaces are connected and distinguished spaces. Nevertheless, note that $\mathbf{B}_T(X)$ is not a distinguished space for every set X and every totally ordered set T. For example, $\mathbf{B}_{[0,1]}(\{a,b\})$ is not a distinguished space, since $\{f \in \{a,b\}^{[0,1]}, \exists \epsilon \in]0,1], t < \epsilon \Rightarrow f(t) = a\}$ is a connected subset which is reducible.

Definition 4. *Let X be a connectivity space. Its* Brunnian closure *is* $\overline{X} = (|X|, \kappa(X) \cup \{|X|\})$.

Example 14. $\mathbf{B}_n$ is the Brunnian closure of the n-points discrete integral space. $\mathbf{HB}_n$ is the Brunnian closure of the disjoint union (*cf. infra*, section 3.2) of n copies of itself.

The next proposition is obvious.

Proposition 4. *If X is a nonempty irreducible space, then $(|X|, \kappa(X) \setminus \{X\})$ is a connectivity space. If X is a non-connected connectivity space, then $\overline{X}$ is an irreducible connected space.*

Because of the next proposition, the notion of irreducibility will play a fundamental role in the case of finite connectivity spaces.

Proposition 5. *A connectivity structure on a given finite set is characterised by the set of the irreducible connected subsets, which is the minimal set of subsets which generates this structure.*

Proof. For any connectivity space X, let $\iota(X)$ denote the set of the irreducible connected subsets of X. Then, for any $\mathcal{A} \in \mathcal{Q}_X$ such that $[\mathcal{A}]_0 = \kappa(X)$, one has $\mathcal{A} \supseteq \iota(X)$ since, by construction, each set $C \in [\mathcal{A}]_0$ which is not in $\mathcal{A}$ is reducible. On the other hand, an easy induction shows that, for every integer k, every reducible connected subset of X with cardinal smaller than k is an element of $[\iota(X)]_0$. Thus, if X is finite, $\kappa(X) = [\iota(X)]_0$.

□

2.3 Connectivity Spaces and Hypergraphs

A hypergraph is a set of vertices endowed with a set of nonempty sets of vertices, these sets of vertices being considered as generalized edges, the so-called *hyperedges*. There is some similarity between hypergraphs and connectivity spaces — for example it is possible to consider Borromean structures in both cases — but

- the union of two hyperedges with a nonempty intersection is not necessarily an hyperedge, so hyperedges are not the same as connected subsets,
- the union of two hyperedges with a nonempty intersection can be an hyperedge, so hyperedges are not the same as irreducible connected subsets.

To clarify the relation between the two concepts, let us consider the category **HypG** of hypergraphs, that is the category whose objects are the pairs $(X, \mathcal{H})$ with X a set and $\mathcal{H} \in \mathcal{Q}_X$ a set whose elements are called hyperedges, and whose morphisms $f : (X, \mathcal{H}) \to (X', \mathcal{H}')$ are functions $X \to X'$ which preserve hyperedges : $H \in \mathcal{H} \Rightarrow f(H) \in \mathcal{H}'$. Then the proposition 2 implies

Corollary 6. *The category* **Cnc** *is concrete on* **HypG** *with a forgetful functor admiting as a left adjoint the functor* **HypG** $\to$ **Cnc** *which associates with each hypergraph* $(X, \mathcal{H})$ *the space whose connectivity structure is generated by* $\mathcal{H}$*, i.e.* $(X, [\mathcal{H}]_0)$*, and with each morphism itself as a connectivity morphism. Similarly, the generation of integral connectivity structures* $[\mathcal{H}]$ *from sets* $\mathcal{H} \in \mathcal{Q}_X$ *defines a left adjoint to the forgetful functor* **Cnct** $\to$ **HypG***, and the situation is the same between finite hypergraphs and finite connectivity spaces.*

3 Limits and Colimits

3.1 Categories with Lattices of Structures

Let **JCPos** be the category of complete (small) lattices and join-preserving maps. If S is a functor from a category $\mathbf{X}$ to **JCPos**, $S(X)$ or S_X will denote the lattice associated by S with an object X, and (while it is unambigous) $f_!$ the map between lattices associated by S with a morphism f. The elements of the lattice S_X will be called the S-structures on X.

Definition 5. *Let* $\mathbf{X}$ *be a category and* $S : \mathbf{X} \to \mathbf{JCPos}$ *a functor. The category* $\mathbf{X}_S$*, which we shall refer to as* the category with lattices of structures associated with S *or more briefly as* the category structured by S, *is defined as follows. Its objects are pairs* $(X, \mathcal{X})$ *with* X *an object of* $\mathbf{X}$ *and* $\mathcal{X} \in S_X$ *an* S*-structure. A morphism* $f : (X, \mathcal{X}) \to (Y, \mathcal{Y})$ *is an* $\mathbf{X}$*-morphism* $f : X \to Y$ *such that* $f_!(\mathcal{X}) \leq \mathcal{Y}$ *in the lattice* S_Y.

In the category $\mathbf{X}_S$, spaces $(X, 1_{S_X})$ are called *indiscrete* spaces, and spaces $(X, 0_{S_X})$ are called *discrete* spaces. If, in the lattice S_X, we have $\mathcal{X} \leq \mathcal{X}'$, then the structure $\mathcal{X}$ is said to be *finer* than $\mathcal{X}'$ and the latter is said to be *coarser* than the former.

Remark 7. An equivalent definition is given by considering *contravariant* functors from the basis category $\mathbf{X}$ to the category **MCPos** of complete (small) lattices and meet-preserving maps: an object of the category defined by such a functor T is a pair $(X, \mathcal{X})$ with $\mathcal{X} \in T_X$, and a morphism $f : (X, \mathcal{X}) \to (Y, \mathcal{Y})$ is a $\mathbf{X}$-morphism $f : X \to Y$ such that $\mathcal{X} \leq f^*(\mathcal{Y})$, where $f^* = T(f)$. Then for each covariant $S : \mathbf{X} \to \mathbf{JCPos}$, there is an *associated contravariant functor* T defining the category $\mathbf{X}_S$ in this way. This functor T is defined on objects X by $T_X = S_X$ and on $\mathbf{X}$-morphisms $f : X \to Y$ by $T(f) = f^*$ with, for each $\mathcal{Y} \in T_Y$,

$$f^*\mathcal{Y} = \bigvee\{\mathcal{X} \in T_X, f_!\mathcal{X} \leq \mathcal{Y}\}.$$

In the next proposition, we use the definition of a topological category given in [1] : a topological category on $\mathbf{X}$ is a concrete category $U : \mathbf{A} \to \mathbf{X}$ (that is, a faithful functor U), such that every U-source $(X \to UA_i)_{i\in I}$ in $\mathbf{X}$ has a unique U-initial lift $(A \to A_i)_{i\in I}$ in $\mathbf{A}$.

Proposition 7. *A category is a small-fibred topological one if and only if it is a category with lattices of structures. More precisely :*

- *For each functor $S : \mathbf{X} \to \mathbf{JCPos}$, the functor $U : \mathbf{X}_S \to \mathbf{X}$ defined by $U(X, \mathcal{X}) = X$ and $Uf = f$ is a small-fibred topological category.*
- *Each small-fibred topological category $U : \mathbf{A} \to \mathbf{X}$ is isomorphic to the category $\mathbf{X}_S$ with S the functor defined for each object X of $\mathbf{X}$ by the fibre $S_X = \{A \in \mathbf{A}, UA = X\}$ with the usual order (i.e. $A_1 \leq A_2$ iff id_X has a lift $A_1 \to A_2$), and for each arrow $f : X \to Y$ in $\mathbf{X}$ and each $A \in S_X$ by $f_!(A) = \wedge\{B \in S_Y, f$ has lift $A \to B\}$.*

Proof. Let $S : \mathbf{X} \to \mathbf{JCPos}$ be any functor. The functor $U : \mathbf{X}_S \to \mathbf{X}$ defined by $U(f : (X, \mathcal{X}) \to (Y, \mathcal{Y})) = (f : X \to Y)$ is trivially faithful, its fibres are the sets S_X, and it is topological : each U-source $(f_i : X \to UA_i)_{i \in I}$ has a unique U-initial lift, that is $(f_i : (X, \mathcal{X}_0) \to A_i)_{i \in I}$, where $\mathcal{X}_0$ is the coarsest S-structure on X such that all f_i be (have lifts as) $\mathbf{X}_S$-morphisms, that is $\mathcal{X}_0 = \wedge_i f_i^*(\mathcal{Y}_i)$ where $\mathcal{Y}_i$ is the S-structure of A_i and, for each $f : X \to Y$ and each $\mathcal{Y} \in S_Y, f^*(\mathcal{Y})$ is the coarsest S-structure $\mathcal{X}$ on X such that f is an $\mathbf{X}_S$-morphism $(X, \mathcal{X}) \to (Y, \mathcal{Y})$, that is $f^*(\mathcal{Y}) = \vee\{\mathcal{X} \in S_X, f_!(\mathcal{X}) \leq \mathcal{Y}\}$.

On the other hand, let now $U : \mathbf{A} \to \mathbf{X}$ be a topological category with small fibres. One knows (see [1]) that such fibres S_X are then complete lattices. We can remark also that, for a given $f : X \to Y$ in $\mathbf{X}$ and an object $A \in S_X$, the set $\{B \in S_Y, f$ has a lift $A \to B\}$ is nonempty, because Y has an indiscrete lift. Then $f_!$ is well-defined as a function. Now, if $(A_i)_{i \in I}$ is any family in the fibre S_X, and $B \in S_Y$ is such that $f : X \to Y$ has a lift $\vee_i A_i \to B$, then id_X has a lift $A_i \to \vee_i A_i$ for each i, so f has a lift $A_i \to B$ for each i. On the other hand, if $f : X \to UB$ has a lift $A_i \to B$ for each i, then $\forall i \in I, A_i \leq A$, where the U-initial lift of f is $A \to B$; but $\vee_i A_i \leq A$, so id_X has a lift $\vee_i A_i \to A$ and f has a lift $\vee_i A_i \to B$. Thus, for a given $f : X \to Y$ and a given family $(A_i)_{i \in I}$ in S_X, we have $\{B \in S_Y, f$ has a lift $\vee_i A_i \to B\} = \{B \in S_Y, \forall i \in I, f$ has a lift $A_i \to B\}$. Let $\beta_i = f_!(A_i) = \wedge\{B \in S_Y, f$ has a lift $A_i \to B\}$. Then

$$f_!(\vee_i A_i) = \wedge\{B \in S_Y, \forall i \in I, f \text{ has a lift } A_i \to B\}$$

$$= \wedge\{B \in S_Y, \forall i \in I, B \geq \beta_i\} = \vee_i \beta_i,$$

so $f_!(\vee_i A_i) = \vee_i f_!(A_i)$: $f_!$ is a **JCPos**-morphism, and the functor S is well-defined. It is then easy to verify that the functor $\mathbf{A} \to \mathbf{X}_S$ defined by

$$(f : A \to B) \mapsto (Uf : (UA, A) \to (UB, B))$$

is an isomorphism of categories, with inverse

$$(f : (X, \mathcal{X}) \to (Y, \mathcal{Y})) \mapsto (\tilde{f} : \mathcal{X} \to \mathcal{Y}),$$

where $\tilde{f}$ is the lift of f, which exists since $f_!(\mathcal{X}) \leq \mathcal{Y}$.

□

By proposition 21.15, theorem 21.16 and corollary 21.17 of [1], we have then

Corollary 8. *If* $\mathbf{X}$ *denotes the category* **Set** *of sets (resp. the category* **fSet** *of finite sets),* $S : \mathbf{X} \to \mathbf{JCPos}$ *any functor,* $T : \mathbf{X}^{op} \to \mathbf{MCPos}$ *the contravariant functor associated with* S *and* $U : \mathbf{A} = \mathbf{X}_S \to \mathbf{X}$ *the construct*[4] *(resp. "finite" construct) defined by* S*, then the following hold*

1. $\mathbf{A}$ *is (co)complete (resp. finitely (co)complete),*
2. U *has a left adjoint* O *(the discrete structure) and a right adjoint* I *(the indiscrete structure) :* $O \dashv U \dashv I$*, so* U *preserves (co)limits,*
3. *the limit* $(l_i : L \to D_i)_{i \in \mathbf{I}}$ *of a small (resp. finite) diagram* $D : \mathbf{I} \to \mathbf{A}$ *is the initial lift of the underlying limit in* $\mathbf{X}$*, that is: if* $(l_i : |L| \to UD_i)_{i \in \mathbf{I}}$ *is the limit of* UD*, then* $L = (|L|, \bigwedge_{i \in \mathbf{I}} l_i^*(\mathcal{X}_i))$*, where* $D_i = (X_i, \mathcal{X}_i)$ *and* $l_i^* = T(l_i)$*,*
4. *colimits are given in the same way, as final lifts: if* $(c_i : |C| \leftarrow UD_i)_{i \in \mathbf{I}}$ *is the colimit of* UD*, then the colimit of* D *in* $\mathbf{A}$ *is* $(c_i : C \leftarrow D_i)_{i \in \mathbf{I}}$ *with* $C = (|C|, \bigvee_{i \in \mathbf{I}} c_{i!}(\mathcal{X}_i))$*, where* $D_i = (X_i, \mathcal{X}_i)$ *and* $c_{i!} = S(c_i)$*,*

[4]See *supra* the note 2.

5. **A** *is wellpowered and cowellpowered,*

6. **A** *is an* $(Epi, ExtremalMonoSource)$*-category,*

7. **A** *has regular factorizations, i.e. is an* $(RegEpi, MonoSource)$*-category (and thus is, in particular, a* $(RegEpi, Mono)$*-category),*

8. *in* **A***, the classes of embeddings (i.e. initial monomorphisms), of extremal monomorphisms and of regular monomorphisms coincide,*

9. *in* **A***, the classes of quotient morphisms (i.e. final epimorphisms), of extremal epimorphisms and of regular epimorphisms coincide,*

10. **A** *has separators and coseparators.*

Example 15. Let $\mathcal{P} : \mathbf{Set} \to \mathbf{JCPos}$ be the (covariant) functor which associates with each set the complete lattice of its subsets. For any functor $T : \mathbf{X} \to \mathbf{Set}$, the category $\mathbf{X}_{\mathcal{P}T}$ structured by the functor $\mathcal{P} \circ T : \mathbf{X} \to \mathbf{JCPos}$ coincides with the topological category $\mathbf{Spa}(T)$ of T-spaces on $\mathbf{X}$ ([1], p. 76). Thus, the "functor-structured categories" $\mathbf{Spa}(T)$ are special cases of the categories structured by functors $\mathbf{X} \to \mathbf{JCPos}$. In particular, for $T = \mathcal{P}$, we obtain $\mathbf{Spa}(\mathcal{P}) = \mathbf{Set}_{\mathcal{Q}} = \mathbf{HypG}$.

3.2 (Co)limits in the Categories of Connectivity spaces

In [2], Börger showed that

Proposition 9. Cnct *is a topological category. It is not cartesian closed.*

It is easy to check that, as a category with lattice of structures, **Cnct** is defined by the covariant functor $Cnct : \mathbf{Set} \to \mathbf{JCPos}$ such that $Cnct_X$ is the lattice of all integral connectivity structures on X and, for every $f : X \to X'$, $Cnct(f) = f_!$ is the **JCPos**-morphism $Cnct_X \to Cnct_{X'}$ such that, for all $\mathcal{K} \in Cnct_X$,

$$f_!(\mathcal{K}) = [\{f(K), K \in \mathcal{K}\}]. \tag{1}$$

Equivalently, the contravariant definition of **Cnct** is given, for all $\mathcal{K}' \in Cnct_{X'}$, by

$$f^*(\mathcal{K}') = \{K \in \mathcal{P}_X, f(K) \in \mathcal{K}'\}. \tag{2}$$

The same formulas hold on **fSet**, defining a functor $fCnct$ such that $\mathbf{fCnct} = \mathbf{fSet}_{fCnct}$, which is thus a topological category on **fSet**. For **Cnc**, it suffices to use $[\{f(K), K \in \mathcal{K}\}]_0$ instead of $[\{f(K), K \in \mathcal{K}\}]$ in the expression of $f_!$ to define a functor Cnc such that $\mathbf{Cnc} = \mathbf{Set}_{Cnc}$, which is thus a topological construct[5].

From the formula (1) and the corollary 8, we deduce that the connectivity structure $\kappa(C)$ of the colimit C of a small diagram $D : \mathbf{I} \to \mathbf{Cnct}$ is given by $\kappa(C) = \bigvee_{i \in \mathbf{I}} [\{c_i(K), K \in \kappa(D_i)\}]$ and then

$$\kappa(C) = [\{c_i(K), i \in \mathbf{I}, K \in \kappa(D_i)\}], \tag{3}$$

where the $c_i : |D_i| \to |C|$ are the coprojections. The same formula holds for colimits of finite diagrams in **fCnct**, and, using $[-]_0$ instead of $[-]$, for small diagrams in **Cnc**.

From the formula (2), one likewise deduces the connectivity structure $\kappa(L)$ of the limit L of a small diagram $D : \mathbf{I} \to \mathbf{Cnct}$,

$$\kappa(L) = \bigcap_{i \in \mathbf{I}} \{K \in \mathcal{P}_{|L|}, l_i(K) \in \kappa(D_i)\}, \tag{4}$$

where the $l_i : |L| \to |D_i|$ are the projections. The same formula holds for limits of small diagrams in **Cnc** and of finite diagrams in **fCnct**.

For example, the cartesian product $C_1 \times C_2$ of two connectivity spaces is characterised by $|C_1 \times C_2| = |C_1| \times |C_2|$ and

$$\kappa(C_1 \times C_2) = \{A \in \mathcal{P}(|C_1| \times |C_2|), \pi_i(A) \in \kappa(C_i) \text{ for } i \in \{1, 2\}\},$$

where the π_i are the projections, whereas the coproduct, or disjoint union, satisfies $|C_1 \amalg C_2| = |C_1| \amalg |C_2|$ and $\kappa(C_1 \amalg C_2) = \kappa(C_1) \amalg \kappa(C_2)$.

With those formulas, it is easy to check that none of the three categories considered here is cartesian closed. It suffices to exhibit a colimit

[5] **Cnc** is not *well-fibred*, so it is not a topological category according to the definition given in 1983 by Herrlich [10], but, as we said, we use here the less restrictive definition finally retained by Herrlich, Adámek and Strecker in [1].

which is not preserved by a product, and this can be done simultaneously in the three categories. For example, let $\{a, *, b\}$ be a set with three distinct elements, A_u be the indiscrete connectivity space defined for each $u \in \{a, b\}$ by its carrier $|A_u| = \{*, u\}$, and B the space with carrier $\{1, 2, 3\}$ and with structure $[\{\{1, 2\}, \{2, 3\}\}]$. Then, in each of the categories concerned, the colimit C of the diagram $A_a \hookleftarrow \{*\} \hookrightarrow A_b$ (with arrows the inclusions) is $C = (\{a, *, b\}, [\{\{a, *\}, \{*, b\}\}])$, its product $C \times B$ with B is the cartesian product $\{a, *, b\} \times \{1, 2, 3\}$ endowed with the integral connectivity structure including all subsets having their two projections connected. For example, the set $\{(a, 1), (*, 3), (b, 2)\}$ is connected in $C \times B$; but it is easy to verify that the same set is not connected in the colimit of the diagram $A_a \times B \hookleftarrow \{*\} \times B \hookrightarrow A_b \times B$. Thus, in each of the categories considered, the endofunctor $- \times B$ does not preserve colimits. We thus proved

Proposition 10. **Cnc** *and* **fCnct** *are topological categories; they are not cartesian closed.*

3.3 Quotients and Embeddings

This section gives trivial but useful consequences of the corollary 8 and of the formulas (1) and (2).

Proposition 11. *In* **Cnct** *and* **fCnct** *(resp.* **Cnc***), a morphism* $f : A \to B$ *is a* regular epimorphism *iff* $|f|$ *is surjective and* $\kappa(B) = [f(\kappa(A))]$ *(resp.* $\kappa(B) = [f(\kappa(A))]_0$*). In* **fCnct***,* **fCnct** *and* **Cnc***, a morphism* $f : A \to B$ *is a* regular monomorphism *iff* $|f|$ *is injective and* $\kappa(A) = \{K \in \mathcal{P}_{|A|}, f(K) \in \kappa(B)\}$.

Now, in every topological construct, a regular epimorphism, *i.e.* a coequalizer, is the same as a quotient morphism, *i.e.* a final morphism which is surjective as a function, and can also be viewed as (the unique final lift of) the canonical map associated with an equivalence relation. This remark results in the definition of the quotient of a connectivity space by an equivalence relation.

Definition 6 (Quotient by an equivalence relation)**.** *If* C *is a connectivity space and* $\sim$ *is an equivalence relation on* $|C|$*, the* quotient space

$C/\sim$ is defined by $|C/\sim| = |C|/\sim$ and

$$\kappa(C/\sim) = s_!(\kappa(C)) = [s(\kappa(C))]_0 \tag{5}$$

where s is the canonical map $s : |C| \twoheadrightarrow |C|/\sim$. In particular, if T is a subset of $|C|$, C/T denotes the space $C/\sim_T$.

Remark 8. Note that if C is an integral connectivity space, then for any surjective map $s : |C| \twoheadrightarrow Y$ we have $[s(\kappa(C))]_0 = [s(\kappa(C))]$.

Likewise, in every topological construct, a regular monomorphism, *i.e.* an equalizer, is the same as an embedding, *i.e.* an initial morphism which is injective as a function, and can also be viewed as (the unique initial lift of) the inclusion map of a subspace. This leads to the definition of the connectivity structure induced by a connectivity space on a subset of its carrier.

Definition 7 (Structure induced on a subset)**.** *If C is a connectivity space and S is a subset of $|C|$, the* connectivity space induced *on S by C is the space $C_{|S}$ defined by $|C_{|S}| = S$ and*

$$\kappa(C_{|S}) = i^*(\kappa(C)) = \mathcal{P}_S \cap \kappa(C) \tag{6}$$

where i is the inclusion map $i : S \hookrightarrow |C|$.

4 Tensor Product of Connectivity Spaces

The formula (4) suggests that the cartesian product of connectivity spaces is in some way "too coarse" to be really useful in algebra. For example, let $\mathbf{N}$ be the set of natural numbers with the integral connectivity structure generated by the subsets $\{n, n+1\}$; it is easy to check that the addition $+ : \mathbf{N}^2 \to \mathbf{N}$ is not a connectivity morphism (when $\mathbf{N}^2$ is endowed with the cartesian square structure of $\mathbf{N}$). Likewise for the addition of real numbers. This section presents a more interesting connectivity product than the cartesian one for algebraic structures.

Let X_i $(i = 1, 2)$ and Y be connectivity spaces. For each $x_1 \in |X_1|$ (resp. $x_2 \in |X_2|$), we denote by $f(x_1, -)$ (resp. $f(-, x_2)$) the partial function associated with a given function $f : |X_1| \times |X_2| \to |Y|$.

Definition 8. *A function* $f : |X_1| \times |X_2| \to |Y|$ *is said to be* partially connecting *from* $X_1 \times X_2$ *to* Y *if* $f(x_1, -) : X_2 \to Y$ *and* $f(-, x_2) : X_1 \to Y$ *are connectivity morphisms for all* $x_1 \in |X_1|$ *and all* $x_2 \in |X_2|$.

Definition 9. *The* connectivity tensor product $X_1 \boxtimes X_2$ *of two connectivity spaces* X_i *(*$i = 1, 2$*) is the space with carrier* $|X_1 \boxtimes X_2| = |X_1| \times |X_2|$ *and with connectivity structure* $\kappa(X_1 \boxtimes X_2) = [\{K_1 \times K_2, (K_1, K_2) \in \kappa(X_1) \times \kappa(X_2)\}]_0$.

For every connectivity space X_i, $\kappa(X_1 \boxtimes X_2)$ is a finer connectivity structure on the set $|X_1| \times |X_2|$ than the one given by the connectivity cartesian product, since $K_1 \times K_2 \in \kappa(X_1 \times X_2)$ for each connected subsets K_1 and K_2. Thus, $id : X_1 \boxtimes X_2 \to X_1 \times X_2$ is a bijective connectivity morphism (but it is of course not an isomorphism in general). If X_1 and X_2 are *integral* connectivity spaces, then its inverse function, that is the function from $X_1 \times X_2$ to $X_1 \boxtimes X_2$ defined by $\tau(x_1, x_2) = (x_1, x_2)$, is a partially connecting function.

Theorem 12. *Let* X_1 *and* X_2 *be integral connectivity spaces,* Y *a connectivity space, and* $f : |X_1| \times |X_2| \to |Y|$ *a function. Then* f *is a partially connecting function from* $X_1 \times X_2$ *to* Y *if and only if it is a connectivity morphism from* $X_1 \boxtimes X_2$ *to* Y*,* i.e. *there exists a unique connectivity morphism* $\tilde{f} : X_1 \boxtimes X_2 \to Y$ *such that* $\tilde{f} \circ \tau = f$*.*

Proof. If $\tilde{f}$ is a connectivity morphism, then $\tilde{f} \circ \tau = f$ is a partially connecting function since τ is such a function. On the other hand, let f be a partially connecting function from $X_1 \times X_2$ to Y. Unicity of $\tilde{f}$ being obvious, since necessarily $\tilde{f}(x_1, x_2) = f(x_1, x_2)$, it suffices to check that this function is a connectivity morphism on $X_1 \boxtimes X_2$. Then, according to the proposition 2, it suffices to check that for every $K_i \in \kappa(X_i)$, $f(K_1 \times K_2) \in \kappa(Y)$. Let $K_1 \times K_2$ be such nonempty subset of $|X_1| \times |X_2|$, and let $x_1^0 \in K_1$. f being partially connecting, the sets $V = \{f(x_1^0, x_2), x_2 \in K_2\}$ and $H_{x_2} = \{f(x_1, x_2), x_1 \in K_1\}$ are, for all $x_2 \in K_2$, in $\kappa(Y)$. So are the sets $V \cup H_{x_2}$ (as $V \cap H_{x_2} \neq \emptyset$), and $\bigcup_{x_2 \in K_2} (V \cup H_{x_2})$; that is: $\tilde{f}(K_1 \times K_2) \in \kappa(Y)$.

□

Example 16. Let $f : \mathbf{R}_+^2 \to \mathbf{R}$ defined by

- $f(0,0) = 0$,
- for all x and y, $f(x,y) = f(y,x)$,
- $\forall x > 0, \forall y \in [0,x], f(x,y) = y/x$.

Then f is a partially connecting map since it is "partially continuous", but it is not continuous, and neither $\Delta = \{(x,x), x \geq 0\}$ nor $f(\Delta) = \{0,1\}$ are connected subsets of, respectively, $\mathbf{R}_+ \boxtimes \mathbf{R}_+$ and $\mathbf{R}$.

Note that for each integral connectivity space X, one has an endofunctor $X \boxtimes - : \mathbf{Cnct} \to \mathbf{Cnct}$ defined for each integral connectivity space Y by $X \boxtimes Y$ and for each connectivity morphism $g : Y_1 \to Y_2$ between integral connectivity spaces by $(X \boxtimes g)(x, y_1) = (x, g(y_1))$.

Now, let us define another endofunctor on **Cnct**. For every subset M of the set $Hom(X,Y)$ of connectivity morphisms from a connectivity space X to a connectivity space Y, and for every subset A of the set $|X|$, let $\langle M, A\rangle$ denotes $\bigcup_{f\in M} f(A)$. Then, for each integral connectivity space X, there is an endofunctor $\mathbf{Cnct}(X,-) : \mathbf{Cnct} \to \mathbf{Cnct}$ defined for every integral connectivity space Y by

- $|\mathbf{Cnct}(X,Y)| = Hom(X,Y)$,
- $\kappa(\mathbf{Cnct}(X,Y)) = \{M \in \mathcal{P}(Hom(X,Y)), \forall K \in \kappa(X), \langle M,K\rangle \in \kappa(Y)\}$,

and for every connectivity morphism $g : Y_1 \to Y_2$ by $\mathbf{Cnct}(X,g) = g_*$ such that

$$\forall \varphi \in \mathbf{Cnct}(X,Y_1), g_*(\varphi) = g \circ \varphi.$$

Remark 9. A set M of connectivity morphisms between two integral connectivity spaces X and Y is connected, that is belongs to $\kappa(\mathbf{Cnct}(X,Y))$, if (and only if) for all $x \in X$, $\langle M, \{x\}\rangle \in \kappa(Y)$. Indeed, if this condition is satisfied, then for every nonempty connected subset K of X and any $x \in K$, one has $\langle M,K\rangle = \bigcup_{f\in M}(f(K) \cup \langle M,\{x\}\rangle) \in \kappa(Y)$.

Theorem 13. *For every integral connectivity space X, the endofunctor $X \boxtimes -$ is left adjoint to the endofunctor* $\mathbf{Cnct}(X,-)$. *Thus,* $(\mathbf{Cnct}, \boxtimes)$ *is a closed symmetric monoidal category.*

Proof. The product $\boxtimes$ is obviously symmetric. Let X, Y and Z be integral connectivity spaces. For every connectivity morphism $\psi : X \boxtimes Y \to Z$, one has a morphism $\rho(\psi) : Y \to \mathbf{Cnct}(X, Z)$ defined for all $y \in Y$ by $\rho(\psi)(y) = \psi(-, y)$. Then ρ is clearly a bijection between the sets $Hom(X \boxtimes Y, Z)$ and $Hom(Y, \mathbf{Cnct}(X, Z))$, and it is natural since for all integral connectivity spaces Y, Y', Z and Z' and for all connectivity morphisms $u : Y \to Y'$, $v : Z \to Z'$ and $\psi : X \boxtimes Y' \to Z$, one has $\rho(v \circ \psi \circ (X \boxtimes u)) = \rho((x, y) \mapsto v(\psi(x, u(y)))) = (y \mapsto v \circ \psi(-, u(y)) = \mathbf{Cnct}(X, v) \circ \rho(\psi) \circ u$.

□

5 Homotopy

Let $\overrightarrow{I}$ be a triple $(I, 0, 1)$ with I a nonempty integral connectivity space, and 0 and 1 some elements of $|I|$. In particular, let $\mathbf{I}$ be the connectivity space associated with the usual topological space $[0, 1]$, and $\overrightarrow{\mathbf{I}} = (\mathbf{I}, 0, 1)$.

Definition 10 (Homotopy)**.** *Let X and Y be integral connectivity spaces, and $f, g : X \to Y$ some connectivity morphisms. The function g is said to be $\overrightarrow{I}$-homotopic to f provided there exists a connectivity morphism*

$$h : I \to \mathbf{Cnct}(X, Y)$$

such that $h(0) = f$ and $h(1) = g$. In particular, in the case of $\overrightarrow{I} = \overrightarrow{\mathbf{I}}$, g is simply said to be homotopic to f.

We denote by $f \sim g$ the homotopy relation between connectivity morphisms. Like in the topological case, it is obviously an equivalence relation. The adjoint situation $(X \boxtimes -) \dashv \mathbf{Cnct}(X, -)$ leads to an alternative definition of homotopy for connectivity morphisms.

Definition 11 (Alternative definition of homotopy)**.** *Let X and Y be integral connectivity spaces. A function $g : X \to Y$ is $\overrightarrow{I}$-homotopic to $f : X \to Y$ provided there exists a connectivity morphism $h : I \boxtimes X \to Y$ such that $h(0, -) = f$ and $h(1, -) = g$, that is a function $h : I \times X \to Y$ such that*

- $h(0,-) = f$ *and* $h(1,-) = g$,
- $\forall t \in I, \forall K \in \kappa(X), h(t,K) \in \kappa(Y)$,
- $\forall D \in \kappa(I), \forall x \in X, h(D,x) \in \kappa(Y)$.

Definition 12 (Contractibility)**.** *An integral connectivity space X is said to be* contractible *provided the identity map $id : X \to X$ of the space be homotopic to a constant map $c : X \to X$.*

Examples. The connectivity space associated with the usual topological circle $S^1 = \{e^{i\theta}, \theta \in [0, 2\pi]\} \subset \mathbf{C}$ is contractible. Indeed, the function $h : \mathbf{I} \times S^1 \to S^1$ defined by

- for $t \in [0,1[$ and $z \in S^1$, $h(t,z) = z.e^{i\frac{t}{1-t}}$,
- $\forall z \in S^1, h(1,z) = 1$,

realizes an homotopy between the identity of the circle and the constant function $z \mapsto 1 \in S^1$.

More generally, the same kind of argument shows that every n-sphere is contractible. On the other hand, there exist a connected connectivity space X such that no two distinct connectivity endomorphisms $X \to X$ are homotopic. For example, if $X = \mathcal{P}(\mathbf{R})$ is endowed with the integral connectivity structure for which non trivial connected subsets are subsets with a cardinal greater than the one of $\mathbf{R}$, then non-trivial connected subsets of $\mathbf{Cnct}(X,X)$ also have such a cardinal, and then every connectivity morphism from $\mathbf{I}$ to $\mathbf{Cnct}(X,X)$ is a constant function.

Those examples show that any theory of homotopy in the connectivity framework should be very different from the topological one. In particular, it could be interesting to use different kinds of discrete times instead of $\mathbf{I}$.

6 Pointed Connectivity Spaces

6.1 Pointed Sets

The category **pSet** of pointed sets and based maps is a concrete category on **Set**. The forgetful functor $\mathbf{pSet} \to \mathbf{Set}$ will be denoted by $|-|$, and the base-point of a pointed set P by $\beta(P)$, so $P = (|P|, \beta(P))$.

pSet has a zero object, $(\{*\}, *)$, it is complete and cocomplete. In particular, the cartesian product of two pointed sets P_1 and P_2 is defined by $|P_1 \times P_2| = |P_1| \times |P_2|$ and $\beta(P_1 \times P_2) = (\beta(P_1), \beta(P_2))$. The class of coequalizers coincides with the class of all epimorphisms, *i.e.* surjective based maps, and with the class of quotient morphisms (in **pSet** every morphism is final). If $\sim$ is an equivalence relation on $|P|$, the quotient pointed set $P/\sim$ is defined by $|P/\sim| = |P|/\sim$ and $\beta(P/\sim) = \widetilde{\beta(P)}$. In particular, if T is a subset of $|P|$, P/T denotes the pointed set $P/\sim_T$. The coproduct of P_1 and P_2 is denoted by $P_1 \vee P_2$. It can be defined either as the quotient of the set $|P_1| \amalg |P_2|$ by the equivalence relation which identifies $\beta(P_1)$ and $\beta(P_2)$ or alternatively by the formulas

$$|P_1 \vee P_2| = (|P_1| \times \{\beta(P_2)\}) \cup (\{\beta(P_1)\} \times |P_2|) \tag{7}$$

and

$$\beta(P_1 \vee P_2) = (\beta(P_1), \beta(P_2)).$$

The category **pSet** is not cartesian closed since, for example, if P is a pointed set with two elements and Q is the zero object, then $P \times (Q \vee Q) \simeq P$ whereas $(P \times Q) \vee (P \times Q)$ has three elements. Nevertheless, the set of based maps from a pointed set P to a pointed set Q has a "natural" special point, that is the constant map $x \mapsto \beta(Q)$, so there is a "natural" object in **pSet** representing $Hom(P, Q)$. Let

$$\mathbf{pSet}(P, Q) = (Hom(P, Q), x \mapsto \beta(Q))$$

denotes this object. For each pointed set P, we then have an endofunctor $\mathbf{pSet}(P, -)$ on **pSet**, with $\mathbf{pSet}(P, f) = f \circ -$. One knows that this functor has a left adjoint $P \wedge -$, the so-called *smash product*, defined on objects by

$$P \wedge Q = (P \times Q)/|P \vee Q|,$$

where the set $|P \vee Q|$ is defined by the formula (7), and on based maps $f : Q \to R$ by

$$\forall (p, q) \in |P| \times |Q|, (P \wedge f)(\widetilde{(p, q)}) = \widetilde{(p, f(q))}. \tag{8}$$

Then, endowed with the smash product, **pSet** is a closed symmetric monoidal category. Note that there are no projections associated with the smash product, and that the two-elements pointed set is a unit for it.

6.2 Pointed Integral Connectivity Spaces

Definition 13. *A* pointed integral connectivity space X *is a triple* $(S, \mathcal{K}, b)$*, where* $(S, \mathcal{K})$ *is an integral connectivity space and* b *a point of* S*, called the base-point of* X*.*

For every pointed connectivity space X, we will denote $|X|$ its underlying carrier set, $\kappa(X)$ its connectivity structure and $\beta(X)$ its base-point, so $X = (|X|, \kappa(X), \beta(X))$.

The category whose objects are the pointed integral connectivity spaces and whose morphisms are connectivity morphisms preserving base-points will be denoted by **pCnct**. It can be viewed as a category with lattices of structures on the base category **pSet** of pointed sets. Indeed, the choice of a base-point does not have any effect on the lattice of (integral) connectivity structures on a given set, and connectivity morphisms between pointed spaces are just based maps between underlying pointed sets which preserve connected subsets, so $\mathbf{pCnct} = \mathbf{pSet}_{pCnct}$ with $pCnct = Cnct \circ |-| : \mathbf{pSet} \to \mathbf{JCPos}$. Thus,

Proposition 14. pCnct *is a topological category on* **pSet**. *It is thus complete and cocomplete.*

The topological forgetful functor $\mathbf{pCnct} \to \mathbf{pSet}$ will be denoted $|-|_p$, so that $|X|_p = (|X|, \beta(X))$. The category **pCnct** can also be viewed as a concrete category on **Cnct**, and we will denote $|-|_\kappa$ the corresponding forgetful functor, so that $|X|_\kappa = (|X|, \kappa(X))$. Then, the product of two pointed integral connectivity spaces X_1 and X_2 is characterised by $|X_1 \times X_2|_p = |X_1|_p \times |X_2|_p$ and $|X_1 \times X_2|_\kappa = |X_1|_\kappa \times |X_2|_\kappa$. If $\sim$ is an equivalence relation on $|X|$, the quotient pointed space $X/_\sim$ is likewise characterised by $|X/_\sim|_p = |X|_p/_\sim$ and $|X/_\sim|_\kappa = |X|_\kappa/_\sim$. This gives in particular the definition of X/T with $T \subseteq |X|$. The coproduct satisfies $|X_1 \vee X_2|_p = |X_1|_p \vee |X_2|_p$, and its connectivity part $|X_1 \vee X_2|_\kappa$ can be defined either as the quotient of $|X_1|_\kappa \amalg |X_2|_\kappa$ by the relation $\beta(X_1) \sim \beta(X_2)$, or as induced by the space $|X_1|_\kappa \boxtimes |X_2|_\kappa$ on $|X_1 \vee X_2|$ seen as a subset of $|X_1| \times |X_2|$ according to the formula (7), the X_i replacing there the P_i. In the sequel, the expression $|X_1 \vee X_2|$ will keep this last meaning. Now, the same argument as for **pSet** shows that **pCnct** is not cartesian closed.

6.3 The Smash Product

Definition 14. *Let X_1 and X_2 be pointed integral connectivity spaces. Then,*

- *the* tensor product $X_1 \boxtimes X_2$ *is defined by the relations*
  1. $|X_1 \boxtimes X_2|_p = |X_1|_p \times |X_2|_p$,
  2. $|X_1 \boxtimes X_2|_\kappa = |X_1|_\kappa \boxtimes |X_2|_\kappa$,
- *the* smash product *is defined by* $X_1 \wedge X_2 = (X_1 \boxtimes X_2)/|X_1 \vee X_2|$,
- $\mathbf{pCnct}(X_1, X_2)$, *the* pointed connectivity space of connecting based maps *from X_1 to X_2, is defined by*
  1. $|\mathbf{pCnct}(X_1, X_2)| = |\mathbf{Cnct}(|X_1|_\kappa, |X_2|_\kappa)| \cap |\mathbf{pSet}(|X_1|_p, |X_2|_p)|$,
  2. $\kappa(\mathbf{pCnct}(X_1, X_2)) = i^*(\kappa(\mathbf{Cnct}(|X_1|_\kappa, |X_2|_\kappa)))$, *where i is the inclusion map* $i : |\mathbf{pCnct}(X_1, X_2)| \hookrightarrow |\mathbf{Cnct}(|X_1|_\kappa, |X_2|_\kappa)|$,
  3. $\beta(\mathbf{pCnct}(X_1, X_2))$ *is the constant map* $x \mapsto \beta(X_2)$.

Now, with those objects we can define, for every pointed integral connectivity space X, the endofunctors $\mathbf{pCnct}(X, -)$ and $X \wedge -$ on the category **pCnct**. In fact, for every morphism f, the morphisms $X \wedge f$ and $\mathbf{pCnct}(X, f)$ are given by the same formulas as for the corresponding endofunctors on **pSet**.

Theorem 15. *For every pointed integral connectivity space X, the endofunctor $(X \wedge -)$ on* **pCnct** *is left adjoint to the endofunctor* $\mathbf{pCnct}(X, -)$.

Proof. Let X, Y and Z be pointed integral connectivity spaces. For every based connecting map $\psi : X \wedge Y \to Z$, one has a based connecting map $\rho(\psi) : Y \to \mathbf{pCnct}(X, Z)$ defined for all $y \in Y$ by

$$\rho(\psi)(y) = \psi(\widetilde{(-, y)}).$$

Indeed, for every $y \in Y$, $\psi(\widetilde{(-, y)}) \in \mathbf{pCnct}(X, Z)$ since

- ψ is defined on classes $\widetilde{(x, y)}$, so $\psi(\widetilde{(-, y)})$ is a function from $|X|$ to $|Z|$,

- $\psi(\widetilde{(-,y)})(\beta(X)) = \psi(\beta(X \wedge Y)) = \beta(Z)$,
- for every $K \in \kappa(X)$, $s(K \times \{y\}) \in \kappa(X \wedge Y)$ so $\psi(\widetilde{(-,y)}(K) \in \kappa(Z)$,

where $s : X \boxtimes Y \twoheadrightarrow X \wedge Y$ denotes the canonical map. And the function $y \mapsto \psi(\widetilde{(-,y)})$ is a based connecting map from Y to $\mathbf{pCnct}(X,Z)$, since

- $\psi(\widetilde{(-,\beta(Y)})) = (x \mapsto \beta(Z)) = \beta(\mathbf{pCnct}(X,Z))$,
- for every $L \in \kappa(Y)$, $\{\psi(\widetilde{(-,y)}), y \in L\} \in \kappa(\mathbf{pCnct}(X,Z))$, since for every $x \in |X|$ one has $< \{\psi(\widetilde{(-,y)}), y \in L\}, x >= \psi(\widetilde{(x,L)}) \in \kappa(Z)$.

Now, one verifies as well that the formula

$$\theta(\varphi)(\widetilde{(x,y)}) = \varphi(y)(x)$$

defines a map θ from $Hom(Y, \mathbf{pCnct}(X,Z))$ to $Hom(X \wedge Y, Z)$, and that θ and ρ are inverses of each other. Finally, ρ is natural since for all pointed integral connectivity spaces Y, Y', Z and Z' and for all based connecting maps $u : Y \to Y'$, $v : Z \to Z'$ and $\psi : X \wedge Y' \to Z$, one has $\rho(v \circ \psi \circ (X \wedge u)) = (y \mapsto v \circ \psi(\widetilde{(-,u(y)})))) = \mathbf{pCnct}(X,v) \circ \rho(\psi) \circ u$.

$\square$

7 Finite Integral Connectivity Spaces

7.1 Generic Graphs

Definition 15. *Let X be a finite integral connectivity space. A* generic point *of X is a non-empty irreducible connected subset of X. The* generic graph G_X *of X is the directed graph whose vertices are the generic points of X and such that $g \to h$ is a directed edge of G_X if and only if $g \supsetneq h$ and there is no generic point k such that $g \supsetneq k \supsetneq h$.*

Associated with a partial order, the directed graph G_X is a so-called *directed acyclic graph*, that is a directed graph with no *directed* cycle; note that cycles are allowed in the undirected graph obtained by forgetting orientation of the edges. On the other hand, not every finite acyclic directed graph is a G_X for some finite integral connectivity space X. For example, the directed acyclic graph $a \to b$ is not such a G_X.

Notation. For the sake of simplicity, if G is a directed graph, $a \in G$ will express that a is a vertex of G and $(a \to b) \in G$ will express that $a \to b = (a, b)$ is a directed edge of this graph.

Proposition 16. *A finite integral connectivity space X is characterised, up to isomorphism, by its generic graph G_X (defined up to isomorphism).*

Proof. The space X being integral, every singleton is an irreducible connected subset, and appears in G_X as a sink, *i.e.* a vertex with no outgoing edges. Thus, the carrier $|X|$ of the space is given, up to bijection, by the set of sinks of G_X. Now, the connectivity structure is given by G_X as a consequence of the proposition 5.

□

Proposition 17. *If X is a non-empty finite integral connectivity space, then*

1. *X is connected iff G_X is connected,*
2. *there is a bijection between connected components of X and those of G_X,*
3. *X is irreducible iff G_X has exactly one source, i.e. a vertex with no incoming edges,*
4. *X is distinguished iff there is no triple (a, b, c) of distinct vertices in G_X such that $(a \to b)$ and $(b \leftarrow c)$ are in G_X.*
5. *X is connected and distinguished iff G_X is a directed tree.*

Proof.

1. If there is an arrow $(a \to b)$ in G_X then a and b, as subsets of $|X|$, are contained in the same connected component of X; thus, if G_X is connected then X is also connected. On the other hand, let (C_i) be the family of G_X connected components and, for each i, let $\sigma(C_i)$ be the union of sinks belonging to C_i; then, every connected subset produced at any step of the process described in theorem 3 stays in one of the $\sigma(C_i)$, otherwise there should be two irreducible connected subsets of X contained respectively in two distinct $\sigma(C_i)$ and with a non-empty intersection, which is not possible. Thus, if G_X is not connected, neither is X.

2. The generic graph G_X of the disjoint union X of any finite family of finite spaces X_i is clearly the disjoint union of the G_{X_i}, thus the connected components of any finite space X are the $\sigma(C_i)$ associated with the connected components C_i of G_X.

3. If X is irreducible then $|X|$ is a generic point which contains all other generic points so it is the only source in G_X.

   If G_X has only one source, then each irreducible connected proper subset of X is contained in a larger irreducible subset, so, X being finite and the set of irreducible connected sets being nonempty, $|X|$ is itself an irreducible connected subset.

4. If there is a triple (a, b, c) with $a \neq c$ and $a \to b \leftarrow c$ in G_X, then $a \cup c$ is a reducible connected subset of X which is thus not distinguished.

   If two irreducible connected subsets of X not included one in the other have a common point, then there must exist in G_X a triple of distinct points (a, b, c) with $a \to b \leftarrow c$ in G_X; thus, if G_X does not admit such a triple, then the inductive generation of connected subsets from irreducible ones (theorem 3) will not produce any new connected subsets.

5. The last affirmation is a direct consequence of the others.

□

Definition 16. *Let X be a non-empty finite integral connectivity space. The* order *of any irreducible subset of X is its height as a vertex of the directed acyclic graph G_X (i.e. the length of the longest path from that vertex to a sink of G_X). The order $\omega(X)$ of X is the maximum of orders of its irreducible connected subsets, that is the length of G_X.*

Example 17. A finite space of order 0 is totally disconnected, *i.e.* its structure is the discrete one.

Example 18. One has $\omega(U_G(S)) \leq 1$ for any finite simple undirected graph S.

The definition of the order of a finite integral connectivity space results in the definition of a new numerical invariant for links:

Definition 17. *The connectivity order of a tame link L in $\mathbf{R}^3$ (or $\mathbf{S}^3$) is $\omega(L) = \omega(S_L)$.*

Example 19. The connectivity order of the Borromean link or, more generally, of any Brunnian link, is $\omega(\mathbf{B}_n) = 1$.

Remark 10. The connectivity order is not a Vassiliev finite type invariant for links. For example, it is easy to check that the connectivity order of the *singular* link with two components, a circle and another component crossing this circle at $2n$ double-points, is greater than 2^n.

Proposition 18. *One has $\omega(X) \leq \mathrm{card}(X) - 1$ for every finite integral space X; and the integral connectivity space $\mathbf{V}_n$ defined by $|\mathbf{V}_n| = n$ and $\kappa(\mathbf{V}_n)^\bullet = \{2, 3, \cdots, n\}$ is, up to isomorphism, the only integral connectivity space such that $\mathrm{card}(\mathbf{V}_n) = n$ and $\omega(\mathbf{V}_n) = n - 1$.*

Proof. A trivial induction results in the first claim. The second one is obvious if $n = 1$. Suppose that it is true for an integer n, and let X be an integral connectivity space with $n+1$ points and with order n. Then there must exist an irreducible connected subset K of X with order $n-1$, and one has necessarily $\mathrm{card}(K) \geq n$, so $\mathrm{card}(K) = n$. By induction, $K \simeq \mathbf{V}_n$. Let x be the unique element of $X \setminus K$. $|X|$ is necessarily the only non-trivial connected subset which contains x, otherwise X would be of order smaller than n, then $\kappa(X) = \{\{x\}\} \cup \kappa(K) \cup \{|X|\}$, and thus $X \simeq \mathbf{V}_{n+1}$.

□

Let us now describe two ways to produce finite spaces from two given non-empty finite integral connectivity spaces X and Y, Y being supposed irreducible.

1. Let x be a point of $|X|$. We denote by $X \rhd_x Y$ the connectivity space whose generic graph is obtained by replacing in G_X the sink $\{x\}$ by (a copy of) G_Y, arrows to x in G_X being replaced by arrows to the unique source of (the copy of) G_Y. In other words, $X \rhd_x Y$ is the integral space such that $|X \rhd_x Y| = |X| \setminus \{x\} \cup |Y'|$ and the set $\kappa_0(X \rhd_x Y)$ of irreducible connected sets is given by

$$\{K \in \kappa_0(X), x \notin K\} \cup \kappa_0(Y') \cup \{K \cup |Y'|, x \in K \in \kappa_0(X)\},$$

where Y' is a copy of Y such that $|X| \cap |Y'| = \emptyset$.

2. We can replace simultaneously every sink of G_X by (a copy of) G_Y to produce a space denoted by $X \rhd Y$. That is, $X \rhd Y$ is the connectivity space such that $|X \rhd Y| = |X| \times |Y|$ and the set $\kappa_0(X \rhd Y)$ of irreducible connected sets is given by

$$\kappa_0(X \rhd Y) = \{\{x\} \times L, x \in |X|, L \in \kappa_0(Y)\} \cup \{K \times |Y|, K \in \kappa_0(X)\}.$$

Example 20. $\mathbf{B}_2 \rhd_x \mathbf{V}_n \simeq \mathbf{V}_{n+1}$, where x is any of the two points of $\mathbf{B}_2$.

Proposition 19. *For any non-empty finite integral connectivity space X and any non-empty irreducible finite integral connectivity space Y, one has $\omega(X \rhd Y) = \omega(X) + \omega(Y)$.*

Proof. By construction, $G_{X \rhd Y}$ is obtained by replacing each sink of G_X by a copy of G_Y, so its length is $\omega(X) + \omega(Y)$.

□

Example 21. The link depicted on figure 1 is a Borromean assembly of three Borromean links. Its generic graph is (isomorphic to) $\mathbf{B}_3 \rhd \mathbf{B}_3$, and its connectivity order is 2.

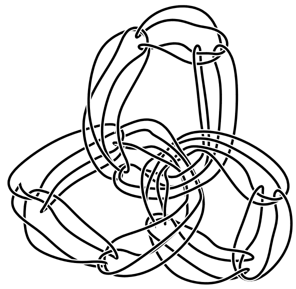

Figure 1: A Borromean ring of borromean rings.

7.2 Representation by Links

In [7, 6], I asked whether every finite connectivity space can be represented by a link, *i.e.* whether there exists a link whose connectivity structure is (isomorphic to) the one given. It turns out that in 1892, Brunn [3] first asked this question, without clearly bringing out the notion of a connectivity space. His answer was positive, and he gave the idea of a proof based on a construction using some of the links now called “Brunnian”. In 1964, Debrunner [5], rejecting Brunn’s “proof”, gave another construction, but proving it only for n-dimensional links with $n \geq 2$. In 1985, Kanenobu [11, 12] seems to be the first to give a proof of the possibility of representing every finite connectivity structure by a classical link, a result which is still little known at this date. The key idea of those different constructions is already in Brunn’s original article; it consists in using some Brunnian structures to successively link the sets of components which are desired to become unsplittable.

Thus already from Brunn’s point of view, the links we now call “Brunnian links” are not so interesting in and of themselves, but rather because they allow one to construct *all* finite connectivity structures from links.

Theorem 20 (Brunn-Debrunner-Kanenobu)**.** *Every finite connectivity structure is the splittability structure of at least one link in* $\mathbf{R}^3$.

Remark 11. Note that the structure of the links used by Brunn is well described by the so-called *Brunnian groups* constituted by the *Brun-*

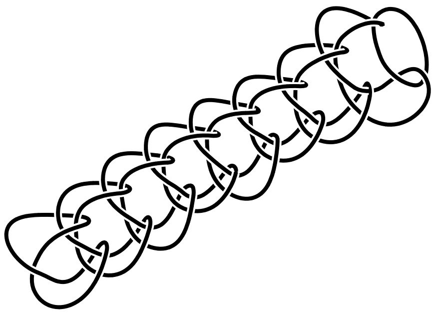

Figure 2: A link with a connectivity order 8.

nian braids introduced as *decomposable braids* by Levinson [13, 14] (see also [16] and [15]) and by the *Brunnian words* studied by Gartside and Greenwood [8].

Example 22. The structure of the connectivity space $\mathbf{V}_9$ with 9 points and maximal connectivity order 8 is the splittability structure of the link depicted on figure 2.

Acknowledgments. Thanks to Jean Bénabou, who introduced me to categories with lattices of structures, to Sergei Soloviev, who asked me about smash products of connectivity spaces, to David C. Ullrich who, on the forum `sci.math`, gave the upper bound ω_0 for the construction I give in the theorem 3. To René Guitart, Mark Weber, Albert Burroni and Quentin Donner for various talks and to Christopher-David Booth, Behrouz Roumizadeh and Anne Richards who helped me to correct my English.

References

[1] Jiri Adamek, Horst Herrlich, and George Strecker. *Abstract and Concrete Categories: The Joy of Cats.* John Wiley and Sons, 1990.

On-line edition, 18th January 2005 : http://katmat.math.uni-bremen.de/acc.

[2] Reinhard Börger. Connectivity spaces and component categories. In *Categorical topology, International Conference on Categorical Topology (1983)*, Berlin, 1984. Heldermann.

[3] Hermann Brunn. Ueber verkettung. *Sitzungsberichte der Bayerische Akad. Wiss., MathPhys. Klasse*, 22:77–99, 1892.

[4] Richard H. Crowell and Ralph H. Fox. *Introduction to knot theory.* Boston : Ginn and Company, 1963.

[5] Hans Debrunner. Über den Zerfall von Verkettungen. *Mathematische Zeitschrift*, 85:154–168, 1964. http://www.digizeitschriften.de.

[6] Stéphane Dugowson. Representation of finite connectivity space. http://arxiv.org/abs/0707.2542v1, 2007.

[7] Stéphane Dugowson. Les frontières dialectiques. *Mathematics and Social Sciences*, 177:87–152, 2007.

[8] Paul Gartside and Sina Greenwood. Brunnian links. *Fundam. Math.*, 193(3):259–276, 2007.

[9] Fritz von Haeseler and Heinz-Otto Peitgen. Newton's method and complex dynamical systems. In *Newton's method and dynamical systems*, pages 3–58. Kluwer Academic Publishers, 1989. Original article published in : *Acta Applicandae Mathematicae: An International Survey Journal on Applying Mathematics and Mathematical Applications*, 13(1):3–58, 1988.

[10] Horst Herrlich. Categorical topology 1971-1981. In J. Novák, editor, *General Topology and its Relations to Modern Analysis and Algebra V, Proceedings of the Fifth Prague Topological Symposium 1981*, pages 279 – 383, Berlin, 1983. Heldermann Verlag.

[11] Taizo Kanenobu. Satellite links with Brunnian properties. *Arch. Math.*, 44(4):369–372, 1985.

[12] Taizo Kanenobu. Hyperbolic links with Brunnian properties. *J. Math. Soc. Japan*, 38:295–308, 1986.

[13] H.W. Levinson. Decomposable braids and linkages. *Transactions of the American Mathematical Society*, 178:111–126, 1973.

[14] H.W. Levinson. Decomposable braids as subgroups of braid groups. *Transactions of the American Mathematical Society*, 202:51–55, 1975.

[15] Jingyan Li and Jie Wu. Boundary brunnian braids, mirror reflection and the homotopy groups. http://www.math.nus.edu.sg/ matwujie/2-5November-Korea.pdf, 2007.

[16] Theodore Stanford. Brunnian braids and some of their generalizations. http://aps.arxiv.org/abs/math.GT/9907072, 1999.

email : stephane.dugowson@supmeca.fr
Institut Supérieur de Mécanique de Paris,
3 Rue Fernand Hainaut
93400 Saint Ouen
France